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RESEARCH

# On the least common multiple of random $q$-integers 

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#### Abstract

For every positive integer $n$ and for every $\alpha \in[0,1]$, let $\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $\mathcal{A} \subseteq\{1, \ldots, n\}$ is constructed by picking independently each element of $\{1, \ldots, n\}$ with probability $\alpha$. Cilleruelo, Rué, Šarka, and Zumalacárregui proved an almost sure asymptotic formula for the logarithm of the least common multiple of the elements of $\mathcal{A}$.Let $q$ be an indeterminate and let $[k]_{q}:=1+q+q^{2}+\cdots$ $+q^{k-1} \in \mathbb{Z}[q]$ be the $q$-analog of the positive integer $k$. We determine the expected value and the variance of $X:=\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right)$, where $[\mathcal{A}]_{q}:=\left\{[k]_{q}: k \in \mathcal{A}\right\}$. Then we prove an almost sure asymptotic formula for $X$, which is a $q$-analog of the result of Cilleruelo et al.


Keywords: Asymptotic formula, Least common multiple, $q$-analog, Random set
Mathematics Subject Classification: Primary: 11N37, Secondary: 11B99

## 1 Introduction

A nice consequence of the Prime Number Theorem is the asymptotic formula

$$
\begin{equation*}
\log \operatorname{lcm}(1,2, \ldots, n) \sim n, \quad \text { as } n \rightarrow+\infty, \tag{1}
\end{equation*}
$$

where 1 cm denotes the least common multiple. Indeed, precise estimates for $\log \operatorname{lcm}(1, \ldots$, $n)$ are equivalent to the Prime Number Theorem with an error term. Thus, a natural generalization is to study estimates for $L_{f}(n):=\log \operatorname{lcm}(f(1), \ldots, f(n))$, where $f$ is a wellbehaved function, for instance, a polynomial with integer coefficients. (We ignore terms equal to 0 in the lcm and we set $\operatorname{lcm} \varnothing:=1$.) When $f \in \mathbb{Z}[x]$ is a linear polynomial, the product of linear polynomials, or an irreducible quadratic polynomial, asymptotic formulas for $L_{f}(n)$ were proved by Bateman et al. [3], Hong et al. [10], and Cilleruelo [6], respectively. In particular, for $f(x)=x^{2}+1$, Rué et al. [15] determined a precise error term for the asymptotic formula. When $f$ is an irreducible polynomial of degree $d \geq 3$, Cilleruelo [6] conjectured that $L_{f}(n) \sim(d-1) n \log n$, as $n \rightarrow+\infty$, but this is still an open problem. However, bounds for $L_{f}(n)$ were proved by Maynard and Rudnick [13], and Sah [16]. Moreover, Rudnick and Zehavi [14] studied the growth of $L_{f}(n)$ along a shifted family of polynomials.
Another direction of research consists in considering the least common multiple of random sets of positive integers. For every positive integer $n$ and every $\alpha \in[0,1]$, let
$\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $\mathcal{A} \subseteq\{1, \ldots, n\}$ is constructed by picking independently each element of $\{1, \ldots, n\}$ with probability $\alpha$. Cilleruelo et al. [9] studied the least common multiple of the elements of $\mathcal{A}$ and proved the following result (see [1] for a more precise version, and [4,5,7,8,12,17-19] for other results of a similar flavor).

Theorem 1.1 Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$. Then, as $\alpha n \rightarrow+\infty$, we have

$$
\log \operatorname{lcm}(\mathcal{A}) \sim \frac{\alpha \log (1 / \alpha)}{1-\alpha} \cdot n,
$$

with probability $1-o(1)$, where the factor involving $\alpha$ is meant to be equal to 1 for $\alpha=1$.
Remark 1.1 In the deterministic case $\alpha=1$, we have $\mathcal{A}=\{1, \ldots, n\}$ (surely) and Theorem 1.1 corresponds to (1).

Let $q$ be an indeterminate. The $q$-analog of a positive integer $k$ is defined by

$$
[k]_{q}:=1+q+q^{2}+\cdots+q^{k-1} \in \mathbb{Z}[q] .
$$

The $q$-analogs of many other mathematical objects (factorial, binomial coefficients, hypergeometric series, derivative, integral...) have been extensively studied, especially in Analysis and Combinatorics $[2,11]$. For every set $\mathcal{S}$ of positive integers, let $[\mathcal{S}]_{q}:=\left\{[k]_{q}: k \in\right.$ $\mathcal{S}\}$.
The aim of this paper is to study the least common multiple of the elements of $[\mathcal{A}]_{q}$ for a random set $\mathcal{A}$ in $\mathcal{B}(n, \alpha)$. Our main results are the following:

Theorem 1.2 Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$ and put $X:=\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right)$. Then, for every integer $n \geq 2$ and every $\alpha \in[0,1]$, we have

$$
\begin{equation*}
\mathbb{E}[X]=\frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}+O\left(\alpha n(\log n)^{2}\right) \tag{2}
\end{equation*}
$$

where $\operatorname{Li}_{2}(z):=\sum_{k=1}^{\infty} z^{k} / k^{2}$ is the dilogarithm and the factor involving $\alpha$ is meant to be equal to 1 when $\alpha=1$. In particular,

$$
\mathbb{E}[X] \sim \frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}
$$

as $n \rightarrow+\infty$, uniformly for $\alpha \in[0,1]$.
Theorem 1.3 Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$ and put $X:=\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right)$. Then there exists a function $\mathrm{v}:(0,1) \rightarrow \mathbb{R}^{+}$such that, as $\alpha n /\left((\log n)^{3}(\log \log n)^{2}\right) \rightarrow+\infty$, we have

$$
\begin{equation*}
\mathbb{V}[X]=(\mathrm{v}(\alpha)+o(1)) n^{3} . \tag{3}
\end{equation*}
$$

Moreover, the upper bound

$$
\begin{equation*}
\mathbb{V}[X] \ll \alpha n^{3}, \tag{4}
\end{equation*}
$$

holds for every positive integer $n$ and every $\alpha \in[0,1]$.
As a consequence of Theorems 1.2 and 1.3 , we obtain the following $q$-analog of Theorem 1.1.

Theorem 1.4 Let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$. Then, as $\alpha n \rightarrow+\infty$, we have

$$
\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right) \sim \frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}
$$

with probability $1-o(1)$, where the factor involving $\alpha$ is meant to be equal to 1 for $\alpha=1$.
Remark 1.2 In the deterministic case $\alpha=1$, we have (see Lemma 4.1 below)

$$
\operatorname{deg} \operatorname{lcm}[\{1,2, \ldots, n\}]_{q}=\sum_{1<d \leq n} \varphi(d)
$$

and Theorem 1.4 corresponds to the well-known asymptotic formula $\sum_{d \leq n} \varphi(d) \sim \frac{3}{\pi^{2}} n^{2}$ (Lemma 3.3 below) for the sum of the first values of the Euler function $\varphi$.

Remark 1.3 In Theorem 1.4 the condition $\alpha n \rightarrow+\infty$ is necessary. Indeed, if $\alpha n \leq C$, for some constant $C>0$, then

$$
\mathbb{P}[\mathcal{A}=\varnothing]=(1-\alpha)^{n} \geq\left(1-\frac{C}{n}\right)^{n} \rightarrow \mathrm{e}^{C}
$$

as $n \rightarrow+\infty$, and so no (nontrivial) asymptotic formula for $\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right)$ can hold with probability $1-o(1)$.

We conclude this section with some possible questions for further research on this topic. Alsmeyer, Kabluchko, and Marynych [1, Corollary 1.5] proved that, for fixed $\alpha \in[0,1]$ and for a random set $\mathcal{A}$ in $\mathcal{B}(n, \alpha)$, an appropriate normalization of the random variable $\log \operatorname{lcm}(\mathcal{A})$ converges in distribution to a standard normal random variable, as $n \rightarrow+\infty$. In light of Theorems 1.2 and 1.3, it is then natural to ask whether the random variable

$$
\frac{\operatorname{deg} \operatorname{lcm}\left([\mathcal{A}]_{q}\right)-\frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}}{\sqrt{\mathrm{v}(\alpha) n^{3}}}
$$

converges in distribution to a normal random variable, or to some other random variable.
Another problem could be considering polynomial values, similarly to the results done in the context of integers, and studying $\operatorname{lcm}\left([f(1)]_{q}, \cdots,[f(n)]_{q}\right)$ for $f \in \mathbb{Z}[x]$ or, more generally, $\operatorname{lcm}\left([f(k)]_{q}: k \in \mathcal{A}\right)$ with $\mathcal{A}$ a random set in $\mathcal{B}(n, \alpha)$.

## 2 Notation

We employ the Landau-Bachmann "Big Oh" and "little oh" notations $O$ and $o$, as well as the associated Vinogradov symbol <<, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables $X$ and $Y$, depending on some parameters, we say that " $X \sim Y$ with probability $1-o(1) "$, as the parameters tend to some limit, if for every $\varepsilon>0$ we have $\mathbb{P}[|X-Y|>$ $\varepsilon|Y|]=o_{\varepsilon}(1)$, as the parameters tend to the limit. We let $(a, b)$ and $[a, b]$ denote the greatest common divisor and the least common multiple, respectively, of two integers $a$ and $b$. As usual, we write $\varphi(n), \mu(n), \tau(n)$, and $\sigma(n)$, for the Euler totient function, the Möbius function, the number of divisors, and the sum of divisors, of a positive integer $n$, respectively.

## 3 Preliminaries

In this section we collect some preliminary results needed in later arguments.

Lemma 3.1 We have

$$
\sum_{m \leq x} \tau(m) \ll x \log x
$$

for every $x \geq 2$.
Proof See, e.g., [20, Part I, Theorem 3.2].
Lemma 3.2 We have

$$
\sum_{\left[e_{1}, e_{2}\right]>x} \frac{1}{e_{1} e_{2}\left[e_{1}, e_{2}\right]} \ll \frac{\log x}{x}
$$

for every $x \geq 2$.
Proof From Lemma 3.1 and partial summation, it follows that

$$
\begin{aligned}
\sum_{m>x} \frac{\tau(m)}{m^{2}} & =\left[\frac{\sum_{m \leq t} \tau(m)}{t^{2}}\right]_{t=x}^{+\infty}+2 \int_{x}^{+\infty} \frac{\sum_{m \leq t} \tau(m)}{t^{3}} \mathrm{~d} t \\
& \ll \int_{x}^{+\infty} \frac{\log t}{t^{2}} \mathrm{~d} t=\left[-\frac{\log t+1}{t}\right]_{t=x}^{+\infty} \ll \frac{\log x}{x}
\end{aligned}
$$

Let $e:=\left(e_{1}, e_{2}\right)$ and $e_{i}^{\prime}:=e_{i} / e$ for $i=1,2$. Then we have

$$
\begin{aligned}
\sum_{\left[e_{1}, e_{2}\right]>x} \frac{1}{e_{1} e_{2}\left[e_{1}, e_{2}\right]} & \leq \sum_{e \geq 1} \frac{1}{e^{3}} \sum_{e_{1}^{\prime} e_{2}^{\prime}>x / e} \frac{1}{\left(e_{1}^{\prime} e_{2}^{\prime}\right)^{2}}=\sum_{e \geq 1} \frac{1}{e^{3}} \sum_{m>x / e} \frac{\tau(m)}{m^{2}} \\
& \ll \sum_{e \leq x / 2} \frac{1}{e^{3}} \frac{\log (x / e)}{x / e}+\sum_{e>x / 2} \frac{1}{e^{3}} \ll \frac{\log x}{x}+\frac{1}{x^{2}} \ll \frac{\log x}{x}
\end{aligned}
$$

as desired.
Let us define

$$
\Phi(x):=\sum_{n \leq x} \varphi(n) \quad \text { and } \quad \Phi\left(a_{1}, a_{2} ; x\right):=\sum_{n \leq x} \varphi\left(a_{1} n\right) \varphi\left(a_{2} n\right)
$$

for every $x \geq 1$ and for all positive integers $a_{1}, a_{2}$.
Lemma 3.3 We have

$$
\Phi(x)=\frac{3}{\pi^{2}} x^{2}+O(x \log x)
$$

for every $x \geq 2$.
Proof See, e.g., [20, Part I, Theorem 3.4].
Lemma 3.4 We have

$$
\begin{equation*}
\Phi\left(a_{1}, a_{2} ; x\right)=C_{1}\left(a_{1}, a_{2}\right) x^{3}+O\left(\sigma\left(a_{1}\right) \sigma\left(a_{2}\right) x^{2}(\log x)^{2}\right) \tag{5}
\end{equation*}
$$

for every $x \geq 2$, where

$$
\begin{equation*}
C_{1}\left(a_{1}, a_{2}\right):=\frac{a_{1} a_{2}}{3} \sum_{d_{1} d_{2} \geq 1} \frac{\mu\left(d_{1}\right) \mu\left(d_{2}\right)}{d_{1} d_{2}\left[d_{1} /\left(a_{1}, d_{1}\right), d_{2} /\left(a_{2}, d_{2}\right)\right]} \tag{6}
\end{equation*}
$$

and the series is absolutely convergent.

Proof From the identity $\varphi(n) / n=\sum_{d \mid n} \mu(d) / d$, it follows that

$$
\begin{aligned}
\sum_{n \leq x} \frac{\varphi\left(a_{1} n\right)}{a_{1} n} \frac{\varphi\left(a_{2} n\right)}{a_{2} n} & =\sum_{n \leq x}\left(\sum_{d_{1} \mid a_{1} n} \frac{\mu\left(d_{1}\right)}{d_{1}} \sum_{d_{2} \mid a_{2} n} \frac{\mu\left(d_{2}\right)}{d_{2}}\right) \\
& =\sum_{\substack{d_{1} \leq a_{1} x \\
d_{2} \leq a_{2} x}} \frac{\mu\left(d_{1}\right)}{d_{1}} \frac{\mu\left(d_{2}\right)}{d_{2}} \#\left\{n \leq x: d_{1} \mid a_{1} n \text { and } d_{2} \mid a_{2} n\right\} \\
& =\sum_{\left[\frac{d_{1}}{\left(a_{1}, d_{1}\right)}, \frac{d_{2}}{\left(a_{2}, d_{2}\right)}\right] \leq x} \frac{\mu\left(d_{1}\right)}{d_{1}} \frac{\mu\left(d_{2}\right)}{d_{2}}\left(\frac{x}{\left[d_{1} /\left(a_{1}, d_{1}\right), d_{2} /\left(a_{2}, d_{2}\right)\right]}+O(1)\right)
\end{aligned}
$$

Let $c_{i}:=\left(a_{i}, d_{i}\right)$ and $e_{i}:=d_{i} / c_{i}$, for $i=1,2$. On the one hand, we have

$$
E_{1}:=\sum_{\left[\frac{d_{1}}{\left(a_{1}, d_{1}\right.}, \frac{d_{2}}{\left(a_{2}, d_{2}\right)}\right] \leq x} \frac{1}{d_{1} d_{2}} \leq \sum_{c_{1} \mid a_{1}} \frac{1}{c_{1}} \sum_{c_{2} \mid a_{2}} \frac{1}{c_{2}} \sum_{e_{1} \leq x} \frac{1}{e_{1}} \sum_{e_{2} \leq x} \frac{1}{e_{2}} \ll \frac{\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)}{a_{1} a_{2}}(\log x)^{2} .
$$

On the other hand, thanks to Lemma 3.2, we have

$$
\begin{aligned}
E_{2} & :=\sum_{\left[\frac{d_{1}}{\left(a_{1} d_{1}\right)}, \frac{d_{2}}{\left(a_{2}, d_{2}\right)}\right]>x} \frac{1}{d_{1} d_{2}\left[d_{1} /\left(a_{1}, d_{1}\right), d_{2} /\left(a_{2}, d_{2}\right)\right]} \\
& \leq \sum_{c_{1} \mid a_{1}} \frac{1}{c_{1}} \sum_{c_{2} \mid a_{2}} \frac{1}{c_{2}} \sum_{\left[e_{1}, e_{2}\right]>x} \frac{1}{e_{1} e_{2}\left[e_{1}, e_{2}\right]} \ll \frac{\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)}{a_{1} a_{2}} \frac{\log x}{x},
\end{aligned}
$$

which, in particular, implies that the series

$$
C_{0}\left(a_{1}, a_{2}\right):=\sum_{d_{1}, d_{2} \geq 1} \frac{\mu\left(d_{1}\right) \mu\left(d_{2}\right)}{d_{1} d_{2}\left[d_{1} /\left(a_{1}, d_{1}\right), d_{2} /\left(a_{2}, d_{2}\right)\right]}
$$

is absolutely convergent. Therefore, we obtain

$$
\begin{align*}
\sum_{n \leq x} \frac{\varphi\left(a_{1} n\right)}{a_{1} n} \frac{\varphi\left(a_{2} n\right)}{a_{2} n} & =\left(C_{0}\left(a_{1}, a_{2}\right)+O\left(E_{2}\right)\right) x+O\left(E_{1}\right) \\
& =C_{0}\left(a_{1}, a_{2}\right) x+O\left(\frac{\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)}{a_{1} a_{2}}(\log x)^{2}\right) \tag{7}
\end{align*}
$$

Now (5) follows from (7) by partial summation and since $C_{1}\left(a_{1}, a_{2}\right)=\frac{a_{1} a_{2}}{3} C_{0}\left(a_{1}, a_{2}\right)$.
Remark 3.1 The obvious bound $\varphi(m) \leq m$ yields $C_{1}\left(a_{1}, a_{2}\right) \leq \frac{a_{1} a_{2}}{3}$ (which is not so obvious from (6)).

We end this section with an easy observation that will be useful later.
Remark 3.2 It holds $1-(1-x)^{k} \leq k x$, for all $x \in[0,1]$ and for all integers $k \geq 0$.

## 4 Proofs

Henceforth, let $\mathcal{A}$ be a random set in $\mathcal{B}(n, \alpha)$, let $[\mathcal{A}]_{q}$ be its $q$-analog, and put $L:=$ $\operatorname{lcm}\left([\mathcal{A}]_{q}\right)$ and $X:=\operatorname{deg} L$. For every positive integer $d$, let us define

$$
I_{\mathcal{A}}(d):= \begin{cases}1 & \text { if } d \mid k \text { for some } k \in \mathcal{A} \\ 0 & \text { otherwise }\end{cases}
$$

The following lemma gives a formula for $X$ in terms of $I_{\mathcal{A}}$ and the Euler function.
Lemma 4.1 We have

$$
\begin{equation*}
X=\sum_{1<d \leq n} \varphi(d) I_{\mathcal{A}}(d) \tag{8}
\end{equation*}
$$

Proof For every positive integer $k$, it holds

$$
[k]_{q}=\frac{q^{k}-1}{q-1}=\prod_{\substack{d \mid k \\ d>1}} \Phi_{d}(q)
$$

where $\Phi_{d}(q)$ is the $d$ th cyclotomic polynomials. Since, as it is well known, every cyclotomic polynomial is irreducible over $\mathbb{Q}$, it follows that $L$ is the product of the polynomials $\Phi_{d}(q)$ such that $d>1$ and $d \mid k$ for some $k \in \mathcal{A}$. Finally, the equality $\operatorname{deg}\left(\Phi_{d}(q)\right)=\varphi(d)$ and the definition of $I_{\mathcal{A}}$ yield (8).

Let $\beta:=1-\alpha$. The next lemma provides two expected values involving $I_{\mathcal{A}}$.
Lemma 4.2 For all positive integers $d, d_{1}, d_{2}$, we have

$$
\begin{equation*}
\mathbb{E}\left[I_{\mathcal{A}}(d)\right]=1-\beta^{\lfloor n / d\rfloor} \tag{9}
\end{equation*}
$$

and

$$
\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right) I_{\mathcal{A}}\left(d_{2}\right)\right]=1-\beta^{\left\lfloor n / d_{1}\right\rfloor}-\beta^{\left\lfloor n / d_{2}\right\rfloor}+\beta^{\left\lfloor n / d_{1}\right\rfloor+\left\lfloor n / d_{2}\right\rfloor-\left\lfloor n /\left[d_{1} d_{2}\right]\right\rfloor}
$$

Proof On the one hand, by the definition of $I_{\mathcal{A}}$, we have

$$
\mathbb{E}\left[I_{\mathcal{A}}(d)\right]=\mathbb{P}[\exists k \in \mathcal{A}: d \mid k]=1-\mathbb{P}\left[\bigwedge_{m \leq\lfloor n / d\rfloor}(d m \notin \mathcal{A})\right]=1-\beta^{\lfloor n / d\rfloor}
$$

which is (9). On the other hand, by linearity of the expectation and by (9), we have

$$
\begin{aligned}
\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right) I_{\mathcal{A}}\left(d_{2}\right)\right] & =\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right)+I_{\mathcal{A}}\left(d_{2}\right)-1+\left(1-I_{\mathcal{A}}\left(d_{1}\right)\right)\left(1-I_{\mathcal{A}}\left(d_{2}\right)\right)\right] \\
& =\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right)\right]+\mathbb{E}\left[I_{\mathcal{A}}\left(d_{2}\right)\right]-1+\mathbb{E}\left[\left(1-I_{\mathcal{A}}\left(d_{1}\right)\right)\left(1-I_{\mathcal{A}}\left(d_{2}\right)\right)\right] \\
& =1-\beta^{\left\lfloor n / d_{1}\right\rfloor}-\beta^{\left\lfloor n / d_{2}\right\rfloor}+\mathbb{E}\left[\left(1-I_{\mathcal{A}}\left(d_{1}\right)\right)\left(1-I_{\mathcal{A}}\left(d_{2}\right)\right)\right]
\end{aligned}
$$

where the last expected value can be computed as

$$
\begin{aligned}
\mathbb{E}\left[\left(1-I_{\mathcal{A}}\left(d_{1}\right)\right)\left(1-I_{\mathcal{A}}\left(d_{2}\right)\right)\right] & =\mathbb{P}\left[\forall k \in \mathcal{A}: d_{1} \nmid k \text { and } d_{2} \nmid k\right] \\
& =\mathbb{P}\left[\bigwedge_{\substack{k \leq n \\
d_{1} \mid k \text { or } d_{2} \mid k}}(k \notin \mathcal{A})\right]=\beta^{\left\lfloor n / d_{1}\right\rfloor+\left\lfloor n / d_{2}\right\rfloor-\left\lfloor n /\left[d_{b}, d_{2}\right]\right\rfloor},
\end{aligned}
$$

and second claim follows.

We are ready to compute the expected value of $X$.

Proof of Theorem 1.2 From Lemmas 4.1 and 4.2, it follows that

$$
\begin{equation*}
\mathbb{E}[X]=\sum_{1<d \leq n} \varphi(d) \mathbb{E}\left[I_{\mathcal{A}}(d)\right]=\sum_{1<d \leq n} \varphi(d)\left(1-\beta^{\lfloor n / d\rfloor}\right) . \tag{10}
\end{equation*}
$$

Moreover, since $\lfloor n / d\rfloor=j$ if and only if $n /(j+1)<d \leq n / j$, we get that

$$
\begin{align*}
\sum_{d \leq n} \varphi(d)\left(1-\beta^{\lfloor n / d\rfloor}\right) & =\sum_{j \leq n}\left(1-\beta^{j}\right) \sum_{n /(j+1)<d \leq n / j} \varphi(d) \\
& =\sum_{j \leq n}\left(1-\beta^{j}\right)\left(\Phi\left(\frac{n}{j}\right)-\Phi\left(\frac{n}{j+1}\right)\right) \\
& =\alpha \sum_{j \leq n} \beta^{j-1} \Phi\left(\frac{n}{j}\right) \\
& =\frac{3}{\pi^{2}} \cdot \alpha \sum_{j \leq n} \frac{\beta^{j-1}}{j^{2}} \cdot n^{2}+O\left(\alpha \sum_{j \leq n} \frac{n}{j} \log \left(\frac{n}{j}\right)\right) \\
& =\frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}+O\left(\alpha n(\log n)^{2}\right), \tag{11}
\end{align*}
$$

where we used Lemma 3.3. Putting together (10) and (11), and noting that, by Remark 3.2, the addend of (11) corresponding to $d=1$ is $1-\beta^{n}=O(\alpha n)$, we get (2). The proof is complete.

Now we consider the variance of $X$.
Proof of Theorem 1.3 From Lemmas 4.1 and 4.2, it follows that

$$
\begin{align*}
\mathbb{V}[X] & =\mathbb{E}\left[X^{2}\right]-\mathbb{E}[X]^{2} \\
& =\sum_{1<d_{1}, d_{2} \leq n} \varphi\left(d_{1}\right) \varphi\left(d_{2}\right)\left(\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right) I_{\mathcal{A}}\left(d_{2}\right)\right]-\mathbb{E}\left[I_{\mathcal{A}}\left(d_{1}\right)\right] \mathbb{E}\left[I_{\mathcal{A}}\left(d_{2}\right)\right]\right) \\
& =\sum_{1<d_{\mathrm{J}}, d_{2} \leq n} \varphi\left(d_{1}\right) \varphi\left(d_{2}\right) \beta^{\left\lfloor n / d_{1}\right\rfloor+\left\lfloor n / d_{2}\right\rfloor-\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\left(1-\beta^{\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\right) . \tag{12}
\end{align*}
$$

Let us define

$$
V_{n}(\alpha):=\frac{1}{n^{3}} \sum_{d_{\mathrm{b}} d_{2} \leq n} \varphi\left(d_{1}\right) \varphi\left(d_{2}\right) \beta^{\left\lfloor n / d_{1}\right\rfloor+\left\lfloor n / d_{2}\right\rfloor-\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\left(1-\beta^{\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\right)
$$

Clearly, we have

$$
V_{n}(\alpha)-\frac{\mathbb{V}[X]}{n^{3}} \ll \frac{1}{n^{3}} \sum_{d \leq n} \varphi(d) \beta^{n}\left(1-\beta^{\lfloor n / d\rfloor}\right) \leq \frac{1}{n^{3}} \sum_{d \leq n} d \ll \frac{1}{n} .
$$

Hence, in order to prove (3), it suffices to show that $V_{n}(\alpha)=\mathrm{v}(\alpha)+o(1)$.
For all vectors $\boldsymbol{a}:=\left(a_{1}, a_{2}\right)$ and $\boldsymbol{j}:=\left(j_{1}, j_{2}, j_{3}\right)$ with components in the set of positive integers, define the quantities

$$
\rho_{1}(\boldsymbol{a}, \boldsymbol{j}):=\max \left(\frac{1}{a_{1}\left(j_{1}+1\right)}, \frac{1}{a_{2}\left(j_{2}+1\right)}, \frac{1}{a_{1} a_{2}\left(j_{3}+1\right)}\right)
$$

and

$$
\rho_{2}(\boldsymbol{a}, \boldsymbol{j}):=\min \left(\frac{1}{a_{1} j_{1}}, \frac{1}{a_{2} j_{2}}, \frac{1}{a_{1} a_{2} j_{3}}\right) .
$$

Let $d:=\left(d_{1}, d_{2}\right)$ and $a_{i}:=d_{i} / d$ for $i=1,2$. Then the equalities

$$
j_{1}=\left\lfloor\frac{n}{d_{1}}\right\rfloor, \quad j_{2}=\left\lfloor\frac{n}{d_{2}}\right\rfloor, \quad j_{3}=\left\lfloor\frac{n}{\left[d_{1}, d_{2}\right]}\right\rfloor
$$

are equivalent to

$$
j_{1} \leq \frac{n}{a_{1} d}<j_{1}+1, \quad j_{2} \leq \frac{n}{a_{2} d}<j_{2}+1, \quad j_{3} \leq \frac{n}{a_{1} a_{2} d}<j_{3}+1
$$

which in turn are equivalent to

$$
\frac{n}{a_{1}\left(j_{1}+1\right)}<d \leq \frac{n}{a_{1} j_{1}}, \quad \frac{n}{a_{2}\left(j_{2}+1\right)}<d \leq \frac{n}{a_{2} j_{2}}, \quad \frac{n}{a_{1} a_{2}\left(j_{3}+1\right)}<d \leq \frac{n}{a_{1} a_{2} j_{3}}
$$

that is,

$$
\rho_{1}(\boldsymbol{a}, \boldsymbol{j}) n<d \leq \rho_{2}(\boldsymbol{a}, \boldsymbol{j}) n
$$

Therefore, letting

$$
\mathcal{S}_{n}:=\left\{(\boldsymbol{a}, \boldsymbol{j}) \in \mathbb{N}^{5}:\left(a_{1}, a_{2}\right)=1, \exists d \in \mathbb{N} \text { s.t. } \rho_{1}(\boldsymbol{a}, \boldsymbol{j}) n<d \leq \rho_{2}(\boldsymbol{a}, \boldsymbol{j}) n\right\}
$$

and

$$
S(\boldsymbol{a}, \boldsymbol{j} ; n):=\frac{1}{n^{3}} \sum_{\rho_{1}(\boldsymbol{a}, \boldsymbol{j}) n<d \leq \rho_{2}(\boldsymbol{a}, \boldsymbol{j}) n} \varphi\left(a_{1} d\right) \varphi\left(a_{2} d\right)
$$

we have

$$
V_{n}(\alpha)=\sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{n}} \beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right) S(\boldsymbol{a}, \boldsymbol{j} ; n)
$$

Now let us define

$$
\begin{equation*}
\mathrm{v}(\alpha):=\sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{\infty}} \beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right) D(\boldsymbol{a}, \boldsymbol{j}) \tag{13}
\end{equation*}
$$

where

$$
\mathcal{S}_{\infty}:=\bigcup_{m \geq 1} \mathcal{S}_{m}=\left\{(\boldsymbol{a}, \boldsymbol{j}) \in \mathbb{N}^{5}:\left(a_{1}, a_{2}\right)=1, \rho_{1}(\boldsymbol{a}, \boldsymbol{j})<\rho_{2}(\boldsymbol{a}, \boldsymbol{j})\right\}
$$

and

$$
D(\boldsymbol{a}, \boldsymbol{j}):=C_{1}\left(a_{1}, a_{2}\right)\left(\rho_{2}(\boldsymbol{a}, \boldsymbol{j})^{3}-\rho_{1}(\boldsymbol{a}, \boldsymbol{j})^{3}\right)
$$

recalling that $C_{1}\left(a_{1}, a_{2}\right)$ is defined by (6). The convergence of series (13) follows easily from Remark 3.1, $\rho_{2}(\boldsymbol{a}, \boldsymbol{j}) \leq 1 /\left(a_{1} a_{2} j_{3}\right)$, and the fact that $\min \left(j_{1}, j_{2}\right) \geq j_{3}$ for all $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{\infty}$.

Thanks to Lemma 3.4, for each $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{n}$ we have

$$
S(\boldsymbol{a}, \boldsymbol{j} ; n)=D(\boldsymbol{a}, \boldsymbol{j})+O\left(\sigma\left(a_{1}\right) \sigma\left(a_{2}\right) \rho_{2}(\boldsymbol{a}, \boldsymbol{j})^{2} \cdot \frac{(\log n)^{2}}{n}\right)
$$

Consequently, we get that

$$
\begin{equation*}
V_{n}(\alpha)=\mathrm{v}(\alpha)-\Sigma_{1}+O\left(\Sigma_{2} \cdot \frac{(\log n)^{2}}{n}\right) \tag{14}
\end{equation*}
$$

where

$$
\Sigma_{1}:=\sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{\infty} \backslash \mathcal{S}_{n}} \beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right) D(\boldsymbol{a}, \boldsymbol{j})
$$

and

$$
\Sigma_{2}:=\sum_{(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{n}} \beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right) \sigma\left(a_{1}\right) \sigma\left(a_{2}\right) \rho_{2}(\boldsymbol{a}, \boldsymbol{j})^{2}
$$

Now we have to bound both $\Sigma_{1}$ and $\Sigma_{2}$.
If $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{\infty} \backslash \mathcal{S}_{n}$ then $\left(\rho_{2}(\boldsymbol{a}, \boldsymbol{j})-\rho_{1}(\boldsymbol{a}, \boldsymbol{j})\right) n<1$ and consequently, also by Remark 3.1,

$$
\begin{align*}
D(\boldsymbol{a}, \boldsymbol{j}) \ll a_{1} a_{2}\left(\rho_{2}^{3}-\rho_{1}^{3}\right) & =a_{1} a_{2}\left(\rho_{1}^{2}+\rho_{1} \rho_{2}+\rho_{2}^{2}\right)\left(\rho_{2}-\rho_{1}\right) \ll \frac{a_{1} a_{2} \rho_{2}^{2}}{n} \\
& \leq \frac{1}{a_{1} a_{2} j_{3}^{2} n}, \tag{15}
\end{align*}
$$

where, for brevity, we wrote $\rho_{i}:=\rho_{i}(\boldsymbol{a}, \boldsymbol{j})$ for $i=1,2$.
If $(\boldsymbol{a}, \boldsymbol{j}) \in \mathcal{S}_{\infty}$ then, as we already noticed, $\min \left(j_{1}, j_{2}\right) \geq j_{3}$ and, moreover,

$$
\frac{j_{2}}{j_{3}+1}<a_{1}<\frac{j_{2}+1}{j_{3}} \quad \text { and } \quad \frac{j_{1}}{j_{3}+1}<a_{2}<\frac{j_{1}+1}{j_{3}}
$$

Hence, we have

$$
\begin{align*}
\sum_{(a, j) \in \mathcal{S}_{\infty}} \frac{\beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right)}{a_{1} a_{2} j_{3}^{2}} & \leq \sum_{j_{3} \geq 1} \frac{1-\beta^{j_{3}}}{j_{3}^{2}} \sum_{j_{1}, j_{2} \geq j_{3}} \beta^{j_{1}+j_{2}-j_{3}} \sum_{\substack{j_{2} /\left(j_{3}+1\right)<a_{1}<\left(j_{2}+1\right) / j_{3} \\
j_{1} /\left(j_{3}+1\right)<a_{2}<\left(j_{1}+1\right) / j_{3}}} \frac{1}{a_{1} a_{2}} \\
& \ll \sum_{j_{3} \geq 1} \frac{1-\beta^{j_{3}}}{j_{3}^{2}} \sum_{j_{1}, j_{2} \geq j_{3}} \beta^{j_{1}+j_{2}-j_{3}}=\frac{1}{\alpha^{2}} \sum_{j \geq 1} \frac{\left(1-\beta^{j}\right) \beta^{j}}{j^{2}} \\
& \leq \frac{1}{\alpha} \sum_{j \leq 1 / \alpha} \frac{1}{j}+\frac{1}{\alpha^{2}} \sum_{j>1 / \alpha} \frac{1}{j^{2}} \ll \frac{\log (1 / \alpha)+1}{\alpha} \tag{16}
\end{align*}
$$

where we used the inequality $1-\beta^{j} \leq \alpha j$, which follows from Remark 3.2.
On the one hand, from (15) and (16) it follows that

$$
\begin{equation*}
\Sigma_{1} \ll \frac{\log (1 / \alpha)+1}{\alpha n}=o(1), \tag{17}
\end{equation*}
$$

as $\alpha n /\left((\log n)^{3}(\log \log n)^{2}\right) \rightarrow+\infty$ (actually, $\alpha n / \log n \rightarrow+\infty$ is sufficient).
On the other hand, from $\rho_{2}(\boldsymbol{a}, \boldsymbol{j}) \leq 1 /\left(a_{1} a_{2} j_{3}\right)$, (16), and the bound $\sigma(m) \ll m \log \log m$ (see, e.g., [20, Part I, Theorem 5.7]) it follows that

$$
\begin{align*}
\Sigma_{2} & \leq \sum_{(a, j) \in \mathcal{S}_{n}} \frac{\beta^{j_{1}+j_{2}-j_{3}}\left(1-\beta^{j_{3}}\right)}{a_{1} a_{2} j_{3}^{2}} \cdot \frac{\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)}{a_{1} a_{2}} \ll \frac{(\log (1 / \alpha)+1)(\log \log n)^{2}}{\alpha} \\
& =o\left(\frac{n}{(\log n)^{2}}\right) \tag{18}
\end{align*}
$$

as $\alpha n /\left((\log n)^{3}(\log \log n)^{2}\right) \rightarrow+\infty$.
At this point, putting together (14), (17), and (18), we obtain $V_{n}(\alpha)=\mathrm{v}(\alpha)+o(1)$, as desired. The proof of (3) is complete.

It remains only to prove the upper bound (4). From (12) it follows that

$$
\begin{aligned}
\mathbb{V}[X] & \leq \sum_{\left[d_{1}, d_{2}\right] \leq n} \varphi\left(d_{1}\right) \varphi\left(d_{2}\right) \beta^{\left\lfloor n / d_{1}\right\rfloor+\left\lfloor n / d_{2}\right\rfloor-\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\left(1-\beta^{\left\lfloor n /\left[d_{1}, d_{2}\right]\right\rfloor}\right) \\
& \leq \sum_{\left[d_{\mathrm{l}}, d_{2}\right] \leq n} d_{1} d_{2} \cdot \frac{\alpha n}{\left[d_{1}, d_{2}\right]}=\alpha n \sum_{\left[d_{1}, d_{2}\right] \leq n}\left(d_{1}, d_{2}\right) \leq \alpha n \sum_{d \leq n} d \sum_{a_{1} a_{2} \leq n / d} 1 \\
& =\alpha n \sum_{d \leq n} d \sum_{m \leq n / d} \tau(m) \ll \alpha n^{2} \sum_{d \leq n} \log \left(\frac{n}{d}\right)=\alpha n^{2}(n \log n-\log (n!))<\alpha n^{3},
\end{aligned}
$$

where we used Remark 3.2, Lemma 3.1, and the bound $n!>(n / \mathrm{e})^{n}$. Thus (4) is proved.

## Proof of Theorem 1.4 By Chebyshev's inequality, Theorems 1.2 and 1.3, we have

$$
\mathbb{P}[|X-\mathbb{E}[X]|>\varepsilon \mathbb{E}[X]] \leq \frac{\mathbb{V}[X]}{(\varepsilon \mathbb{E}[X])^{2}} \ll \frac{\alpha n^{3}}{(\varepsilon \alpha n)^{2}}=\frac{1}{\varepsilon^{2} \alpha n}=o_{\varepsilon}(1)
$$

as $\alpha n \rightarrow+\infty$. Hence, using again Theorem 1.2, we get

$$
X \sim \frac{3}{\pi^{2}} \cdot \frac{\alpha \operatorname{Li}_{2}(1-\alpha)}{1-\alpha} \cdot n^{2}
$$

with probability $1-o(1)$, as $\alpha n \rightarrow+\infty$.

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