G-deformations of maps into projective space

G-DEFORMATIONS OF MAPS INTO PROJECTIVE SPACE

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ABSTRACT. G-deformability of maps into projective space is characterised by the existence of certain Lie algebra valued 1-forms. This characterisation gives a unified way to obtain well known results regarding deformability in different geometries.

1. Introduction

It is well known that isothermic surfaces are the only surfaces in conformal geometry that admit non-trivial second order deformations [13] and that R- and R_0-surfaces are the only surfaces in projective geometry that admit non-trivial second order deformations [11, 17]. In [27] it is shown that Ω- and Ω_0-surfaces are the only surfaces in Lie sphere geometry that admit non-trivial second order deformations. Motivated by these results we investigate G-deformations of smooth maps into G-invariant submanifolds of projective space $\mathbb{P}(V)$, where G is a group acting linearly on V. This method quickly recovers the aforementioned results regarding deformability in the context of gauge theory.

The examples studied in this paper are all examples of R-spaces [33]. The author believes that the main theorem of this paper can be used to study deformations in general R-spaces and intends to do so in subsequent work.

It should be noted that Cartan's method of moving frames was utilised in [19, 22] to outline methods for considering deformations of submanifolds of general homogeneous spaces. A different approach is used in this paper that is more suited to recovering gauge-theoretic characterisations of certain classes of surfaces.

We start by recalling the definition of k-th order deformations of maps into homogeneous spaces [19, 22]. Let N be a manifold on which a Lie group G, with Lie algebra $\mathfrak{g}$, acts smoothly and let $F : \Sigma \to N$ be a smooth map from a manifold Σ into N.

Definition 1.1. Let $k \in \mathbb{N} \cup \{0\}$. We say that $\hat{F} : \Sigma \to N$ is a k^{th}*-order* G*-deform of* F if there exists a smooth map $g : \Sigma \to G$ such that for all $p \in \Sigma$

$$g^{-1}(p)\hat{F} \quad \text{and} \quad F$$

agree to order k at p. The map g is called a *k-th order G-deformation of F*.

If F and $\hat{F}$ are congruent, i.e., $\hat{F} = AF$ for some $A \in G$, we say that the deformation is trivial. A map $F : \Sigma \to N$ is said to be *G-deformable of order k* if it admits a non-trivial k-th order G-deformation, otherwise F is said to be *G-rigid to k-th order.*

Remark 1.2. Note that the notion of "agreeing to order k" means that the projections into any chart agree to order k.

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Remark 1.3. k-th order contact at a point is transitive, i.e., if ϕ_1 and ϕ_2 agree to k-th order at a point p and ϕ_2 and ϕ_3 agree to k-th order at p, then ϕ_1 and ϕ_3 agree to k-th order at p.

Clearly, if $\hat{F}$ is a k-th order G-deform of F then we may write $\hat{F} = gF$ for the given k-th order G-deformation $g : \Sigma \to G$. In this way we may recover $\hat{F}$ from the deformation g. Furthermore, for any $A \in G$, it is clear that Ag is a k-th order deformation of F if and only if g is a k-th order deformation of F. This leads us to the following definition:

Definition 1.4. $\eta \in \Omega^1(\mathfrak{g})$ is a *k-th order infinitesimal deformation of F* if η satisfies the Maurer-Cartan equation and g is a k-th order G-deformation of F for any $g : \Sigma \to G$ satisfying $g^{-1}dg = \eta$.

The following lemma concerns the uniqueness of the map $g : \Sigma \to G$ defining a G-deform:

Lemma 1.5. *Let $\hat{F} : \Sigma \to S$ be a k-th order G-deform of F of each other via $g : \Sigma \to G$ and. Then $\hat{F}$ is a k-th order G-deform of F via $\tilde{g} : \Sigma \to G$ as well if and only if F is a k-th order deform of itself via $h := g^{-1}\tilde{g}$.*

Proof. Since $\hat{F}$ is a k-th order G-deform of F via g, we have that for each $p \in \Sigma$, $g^{-1}(p)\hat{F}$ agrees to k-th order with F at p. Let $\tilde{g} : \Sigma \to G$ and define $h := g^{-1}\tilde{g}$. Then since $h^{-1}(p)$ is constant, one has that $h^{-1}(p)g^{-1}(p)\hat{F}$ agrees to order k with $h^{-1}(p)F$ at p. It follows by Remark 1.3 that $h^{-1}(p)F$ agrees to order k with F at p if and only if $\tilde{g}^{-1}(p)\hat{F} = h^{-1}(p)g^{-1}(p)\hat{F}$ agrees to order k with F at p. □

We will only be interested in deformations that are non-trivial. We thus have the following result:

Lemma 1.6. *Suppose that $\hat{F} : \Sigma \to S$ is a k-th order G-deform of F via $g : \Sigma \to G$. Then this is a trivial deformation if and only if $g = Ah$ where $A \in G$ and $h : \Sigma \to G$ such that F is a k-th order G-deform of itself via $h : \Sigma \to G$.*

Proof. This follows by Lemma 1.5 and noting that if $\hat{F} = AF$ for some $A \in G$ then $\hat{F}$ is a k-th order G deform of F via A. □

2. Deformations in projective space

Suppose that V is a vector space with projectivisation $\mathbb{P}(V)$ and suppose that G is a Lie group acting linearly on V.

Proposition 2.1. *$\phi, \hat{\phi} : \Sigma \to \mathbb{P}(V)$ agree to order k at $p \in \Sigma$ if and only if for any $v_0 \in V^*$, the sections $\sigma, \hat{\sigma}$ of ϕ and $\hat{\phi}$, respectively, such that*

$$v_0(\sigma) = v_0(\hat{\sigma}) = 1$$

agree to order k at p on the open set where they are defined.

Proof. ϕ and $\hat{\phi}$ agree to order k at p if and only if in any chart of $\mathbb{P}(V)$ they agree to order k at p. Let $U := \mathbb{P}(V)\backslash\mathbb{P}(\ker v_0)$. Then U is an open subset of $\mathbb{P}(V)$ and

$$\psi : U \to V, \quad [u] \mapsto u,$$

where $u \in [u]$ satisfies $v_0(u) = 1$, defines a chart (U, ψ) on $\mathbb{P}(V)$. Thus, ϕ and $\hat{\phi}$ agreeing to order k at p in this chart is equivalent to $\sigma := \psi(\phi)$ and $\hat{\sigma} := \psi(\hat{\phi})$

agreeing to order k at p. The result follows as the collection of charts defined by all $v_0 \in V^*$ is an atlas for $\mathbb{P}(V)$. □

Let S be a G-invariant submanifold of $\mathbb{P}(V)$. k-th order contact of two maps in S is equivalent to k-th order contact as maps into $\mathbb{P}(V)$. Therefore we may use Proposition 2.1 to study contact in S. Let $F : \Sigma \to S$ be a smooth map from a manifold Σ into S.

To simplify our exposition in this section, we shall use the following notation: let $j, k \in \mathbb{Z}$ and define $S_{j,k} := \{j, ..., k\}$ if $j \leq k$ and $S_{j,k} := \emptyset$ if $k < j$. Let W be a vector bundle over Σ, suppose that $X_j, ..., X_k \in \Gamma T\Sigma$ and let $\sigma \in \Gamma W$. Then for $J \subset S_{j,k}$ with $J = \{j_1 < ... < j_l\}$ we let

$$d_{X_J}\sigma := d_{X_{j_1}}(d_{X_{j_2}}...(d_{X_{j_l}}\sigma)),$$

and

$$d_{X_\emptyset}\sigma := \sigma.$$

We will repeatedly use the Leibniz rule, i.e., if $\sigma, \xi \in \Gamma W$ and $J \subset S_{j,k}$, then

$$d_{X_J}(\sigma \otimes \xi) = \sum_{K \subset J} (d_{X_K}\sigma) \otimes (d_{X_{J\setminus K}}\xi).$$

The following lemma allows us to characterise deformability of a map $g : \Sigma \to G$ in terms of its Maurer-Cartan form:

Lemma 2.2. *Let $k \in \mathbb{N}$ and suppose that g is a $(k-1)$-th order deformation of F. Then F and $g^{-1}(p)gF$ agree to order k at $p \in \Sigma$ if and only if for any $v_0 \in V^*$ and $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$,*

$$\theta(Y)d_{X_{S_{1,k-1}}}\sigma = \sum_{K \subset S_{1,k-1}} v_0(\theta(Y)d_{X_K}\sigma)d_{X_{S_{1,k-1}\setminus K}}\sigma,$$

at p, where $\theta = g^{-1}dg$ and $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$.

Proof. We shall use strong induction on k. Consider the case $k = 1$: F and $g^{-1}(p)gF$ agree to order 1 at p if and only if for any $v_0 \in V^*$, $v_0(g^{-1}(p)g\sigma)\sigma$ and $g^{-1}(p)g\sigma$ agree to order 1 at p where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$. This holds if and only if for any $Y \in T_p\Sigma$,

$$g^{-1}(p)d_Y(g\sigma) = d_Y(v_0(g^{-1}(p)g\sigma)\sigma).$$

Now using the Leibniz rule and that $\theta_p(Y) = g^{-1}(p)d_Yg$, this holds if and only if

$$\theta_p(Y)\sigma + d_Y\sigma = v_0(\theta_p(Y)\sigma)\sigma + d_Y\sigma.$$

Noting that $d_\emptyset\sigma = \sigma$, we see that the proposition holds when $k = 1$.

Let $n \in \mathbb{N}$ and assume that the proposition holds for all $k < n$ and assume that F and $\hat{F}$ are $(n-1)$-th order deformations of each other. Let $Y, X_1, ..., X_{n-1} \in \Gamma T\Sigma$. Then for any $K \subset \{1, ..., n-1\}$ with $|K| < n-1$ we have, by our inductive hypothesis,

$$\theta(Y)d_{X_K}\sigma = \sum_{L \subset K} v_0(\theta(Y)d_{X_L}\sigma)d_{X_{K\setminus L}}\sigma. \tag{1}$$

Since F and $\hat{F}$ are $(n-1)$-th order deformations of each other we have that for any $v_0 \in V^*$ and $X_1, ..., X_{n-1} \in \Gamma T\Sigma$,

$$g^{-1}d_{X_{S_{1,n-1}}}g\sigma - \sum_{K \subset S_{1,n-1}} v_0(g^{-1}d_{X_K}g\sigma)d_{X_{S_{1,n-1}\setminus K}}\sigma = 0,$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. Differentiating at p with respect to $X_0 \in \Gamma T\Sigma$ we get, using the Leibniz rule and that $d_Y g^{-1} = -\theta(Y)g^{-1}$,

$$\begin{aligned}
0 &= -\theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma + g^{-1}(p)d_{X_0}d_{X_{S_{1,n-1}}}g\sigma \\
&+ \sum_{K\subset S_{1,n-1}} [v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma \\
&- v_0(g^{-1}(p)d_{X_0X_K}g\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma - v_0(g^{-1}(p)d_{X_K}g\sigma)d_{X_0X_{S_{1,n-1}\backslash K}}\sigma] \\
&= -\theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma + d_{X_{S_{0,n-1}}}(g^{-1}(p)g\sigma) \\
&+ \sum_{K\subset S_{1,n-1}} v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma - d_{X_{S_{0,n-1}}}(v_0(g^{-1}(p)g\sigma)\sigma).
\end{aligned}$$

Thus, $v_0(g^{-1}(p)g\sigma)\sigma$ and $g^{-1}(p)g\sigma$ agree to order n at p if and only if

$$\theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma = \sum_{K\subset S_{1,n-1}} v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma. \tag{2}$$

Now, $v_0(g^{-1}(p)g\sigma)\sigma$ and $g^{-1}(p)g\sigma$ agree up to order $n-1$ at p, thus for any $K \subset S_{1,n-1}$,

$$g^{-1}(p)d_{X_K}g\sigma = d_{X_K}(v_0(g^{-1}(p)g\sigma)\sigma) = \sum_{L\subset K} v_0(g^{-1}(p)d_{X_L}g\sigma)d_{X_{K\backslash L}}\sigma.$$

Thus, (2) becomes

$$\begin{aligned}
0 &= -\theta_p(X_0)\sum_{K\subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma \\
&+ \sum_{K\subset S_{1,n-1}}\sum_{L\subset K} v_0(\theta_p(X_0)v_0(g^{-1}(p)d_{X_L}g\sigma)d_{X_{K\backslash L}}\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma \\
&= -\sum_{K\subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)\theta_p(X_0)d_{X_{S_{1,n-1}\backslash K}}\sigma \\
&+ \sum_{K\subset S_{1,n-1}}\sum_{L\subset K} v_0(g^{-1}(p)d_{X_L}g\sigma)v_0(\theta_p(X_0)d_{X_{K\backslash L}}\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma.
\end{aligned}$$

After relabelling we have that

$$\begin{aligned}
0 = \sum_{K\subset S_{1,n-1}} & v_0(g^{-1}(p)d_{X_K}g\sigma)(-\theta_p(X_0)d_{X_{S_{1,n-1}\backslash K}}\sigma \\
&+ \sum_{L\subset (S_{1,n-1}\backslash K)} v_0(\theta_p(X_0)d_{X_L}\sigma)d_{X_{(S_{1,n-1}\backslash K)\backslash L}}\sigma).
\end{aligned}$$

Using the inductive hypothesis (1) we then have

$$0 = -\theta_p(X_0)d_{X_{S_{1,n-1}}}\sigma + \sum_{K\subset S_{1,n-1}} v_0(\theta_p(X_0)d_{X_K}\sigma)d_{X_{S_{1,n-1}\backslash K}}\sigma.$$

Hence, the result holds for the case $k = n$. Therefore, by induction the result is proved. □

Applying Lemma 2.2 recursively, one obtains the following theorem:

Theorem 2.3. *$\eta \in \Omega^1(\mathfrak{g})$ is a k-th order infinitesimal deformation of F if and only if η satisfies the Maurer Cartan equation and for all $r \in \{0, ..., k-1\}$, $v_0 \in V^*$ and $Y, X_1, ..., X_r \in \Gamma T\Sigma$,*

$$\eta(Y) d_{X_{S_{1,r}}} \sigma = \sum_{K \subset S_{1,r}} v_0(\eta(Y) d_{X_K} \sigma) d_{X_{S_{1,r} \setminus K}} \sigma,$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$.

We now wish to find an invariant characterisation of deformability in terms of the Maurer-Cartan form, i.e., a characterisation that does not require charts. Essentially this achieved by taking the characterisation of Theorem 2.3 and successively applying the Leibniz rule. Let $r \in \{0, ..., k-1\}$, $Y, X_1, ..., X_r \in \Gamma T\Sigma$ and $v_0 \in V^*$. For $I, J \subset \{1, ..., r\}$, contemplate the following equation:

$$(d_{X_I} \eta(Y)) d_{X_J} \sigma = \sum_{K \subset J} v_0((d_{X_I} \eta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma, \tag{3}$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$.

Lemma 2.4. *Suppose that for all $I, J \subset \{1, ..., r\}$ with $|I| + |J| < r$, (3) holds. Then (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = i \in \{0, , ..., r\}$ and $|I| + |J| = r$ if and only if (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = i + 1$ and $|I| + |J| = r$.*

Proof. Suppose that (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = i \in \{0, , ..., r\}$ and $|I| + |J| = r$. Let $I, J \subset \{1, ..., r\}$ with $|I| = i+1$ and $|I| + |J| = r$. Without loss of generality, assume that $\min I < \min J$. Let a denote the smallest element of I and $\hat{I} := I \backslash \{a\}$. Then by our assumption

$$(d_{X_{\hat{I}}} \eta(Y)) d_{X_J} \sigma = \sum_{K \subset J} v_0((d_{X_{\hat{I}}} \eta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma.$$

Differentiating this with respect to X_a at p and using the Leibniz rule we have that

$$\begin{aligned}
&(d_{X_I} \eta(Y)) d_{X_J} \sigma + (d_{X_{\hat{I}}} \eta(Y)) d_{X_{\{a\} \cup J}} \sigma \\
&= \sum_{K \subset J} (v_0((d_{X_I} \eta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma + v_0((d_{X_{\hat{I}}} \eta(Y)) d_{X_{\{a\} \cup K}} \sigma) d_{X_{J \setminus K}} \sigma \\
&+ \sum_{K \subset J} v_0((d_{X_{\hat{I}}} \eta(Y)) d_{X_K} \sigma) d_{X_{\{a\} \cup J \setminus K}} \sigma) \\
&= \sum_{K \subset J} v_0((d_{X_I} \eta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma + \sum_{L \subset \{a\} \cup J} v_0((d_{X_{\hat{I}}} \eta(Y)) d_{X_L} \sigma) d_{X_{\{a\} \cup J \setminus L}} \sigma.
\end{aligned}$$

By our supposition,

$$(d_{X_{\hat{I}}} \eta(Y)) d_{X_{\{a\} \cup J}} \sigma = \sum_{L \subset \{a\} \cup J} v_0((d_{X_{\hat{I}}} \eta(Y)) d_{X_L} \sigma) d_{X_{\{a\} \cup J \setminus L}} \sigma.$$

Thus,

$$(d_{X_I} \eta(Y)) d_{X_J} \sigma = \sum_{K \subset J} (v_0((d_{X_I} \theta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma.$$

A similar argument can be used to prove the converse. □

Corollary 2.5. *Suppose that for all $I, J \subset \{1, ..., r\}$ with $|I| + |J| < r$, (3) holds. Then if (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = i \in \{0, , ..., r\}$ and $|I| + |J| = r$, then (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| + |J| = r$.*

We are now in a position to state the following invariant version of Theorem 2.3:

Theorem 2.6. *$\eta \in \Omega^1(\mathfrak{g})$ is a k-th order infinitesimal deformation of F if and only if η satisfies the Maurer-Cartan equation and*

$$\eta(Y)F \le F, \quad (d_{X_1}\eta(Y))F \le F, \quad \dots \quad ,(d_{X_1\dots X_{k-1}}\eta(Y))F \le F, \tag{4}$$

for all $Y, X_1, ..., X_{k-1}, \in \Gamma T\Sigma$.

Proof. Firstly, notice that (4) is equivalent to (3) with $|I| = r \in \{0, ..., k-1\}$ and $|J| = 0$, for any choice of $v_0 \in V^*$.

Suppose that η is a k-th order infinitesimal deformation of F and let $g : \Sigma \to G$ such that $g^{-1}dg = \eta$. Then by Theorem 2.3, for any $r \in \{0, ..., k-1\}$, $Y, X_1, ..., X_r \in \Gamma T\Sigma$ and $v_0 \in V^*$, we have that (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = 0$ and $|J| = r$. By Corollary 2.5 it then follows that (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = r$ and $|J| = 0$.

Conversely, suppose that η satisfies the Maurer-Cartan equation and, for any $r \in \{0, ..., k-1\}$, $Y, X_1, ..., X_r \in \Gamma T\Sigma$ and $v_0 \in V^*$, (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = r$ and $|J| = 0$. Then by Corollary 2.5, (3) holds for all $I, J \subset \{1, ..., r\}$ with $|I| = 0$ and $|J| = r$. By Theorem 2.3 it then follows that η is a k-th order infinitesimal deformation of F. □

3. Projective 3-space

Cartan [11] investigated projective deformability and rigidity of surfaces in projective 3-space. Modern references on this topic include [1, 17, 20, 23]. It was shown in [17] that the class of second order deformable surfaces in projective 3-space can be split naturally into two subclasses: R- and R_0-surfaces. A modern account of this can be found in [15] and a gauge theoretic approach for these surfaces was developed in [14]. In this section we will use the results from Section 2 to study these notions.

So let us consider projective 3-space $\mathbb{P}(\mathbb{R}^4)$ with transformation group SL(4). Suppose that Σ is a 2-dimensional manifold and let $F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ be a smooth map. We can view F as a rank 1 subbundle of the trivial bundle $\underline{\mathbb{R}}^4 := \Sigma \times \mathbb{R}^4$. Let $F^{(1)}$ denote derived bundle of F, i.e., the set of sections of F and derivatives of sections of F. Assuming that F is an immersion is equivalent to assuming that $F^{(1)}$ is a rank 3 subbundle of the trivial bundle. Let T_1, T_2 denote the (possibly complex conjugate) asymptotic directions of F, i.e., for any $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $\sigma \in \Gamma F$,

$$d_X d_X \sigma, d_Y d_Y \sigma \in \Gamma F^{(1)}.$$

We will make the further assumption that the derived bundle $F^{(2)}$ of $F^{(1)}$ satisfies $F^{(2)} = \underline{\mathbb{R}}^4$. In other words, for $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $\sigma \in \Gamma F$, $d_X d_Y \sigma$ never belongs to $F^{(1)}$.

3.1. Second order deformations.

We will now investigate when F admits non-trivial second order SL(4)-deformations. By Theorem 2.3, $\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ is a second order infinitesimal deformation of F if and only if η satisfies the Maurer-Cartan equation and for all $v_0 \in (\mathbb{R}^4)^*$ and $X, Y \in \Gamma T\Sigma$

$$\eta\sigma = v_0(\eta\sigma)\sigma \tag{5}$$

and

$$\eta(X)d_Y\sigma = v_0(\eta(X)\sigma)d_Y\sigma + v_0(\eta(X)d_Y\sigma)\sigma, \tag{6}$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$.

Suppose that η is such a second order infinitesimal deformation. Let $X \in \Gamma T_1$ and $Y \in \Gamma T_2$. By equation (6) we have that

$$\eta(X) d_X \sigma = v_0(\eta(X)\sigma) d_X \sigma + v_0(\eta(X) d_X \sigma)\sigma.$$

Differentiating this in the Y direction gives

$$\begin{aligned}(d_Y \eta(X)) d_X \sigma + \eta(X) d_{YX} \sigma = d_Y (v_0(\eta(X)\sigma)) d_X \sigma + v_0(\eta(X)\sigma) d_{YX} \sigma \\ + d_Y (v_0(\eta(X) d_X \sigma))\sigma + v_0(\eta(X) d_X \sigma) d_Y \sigma.\end{aligned}$$

Since η satisfies the Maurer-Cartan equation, one deduces that the left hand side of this equation is

$$\eta(X) d_{YX} \sigma \, mod \, F^{(1)}.$$

Whereas the right hand side is

$$v_0(\eta(X)\sigma) d_{YX} \sigma \, mod \, F^{(1)}.$$

Similarly, one can show that

$$\eta(Y) d_{YX} \sigma = v_0(\eta(Y)\sigma) d_{YX} \sigma \, mod \, F^{(1)}.$$

Using that $\{\sigma, d_X \sigma, d_Y \sigma, d_{YX} \sigma\}$ forms a basis for $\mathbb{P}(\mathbb{R}^4)$ and that η takes values in $\mathfrak{sl}(4)$ and is thus trace free, we must have that $v_0(\eta\sigma) = 0$. Therefore,

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F).$$

Conversely if η satisfies

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F)$$

then clearly (5) and (6) hold and thus η is a second order infinitesimal deformation of F.

One can show (see [31, Lemma 3.21]) that an $\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ of the above form satisfies the Maurer-Cartan equation if and only if η is closed. Thus, we have arrived at the following proposition:

Proposition 3.1. *$\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ is a second order infinitesimal deformation of F if and only if η is closed and satisfies $\eta F = 0$ and $\eta F^{(1)} \leq \Omega^1(F)$.*

We will now investigate the uniqueness and triviality of second order deformations. According to Lemma 1.5 and Lemma 1.6, this is determined by second order deformations, $h : \Sigma \to G$, between F and itself. By Proposition 3.1, such a h satisfies

$$hF = F, \quad \theta_h F = 0 \quad \text{and} \quad \theta_h F^{(1)} \leq \Omega^1(F), \tag{7}$$

where $\theta_h := h^{-1} dh$. Now $hF = F$ implies that for any $\sigma \in \Gamma F$,

$$h\sigma = \lambda\sigma$$

for a smooth function λ. Thus, for any $X \in \Gamma T\Sigma$

$$(d_X h)\sigma + h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Using that $\theta_h F = 0$

$$h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Differentiating this condition with respect to $Y \in \Gamma T\Sigma$ we have that

$$h d_{YX} \sigma = \lambda d_{YX} \sigma + (d_Y \lambda) d_X \sigma + (d_X \lambda) d_Y \sigma + (d_{YX} \lambda)\sigma - (d_Y h) d_X \sigma.$$

Then, since h takes values in $\mathrm{SL}(4)$ and $\theta_h F^{(1)} \leq \Omega^1(F)$, we must have that $\lambda = \pm 1$. Furthermore,

$$h|_{F^{(1)}} = \pm id|_{F^{(1)}} \quad \text{and} \quad h|_{\underline{\mathbb{R}}^4/F} = \pm id|_{\underline{\mathbb{R}}^4/F}.$$

Thus, we may write

$$h = \pm(id + \xi),$$

where ξ satisfies $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$. Clearly ξ is trace-free, so $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$. Hence, $h = \pm\exp(\xi)$. Conversely, given an h of such a form, one can easily check that (7) is satisfied. Thus we obtain the following lemmata:

Lemma 3.2. *Second order deformations between two maps $F, \hat{F} : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ are determined up to right multiplication by $\pm\exp(\xi)$, for any $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.*

Lemma 3.3. *η is a trivial second order infinitesimal deformation of F if and only if $\eta = d\xi$, where $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.*

We have therefore proved the main theorem of this subsection:

Theorem 3.4. *$F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ is deformable of order two if and only if there exists $\eta \in \Omega^1(\mathfrak{sl}(4))$, such that η is closed,*

$$\eta F = 0, \quad \eta F^{(1)} \leq \Omega^1(F)$$

and $\eta \neq d\xi$ for any $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.

In Section 6 we shall see that the deformability of a map into $\mathbb{P}(\mathbb{R}^4)$ coincides with deformability of its contact lift. In that setting the triviality of deformations can be identified by the vanishing of a certain two-tensor.

By using the gauge theoretic definition of R-/R_0-surfaces given in [14], one recovers the following classical result:

Corollary 3.5 ([11, 17])**.** *R-surface and R_0-surfaces are the only second order deformable surfaces of projective geometry.*

3.2. **Third order deformations.** We shall now show that rigidity occurs at third order in projective 3-space. Suppose that η is a third order infinitesimal deformation of F. Then by Theorem 3.4, η is closed and satisfies

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F).$$

Furthermore, by Theorem 2.3, for any $v_0 \in (\mathbb{R}^4)^*$ and $X, Y, Z \in \Gamma T\Sigma$,

$$\begin{aligned} \eta(X)d_{YZ}\sigma &= v_0(\eta(X)d_{YZ}\sigma)\sigma + v_0(\eta(X)d_Y\sigma)d_Z\sigma \\ &\quad + v_0(\eta(X)d_Z\sigma)d_Y\sigma + v_0(\eta(X)\sigma)d_{YZ}\sigma, \end{aligned}$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. Now suppose that Y is an asymptotic direction of F and $Z = Y$. Then $d_{YZ}\sigma \in \Gamma F^{(1)}$ and thus $\eta(X)d_{YZ}\sigma \in \Gamma F$. Hence, $v_0(\eta(X)d_Y\sigma) = 0$. Therefore, $\eta F^{(1)} = 0$. We will now use that η is closed to show that $\eta = 0$: suppose that $X, Y, Z \in \Gamma T\Sigma$. Then, as η is closed, we have that for any $\sigma \in \Gamma F$

$$d\eta(X, Y)d_Z\sigma = 0.$$

Since $\eta|_{F^{(1)}} = 0$, this is equivalent to

$$\eta(X)d_{YZ}\sigma - \eta(Y)d_{XZ}\sigma = 0.$$

Assume now that X and Y are distinct asymptotic directions of F. Then setting $Z = Y$ implies that $\eta(Y)d_{XY}\sigma = 0$, since $d_{YY}\sigma \in \Gamma F^{(1)}$. Similarly, setting $Z = X$ implies that $\eta(X)d_{YX}\sigma = 0$, which in turn implies that $\eta(X)d_{XY}\sigma = 0$. Therefore as $\{\sigma, d_X\sigma, d_Y\sigma, d_{XY}\sigma\}$ is a basis for $\underline{\mathbb{R}}^4$, $\eta = 0$. Thus we have proved the following classically known theorem:

Theorem 3.6. *Surfaces in projective* 3*-space are rigid to third order.*

4. Hypersurfaces in the conformal n-sphere

In this section we will apply the results of Section 2 to examine deformations of hypersurfaces in conformal geometry. For a modern treatment of conformal geometry see for example [2, 3, 6, 7, 21, 25, 24].

Let $n \in \mathbb{N}$. Then we may view $\mathbb{S}^n$ as the projective light cone $\mathbb{P}(\mathcal{L})$ of $\mathbb{R}^{n+1,1}$, which is acted upon transitively by the orthogonal group $\mathrm{O}(n+1,1)$. Suppose that $F : \Sigma \to \mathbb{P}(\mathcal{L})$ is an immersion, where Σ is an $(n-1)$-dimensional manifold. We will view F as a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$. Note that as F is an immersion, the derived bundle $F^{(1)}$ of F is a codimension 1 subbundle of $F^{\perp}$. Let V be a sphere congruence enveloped by F, i.e., V is a bundle of $(n,1)$-planes such that $F^{(1)} \leq V$. Then let $\tilde{F}$ be a null-line subbundle of V complementary to F, i.e., $F \oplus \tilde{F}$ is a $(1,1)$-subbundle of V. Let $U := (F \oplus \tilde{F})^{\perp} \cap V$. Then $F^{(1)} = F \oplus U$ and $F^{\perp} = F \oplus U \oplus V^{\perp}$. We now have a splitting

$$\underline{\mathbb{R}}^{n+1,1} = F \oplus \tilde{F} \oplus U \oplus V^{\perp},$$

and thus a splitting of $\wedge^2\underline{\mathbb{R}}^{n+1,1}$:

$$\wedge^2\underline{\mathbb{R}}^{n+1,1} = F \wedge U \oplus F \wedge V^{\perp} \oplus U \wedge U \oplus U \wedge V^{\perp} \oplus F \wedge \tilde{F} \oplus \tilde{F} \wedge U \oplus \tilde{F} \wedge V^{\perp}.$$

4.1. **Second order deformations.** By Theorem 2.3, $\eta \in \Omega^1(\underline{\mathfrak{o}(n+1,1)})$ is a second order infinitesimal deformation of F if and only if η satisfies the Maurer-Cartan equation, and for all $v_0 \in (\mathbb{R}^{n+1,1})^*$ and $X, Y \in \Gamma T\Sigma$

$$(8) \quad \eta\sigma = v_0(\eta\sigma)\sigma \quad \text{and} \quad \eta(X)d_Y\sigma = v_0(\eta(X)\sigma)d_Y\sigma + v_0(\eta(X)d_Y\sigma)\sigma,$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$. From the skew-symmetry of η it follows that $v_0(\eta\sigma) = 0$. Thus, (8) holds if and only if

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F),$$

or equivalently

$$\eta F = 0 \quad \text{and} \quad \eta U \leq \Omega^1(F).$$

This clearly holds if and only if

$$\eta \in \Omega^1(F \wedge U \oplus F \wedge V^{\perp}) = \Omega^1(F \wedge F^{\perp}).$$

Now $F \wedge F^{\perp}$ is a bundle of abelian subalgebras of $\underline{\mathfrak{o}(n+1,1)}$. Therefore, $[\eta \wedge \eta] = 0$ and the condition that η satisfies the Maurer-Cartan equation reduces to η being closed.

We shall now investigate the uniqueness and triviality of second order deformations. According to Lemma 1.5 and Lemma 1.6, this is determined by second order deformations, $h : \Sigma \to G$, between F and itself, i.e., h satisfies $hF = F$ and $\theta_h := h^{-1}dh \in \Omega^1(F \wedge F^{\perp})$. Thus, for any section $\sigma \in \Gamma F$, $h\sigma = \lambda\sigma$, for some smooth function λ. Differentiating this along $X \in \Gamma T\Sigma$ gives

$$(d_Xh)\sigma + hd_X\sigma = (d_X\lambda)\sigma + \lambda d_X\sigma.$$

But since $\theta_h F = 0$, we have that

$$hd_X\sigma = (d_X\lambda)\sigma + \lambda d_X\sigma.$$

The orthogonality of h then gives that $\lambda = \pm 1$. Furthermore $h|_{F^{(1)}} = \pm id|_{F^{(1)}}$ and so for any $\nu \in \Gamma F^{(1)}$, $h\nu = \pm\nu$. Differentiating this condition along $Y \in \Gamma T\Sigma$ gives that

$$(d_Y h)\nu + hd_Y\nu = \pm d_Y\nu.$$

Then since $\theta_h F^\perp \le F$, we have that $h|_{F^{(2)}} \equiv \pm id|_{F^{(2)}} \, mod\, F$. Now, $F^{(2)} := (F^{(1)})^{(1)} = \underline{\mathbb{R}}^{n+1,1}$, so we may write

$$h = \pm id + \xi,$$

where $\xi|_{F^{(1)}} = 0$ and $im\xi \le F$. From the orthogonality of h one may deduce that ξ is skew-symmetric. Combined with $\xi|_{F^{(1)}} = 0$ and $im\xi \le F$, this can only hold if $\xi = 0$. We therefore have the following lemmata:

Lemma 4.1. *Suppose that g_1 and g_2 are second order deformations between F and $\hat{F}$. Then $g_1 = \pm g_2$.*

Lemma 4.2. *η is a trivial second order infinitesimal deformation of F if and only if $\eta = 0$.*

We have thus arrived at the main theorem of this subsection:

Theorem 4.3. *$F : \Sigma \to \mathbb{P}(\mathcal{L})$ is deformable of order two if and only if there exists a closed non-zero one-form η taking values in $F \wedge F^\perp$.*

In [5] it is shown that an η satisfying the conditions of Theorem 4.3 does not exist for $n > 3$. In the case of $n = 3$, using the gauge-theoretic definition of isothermic surfaces (see for example [7, 10]), one recovers the classically known result:

Corollary 4.4 ([13])**.** *Isothermic surfaces are the only second order deformable surfaces in the conformal 3-sphere.*

Remark 4.5. In [12, 28], the deformability of submanifolds in the conformal n-sphere with codimension greater that one was considered. In this case it is shown that, although isothermic surfaces are deformable to second order, a generic second order deformable surface is not isothermic.

In [32] it was proved that more can be said about where η takes values:

Proposition 4.6. *If $\eta \in \Omega^1(F \wedge F^\perp)$ is closed then $\eta \in \Omega^1(F \wedge F^{(1)})$.*

4.2. **Third order deformations.** We will now show that rigidity occurs at third order in the conformal 3-sphere. Suppose that η is a third order infinitesimal deformation of F. Then by Proposition 4.6, $\eta \in \Omega^1(F \wedge F^{(1)})$. Furthermore, by Theorem 2.6, for all $X, Y, Z \in \Gamma T\Sigma$,

$$(d_Y\eta(Z))\sigma = \xi\,\sigma \quad \text{and} \quad (d_X d_Y\eta(Z))\sigma \in \Gamma F,$$

for some smooth function ξ. Using the Leibniz rule, one then deduces that

$$(d_Y\eta(Z))d_X\sigma = \xi\, d_X\sigma \, mod\, F,$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$. The skew-symmetry of $(d_Y\eta(Z))$ implies that $\xi = 0$. Hence, $(d_Y\eta(Z))\sigma = 0$. By the Leibniz rule this implies that $\eta(Z)d_Y\sigma = 0$ and thus $\eta F^{(1)} = 0$. Therefore, $\eta = 0$ and it follows that:

Theorem 4.7. *A surface in the conformal 3-sphere is rigid to third order.*

5. Legendre maps

In this section we study the deformability of contact elements in Lie sphere geometry and projective geometry. This problem has been studied in [4, 15, 16, 18, 27].

Let $s, t \in \mathbb{N}$ such that $(s, t) = (3, 3)$ or $(s, t) = (4, 2)$. Consider $\mathbb{R}^{s,t}$ and let $\mathcal{L}^5$ denote the 5-dimensional lightcone of this space. Let $\mathcal{Z}$ denote the Grassmannian of null two dimensional subspaces of $\mathbb{R}^{s,t}$. $\mathcal{Z}$ is acted upon transitively by $G = \mathrm{O}(s, t)$. We say that a smooth map $f : \Sigma \to \mathcal{Z}$ is a *Legendre map* if $f^{(1)} \leq f^{\perp}$ and at every $p \in \Sigma$, if $X \in T_p\Sigma$ such that $d_X\sigma \in f(p)$ for all sections $\sigma \in \Gamma f$, then $X = 0$. We may view a Legendre map as rank 2 null subbundle on the trivial bundle $\underline{\mathbb{R}}^{s,t} := \Sigma \times \mathbb{R}^{s,t}$.

It was shown in [8] that a Legendre map naturally equips $T\Sigma$ with a conformal structure. In the case that $(s, t) = (4, 2)$ this conformal structure at each point either vanishes or has signature $(1, 1)$, however in the case of $(s, t) = (3, 3)$, any signature is possible. From this point onwards we shall make the assumption that the signature of this conformal structure is $(1, 1)$ at each point. In this case we may denote by T_1 and T_2 the null subbundles of this conformal structure. Our Legendre map then admits two special rank 1 subbundles s_1 and s_2, called the *curvature sphere congruences of* f, such that

$$d_X\sigma_1, d_Y\sigma_2 \in \Gamma f,$$

for all $\sigma_1 \in \Gamma s_1$, $\sigma_2 \in \Gamma s_2$, $X \in \Gamma T_1$ and $Y \in \Gamma T_2$. We may then form a splitting of the trivial bundle $\underline{\mathbb{R}}^{s,t}$ as $\underline{\mathbb{R}}^{s,t} = S_1 \oplus_\perp S_2$, where

$$S_1 := \langle \sigma_1, d_Y\sigma_1, d_Y d_Y\sigma_1 \rangle \quad \text{and} \quad S_2 := \langle \sigma_2, d_X\sigma_2, d_X d_X\sigma_2 \rangle. \tag{9}$$

This is called the *Lie cyclide splitting*. For $i \in \{1, 2\}$, let f_i denote the set of sections of f and derivatives of f along T_i. One then has that f_i is a rank 3 subbundle of $f^{\perp}$ and furthermore

$$f^{\perp}/f = f_1/f \oplus_\perp f_2/f,$$

with each f_i/f inheriting a non-degenerate metric from that of $\mathbb{R}^{s,t}$.

We identify f with the map $F : \Sigma \to Z$, defined by $F = \wedge^2 f$, where Z is the subset of $\mathbb{P}(\wedge^2\mathbb{R}^{s,t})$ defined by

$$Z := \{[v \wedge w] : v, w \in \mathcal{L} \text{ and } (v, w) = 0\}.$$

Z is acted upon smoothly and transitively by $O(s, t)$ via

$$A[v \wedge w] = [Av \wedge Aw].$$

Let $\tilde{f} : \Sigma \to \mathcal{Z}$ be complementary to f, i.e., $f \oplus \tilde{f}$ is a rank 4 bundle with signature $(2, 2)$. Let $U = (f \oplus \tilde{f})^{\perp}$. Then we have a splitting of $\underline{\mathbb{R}}^{s,t}$:

$$\underline{\mathbb{R}}^{s,t} = (f \oplus \tilde{f})^{\perp} \oplus_\perp U.$$

This induces a splitting of $\wedge^2\underline{\mathbb{R}}^{s,t}$:

$$\wedge^2\underline{\mathbb{R}}^{s,t} = \wedge^2 f \oplus f \wedge U \oplus f \wedge \tilde{f} \oplus \wedge^2 U \oplus \tilde{f} \wedge U \oplus \wedge^2 \tilde{f}.$$

5.1. **Second order deformations.** By Theorem 2.6, $\eta \in \Omega^1(\underline{\mathfrak{o}(s,t)})$ is a second order infinitesimal deformation if and only if η satisfies the Maurer-Cartan equation and

$$\eta F \leq \Omega^1(F) \quad \text{and} \quad (d_X\eta(Y))F \leq F, \tag{10}$$

for all $X, Y \in \Gamma T\Sigma$. Now $\eta F \leq \Omega^1(F)$ if and only if for linearly independent $\sigma, \xi \in \Gamma f$,

$$(\eta\sigma) \wedge \xi + \sigma \wedge (\eta\xi) = \eta(\sigma \wedge \xi) \in \Omega^1(F).$$

Since σ and ξ are linearly independent this is equivalent to

$$\eta f \leq \Omega^1(f).$$

Similarly, one can show that $(d_X\eta(Y))F \leq F$ is equivalent to $(d_X\eta(Y))f \leq f$. By the Leibniz rule, this holds if and only if for any section $\sigma \in \Gamma f$,

$$d_X(\eta(Y)\sigma) - \eta(Y)d_X\sigma \in \Gamma f. \tag{11}$$

Now, if we assume that X is a curvature direction, i.e., $X \in \Gamma T_i$ for some $i \in \{1,2\}$, then $\eta f \leq \Omega^1(f)$ implies that $d_X(\eta(Y)\sigma) \in \Gamma f_i$. Furthermore, $\eta(Y)d_X\sigma$ is orthogonal to $d_X\sigma$. Therefore, as the metric on $\underline{\mathbb{R}}^{s,t}$ restricts to a non-degenerate metric on f_i/f, we can deduce that

$$d_X(\eta(Y)\sigma),\, \eta(Y)d_X\sigma \in \Gamma f.$$

Now, $d_X(\eta(Y)\sigma) \in \Gamma f$ if and only if $\eta(Y)\sigma \in \Gamma s_i$. Since this holds for all $i \in \{1,2\}$, one has that $\eta f \equiv 0$. Also, $\eta(X)d_Y\sigma \in \Gamma f$ implies that $\eta f^{(1)} \leq \Omega^1(f)$. Thus, $\eta U \leq \Omega^1(f)$. Finally,

$$\eta f \equiv 0 \quad \text{and} \quad \eta U \leq \Omega^1(f)$$

if and only if

$$\eta \in \Omega^1(\wedge^2 f \oplus f \wedge U) = \Omega^1(f \wedge f^{\perp}).$$

One can easily check that the converse is true, i.e., given $\eta \in \Omega^1(f \wedge f^{\perp})$ satisfying the Maurer-Cartan equation, (10) holds.

The following proposition was proved in [30] in the case that $(s,t) = (4,2)$. Using analogous arguments one can show that it holds in the case that $(s,t) = (3,3)$ as well.

Proposition 5.1. *Suppose that $\eta \in \Omega^1(f \wedge f^{\perp})$. Then η satisfies the Maurer-Cartan equation if and only if it is closed. Furthermore, $\eta(T_i) \leq f \wedge f_i$ and $[\eta \wedge \eta] = 0$.*

Thus, we have arrived at the following proposition:

Proposition 5.2. *$\eta \in \Omega^1(\underline{\mathfrak{o}(s,t)})$ is a second order infinitesimal deformation of f if and only if η is closed and takes values in $f \wedge f^{\perp}$.*

We now wish to determine the uniqueness and triviality of such deformations. Following Lemma 1.5 and Lemma 1.6, we investigate second order deformations $h : \Sigma \to O(s,t)$ between F and itself. By Proposition 5.2, such a h is characterised by

$$hF = F \quad \text{and} \quad \theta_h := h^{-1}dh \in \Omega^1(f \wedge f^{\perp}). \tag{12}$$

Furthermore, $hF = F$ if and only if $hf = f$. Let $\sigma_i \in \Gamma s_i$ be a lift of one of the curvature spheres of f. Then, since $hf = f$ we have that

$$h\sigma_i = \nu,$$

for some $\nu \in \Gamma f$. Differentiating this condition with respect to the curvature direction $X \in \Gamma T_i$ yields

$$(d_X h)\sigma_i + h d_X \sigma_i = d_X \nu.$$

Since $\theta_h \in \Omega^1(f \wedge f^\perp)$, we have that $(d_X h)\sigma_i = 0$ and thus

$$h d_X \sigma_i = d_X \nu.$$

Since $d_X \sigma_i \in \Gamma f$ and $hf = f$, we must have that $d_X \nu \in \Gamma f$. Thus, $\nu \in \Gamma s_i$. Therefore, for some smooth function λ we have that $h\sigma_i = \lambda\sigma_i$. Differentiating this condition gives for all $Z \in \Gamma T\Sigma$,

$$(d_Z h)\sigma_i + h d_Z \sigma_i = (d_Z \lambda)\sigma_i + \lambda d_Z \sigma_i. \tag{13}$$

Then the orthogonality of h and that $\theta_h f \equiv 0$ implies that $\lambda = \pm 1$. Therefore, $h|_{s_i} = \pm id|_{s_i}$. We then have two cases to consider either $h|_f = \pm id|_f$ or $h|_{s_1} = \pm id|_{s_1}$ and $h|_{s_2} = \mp id|_{s_2}$.

Lemma 5.3. *Suppose that $h|_{s_1} = \pm id|_{s_1}$ and $h|_{s_2} = \mp id|_{s_2}$. Then S_1 and S_2 are constant.*

Proof. Let $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ and let $X \in \Gamma T_1$ and $Y \in \Gamma T_2$. Then

$$d_X \sigma_1 = \alpha_1 \sigma_1 + \beta_1 \sigma_2 \quad \text{and} \quad d_Y \sigma_2 = \alpha_2 \sigma_1 + \beta_2 \sigma_2,$$

for smooth functions $\alpha_1, \alpha_2, \beta_1, \beta_2$. Now

$$\pm(\alpha_1\sigma_1 + \beta_1\sigma_2) = \pm d_X\sigma_1 = d_X(h\sigma_1) = (d_X h)\sigma_1 + h d_X \sigma_1 = \pm\alpha_1\sigma_1 \mp \beta_1\sigma_2,$$

since $\theta_h f \equiv 0$. Thus $\beta_1 = 0$. Similarly, one can show that $\alpha_2 = 0$. Then, since $X \in \Gamma T_1$ and $Y \in \Gamma T_2$ are arbitrary. Thus, $d_X \sigma_1 \in \Gamma s_1$ and $d_Y \sigma_2 \in \Gamma s_2$ and one deduces from (9) that S_1 and S_2 are constant. □

S_1 and S_2 can only be constant if f is a Dupin cyclide. In that case we may define $\rho \in \mathrm{O}(s,t)$ such that ρ restricts to the identity on S_1 and minus the identity on S_2. One then has that $\tilde{h} := \rho h$ is a second order deformation between F and itself satisfying $\tilde{h}|_f = \pm id|_f$.

So let us now assume that $h|_f = \pm id|_f$. Then by (13), $h|_{f^{(1)}} = \pm id|_{f^{(1)}}$. By differentiating this condition again one finds that $h|_{f^{(2)}/f} = \pm id|_{f^{(2)}/f}$. Therefore we may write

$$h = \pm(id + \xi),$$

where ξ satisfies $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$ and $\xi f^\perp \equiv 0$. Since $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$, we have that $(\xi v, \xi w) = 0$ for all $v, w \in \Gamma\underline{\mathbb{R}}^{s,t}$.The orthogonality of h then implies that ξ is skew-symmetric. Combining this with the fact that $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$ and $\xi f^\perp \equiv 0$ gives that $\xi \in \Gamma(\wedge^2 f)$. Hence, $h = \pm\exp(\xi)$.

Conversely, it is straightforward to check that if $h = \pm\exp(\xi)$, for some $\xi \in \Gamma(\wedge^2 f)$, then h satisfies (12). We have thus arrived at the following lemmata:

Lemma 5.4. *Suppose that f and $\hat{f}$ are second order deformations of each other via g_1 and g_2. Then in the case that f is not a Dupin cyclide we have that $g_2 = \pm g_1 \exp(\xi)$ for some $\xi \in \Gamma(\wedge^2 f)$. In the case that f is a Dupin cyclide, either $g_2 = \pm g_1 \exp(\xi)$ or $g_2 = \pm\rho g_1 \exp(\xi)$.*

Lemma 5.5. *η is a trivial second order infinitesimal deformation of f if and only if $\eta = d\xi$ for some $\xi \in \Gamma(\wedge^2 f)$.*

As shown in [30], since $\sigma \mapsto \eta(X)d_Y\sigma$ defines an endomorphism $f \to f$, there is a quadratic differential

$$q(X,Y) = \mathrm{tr}(\sigma \mapsto \eta(X)d_Y\sigma)$$

associated to closed one-forms taking values in $f \wedge f^\perp$. It turns out that we may use q to determine the triviality of η:

Lemma 5.6. *$q = 0$ if and only if $\eta = d\xi$ for some $\xi \in \Gamma(\wedge^2 f)$.*

Proof. We may write an arbitrary closed one-form $\eta \in \Omega^1(f \wedge f^\perp)$ as

$$\eta = \alpha\,\sigma_1 \wedge d\sigma_1 + \beta\,\sigma_2 \wedge d\sigma_1 + \gamma\,\sigma_1 \wedge d\sigma_2 + \delta\,\sigma_2 \wedge d\sigma_2 \; mod\, \Omega^1(\wedge^2 f)$$

for $\sigma_1 \in \Gamma s_1$, $\sigma_2 \in \Gamma s_2$ and some smooth functions $\alpha, \beta, \gamma, \delta$. The quadratic differential of η is then

$$q = -\alpha(d\sigma_1, d\sigma_1) - \delta(d\sigma_2, d\sigma_2).$$

Since $(d\sigma_1, d\sigma_1) \in \Gamma(T_2^*)^2$ and $(d\sigma_2, d\sigma_2) \in \Gamma(T_1^*)^2$, one has that $q = 0$ if and only if $\alpha = \delta = 0$. One can clearly see that if $\eta = d\xi$, for some $\xi := \lambda\sigma_1 \wedge \sigma_2$, then $\alpha = \delta = 0$. On the other hand, if $\alpha = \delta = 0$, then the closure of η implies that $\beta = -\gamma$ and moreover $\eta = d(\beta\sigma_2 \wedge \sigma_1)$. Hence $\eta = d\xi$ for $\xi := \beta\sigma_2 \wedge \sigma_1$. □

We thus obtain the main theorem of this section:

Theorem 5.7. *$f : \Sigma \to \mathcal{Z}$ is deformable to second order if and only if there exists a closed one-form η taking values in $f \wedge f^\perp$ such that $q \neq 0$.*

Using the gauge theoretic definition of Ω- and Ω_0-surfaces of [30], one recovers the following result:

Corollary 5.8 ([27])**.** *Ω- and Ω_0-surfaces are the only second order deformable surfaces of Lie sphere geometry.*

Remark 5.9. In [9, 27] it was shown how second order deformable maps in Lie sphere geometry yield deformable maps in conformal and Laguerre geometry. For more information about deformability in Laguerre geometry, see [26, 29].

5.2. **Third order deformations.** In this subsection we shall show that rigidity occurs at third order for Legendre maps. Suppose that η is a third order infinitesimal deformation of F. Then by Theorem 5.7, $\eta \in \Omega^1(f \wedge f^\perp)$ and η is closed. Now by Theorem 2.6, for $X, Y, Z \in \Gamma T\Sigma$,

$$(d_X d_Y \eta(Z))F \leq F.$$

or, equivalently,

$$(d_X d_Y \eta(Z))f \leq f. \tag{14}$$

Let $\sigma \in \Gamma f$ and assume that X is a curvature direction of f, i.e, $X \in \Gamma T_i$ for $i \in \{1, 2\}$. Then by the Leibniz rule, equation (14) implies that

$$d_X((d_Y\eta(Z))\sigma) - (d_Y\eta(Z))d_X\sigma \in \Gamma f. \tag{15}$$

Now since $(d_Y\eta(Z))\sigma \in \Gamma f$, we have that $d_X((d_Y\eta(Z))\sigma) \in \Gamma f_i$. Furthermore, as $d_Y\eta(Z)$ is skew-symmetric, $(d_Y\eta(Z))d_X\sigma$ is orthogonal to $d_X\sigma$. Thus, equation (15) holds if and only if

$$d_X((d_Y\eta(Z))\sigma) \in \Gamma f \quad \text{and} \quad (d_Y\eta(Z))d_X\sigma \in \Gamma f.$$

Now $d_X((d_Y\eta(Z))\sigma) \in \Gamma f$ implies that

$$(d_Y\eta(Z))\sigma \in \Gamma s_i.$$

Since i was arbitrary, we then have that $(d_Y\eta(Z))\sigma = 0$. By the Leibniz rule this implies that

$$d_Y(\eta(Z)\sigma) - \eta(Z)d_Y\sigma = 0,$$

and since $\eta(Z)f = 0$, we have that

$$\eta(Z)d_Y\sigma = 0.$$

Hence, $\eta f^\perp \equiv 0$ and thus $\eta \in \Omega^1(\wedge^2 f)$. One can then check that η being closed implies that $\eta \equiv 0$. We have thus arrived at the following result:

Theorem 5.10. *Legendre maps are rigid to third order.*

6. Projective applicability revisited

It is well known that surfaces in projective space $F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ can be represented by their contact lifts in $\mathbb{R}^{3,3}$:

$$f = F \wedge F^{(1)}.$$

The derived bundle of this contact lift is

$$f^{(1)} = F^{(1)} \wedge F^{(1)} + F \wedge \underline{\mathbb{R}}^4.$$

Recall also that there is an isomorphism $\phi : \mathfrak{sl}(4) \to \mathfrak{o}(3,3)$, defined by

$$\phi(A)\,(v \wedge w) = Av \wedge w + v \wedge Aw.$$

Since ϕ is constant, ϕ intertwines the trivial connections on $\underline{\mathfrak{sl}(4)}$ and $\underline{\mathfrak{o}(3,3)}$. Let $\Theta \leq \underline{\mathfrak{sl}(4)}$ denote the subbundle of $\underline{\mathfrak{sl}(4)}$ such that $A \in \Gamma\Theta$ if and only if

$$AF = 0 \quad \text{and} \quad AF^{(1)} \leq F.$$

Then ϕ yields an isomorphism between Θ and $f \wedge f^\perp$. Since ϕ is constant one has that closed 1-forms taking values in Θ are in one-to-one correspondence with closed one forms taking values in $f \wedge f^\perp$. Furthermore, if we let Ψ denote the subbundle of Θ defined by $A \in \Gamma\Psi$ if and only if

$$AF^{(1)} = 0 \quad \text{and} \quad imA \leq F,$$

then ϕ yields an isomorphism between Ψ and $\wedge^2 f$. Thus, one deduces that the triviality of second order infinitesimal deformations is preserved by ϕ. We have thus recovered the classical result of Fubini [18]:

Theorem 6.1. *A surface in projective* 3*-space is deformable of order two if and only if its contact lift is deformable of order two.*

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