Poissonian Distributions in Physics: Counting Electrons and Photons

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Abstract: Here we discuss physics containing Poissonian distributions. Among the phenomena, we find thermoionic emission of electrons, photodetection and Poisson noise. After a description of the Hanbury Brown - Twiss effect, we will consider the second order correlation of intensity and Poissonian, super- and sub-Poissonian lights. The quadrature and squeezed states will be also considered.

Keywords: Optics, Quantum Optics, Photons, Coherent Photons, Hanbury Brown - Twiss effect, Poissonian Light, Super-Poissonian Light, Sub-Poissonian light, Squeezed States.

Introduction

Poisson distribution is a distribution used in probability theory, statistics, and physics. It is a discrete distribution that gives the probability of a number of events. Poisson distribution is used as a model to have the number of times an event occurs in an interval of time or space.

A discrete random variable is said to have a Poisson distribution with parameter $\lambda > 0$, for $n = 0, 1, 2, ...$, if its probability mass function is given by:

$$f(n;\lambda)=\frac{\lambda^n e^{-\lambda}}{n!}$$

where e is Euler's number, n is the number of occurrences, and $n!$ is the factorial of n. The positive real number λ is equal to the expected value of n and also to its variance. The Poisson distribution can be applied to systems having a large number of possible events, each of which is rare. The number of such events that occur during a fixed time interval is, under the right circumstances, a random number with a Poisson distribution.

The Poisson distribution may be useful to model events such as the radioactive decays [1] for instance. Also the number of laser photons hitting a detector in a particular time interval is described by a Poisson distribution.

The distribution is based on the following assumptions: 1) an event is described by integers *n;* 2) occurrence of an event does not affect the probability that a second event will occur, and this means that events are occurring independently; 3) the average rate at which events occur is independent of any occurrences.

If these conditions are true, then *n* is a Poisson random variable, and the distribution of *n* is Poissonian. The Poisson distribution is also the limit of a binomial distribution (see Appendix), for which the probability of success for each trial equals λ divided by the number of trials, as the number of trials approaches infinity.

Here we discuss the presence of Poissonian distribution in the physics of electrons and photons, in particular in their counting.

A calculus on the thermoionic emission of electrons

In [2], we can find the following problem.

During thermoionic emission, electrons leave the surface of a metal or a semiconductor. Assuming that 1) emissions of electrons are statistically independent event and 2) the probability of emission of an electron in a small time interval dt is equal to $\lambda\, dt$ (λ being a constant), determine the probability of emissions of electrons in a time interval *t*.

$P_n(t)$ = *probability of emission of electrons in time t*

$P_0(t)$ = *probability of no emission in the same time t*

Assuming 1), let us use the rule of the probability for two consecutive events. There are two ways of having *n* events in the time from 0 to $t+\Delta t$. One way is that of having *n* events from 0 to t , and none from *t* to $t+\Delta t$. The other way is that of having $n-1$ events from 0 to t and one event from t to $t+\Delta t$. Then:

$$P_n(t+dt)=P_{n-1}(t)P_1+P_n(t)(1-P_1)$$

$$P_0(t+dt)=P_0(t)(1-P_1)$$

$P_1=\lambda\, dt$ *probability of emission of one electron in the time dt.*

After expansion:

$$P_n(t)+\frac{dP_n}{dt}dt=P_{n-1}P_1+P_n-P_nP_1$$

$$P_0(t)+\frac{dP_0}{dt}dt=-P_0P_1+P_0$$

Then:

$$\frac{dP_n}{dt}dt = P_{n-1}P_1 - P_n P_1 = P_1(P_{n-1} - P_n) = \lambda\, dt\,(P_{n-1} - P_n)$$

$$\frac{dP_0}{dt}dt = -\lambda\, dt\, P_0$$

Pose initial conditions: $P_n(0) = 1, n = 0$, $P_n(0) = 0$, $n \neq 0$. The solution is;

$$P_n(t) = \frac{(\lambda t)^n}{n!}\exp(-\lambda t)$$

In fact:

$$\frac{d P_n(t)}{dt} = \lambda n \frac{(\lambda t)^{n-1}}{n!}\exp(-\lambda t) - \lambda \frac{(\lambda t)^n}{n!}\exp(-\lambda t) \quad =$$

$$\lambda \frac{(\lambda t)^{n-1}}{(n-1)!}\exp(-\lambda t) - \lambda \frac{(\lambda t)^n}{n!}\exp(-\lambda t)$$

$$\frac{d P_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$$

Let us evaluate $\langle \Delta n^2 \rangle = \langle (n - \langle n \rangle)^2 \rangle$, if on the average n_0 electrons are emitted in a second. First, let us calculate $\langle n \rangle$.

$$\sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \lambda t \exp(-\lambda) \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

that is: $\displaystyle\sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \lambda t \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{(n)!}$

$$\sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \lambda t \exp(-\lambda t)\exp(\lambda t) = \lambda t$$

Actually: $\displaystyle\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \exp(\lambda t)$.

$$\langle \Delta n^2 \rangle = \sum_{n=0}^{\infty} n^2 P_n(t) - [\sum_{n=0}^{\infty} n P_n(t)]^2 \;=\; \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!} - \lambda^2 t^2$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{(\lambda t)^n e^{-\lambda t}}{n!} + \sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} - \lambda^2 t^2$$

$$= \sum_{n=2}^{\infty} n(n-1) \frac{(\lambda t)^n e^{-\lambda t}}{n!} + \lambda t - \lambda^2 t^2$$

$$\langle \Delta n^2 \rangle = \langle (n - \langle n \rangle)^2 \rangle = \lambda^2 t^2 \exp(-\lambda t) \exp(\lambda t) + \lambda t - \lambda^2 t^2 = \lambda t$$

We have that: $\lambda t = \sum_{n=0}^{\infty} n P_n(t) = \langle n \rangle = n_0 t$, so $\langle \Delta n^2 \rangle = n_0 t$.

Photodetection

In Ref.3 it is discussed the semi-classical theory of the photo-detection. Let us consider a photon-counting detector such as a photomultiplier illuminated by a faint light beam. Let us assume the light beam described by a classical electromagnetic wave of intensity *I*.

Let us suppose: 1) The probability of the emission of a photo-electron in a short time interval being proportional to *I*, to *A* the illuminated area, and to time interval Δt . 2) If Δt is sufficiently small, the probability of emitting two photo-electrons is negligible. 3) The events of photoemission which are registered in different time intervals are statistically independent of each other.

In [3], the approach to probability is the same as given before for the discussion of the thermoionic emission. Let us consider $P_1 = \xi I(t) dt$, where ξ is proportional to the illuminated area. The probability is given as:

$$P_n(t) = \frac{\left[\int_0^t \xi I(t') dt'\right]^n}{n!} \exp\left(-\int_0^t \xi I(t') dt'\right)$$

Let us suppose $I(t)$ constant, so that $\xi I(t) = C$. As we have previously seen:

$$\frac{dP_n(t)}{dt} + C P_n(t) = C P_{n-1}(t)$$

For $n = 0$, $P_{n-1}(t) = 0$.

$$\frac{dP_0(t)}{dt} = -C P_0(t) \quad ; \quad P_0(0) = 1$$

Multiplying by e^{Ct}, we have that:

$$e^{Ct}\frac{dP_n(t)}{dt}+e^{Ct}C\,P_n(t)=e^{Ct}C\,P_{n-1}(t)$$

For $n\geq 1$:

$$\frac{d}{dt}\left(e^{Ct}P_n(t)\right)=e^{Ct}C\,P_{n-1}(t)$$

$$P_n(t)=e^{-Ct}\int_0^t C\,e^{Ct'}P_{n-1}(t')\,dt'$$

The recurrence is the following:

$$P_1(t)=e^{-Ct}\int_0^t C\,e^{Ct'}P_0(t')\,dt'=(Ct)e^{-Ct}$$

$$P_2(t)=e^{-Ct}\int_0^t C\,e^{Ct'}P_1(t')\,dt'=\frac{(Ct)^2}{2!}e^{-Ct}$$

Then: $P_n(t)=\frac{(Ct)^n}{n!}e^{-Ct}$.

We have: $\langle n\rangle=\xi\,I\,T\equiv Ct$. Therefore

$$P_n(t)=\frac{\langle n\rangle}{n!}e^{-\langle n\rangle}$$

Let us stress that $I(t)$ must be a constant.

Poisson noise

Poisson (shot) noises are an important set of Poissonian models with a wide range of applications [4-7]. For an input random variable $X\geq 0$ the Poisson noise is described by the conditional probability mass function PMF of the output random variable *Y* [4-7]:

$$P_{Y|X}(y|x)=\frac{1}{y!}(ax+\lambda)^y\exp-(ax+\lambda)\ ,\ x\geq 0,\ y=0,1,\ldots$$

where $a>0$ is a scaling factor and $\lambda\geq 0$ is a constant. Moreover, the convention $0^0=1$ is assumed. As explained by A, Dytso, at www.princeton.edu, this constant is called the dark current parameter. Conditioned by a non-negative input $X=x$, **,** the output of the Poisson channel is a non-negative integer-valued random variable *Y*, with a Poissonian distribution.

The random transformation of the input random variable X to an output random variable Y will be denoted by $Y = O(aX + \lambda)$. It is important to note that operator O is not linear. Using the language of lasers, the term aX represents the intensity of a laser beam and Y represents the number of photons that arrive at the receiver with a particle counter, that is a photodetector. The dark current parameter λ represents the intensity of an additional source of noise or interference, which produces extra photons at a particle counter.

A dark current is a relatively small electric current that is flowing in photosensitive devices (photomultiplier tubes, photodiodes, charge-coupled devices CCDs) even when no photons are entering them. The current is due to charges generated inside the detector, when outside radiation is absent. Dark current is originated by a random generation of electrons and holes within the depletion region of the device.

Oscillators and quanta

Let us consider a particle, described by canonical variable q, p , in a parabolic potential. Its Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

where $p = -i\hbar\partial_q$ and $[q, p] = i\hbar$. ω is the angular frequency.

The ground state width can be found by minimizing energy:

$$\langle H \rangle = \frac{\hbar^2}{2m\lambda} + \frac{m\omega^2\lambda^2}{2} \to min$$

which gives $\lambda = (\hbar/m\omega)^{1/2}$. It will be convenient to use dimensionless variables $\tilde{q}$, $\tilde{p}$. In this manner the classical phase volume is scaled by $\hbar$. Thus we obtain:

$$H = \frac{\hbar\omega}{2}(\tilde{p}^2 + \tilde{q}^2) \ , \text{ where } \ \tilde{p} = -i\partial_{\tilde{q}} \ \text{ and } \ [\tilde{q}, \tilde{p}] = i \ .$$

The canonical creation and annihilation operators are defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\tilde{q} + i\tilde{p}) \ , \ \hat{a}^+ = \frac{1}{\sqrt{2}}(\tilde{q} - i\tilde{p})$$

They can be used to express q, p and H as follows:

$$\hat{q} = \frac{\lambda}{\sqrt{2}}(\hat{a} + \hat{a}^+) \ , \ \hat{p} = i\frac{\hbar}{\sqrt{2}\lambda}(\hat{a}^+ - \hat{a}) \ , \ H = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$$

Operators $\hat{a}$ and $\hat{a}^+$ obey the commutation relation $[\hat{a}, \hat{a}^+] = 1$, as we can easily see. By means of operators $\hat{a}$ and $\hat{a}^+$, we can study the harmonic oscillator quantum mechanics. Let us just remember that the normalized oscillator eigenstates are:

$$|n>=\frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0>$$

These eigenstates are forming an orthonormal complete set of functions, providing a basis in the oscillator Hilbert space. The ground state $|0\rangle$ is also known as *the vacuum state*.

The operators $\hat{a}$ and $\hat{a}^{+}$ given as matrices in the basis of states have nonzero matrix elements only between the states $|n>$ and $|n\pm 1>$, and they are used to have the *number operator* $\hat{n}=\hat{a}^{+}\hat{a}$. This operator is counting the number of energy quanta. In the energy basis $|n\rangle$, the number operator is diagonal:

$$\hat{n}|n>=\hat{a}^{+}\hat{a}|n>=n|n> \quad , \quad H=\hbar\,\omega\left(\hat{n}+\frac{1}{2}\right)$$

Definition of coherent states

The coherent states are defined as eigenstates of operator $\hat{a}$: $\hat{a}|v>=v|v>$, where v is a complex parameter. That is, they are the eigenstates of the annihilation operator. Let us expand in the basis (Fock states):

$$|v>=\sum_{n=0}^{\infty} c_n|n>$$

where $|n>$ are energy (number) eigenvectors of the Hamiltonian (number and energy operators commute).

Properties of the states are:

$$<n|m>=\delta_{n,m} \quad ; \quad |n>=\frac{(\hat{a}^{+})^{n}}{\sqrt{n!}}|0>$$

$$\hat{a}^{+}|n>=\sqrt{n+1}|n+1> \quad ; \quad \hat{a}|n>=\sqrt{n}|n-1>$$

The coherent state can be reconstructed from recurrence:

$$\hat{a}|v>=\sum_{n=0}^{\infty} c_n\sqrt{n}|n-1>=\sum_{n=0}^{\infty} v\,c_n|n>$$

Comparing the coefficients, we obtain a recursion relation $c_n=(v/\sqrt{n})c_{n-1}$, leading to coefficients:

$$c_n=\frac{v^n}{\sqrt{n!}}c_0$$

The coefficient c_0 is determined from normalization:

$$1=\sum_{n=0}^{\infty}|c_n|^2=\sum_{n=0}^{\infty}\frac{|v|^{2n}}{n!}|c_0|^2=\exp(|v|^2)|c_0|^2$$

Finally: $|v>=\exp(-|v|^2/2)\sum_{n=0}^{\infty}\frac{v^n}{\sqrt{n!}}|n>$. We have also:

$$|v>=\exp(-|v|^2/2)\exp(v\hat{a}^+)\exp(-v^*\hat{a})|0>$$

Using the Baker–Campbell–Hausdorff formula, we can see that this is in agreement with the expression of the unitary displacement operator:

$$|v>=\exp(v\hat{a}^+-v^*\hat{a})|0>=D(v)|0>$$

As an example, consider the distribution of the number of quanta $\hat{n}=\hat{a}^+\hat{a}$ in a coherent state. Since $\hat{n}|n>=n|n>$, the distribution is given by

$$p_n=|c_n|^2=\frac{|v|^{2n}}{n!}e^{-|v|^2}$$

which is a Poisson distribution having $\langle n\rangle=|v|^2$. Photons emitted by lasers possess a Poissonian distribution.

As told previously, a coherent state is defined to be an eigenstate of the annihilation operator according to $\hat{a}|v>=v|v>$. Operator $\hat{a}$ is not hermitian, and v is a complex number. We can also write $v=|v|e^{i\theta}$, where $|v|$ and θ are the amplitude and phase of the state $|v>$.

Physically, $\hat{a}|v>=v|v>$ means that a coherent state remains unchanged by the annihilation of field excitation or, say, a particle. An eigenstate of the annihilation operator has a Poissonian number distribution when expressed in a basis of energy eigenstates, as shown below. A Poisson distribution is a necessary and sufficient condition that all detections are statistically independent.

Coherent states are obtained from the vacuum by application of a unitary displacement operator:

$$|v>=\exp(v\hat{a}^+-v*\hat{a})|0>=D(v)|0>$$

Let us consider also the uncertainty principle. In the case of an oscillator or optical field, we can use two dimensionless operators X, P. With these operators, the Hamiltonian of either system becomes

$$\hat{H}=\hbar\omega(P^2+X^2) \text{ with } [X,P]=\frac{i}{2}I$$

The quantum state of the harmonic oscillator that minimizes the uncertainty relation with uncertainty equally distributed between X and P satisfies the equation:

$$(X-<X>)>|v>=-i(P-<P>)|v>$$

or, equivalently,

$$(X+iP)|v>=<(X+iP)>|v>$$

Therefore:

$$<v|(X-<X>)^2+(P-<P>)^2|v>=1/2$$

Schrödinger found that the minimum uncertainty states for the linear harmonic oscillator are the eigenstates of $(X+iP)$, therefore are the eigenstates of $\hat{a}$, which is the operator we used to define the coherent state. The uncertainty is minimized, but this does not mean that we must have only equally balanced between X and P, . We can have also unbalanced states, which are called the squeezed coherent states. We will see them at the end of the discussion.

Photon counting: Poissonian, super-Poissonian, sub-Poissonian

In [3] we find the following approach to the Poissonian photon statistics.

Let us start from the probability $P(n)$ of finding n photons within a beam of length L containing N sub-segments. It is the probability of finding n sub-segments containing one photon and $(N-n)$ containing no photons, in any possible order. This probability is described by the binomial distribution:

$$P(n)=\frac{N!}{n!(N-n)!}p^n q^{N-n}$$

where $q=1-p$. Using $p=\langle n\rangle/N$, we have:

$$P(n)=\frac{N!}{n!(N-n)!}\left(\frac{\langle n\rangle}{N}\right)^n\left(1-\frac{\langle n\rangle}{N}\right)^{N-n}$$

By means of the approximation given in Appendix, we find again:

$$P(n)=\frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle} \quad , \quad n=0,1,2,...$$

For a coherent source (lasers), the probability to measure n photons when the mean number of photons is $\langle n\rangle$ can be given by the Poisson law. The standard deviation of the distribution is $\Delta n=\sqrt{\langle n\rangle}$.

For a chaotic light source the numbers of detected photons are distributed according to the probability:

$$P(n)=\frac{\langle n\rangle^n}{(\langle n\rangle+1)^{n+1}} .$$

The standard deviation is larger than that of the Poisson distribution (super Poisson distribution):

$$\Delta n=\sqrt{\langle n\rangle+(\langle n\rangle)^2} .$$

Then, a light governed by super-Poissonian statistics exhibits a statistical distribution with variance $(\Delta n)^2>\langle n\rangle$ [14]. A light that exhibits super-Poissonian statistics is thermal light.

Light can be also governed by sub-Poissonian statistics, but it cannot be described by classical electromagnetic theory. It is defined by $(\Delta n)^2<\langle n\rangle$ [3]. An example of light exhibiting sub-Poissonian statistics is the squeezed light.

Thermal light (super-Poissonian)

The electromagnetic radiation emitted by a thermal light is considered a black-body radiation [3]. It is governed by the laws of statistical mechanics concerning an enclosed cavity at a temperature *T*.

The energy density within the angular frequency range ω to ω + dω is given by Planck's law (*c* is the speed of light):

$$\rho(\omega,T)=\frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\omega/k_B T)-1}d\omega$$

The derivation of this equation is requiring the quantization of energy, so that:

$$E_n=(n+1/2)\hbar\omega \quad .$$

Let us consider a single radiation mode with frequency ω. The probability of having n photons in this mode is:

$$P_\omega(n)=\frac{\exp(-E_n/k_B T)}{\sum_{n=0}^{\infty}\exp(-E_n/k_B T)}$$

Substituting the value of energy:

$$P_\omega(n)=\frac{\exp(-n\hbar\omega/k_B T)}{\sum_{n=0}^{\infty}\exp(-n\hbar\omega/k_B T)}$$

Let us define $x=\exp(-n\hbar\omega/k_B T)$.

$$P_\omega(n)=\frac{x^n}{\sum_{n=0}^{\infty} x^n}$$

We have that [3]: $\sum_{i=1}^{k} r^{i-1} \equiv \sum_{j=0}^{k-1} r^j = \frac{1-r^k}{1-r}$ and then $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$, if $r<1$. As a consequence [3]:

$$P_\omega(n)=x^n(1-x)=(1-\exp(-n\hbar\omega/k_B T))\exp(-n\hbar\omega/k_B T)$$

We can calculate the mean number: $\sum_{n=0}^{\infty} n P_\omega(n)=\frac{x}{1-x}$ and then we obtain the Planck formula:

$$\langle n\rangle=\frac{1}{\exp(\hbar\omega/k_B T)-1}$$

However, $x=\frac{\langle n\rangle}{\langle n\rangle+1}$ so we have:

$$P_\omega(n)=\frac{1}{\langle n\rangle+1}\left(\frac{\langle n\rangle}{\langle n\rangle+1}\right)^n$$

The variance of the distribution is: $(\Delta n)^2=\langle n\rangle+\langle n\rangle^2$. It shows that the variance of the Bose–Einstein distribution is always larger than that of a Poisson distribution. The thermal light is a super-Poissonian light [3].

According to [3], the single-mode variance can be interpreted in an interesting way, in the case that we refer to Einstein's analysis of the energy fluctuations of back-body radiation, as originally given in 1909. Einstein realized that the first term of the variance has origin from the particle nature of the light. The second term originates from the thermal fluctuations of the energy of the electromagnetic radiation, being therefore of classical origin. It is called the wave noise [3].

HBT effect

The vast majority of light sources produce photons by means of random processes. Therefore, when these photons, for instance from a star, arrive at the Earth, they are randomly spaced. The number of photons counted in a short time interval will vary, even if the long-term mean number of photons is constant. This variation is known as shot (Poisson) noise. It represents the irreducible minimum level of noise present in an astronomical observation. However, some effects exist about the counting of photons in intensity interferometry, such as the Hanbury Brown - Twiss (HBT) effect.

HBT effect is including phenomena concerning a variety of correlation and anticorrelation effects in the intensities received by two detectors from a beam of particles. Devices which use the effect are commonly called intensity interferometers. Originally used in astronomy, these interferometers were applied in the field of quantum optics.

It was in 1954 that Robert Hanbury Brown and Richard Q. Twiss introduced the intensity interferometry. In 1956, they published results from an experimental set-up which was using a mercury vapor lamp. Later they applied the technique to measure the size of Sirius. Two photomultiplier tubes, separated by a few meters, were aimed at the star by means of telescopes. A correlation was observed between the two fluctuating intensities. Just as in the radio studies, the correlation dropped away as they increased the separation. They used this information to determine the apparent angular size of Sirius [8-11].

In [3], a classical description of the time-dependent intensity fluctuations in a light beam is given for the HBT effect. From this effect we have naturally the concept of the second-order correlation function, $G^{(2)}(\tau)$ that we can evaluate for different types of light. For this study, it is possible to use a semi-classical approach, in which the light is treated classically and quantum theory in introduced in the photodetection process.

The quantum description of HBT effect is given in the Nobel Lecture by Roy J. Glauber [12]. Again, we can find that he started from Poissonian distribution [13].

From Glauber's Nobel Lecture

Let us start from a real field and split it into two complex conjugate terms:

$$E = E^{(+)} + E^{(-)} \quad , \quad \left(E^{(+)}\right)^* = E^{(-)}$$

Let us define the correlation function:

$$G^{(1)}(\boldsymbol{r_1} t_1 \boldsymbol{r_2} t_2) = < E^{(-)}(\boldsymbol{r_1} t_1) E^{(+)}(\boldsymbol{r_2} t_2) >$$

In the two-pinhole Young experiment, light passing through a pinhole in the first screen falls on two closely spaced pinholes in a second screen. The superposition of the waves radiated by those pinholes at $\mathbf{r}_1$ and $\mathbf{r}_2$ leads to interference fringes seen at points $\mathbf{r}$ on the third screen.

Let us note that the average intensity of the field at a given point is then just:

$$G^{(1)}(\boldsymbol{r} t \boldsymbol{r} t) = < E^{(-)}(\boldsymbol{r} t) E^{(+)}(\boldsymbol{r} t) >$$

The Young experiment measures:

$$G^{(1)}(\boldsymbol{r_1} t_1 \boldsymbol{r_1} t_1) + G^{(1)}(\boldsymbol{r_2} t_2 \boldsymbol{r_2} t_2) + G^{(1)}(\boldsymbol{r_1} t_1 \boldsymbol{r_2} t_2) + G^{(1)}(\boldsymbol{r_2} t_2 \boldsymbol{r_1} t_1)$$

The first two terms are the separate contributions of the two pinholes, that is the intensities as they would contribute individually. These intensity distributions are supplemented by two other terms which are the interference effects.

Interference fringes have the greatest possible contrast and therefore the visibility when the cross correlation terms $G^{(1)}(\boldsymbol{r_1} t_1 \boldsymbol{r_2} t_2), G^{(1)}(\boldsymbol{r_2} t_2 \boldsymbol{r_1} t_1)$ are as large in magnitude as possible. A limitation imposed on the magnitude of such correlations by the Schwarz inequality. Let us indicate $x=(\boldsymbol{r}, t)$. The measure given above becomes:

$$G^{(1)}(x_1 x_1)+G^{(1)}(x_2 x_2)+G^{(1)}(x_1 x_2)+G^{(1)}(x_2 x_1)$$

with Schwarz inequality telling:

$$|G^{(1)}(x_1 x_2)|^2 \leq G^{(1)}(x_1 x_1) G^{(1)}(x_2 x_2)$$

The upper bound is attained if we have:

$$|G^{(1)}(x_1 x_2)|^2 = G^{(1)}(x_1 x_1) G^{(1)}(x_2 x_2)$$

and with it we achieve maximum fringe contrast. The fields at x_1 and x_2 are optically coherent with one another. This is the definition of relative coherence.

The ordinary (amplitude) interferometry measures $G^{(1)}(\boldsymbol{r_1} t_1 \boldsymbol{r_2} t_2)$, but intensity interferometry measures another correlation. Let us define a higher order coherence (e.g. second order):

$$G^{(2)}(\boldsymbol{r} t \boldsymbol{r}' t' \boldsymbol{r}' t' \boldsymbol{r} t) = < E^{(-)}(\boldsymbol{r} t) E^{(-)}(\boldsymbol{r}' t') E^{(+)}(\boldsymbol{r}' t') E^{(+)}(\boldsymbol{r} t) >$$

In the case of a factorization:

$$G^{(2)}(\boldsymbol{r} t \boldsymbol{r}' t' \boldsymbol{r}' t' \boldsymbol{r} t) = G^{(1)}(\boldsymbol{r} t \boldsymbol{r} t) G^{(1)}(\boldsymbol{r}' t' \boldsymbol{r}' t')$$

In this case, we have the product of two average intensities measured separately, but this is not observed in the Hanbury Brown - Twiss experiment. As told by Roy Glauber, ordinary light beams, that is, light from ordinary sources, even extremely monochromatic ones as in the Hanbury Brown - Twiss experiment, do not have any such second order coherence.

Degree of coherence

Correlation functions can be used to characterize the coherence properties of an electromagnetic field. The degree of coherence is given by the normalized correlation of electric fields, that is $G^{(1)}$. As we have previously seen, it is useful for the coherence as measured in a Michelson interferometer. The correlation between pairs of fields, $G^{(2)}$, is used to find the statistical character of intensity fluctuations. First order correlation is actually the amplitude-amplitude correlation and the second order correlation is the intensity-intensity correlation.

Photon bunching

The determination of the photon statistics of a light source is not simple to obtain. A manner to investigate the statistics is the measurement of the second order correlation function, as illustrated before. The correlation function is obtained from the probabilities that photons arrive in coincidence at two photon detectors as a function of the arrival time difference τ and can be defined also in the following manner [3,14]:

$$g^{(2)}(\tau)=\frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle} .$$

In the correlation function, we find the intensity I as a function of time. For coherent radiation, $g^{(2)}(\tau)=1$ for all τ *, F*or a thermal source, there is an augmentation of the coincidence rate when the coincidence (observation) time (τ) is smaller than the coherence time (τ_c). For all classical light sources $g^{(2)}(\tau=0)\geq 1$ and $g^{(2)}(0)>g^{(2)}(\tau)$ for all τ .

The augmentation can be interpreted by the fact that photons emitted by a chaotic source have the tendency to arrive in packets (bunches) to the detector whereas the photon emission by a laser is always regular. Bunched light is generated by chaotic light sources [15]. The clumping of photons can be observed if the observation time is smaller than the time of coherence ($\tau<\tau_c$). At longer time of exposure ($\tau>\tau_c$), the bunching of photons becomes negligible and the photons arrive regularly [15].

Second order correlation function

To explain HBT effect in classical theory, it is useful to introduce the intensity correlations [3]:

$$g^{(2)}(\tau)=\frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle}$$

where E is the electric field, or, as we have previously seen:

$$g^{(2)}(\tau)=\frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}$$

where $I(t)$ is the intensity of the light beam at time t. The brackets indicate the time average computed by integrating over a long time period.

Let us consider a constant average intensity such that $\langle I(t)\rangle = \langle I(t+\tau)\rangle$. The second-order correlation function investigates the temporal coherence of the source. As told in [3], the time-scale of the intensity fluctuations is determined by the coherence time τ_c of the source. If $\tau \gg \tau_c$, the intensity fluctuations at times t and $t+\tau$ will be completely uncorrelated with each other. Therefore:

$$\langle I(t) I(t+\tau)\rangle_{\tau \gg \tau_c} = \langle I\rangle^2$$

As a consequence $g^{(2)}(\tau \gg \tau_c) = 1$. If $\tau \ll \tau_c$, there will be correlations between the fluctuations at the two times. In particular, if $\tau = 0$, the chaotic light has $g^{(2)}(0) \geq 1$ [3]. It can also be shown that $g^{(2)}(0) \geq g^{(2)}(\tau)$. As a consequence we have a different behaviour of chaotic (thermal) and coherent light.

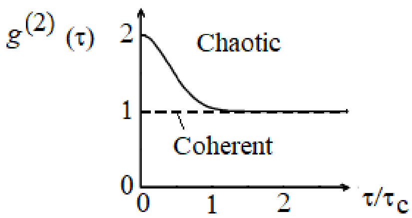

Fig.1 - Second-order correlation function $g^{(2)}(\tau)$ for chaotic and coherent light plotted on the same time-scale, as shown in [3].

We have seen previously the classification of light according to whether the statistics were sub-Poissonian, Poissonian, or super–Poissonian. We can also use a different classification according to the second-order correlation function $g^{(2)}(\tau)$, classification based on its value.

Bunched light: $g^{(2)}(0) > 1$; Coherent light: $g^{(2)}(0) = 1$; Antibunched light: $g^{(2)}(0) < 1$

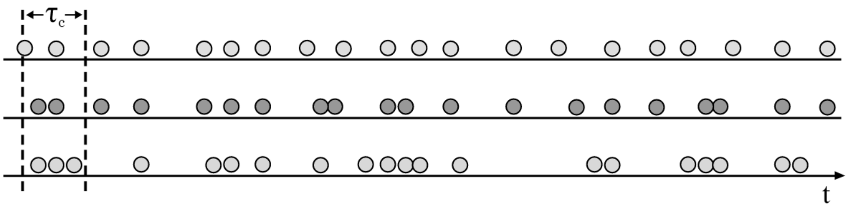

Fig.2 - Comparison of the photon detection for antibunched light, coherent light, and bunched light. For the case of coherent light, the Poissonian photon statistics correspond to random time intervals between the photons.(Image Courtesy Ajbura)

In antibunched light, the photons come out with regular gaps between them, rather than with a random spacing. This is illustrated schematically before. If the flow of photons is regular, then there will be long time intervals between observing photon counting events. Antibunched light has $g^{(2)}(0)<g^{(2)}(\tau)$, $g^{(2)}(0)<1$. This is in violation of the relations which apply to classical light. Hence the observation of photon antibunching is a purely quantum effect with no classical counterpart.

Classification of light according to statistics

In [3], we can find the following classification. *Super-Poissonian*: Classical equivalent are the partially coherent (chaotic), incoherent or thermal light. *Poissonian*: Perfectly coherent light. *Sub-Poissonian*: None (non classical).

Examples of chaotic lights are the Lorentzian chaotic light (e.g. collision broadened) and the Gaussian chaotic light (e.g. Doppler broadened). Then, what is in general a "chaotic light source"? Sometimes we can find that it is the blackbody radiation which is also called "chaotic radiation". However, it is also defined as the light coming from a spectral lamp. In [3], Fox writes that: "The light emitted by a mercury lamp originates from many different atoms. This leads to fluctuations in the light intensity on time-scales comparable to the coherence time. These light intensity fluctuations originate from fluctuations in the number of atoms emitting at a given time, and also from jumps and discontinuities in the phase emitted by the individual atoms. The partially coherent light emitted from such a source is called chaotic to emphasize the underlying randomness of the emission process at the microscopic level".

In any case, it is possible to characterize the light by means of the degree of coherence. For instance, we can use optical interferometers (Michelson interferometer, Mach–Zehnder interferometer, or Sagnac interferometer), where an electric field is split into two components. In this manner, a time delay between the components is introduced. Then, the two beams are recombines. The intensity of resulting field is measured as a function of the time delay. Let us use $G^{(1)}$.

For the light having a single frequency (laser):

$$G^{(1)}(\tau)=\exp(-i\omega_o\tau)$$

For a Lorentzian chaotic light (collision broadened light):

$$G^{(1)}(\tau)=\exp(-i\omega_o\tau-|\tau|/\tau_c)$$

For a Gaussian chaotic light (Doppler broadened light):

$$G^{(1)}(\tau)=\exp(-i\omega_o\tau-|\tau|/\tau_c)$$

ω_o is the central frequency of the light and τ_c is the coherence time of the light.

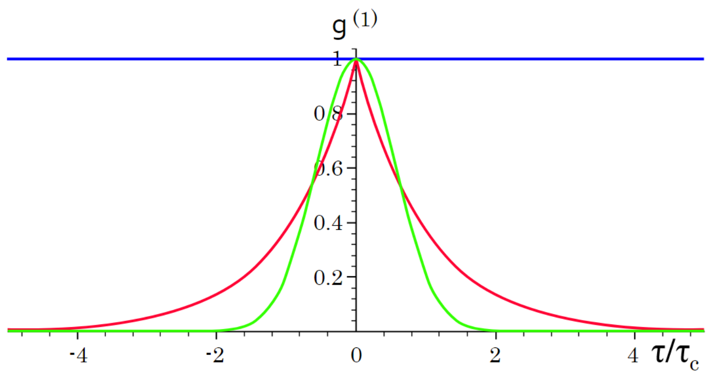

Fig.3 - Absolute value of $G^{(1)}$ as a function of the delay normalized to the coherence length τ/τ_c . The blue curve is for a coherent state. The red curve is for Lorentzian chaotic light. The green curve is for Gaussian chaotic light. (Image courtesy Ajbura).

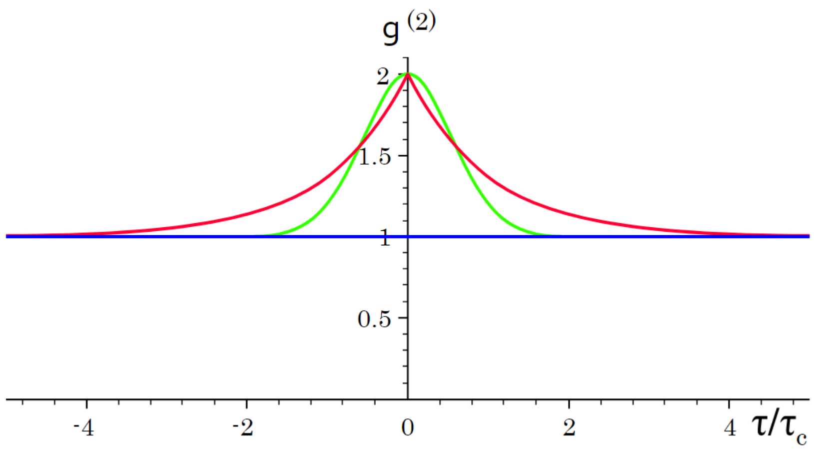

Fig.4 - $G^{(2)}$ as a function of the delay normalized to the coherence length τ/τ_c . The blue curve is for a coherent state. The red curve is for Lorentzian chaotic light. The green curve is for Gaussian chaotic light. The chaotic light is super-Poissonian and bunched.(Image courtesy Ajbura).

We can also use $G^{(2)}$. For the chaotic light of all kinds:

$$G^{(2)}(\tau)=1+|G^{(1)}(\tau)|^2$$

The Hanbury Brown - Twiss effect uses $G^{(2)}$ to find $G^{(1)}$.

Light of a single frequency: $G^{(2)}(\tau)=1$.

Antibunching experiments

The first successful experiment showing of photon antibunching was made by Kimble et al. in 1977 [3,16]. They used the light emitted by sodium atoms. "The basic principle of an antibunching experiment is to isolate an individual emitting species (i.e. an individual atom, molecule, quantum dot, or colour centre) and regulate the rate at which the photons are emitted by fluorescence". A laser is used to excite the emissive species to emit a photon. After a photon has been emitted - Ref.3 explains - it will take a time approximately equal to the radiative lifetime of the transition, τ_R , before the next photon can be emitted. In this manner, time gaps between the photons appear creating an antibunched light.

Abstract of [16] tells: "The phenomenon of antibunching of photoelectric counts has been observed in resonance fluorescence experiments in which sodium atoms are continuously excited by a dye-laser beam. It is pointed out that, unlike photoelectric bunching, which can be given a semiclassical interpretation, antibunching is understandable only in terms of a quantized electromagnetic field. The measurement also provides rather direct evidence for an atom undergoing a quantum jump".

Squeezing and sub-Poissonian

In Ref.17, we find told by its abstract that it is better to distinguish the sub-Poissonian statistics form the squeezing of light. "It is pointed out that, although squeezing and sub-Poissonian photon statistics need not go together, in the sense that an electromagnetic field may exhibit one but not the other, the method that is normally used to detect a squeezed state automatically generates sub-Poissonian photon statistics. However, when these considerations are applied to the fluorescence from a coherently driven atom, which exhibits both squeezing and sub-Poisson fluctuations, one finds that the statistics of the emitted photons show even larger departures from classical field theory than the squeezing". Then, let us see what are the squeezed photons.

Wave quadrature and squeezed states

Let us consider the phase space used in optics. In the quantum theory of light, an electromagnetic oscillator is involved to describe an oscillation of the electric field. The magnetic field too oscillates.

Let $\boldsymbol{u}(\boldsymbol{x},t)$ be a vector describing a single mode of an electromagnetic oscillator. An example is the plane wave given by :

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

where $\boldsymbol{u_0}$ is the polarization vector, $\boldsymbol{k}$ is the wave vector, ω the angular frequency. This is a plane wave describing an electromagnetic oscillator, with a single mode of oscillation. Such an oscillator, when it is quantized, produces a quantum oscillator described by means of creation and annihilation operators $\hat{a}^{+},\hat{a}$. Physical quantities, such as the electric field strength, then become quantum operators. Therefore, the field is given as [18]:

$$\hat{E}_i=u_i^*(\boldsymbol{x},t)\hat{a}^{+}+u_i(\boldsymbol{x},t)\hat{a}$$

Index i indicates a component of the field. The Hamiltonian is given by:

$$\hat{H}=\hbar\omega(\hat{a}^{+}\hat{a}+1/2)$$

The canonical commutation relation is $[\hat{a},\hat{a}^{+}]=1$ as before. Let us consider operators as given in [19]:

$$\hat{q}=\frac{1}{2}(\hat{a}^{+}+\hat{a}) \quad , \quad \hat{p}=\frac{1}{2}(\hat{a}^{+}-\hat{a})$$

There operators are the "quadratures", representing the real and imaginary parts of the complex amplitude. The commutation relation between the two quadratures can easily be calculated [19]:

$$[\hat{q},\hat{p}]=\frac{i}{2}$$

The "quadratures" obey Heisenberg Uncertainty Principle given by:

$$\Delta q\Delta p\geq 1/2$$

where $\Delta q,\Delta p$ are the variances of the distributions of q and p, respectively. Therefore, we have expressed in quantum optics the uncertainty principle.

Let us conclude proposing the squeezing of photons.

A general form of a squeezed coherent state for a quantum harmonic oscillator is given by:

$$|\nu,\zeta>=D(\nu)S(\zeta)|0>$$

where $|0>$ is the vacuum state, D is the displacement operator and S the squeeze operator, given by:

$$D(\nu)=\exp(\nu\hat{a}^{+}-\nu^{*}\hat{a}) \quad \text{and} \quad S(\zeta)=\exp[\frac{1}{2}(\zeta^{*}\hat{a}^{2}-\zeta\hat{a}^{+2})]$$

where $\hat{a},\hat{a}^{+}$ are annihilation and creation operators, respectively.

For a quantum harmonic oscillator of angular frequency ω, we can write these operators as:

$$\hat{a}^{+}=\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right) \quad , \quad \hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)$$

For a real ζ the uncertainties for x and p are:

$$(\Delta x)^{2}=\frac{\hbar}{2m\omega}\exp(-2\zeta) \quad \text{and} \quad (\Delta p)^{2}=\frac{m\hbar\omega}{2}\exp(2\zeta)$$

Therefore, a squeezed coherent state saturates the Heisenberg uncertainty principle $\Delta x\Delta p=\hbar/2$, with reduced uncertainty in one of its quadrature components and increased uncertainty in the other.

Conclusion

We have discussed the Poissonian distributions that we can find in the thermoionic emission of electrons and in the photodetection. After a description of the Hanbury Brown - Twiss effect, we have considered the second order correlation of intensity and Poissonian, super- and sub-Poissonian lights. The quadrature and squeezed states have been considered too. The discussion was proposed in the framework of teaching purposes.

Appendix

Poisson distribution can be obtained from the binomial distribution. If the random variable X follows the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, we tell that $X \sim B(n, p)$. The probability of getting exactly k successes in n independent Bernoulli trials is given by the probability mass function ($q=1-p$):

$$P(k)=\binom{n}{k}p^{k}q^{n-k}=\frac{n!}{k!(n-k)!}p^{k}q^{n-k}$$

Let us define constant $\lambda=np$ and use it in the formula:

$$P(k)=\frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^{k}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

For $n\rightarrow\infty$:

$$P(k)=\lim_{n\to\infty}\frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n(n-1)(n-2)\ldots(n-k+1)(n-k)!}{(n-k)!}\left(\frac{1}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

In the limit, the numerator becomes composed by k factors of n :

$$P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

Let us consider $\lambda/n\to 0$: $P(k)=\frac{\lambda^k}{k!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$

But: $e^{-\lambda}=\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$, so we have: $P(k)=\frac{\lambda^k}{k!}e^{-\lambda}$.

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