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On the least common multiple of random q-integers

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Abstract

For every positive integer n and for every $\alpha \in [0, 1]$, let $\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $\mathcal{A} \subseteq \{1, \ldots, n\}$ is constructed by picking independently each element of $\{1, \ldots, n\}$ with probability α. Cilleruelo, Rué, Šarka, and Zumalacárregui proved an almost sure asymptotic formula for the logarithm of the least common multiple of the elements of $\mathcal{A}$.Let q be an indeterminate and let $[k]_q := 1 + q + q^2 + \cdots + q^{k-1} \in \mathbb{Z}[q]$ be the q-analog of the positive integer k. We determine the expected value and the variance of $X := \deg \operatorname{lcm}([\mathcal{A}]_q)$, where $[\mathcal{A}]_q := \{[k]_q : k \in \mathcal{A}\}$. Then we prove an almost sure asymptotic formula for X, which is a q-analog of the result of Cilleruelo et al.

Keywords: Asymptotic formula, Least common multiple, q-analog, Random set

Mathematics Subject Classification: Primary: 11N37, Secondary: 11B99

1 Introduction

A nice consequence of the Prime Number Theorem is the asymptotic formula

$$\log \operatorname{lcm}(1, 2, \ldots, n) \sim n, \quad \text{as } n \to +\infty, \tag{1}$$

where lcm denotes the least common multiple. Indeed, precise estimates for $\log \operatorname{lcm}(1, \ldots, n)$ are equivalent to the Prime Number Theorem with an error term. Thus, a natural generalization is to study estimates for $L_f(n) := \log \operatorname{lcm}(f(1), \ldots, f(n))$, where f is a well-behaved function, for instance, a polynomial with integer coefficients. (We ignore terms equal to 0 in the lcm and we set $\operatorname{lcm} \varnothing := 1$.) When $f \in \mathbb{Z}[x]$ is a linear polynomial, the product of linear polynomials, or an irreducible quadratic polynomial, asymptotic formulas for $L_f(n)$ were proved by Bateman et al. [3], Hong et al. [10], and Cilleruelo [6], respectively. In particular, for $f(x) = x^2 + 1$, Rué et al. [15] determined a precise error term for the asymptotic formula. When f is an irreducible polynomial of degree $d \geq 3$, Cilleruelo [6] conjectured that $L_f(n) \sim (d-1)\, n \log n$, as $n \to +\infty$, but this is still an open problem. However, bounds for $L_f(n)$ were proved by Maynard and Rudnick [13], and Sah [16]. Moreover, Rudnick and Zehavi [14] studied the growth of $L_f(n)$ along a shifted family of polynomials.

Another direction of research consists in considering the least common multiple of random sets of positive integers. For every positive integer n and every $\alpha \in [0, 1]$, let

$\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $\mathcal{A} \subseteq \{1, \dots, n\}$ is constructed by picking independently each element of $\{1, \dots, n\}$ with probability α. Cilleruelo et al. [9] studied the least common multiple of the elements of $\mathcal{A}$ and proved the following result (see [1] for a more precise version, and [4,5,7,8,12,17–19] for other results of a similar flavor).

Theorem 1.1 *Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$. Then, as $\alpha n \to +\infty$, we have*

$$\log \operatorname{lcm}(\mathcal{A}) \sim \frac{\alpha \log(1/\alpha)}{1-\alpha} \cdot n,$$

with probability $1 - o(1)$, where the factor involving α is meant to be equal to 1 for $\alpha = 1$.

Remark 1.1 In the deterministic case $\alpha = 1$, we have $\mathcal{A} = \{1, \dots, n\}$ (surely) and Theorem 1.1 corresponds to (1).

Let q be an indeterminate. The *q-analog* of a positive integer k is defined by

$$[k]_q := 1 + q + q^2 + \cdots + q^{k-1} \in \mathbb{Z}[q].$$

The q-analogs of many other mathematical objects (factorial, binomial coefficients, hypergeometric series, derivative, integral...) have been extensively studied, especially in Analysis and Combinatorics [2,11]. For every set $\mathcal{S}$ of positive integers, let $[\mathcal{S}]_q := \big\{[k]_q : k \in \mathcal{S}\big\}$.

The aim of this paper is to study the least common multiple of the elements of $[\mathcal{A}]_q$ for a random set $\mathcal{A}$ in $\mathcal{B}(n, \alpha)$. Our main results are the following:

Theorem 1.2 *Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$ and put $X := \deg \operatorname{lcm}\big([\mathcal{A}]_q\big)$. Then, for every integer $n \geq 2$ and every $\alpha \in [0, 1]$, we have*

$$\mathbb{E}[X] = \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2 + O\big(\alpha n (\log n)^2\big), \tag{2}$$

where $\operatorname{Li}_2(z) := \sum_{k=1}^{\infty} z^k/k^2$ is the dilogarithm and the factor involving α is meant to be equal to 1 when $\alpha = 1$. In particular,

$$\mathbb{E}[X] \sim \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2,$$

as $n \to +\infty$, uniformly for $\alpha \in [0, 1]$.

Theorem 1.3 *Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$ and put $X := \deg \operatorname{lcm}\big([\mathcal{A}]_q\big)$. Then there exists a function $\mathrm{v} : (0, 1) \to \mathbb{R}^+$ such that, as $\alpha n / \big((\log n)^3 (\log\log n)^2\big) \to +\infty$, we have*

$$\mathbb{V}[X] = (\mathrm{v}(\alpha) + o(1))\, n^3. \tag{3}$$

Moreover, the upper bound

$$\mathbb{V}[X] \ll \alpha n^3, \tag{4}$$

holds for every positive integer n and every $\alpha \in [0, 1]$.

As a consequence of Theorems 1.2 and 1.3, we obtain the following q-analog of Theorem 1.1.

Theorem 1.4 *Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$. Then, as $\alpha n \to +\infty$, we have*

$$\deg \operatorname{lcm}\big([\mathcal{A}]_q\big) \sim \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2,$$

with probability $1 - o(1)$, where the factor involving α is meant to be equal to 1 for $\alpha = 1$.

Remark 1.2 In the deterministic case $\alpha = 1$, we have (see Lemma 4.1 below)

$$\deg \operatorname{lcm}\big[\{1, 2, \ldots, n\}\big]_q = \sum_{1 < d \leq n} \varphi(d),$$

and Theorem 1.4 corresponds to the well-known asymptotic formula $\sum_{d \leq n} \varphi(d) \sim \frac{3}{\pi^2} n^2$ (Lemma 3.3 below) for the sum of the first values of the Euler function φ.

Remark 1.3 In Theorem 1.4 the condition $\alpha n \to +\infty$ is necessary. Indeed, if $\alpha n \leq C$, for some constant $C > 0$, then

$$\mathbb{P}[\mathcal{A} = \varnothing] = (1-\alpha)^n \geq \left(1 - \frac{C}{n}\right)^n \to \mathrm{e}^C$$

as $n \to +\infty$, and so no (nontrivial) asymptotic formula for $\deg \operatorname{lcm}\big([\mathcal{A}]_q\big)$ can hold with probability $1 - o(1)$.

We conclude this section with some possible questions for further research on this topic. Alsmeyer, Kabluchko, and Marynych [1, Corollary 1.5] proved that, for fixed $\alpha \in [0, 1]$ and for a random set $\mathcal{A}$ in $\mathcal{B}(n, \alpha)$, an appropriate normalization of the random variable $\log \operatorname{lcm}(\mathcal{A})$ converges in distribution to a standard normal random variable, as $n \to +\infty$. In light of Theorems 1.2 and 1.3, it is then natural to ask whether the random variable

$$\frac{\deg \operatorname{lcm}\big([\mathcal{A}]_q\big) - \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2}{\sqrt{\mathrm{v}(\alpha) n^3}}$$

converges in distribution to a normal random variable, or to some other random variable.

Another problem could be considering polynomial values, similarly to the results done in the context of integers, and studying $\operatorname{lcm}\big([f(1)]_q, \cdots, [f(n)]_q\big)$ for $f \in \mathbb{Z}[x]$ or, more generally, $\operatorname{lcm}\big([f(k)]_q : k \in \mathcal{A}\big)$ with $\mathcal{A}$ a random set in $\mathcal{B}(n, \alpha)$.

2 Notation

We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbol $\ll$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables X and Y, depending on some parameters, we say that "$X \sim Y$ with probability $1 - o(1)$", as the parameters tend to some limit, if for every $\varepsilon > 0$ we have $\mathbb{P}\big[\,|X - Y| > \varepsilon |Y|\,\big] = o_\varepsilon(1)$, as the parameters tend to the limit. We let (a, b) and $[a, b]$ denote the greatest common divisor and the least common multiple, respectively, of two integers a and b. As usual, we write $\varphi(n)$, $\mu(n)$, $\tau(n)$, and $\sigma(n)$, for the Euler totient function, the Möbius function, the number of divisors, and the sum of divisors, of a positive integer n, respectively.

3 Preliminaries

In this section we collect some preliminary results needed in later arguments.

Lemma 3.1 *We have*

$$\sum_{m \le x} \tau(m) \ll x \log x,$$

for every $x \ge 2$.

Proof See, e.g., [20, Part I, Theorem 3.2]. □

Lemma 3.2 *We have*

$$\sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} \ll \frac{\log x}{x},$$

for every $x \ge 2$.

Proof From Lemma 3.1 and partial summation, it follows that

$$\begin{aligned}\sum_{m > x} \frac{\tau(m)}{m^2} &= \left[\frac{\sum_{m \le t} \tau(m)}{t^2}\right]_{t=x}^{+\infty} + 2\int_x^{+\infty} \frac{\sum_{m \le t} \tau(m)}{t^3}\, \mathrm{d}t \\ &\ll \int_x^{+\infty} \frac{\log t}{t^2}\, \mathrm{d}t = \left[-\frac{\log t + 1}{t}\right]_{t=x}^{+\infty} \ll \frac{\log x}{x}.\end{aligned}$$

Let $e := (e_1, e_2)$ and $e_i' := e_i / e$ for $i = 1, 2$. Then we have

$$\begin{aligned}\sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} &\le \sum_{e \ge 1} \frac{1}{e^3} \sum_{e_1' e_2' > x/e} \frac{1}{(e_1' e_2')^2} = \sum_{e \ge 1} \frac{1}{e^3} \sum_{m > x/e} \frac{\tau(m)}{m^2} \\ &\ll \sum_{e \le x/2} \frac{1}{e^3} \frac{\log(x/e)}{x/e} + \sum_{e > x/2} \frac{1}{e^3} \ll \frac{\log x}{x} + \frac{1}{x^2} \ll \frac{\log x}{x},\end{aligned}$$

as desired. □

Let us define

$$\Phi(x) := \sum_{n \le x} \varphi(n) \quad \text{and} \quad \Phi(a_1, a_2; x) := \sum_{n \le x} \varphi(a_1 n)\, \varphi(a_2 n),$$

for every $x \ge 1$ and for all positive integers a_1, a_2.

Lemma 3.3 *We have*

$$\Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

for every $x \ge 2$.

Proof See, e.g., [20, Part I, Theorem 3.4]. □

Lemma 3.4 *We have*

$$\Phi(a_1, a_2; x) = C_1(a_1, a_2)\, x^3 + O\big(\sigma(a_1)\, \sigma(a_2)\, x^2 (\log x)^2\big), \tag{5}$$

for every $x \ge 2$, *where*

$$C_1(a_1, a_2) := \frac{a_1 a_2}{3} \sum_{d_1, d_2 \ge 1} \frac{\mu(d_1)\mu(d_2)}{d_1 d_2 \big[d_1/(a_1, d_1), d_2/(a_2, d_2)\big]} \tag{6}$$

and the series is absolutely convergent.

Proof From the identity $\varphi(n)/n = \sum_{d \mid n} \mu(d)/d$, it follows that

$$\begin{aligned}
\sum_{n \le x} \frac{\varphi(a_1 n)}{a_1 n} \frac{\varphi(a_2 n)}{a_2 n} &= \sum_{n \le x} \left(\sum_{d_1 \mid a_1 n} \frac{\mu(d_1)}{d_1} \sum_{d_2 \mid a_2 n} \frac{\mu(d_2)}{d_2} \right) \\
&= \sum_{\substack{d_1 \le a_1 x \\ d_2 \le a_2 x}} \frac{\mu(d_1)}{d_1} \frac{\mu(d_2)}{d_2} \#\big\{ n \le x : d_1 \mid a_1 n \text{ and } d_2 \mid a_2 n \big\} \\
&= \sum_{\left[\frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)}\right] \le x} \frac{\mu(d_1)}{d_1} \frac{\mu(d_2)}{d_2} \left(\frac{x}{\big[d_1/(a_1, d_1), d_2/(a_2, d_2)\big]} + O(1) \right).
\end{aligned}$$

Let $c_i := (a_i, d_i)$ and $e_i := d_i / c_i$, for $i = 1, 2$. On the one hand, we have

$$E_1 := \sum_{\left[\frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)}\right] \le x} \frac{1}{d_1 d_2} \le \sum_{c_1 \mid a_1} \frac{1}{c_1} \sum_{c_2 \mid a_2} \frac{1}{c_2} \sum_{e_1 \le x} \frac{1}{e_1} \sum_{e_2 \le x} \frac{1}{e_2} \ll \frac{\sigma(a_1)\, \sigma(a_2)}{a_1 a_2} (\log x)^2.$$

On the other hand, thanks to Lemma 3.2, we have

$$\begin{aligned}
E_2 &:= \sum_{\left[\frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)}\right] > x} \frac{1}{d_1 d_2 \big[d_1/(a_1, d_1), d_2/(a_2, d_2)\big]} \\
&\le \sum_{c_1 \mid a_1} \frac{1}{c_1} \sum_{c_2 \mid a_2} \frac{1}{c_2} \sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} \ll \frac{\sigma(a_1)\, \sigma(a_2)}{a_1 a_2} \frac{\log x}{x},
\end{aligned}$$

which, in particular, implies that the series

$$C_0(a_1, a_2) := \sum_{d_1, d_2 \ge 1} \frac{\mu(d_1)\mu(d_2)}{d_1 d_2 [d_1/(a_1, d_1), d_2/(a_2, d_2)]}$$

is absolutely convergent. Therefore, we obtain

$$\begin{aligned}
\sum_{n \le x} \frac{\varphi(a_1 n)}{a_1 n} \frac{\varphi(a_2 n)}{a_2 n} &= \big(C_0(a_1, a_2) + O(E_2)\big)\, x + O(E_1) \\
&= C_0(a_1, a_2)\, x + O\!\left(\frac{\sigma(a_1)\, \sigma(a_2)}{a_1 a_2} (\log x)^2 \right). \qquad (7)
\end{aligned}$$

Now (5) follows from (7) by partial summation and since $C_1(a_1, a_2) = \dfrac{a_1 a_2}{3} C_0(a_1, a_2)$. □

Remark 3.1 The obvious bound $\varphi(m) \le m$ yields $C_1(a_1, a_2) \le \dfrac{a_1 a_2}{3}$ (which is not so obvious from (6)).

We end this section with an easy observation that will be useful later.

Remark 3.2 It holds $1 - (1 - x)^k \le kx$, for all $x \in [0, 1]$ and for all integers $k \ge 0$.

4 Proofs

Henceforth, let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$, let $[\mathcal{A}]_q$ be its q-analog, and put $L := \operatorname{lcm}\big([\mathcal{A}]_q\big)$ and $X := \deg L$. For every positive integer d, let us define

$$I_{\mathcal{A}}(d) := \begin{cases} 1 & \text{if } d \mid k \text{ for some } k \in \mathcal{A}; \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma gives a formula for X in terms of $I_{\mathcal{A}}$ and the Euler function.

Lemma 4.1 *We have*

$$X = \sum_{1 < d \leq n} \varphi(d)\, I_{\mathcal{A}}(d). \tag{8}$$

Proof For every positive integer k, it holds

$$[k]_q = \frac{q^k - 1}{q - 1} = \prod_{\substack{d \mid k \\ d > 1}} \Phi_d(q),$$

where $\Phi_d(q)$ is the dth cyclotomic polynomials. Since, as it is well known, every cyclotomic polynomial is irreducible over $\mathbb{Q}$, it follows that L is the product of the polynomials $\Phi_d(q)$ such that $d > 1$ and $d \mid k$ for some $k \in \mathcal{A}$. Finally, the equality $\deg\big(\Phi_d(q)\big) = \varphi(d)$ and the definition of $I_{\mathcal{A}}$ yield (8). □

Let $\beta := 1 - \alpha$. The next lemma provides two expected values involving $I_{\mathcal{A}}$.

Lemma 4.2 *For all positive integers d, d_1, d_2, we have*

$$\mathbb{E}\big[I_{\mathcal{A}}(d)\big] = 1 - \beta^{\lfloor n/d \rfloor} \tag{9}$$

and

$$\mathbb{E}\big[I_{\mathcal{A}}(d_1) I_{\mathcal{A}}(d_2)\big] = 1 - \beta^{\lfloor n/d_1 \rfloor} - \beta^{\lfloor n/d_2 \rfloor} + \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor}.$$

Proof On the one hand, by the definition of $I_{\mathcal{A}}$, we have

$$\mathbb{E}\big[I_{\mathcal{A}}(d)\big] = \mathbb{P}\big[\exists k \in \mathcal{A} : d \mid k\big] = 1 - \mathbb{P}\left[\bigwedge_{m \leq \lfloor n/d \rfloor} (dm \notin \mathcal{A})\right] = 1 - \beta^{\lfloor n/d \rfloor},$$

which is (9). On the other hand, by linearity of the expectation and by (9), we have

$$\begin{aligned}
\mathbb{E}\big[I_{\mathcal{A}}(d_1) I_{\mathcal{A}}(d_2)\big] &= \mathbb{E}\big[I_{\mathcal{A}}(d_1) + I_{\mathcal{A}}(d_2) - 1 + \big(1 - I_{\mathcal{A}}(d_1)\big)\big(1 - I_{\mathcal{A}}(d_2)\big)\big] \\
&= \mathbb{E}\big[I_{\mathcal{A}}(d_1)\big] + \mathbb{E}\big[I_{\mathcal{A}}(d_2)\big] - 1 + \mathbb{E}\big[\big(1 - I_{\mathcal{A}}(d_1)\big)\big(1 - I_{\mathcal{A}}(d_2)\big)\big] \\
&= 1 - \beta^{\lfloor n/d_1 \rfloor} - \beta^{\lfloor n/d_2 \rfloor} + \mathbb{E}\big[\big(1 - I_{\mathcal{A}}(d_1)\big)\big(1 - I_{\mathcal{A}}(d_2)\big)\big],
\end{aligned}$$

where the last expected value can be computed as

$$\begin{aligned}
\mathbb{E}\big[\big(1 - I_{\mathcal{A}}(d_1)\big)\big(1 - I_{\mathcal{A}}(d_2)\big)\big] &= \mathbb{P}\big[\forall k \in \mathcal{A} : d_1 \nmid k \text{ and } d_2 \nmid k\big] \\
&= \mathbb{P}\left[\bigwedge_{\substack{k \leq n \\ d_1 \mid k \text{ or } d_2 \mid k}} (k \notin \mathcal{A})\right] = \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor},
\end{aligned}$$

and second claim follows. □

We are ready to compute the expected value of X.

Proof of Theorem 1.2 From Lemmas 4.1 and 4.2, it follows that

$$\mathbb{E}[X] = \sum_{1 < d \leq n} \varphi(d)\, \mathbb{E}\big[I_{\mathcal{A}}(d)\big] = \sum_{1 < d \leq n} \varphi(d)\big(1 - \beta^{\lfloor n/d \rfloor}\big). \tag{10}$$

Moreover, since $\lfloor n/d \rfloor = j$ if and only if $n/(j+1) < d \leq n/j$, we get that

$$\begin{aligned}\sum_{d \leq n} \varphi(d)\big(1 - \beta^{\lfloor n/d \rfloor}\big) &= \sum_{j \leq n} (1 - \beta^j) \sum_{n/(j+1) < d \leq n/j} \varphi(d) \\ &= \sum_{j \leq n} (1 - \beta^j)\left(\Phi\left(\frac{n}{j}\right) - \Phi\left(\frac{n}{j+1}\right)\right) \\ &= \alpha \sum_{j \leq n} \beta^{j-1} \Phi\left(\frac{n}{j}\right) \\ &= \frac{3}{\pi^2} \cdot \alpha \sum_{j \leq n} \frac{\beta^{j-1}}{j^2} \cdot n^2 + O\left(\alpha \sum_{j \leq n} \frac{n}{j} \log\left(\frac{n}{j}\right)\right) \\ &= \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2 + O\big(\alpha n (\log n)^2\big),\end{aligned} \tag{11}$$

where we used Lemma 3.3. Putting together (10) and (11), and noting that, by Remark 3.2, the addend of (11) corresponding to $d = 1$ is $1 - \beta^n = O(\alpha n)$, we get (2). The proof is complete. □

Now we consider the variance of X.

Proof of Theorem 1.3 From Lemmas 4.1 and 4.2, it follows that

$$\begin{aligned}\mathbb{V}[X] &= \mathbb{E}\big[X^2\big] - \mathbb{E}[X]^2 \\ &= \sum_{1 < d_1, d_2 \leq n} \varphi(d_1)\, \varphi(d_2) \Big(\mathbb{E}\big[I_{\mathcal{A}}(d_1)\, I_{\mathcal{A}}(d_2)\big] - \mathbb{E}\big[I_{\mathcal{A}}(d_1)\big]\, \mathbb{E}\big[I_{\mathcal{A}}(d_2)\big]\Big) \\ &= \sum_{1 < d_1, d_2 \leq n} \varphi(d_1)\, \varphi(d_2)\, \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor} \big(1 - \beta^{\lfloor n/[d_1, d_2] \rfloor}\big).\end{aligned} \tag{12}$$

Let us define

$$V_n(\alpha) := \frac{1}{n^3} \sum_{d_1, d_2 \leq n} \varphi(d_1)\, \varphi(d_2)\, \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor} \big(1 - \beta^{\lfloor n/[d_1, d_2] \rfloor}\big).$$

Clearly, we have

$$V_n(\alpha) - \frac{\mathbb{V}[X]}{n^3} \ll \frac{1}{n^3} \sum_{d \leq n} \varphi(d)\, \beta^n \big(1 - \beta^{\lfloor n/d \rfloor}\big) \leq \frac{1}{n^3} \sum_{d \leq n} d \ll \frac{1}{n}.$$

Hence, in order to prove (3), it suffices to show that $V_n(\alpha) = \mathrm{v}(\alpha) + o(1)$.

For all vectors $\boldsymbol{a} := (a_1, a_2)$ and $\boldsymbol{j} := (j_1, j_2, j_3)$ with components in the set of positive integers, define the quantities

$$\rho_1(\boldsymbol{a}, \boldsymbol{j}) := \max\left(\frac{1}{a_1(j_1+1)}, \frac{1}{a_2(j_2+1)}, \frac{1}{a_1 a_2 (j_3+1)}\right)$$

and

$$\rho_2(\boldsymbol{a}, \boldsymbol{j}) := \min\left(\frac{1}{a_1 j_1}, \frac{1}{a_2 j_2}, \frac{1}{a_1 a_2 j_3}\right).$$

Let $d := (d_1, d_2)$ and $a_i := d_i/d$ for $i = 1, 2$. Then the equalities

$$j_1 = \left\lfloor \frac{n}{d_1} \right\rfloor, \quad j_2 = \left\lfloor \frac{n}{d_2} \right\rfloor, \quad j_3 = \left\lfloor \frac{n}{[d_1, d_2]} \right\rfloor,$$

are equivalent to

$$j_1 \le \frac{n}{a_1 d} < j_1 + 1, \quad j_2 \le \frac{n}{a_2 d} < j_2 + 1, \quad j_3 \le \frac{n}{a_1 a_2 d} < j_3 + 1,$$

which in turn are equivalent to

$$\frac{n}{a_1(j_1+1)} < d \le \frac{n}{a_1 j_1}, \quad \frac{n}{a_2(j_2+1)} < d \le \frac{n}{a_2 j_2}, \quad \frac{n}{a_1 a_2 (j_3+1)} < d \le \frac{n}{a_1 a_2 j_3},$$

that is,

$$\rho_1(\boldsymbol{a}, \boldsymbol{j})\, n < d \le \rho_2(\boldsymbol{a}, \boldsymbol{j})\, n.$$

Therefore, letting

$$\mathcal{S}_n := \left\{(\boldsymbol{a}, \boldsymbol{j}) \in \mathbb{N}^5 : (a_1, a_2) = 1,\ \exists d \in \mathbb{N} \text{ s.t. } \rho_1(\boldsymbol{a}, \boldsymbol{j})\, n < d \le \rho_2(\boldsymbol{a}, \boldsymbol{j})\, n\right\}$$

and

$$S(\boldsymbol{a}, \boldsymbol{j}; n) := \frac{1}{n^3} \sum_{\rho_1(\boldsymbol{a}, \boldsymbol{j})\, n < d \le \rho_2(\boldsymbol{a}, \boldsymbol{j})\, n} \varphi(a_1 d)\, \varphi(a_2 d),$$

we have

$$V_n(\alpha) = \sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_n} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3})\, S(\boldsymbol{a}, \boldsymbol{j}; n).$$

Now let us define

$$\mathrm{v}(\alpha) := \sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_\infty} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3})\, D(\boldsymbol{a}, \boldsymbol{j}), \tag{13}$$

where

$$\mathcal{S}_\infty := \bigcup_{m \ge 1} \mathcal{S}_m = \left\{(\boldsymbol{a}, \boldsymbol{j}) \in \mathbb{N}^5 : (a_1, a_2) = 1,\ \rho_1(\boldsymbol{a}, \boldsymbol{j}) < \rho_2(\boldsymbol{a}, \boldsymbol{j})\right\}$$

and

$$D(\boldsymbol{a}, \boldsymbol{j}) := C_1(a_1, a_2)\big(\rho_2(\boldsymbol{a}, \boldsymbol{j})^3 - \rho_1(\boldsymbol{a}, \boldsymbol{j})^3\big),$$

recalling that $C_1(a_1, a_2)$ is defined by (6). The convergence of series (13) follows easily from Remark 3.1, $\rho_2(\boldsymbol{a}, \boldsymbol{j}) \le 1/(a_1 a_2 j_3)$, and the fact that $\min(j_1, j_2) \ge j_3$ for all $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_\infty$.

Thanks to Lemma 3.4, for each $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_n$ we have

$$S(\boldsymbol{a}, \boldsymbol{j}; n) = D(\boldsymbol{a}, \boldsymbol{j}) + O\left(\sigma(a_1)\, \sigma(a_2)\, \rho_2(\boldsymbol{a}, \boldsymbol{j})^2 \cdot \frac{(\log n)^2}{n}\right).$$

Consequently, we get that

$$V_n(\alpha) = \mathrm{v}(\alpha) - \Sigma_1 + O\left(\Sigma_2 \cdot \frac{(\log n)^2}{n}\right), \tag{14}$$

where

$$\Sigma_1 := \sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_\infty \setminus \mathcal{S}_n} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3})\, D(\boldsymbol{a}, \boldsymbol{j})$$

and

$$\Sigma_2 := \sum_{(\boldsymbol{a},\boldsymbol{j}) \in \mathcal{S}_n} \beta^{j_1+j_2-j_3}(1-\beta^{j_3})\,\sigma(a_1)\,\sigma(a_2)\,\rho_2(\boldsymbol{a},\boldsymbol{j})^2.$$

Now we have to bound both Σ_1 and Σ_2.

If $(\boldsymbol{a},\boldsymbol{j}) \in \mathcal{S}_\infty \setminus \mathcal{S}_n$ then $\big(\rho_2(\boldsymbol{a},\boldsymbol{j}) - \rho_1(\boldsymbol{a},\boldsymbol{j})\big)n < 1$ and consequently, also by Remark 3.1,

$$\begin{aligned} D(\boldsymbol{a},\boldsymbol{j}) &\ll a_1a_2\big(\rho_2^3-\rho_1^3\big) = a_1a_2\big(\rho_1^2+\rho_1\rho_2+\rho_2^2\big)(\rho_2-\rho_1) \ll \frac{a_1a_2\rho_2^2}{n} \\ &\le \frac{1}{a_1a_2j_3^2n}, \end{aligned} \tag{15}$$

where, for brevity, we wrote $\rho_i := \rho_i(\boldsymbol{a},\boldsymbol{j})$ for $i = 1, 2$.

If $(\boldsymbol{a},\boldsymbol{j}) \in \mathcal{S}_\infty$ then, as we already noticed, $\min(j_1, j_2) \ge j_3$ and, moreover,

$$\frac{j_2}{j_3+1} < a_1 < \frac{j_2+1}{j_3} \quad \text{and} \quad \frac{j_1}{j_3+1} < a_2 < \frac{j_1+1}{j_3}.$$

Hence, we have

$$\begin{aligned} \sum_{(\boldsymbol{a},\boldsymbol{j}) \in \mathcal{S}_\infty} \frac{\beta^{j_1+j_2-j_3}(1-\beta^{j_3})}{a_1a_2j_3^2} &\le \sum_{j_3 \ge 1} \frac{1-\beta^{j_3}}{j_3^2} \sum_{j_1, j_2 \ge j_3} \beta^{j_1+j_2-j_3} \sum_{\substack{j_2/(j_3+1) < a_1 < (j_2+1)/j_3 \\ j_1/(j_3+1) < a_2 < (j_1+1)/j_3}} \frac{1}{a_1a_2} \\ &\ll \sum_{j_3 \ge 1} \frac{1-\beta^{j_3}}{j_3^2} \sum_{j_1, j_2 \ge j_3} \beta^{j_1+j_2-j_3} = \frac{1}{\alpha^2} \sum_{j \ge 1} \frac{(1-\beta^j)\beta^j}{j^2} \\ &\le \frac{1}{\alpha} \sum_{j \le 1/\alpha} \frac{1}{j} + \frac{1}{\alpha^2} \sum_{j > 1/\alpha} \frac{1}{j^2} \ll \frac{\log(1/\alpha)+1}{\alpha}, \end{aligned} \tag{16}$$

where we used the inequality $1 - \beta^j \le \alpha j$, which follows from Remark 3.2.

On the one hand, from (15) and (16) it follows that

$$\Sigma_1 \ll \frac{\log(1/\alpha)+1}{\alpha n} = o(1), \tag{17}$$

as $\alpha n/\big((\log n)^3(\log\log n)^2\big) \to +\infty$ (actually, $\alpha n/\log n \to +\infty$ is sufficient).

On the other hand, from $\rho_2(\boldsymbol{a},\boldsymbol{j}) \le 1/(a_1a_2j_3)$, (16), and the bound $\sigma(m) \ll m \log\log m$ (see, e.g., [20, Part I, Theorem 5.7]) it follows that

$$\begin{aligned} \Sigma_2 &\le \sum_{(\boldsymbol{a},\boldsymbol{j}) \in \mathcal{S}_n} \frac{\beta^{j_1+j_2-j_3}(1-\beta^{j_3})}{a_1a_2j_3^2} \cdot \frac{\sigma(a_1)\,\sigma(a_2)}{a_1a_2} \ll \frac{(\log(1/\alpha)+1)(\log\log n)^2}{\alpha} \\ &= o\left(\frac{n}{(\log n)^2}\right), \end{aligned} \tag{18}$$

as $\alpha n/\big((\log n)^3(\log\log n)^2\big) \to +\infty$.

At this point, putting together (14), (17), and (18), we obtain $V_n(\alpha) = \mathrm{v}(\alpha) + o(1)$, as desired. The proof of (3) is complete.

It remains only to prove the upper bound (4). From (12) it follows that

$$\begin{aligned} \mathbb{V}[X] &\le \sum_{[d_1, d_2] \le n} \varphi(d_1)\,\varphi(d_2)\,\beta^{\lfloor n/d_1\rfloor + \lfloor n/d_2\rfloor - \lfloor n/[d_1,d_2]\rfloor}\big(1-\beta^{\lfloor n/[d_1,d_2]\rfloor}\big) \\ &\le \sum_{[d_1, d_2] \le n} d_1d_2 \cdot \frac{\alpha n}{[d_1,d_2]} = \alpha n \sum_{[d_1, d_2] \le n} (d_1,d_2) \le \alpha n \sum_{d \le n} d \sum_{a_1a_2 \le n/d} 1 \\ &= \alpha n \sum_{d \le n} d \sum_{m \le n/d} \tau(m) \ll \alpha n^2 \sum_{d \le n} \log\left(\frac{n}{d}\right) = \alpha n^2\big(n\log n - \log(n!)\big) < \alpha n^3, \end{aligned}$$

where we used Remark 3.2, Lemma 3.1, and the bound $n! > (n/\mathrm{e})^n$. Thus (4) is proved. □

Proof of Theorem 1.4 By Chebyshev's inequality, Theorems 1.2 and 1.3, we have

$$\mathbb{P}\big[\, |X - \mathbb{E}[X]| > \varepsilon\, \mathbb{E}[X] \,\big] \leq \frac{\mathbb{V}[X]}{\big(\varepsilon \mathbb{E}[X]\big)^2} \ll \frac{\alpha n^3}{(\varepsilon \alpha n)^2} = \frac{1}{\varepsilon^2 \alpha n} = o_\varepsilon(1),$$

as $\alpha n \to +\infty$. Hence, using again Theorem 1.2, we get

$$X \sim \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2,$$

with probability $1 - o(1)$, as $\alpha n \to +\infty$. □

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