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Analytic equilibrium of "straight tokamak" plasma bounded by a magnetic separatrix / Porcelli, F.; Yolbarsop, A.. - In: PHYSICS OF PLASMAS. - ISSN 1070-664X. - 26:5(2019), p. 054501.

*Availability:*

This version is available at: 11583/2862228 since: 2021-01-17T19:26:42Z

*Publisher:*

American Institute of Physics Inc.

*Published*

DOI:10.1063/1.5096838

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Cite as: Phys. Plasmas **26**, 054501 (2019); <https://doi.org/10.1063/1.5096838>

Submitted: 20 March 2019 . Accepted: 23 April 2019 . Published Online: 10 May 2019

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## ABSTRACT

Theoretical and experimental considerations suggest that axisymmetric perturbations that are resonant at the X-point(s) of a magnetic divertor separatrix may play a role in the understanding of Edge Localized Modes in tokamak experiments and their active control via so-called *vertical kicks*. With this motivation in mind, the first step in the development of an analytical model for resistive axisymmetric X-point modes is presented, i.e., finding an adequate, but at the same time relatively simple analytical magnetohydrodynamic equilibrium for a plasma column with a noncircular cross section bound by a magnetic separatrix. An early example is Gajewski's equilibrium solution [R. Gajewski, Phys. Fluids **15**, 70 (1972)], which, however, has the shortcoming that infinite external currents placed at an infinite distance from the X-points produce the elliptical elongation of the plasma column. In this article, Gajewski's solution is extended to the case where external currents are located at a finite distance from the boundary of the plasma current density and the latter is distributed uniformly over a domain bound by a nearly elliptical magnetic flux surface.

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The stability of the X-point region of tokamak plasmas with divertor configurations takes on great importance in the realization of sustainable High-confinement (H-mode) regimes<sup>1</sup> with high fusion performance. In particular, instabilities like Edge Localized Modes (ELMs)<sup>2</sup> modulate the heat flux across the magnetic separatrix and can give rise to considerable heat loads on divertor plates.

The toroidal magnetic field line going through the X-point is resonant to axisymmetric MHD perturbations. Indeed, considering a generic perturbation  $\xi(\psi, \vartheta, \phi) \approx \sum_m \xi_m(\psi) e^{im\vartheta + in\phi}$ , the resonant condition  $\mathbf{B} \cdot \nabla \xi = inB_T \xi + iB_P(\psi) \sum_m m \xi_m e^{im\vartheta + in\phi} = 0$  is satisfied at the X-point, where  $B_P(\psi)$  vanishes, for toroidal mode number  $n=0$  and any poloidal mode number  $m$  ( $B_T$  and  $B_P$  are the toroidal and poloidal components of the magnetic field, respectively, and  $\psi$  is the magnetic flux). When resistivity is accounted for, localized toroidal current sheets can be driven unstable, leading to a change in the X-point topological structure. Such a process has been studied extensively in the context of astrophysical plasmas.<sup>3,4</sup> Therefore, one may suspect that in a tokamak, the magnetic X-point region may be strongly influenced by axisymmetric MHD perturbations.

From an experimental viewpoint, observations correlate type-I giant ELMs with  $n=0$  axisymmetric perturbations. For instance, a puzzling experimental fact was the rather large shift of the strike points

on divertor target plates observed during giant ELMs in JET experiments.<sup>5,6</sup> It was suggested<sup>7</sup> that this large shift could be explained by the inferred<sup>6</sup> relatively large currents flowing from the magnetic X-point to the target plates near both strike points. These earlier observations were corroborated by later JET studies.<sup>8,9</sup>

Another case in point is the question of the *vertical kick* experiments for active ELM control.<sup>10–12</sup> In a tokamak, vertical kicks are controlled, axisymmetric magnetic perturbations driven by external coils, which result in rapid upward or downward displacements of the plasma column. It may be argued that vertical kicks could lead to a change in the X-point topological structure.

Yet, in the context of tokamak plasmas, it appears that the theory of resonant, resistive axisymmetric X-point modes has not been developed so far. The reason may be related to the mathematical and numerical difficulties associated with the treatment of these modes. From an analytical viewpoint, the main difficulties concern the proper treatment of a 2D equilibrium with external currents and with the 2D nature of the current sheets that would form around the X-point region. From a numerical viewpoint, none of the codes developed so far for the treatment of resistive-MHD perturbations in realistic tokamak geometry is able to treat properly the divertor X-point region, with the possible exception of the JOEK-STARWALL code,<sup>12</sup> where

free-boundary conditions appropriate to the treatment of axisymmetric perturbations and the vertical instability have been implemented. Furthermore, both analytical and numerical models face the difficulty of the proper treatment of the plasma–scrape-off-layer transition region.

In this article, we are concerned with the first step in the development of an analytical model for resistive axisymmetric X-point modes, i.e., finding an adequate, but at the same time relatively simple analytical MHD equilibrium for a plasma column with a noncircular cross section bound by a magnetic separatrix. In searching the literature, we found that the analytical equilibrium model that comes closest to our needs is the one published by Gajewski in Ref. 13.

Gajewski’s solution describes the equilibrium of an elliptical plasma cylinder bound by a magnetic separatrix with two magnetic X-points, corresponding, in tokamak jargon, to a double-null divertor configuration. It has the distinctive advantage of being a particularly simple model equilibrium and therefore amenable to analytical work; on the other hand, its main shortcoming is that the elliptical elongation of the plasma column is produced by infinite external currents placed at an infinite distance from the X-points. This shortcoming makes Gajewski’s model equilibrium inadequate for the analytical treatment of the effects of axisymmetric modes interacting with externally modulated currents and for the *vertical kick* scenario.

In this article, we extend Gajewski’s equilibrium model by considering finite external currents located at a finite distance from the magnetic X-points. The cylindrical plasma column extends along the  $z$ -axis and no physical quantity depends on the coordinate  $z$ , which mimics the toroidal angle  $\varphi$  of a tokamak configuration. The density  $J_z$  of the longitudinal plasma current is taken to be uniform in space up to a nearly (but not exactly) elliptical cross section with major axis  $a$  and minor axis  $b$ . The external currents are modeled by two equal, parallel current filaments also directed along the  $z$ -axis and placed symmetrically around the plasma column at distance  $l$  from the plasma center, as shown in Fig. 1. The external currents and the X-points of the magnetic separatrix are located along the  $x$ -axis. The coordinates of the two X-points are denoted by  $\pm l_X$ , where clearly  $a < l_X < l$ . Thus, the equilibrium is horizontally elongated, as in Ref. 13.

We introduce a small expansion parameter

$$\varepsilon = \frac{a^2 - b^2}{l^2}. \tag{1}$$

As we shall see, Gajewski’s equilibrium<sup>13</sup> is recovered in the limit  $\varepsilon \rightarrow 0$  and  $l \rightarrow \infty$ , where the external currents diverge as

$$\frac{I_{ext}}{I_p} \sim \frac{l^4}{(a+b)^2(a^2+b^2)} \varepsilon = \frac{a-b}{a+b} \frac{l^2}{(a^2+b^2)}. \tag{2}$$

In Eq. (2),  $I_{ext}$  is the external current carried by either one of the two filaments and  $I_p = \pi ab J_z$  is the plasma current. In Gajewski’s limit, the surface bounding the plasma current becomes exactly elliptical. Equation (2) is consistent with the circular cross section limit  $a = b$ , where no external currents are required for equilibrium. The new equilibrium found in this article is valid up to terms of order  $\varepsilon$ .

To start with, the magnetic field is represented as  $\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi$ , where  $\mathbf{e}_z$  is the unit vector along the direction of the ignorable  $z$  coordinate. As in Ref. 13, c.g.s. units are used in this article. The current density is  $4\pi \mathbf{J} = c \nabla \times \mathbf{B} = c \nabla B_z \times \mathbf{e}_z + c \nabla^2 \psi \mathbf{e}_z$ . The ideal MHD equilibrium condition is  $c \nabla p = \mathbf{J} \times \mathbf{B}$ . From these equations,  $J_z = J_z(\psi)$ , and therefore,

$$c \nabla^2 \psi = 4\pi J_z(\psi) \tag{3}$$

can be viewed as a nonlinear equation for  $\psi$  to be solved subject to appropriate boundary conditions. Furthermore, we find that  $p = p(\psi)$  and  $B_z = B_z(\psi)$ .

In solving the equilibrium problem, the standard procedure<sup>14</sup> is to note that the equilibrium problem contains three functions of  $\psi$ , i.e.,  $J_z(\psi)$ ,  $p = p(\psi)$ , and  $B_z = B_z(\psi)$ , of which two can be chosen arbitrarily and the third is derived consistently with this choice. Following Gajewski,<sup>13</sup> our simplifying assumption is  $J_z(\psi)$  constant inside a domain  $D$  of the Oxy plane centered at the origin,  $x = y = 0$ , extending symmetrically in the  $x$  and  $y$  directions. This domain is bound by the curve  $C(x, y) = 0$  (see Fig. 1).

The conceptual difference between Gajewski’s procedure<sup>13</sup> and the analysis in this article is stated mathematically in the following terms. If we chose the curve  $C(x, y) = 0$  to correspond exactly to an ellipse, as in Ref. 13, then the self-consistent solution would require infinite external currents located at  $x = \pm \infty$ ; indeed,  $B_y(x, y = 0) \rightarrow \infty$  with  $x \rightarrow \pm \infty$  in Ref. 13. Instead, in this article, we assume that the boundary  $C(x, y) = 0$  corresponds only approximately to an ellipse, in the sense that we can set  $C(x, y) = (x^2/a^2) + (y^2/b^2) - 1 + \delta C(x, y) = 0$ , where  $\delta C(x, y)$  represents a correction of order  $\varepsilon$  to the elliptical boundary. Insofar as only the leading order equilibrium solution in  $\varepsilon$  is of interest, exact knowledge of the function  $\delta C(x, y)$  is not required. In this way, finite external currents located at a finite distance from the domain  $D$  are allowed.

The choice  $J_z = J_0 = const$  inside  $D$  converts Eq. (3) into a linear, inhomogeneous partial differential equation. Thus, the equations to be solved are

$$c \nabla^2 \psi = 4\pi J_0 \tag{4}$$

inside domain  $D$  and

$$c \nabla^2 \psi = 4\pi I_{ext} \cdot [\delta(x - l, y) + \delta(x + l, y)] \tag{5}$$

outside domain  $D$ , with boundary conditions

$$\psi = \psi_b = const \tag{6}$$

on the curve  $C(x, y) = 0$  bounding the domain  $D$  and

$$\{\psi\} = 0; \quad \{\partial \psi / \partial n\} = 0, \tag{7}$$

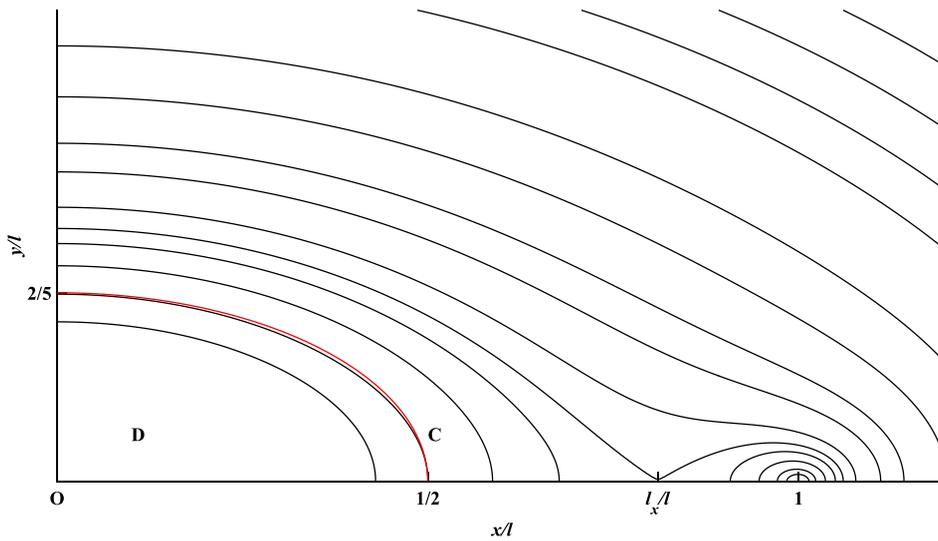
where the angular brackets  $\{\phi\}$  denote the jump of the generic quantity  $\phi$  across the boundary of  $D$  and  $\mathbf{n}$  is the outer normal. The right-hand-side of Eq. (5) is the contribution of the two, equal external currents, approximated by current filaments, i.e., delta-functions in two dimensions, located at  $x = \pm l$  and  $y = 0$ .

Finally, at infinity, we require

$$\psi \sim \ln r \quad \text{for} \quad r = (x^2 + y^2)^{1/2} \rightarrow \infty, \tag{8}$$

which ensures that the equilibrium magnetic field goes to zero as  $r^{-1}$  for  $r \rightarrow \infty$ , as it should when all (plasma plus external) currents flow across a finite region of the Oxy plane.

The difference between our approach and Gajewski’s procedure<sup>13</sup> can be summarized by the following three points: (i)  $\delta C = 0$  in Ref. 13, while  $\delta C(x, y)$  represents a small, order  $\varepsilon$  correction to the elliptical boundary in our approach; (ii)  $\nabla^2 \psi = 0$  outside domain  $D$  in Ref. 13, while Eq. (5) is used in our approach; and (iii) the condition  $\psi \sim \ln r$  for  $r \rightarrow \infty$  is enforced in our work (it is not satisfied in Ref. 13).



**FIG. 1.** Magnetic flux surfaces in the plane of coordinates  $x/l$  and  $y/l$ . Because of symmetry, only the quadrant with positive  $x$  and  $y$  is shown. Parameter values are  $a/l = 1/2$  and  $b/l = 2/5$ , yielding  $\varepsilon = 0.09$ . The red line corresponds to the ellipse with semi-axes  $a$  and  $b$ .

The solution of the equilibrium problem (4)–(8) is presented in the following. To obtain the complete equilibrium solution, we proceed as follows. As we pointed out earlier, we still have the freedom to choose either  $p(\psi)$  or  $B_z(\psi)$ , subject to realistic physical constraints. Let us choose a convenient  $p(\psi)$ ; then,  $B_z(\psi)$  can be obtained from the force balance relation, which can be cast in the form

$$c \frac{d}{d\psi} \left( p + \frac{B_z^2}{8\pi} \right) = -J_z. \tag{9}$$

From the knowledge of  $\psi(x, y)$  and the solution of Eq. (9), we can obtain  $p, B_z$ , and  $J_z$  as functions of  $x$  and  $y$ .

Now, it is convenient to introduce elliptical coordinates  $(\mu, \vartheta)$ , related to Cartesian coordinates  $(x, y)$  by the transformation

$$\begin{cases} x = A \cosh \mu \cos \vartheta, \\ y = A \sinh \mu \sin \vartheta. \end{cases} \tag{10}$$

In this coordinate system, curves of constant  $\mu$  are ellipses. If we set

$$\begin{cases} a = A \cosh \mu_b, \\ b = A \sinh \mu_b, \end{cases} \tag{11}$$

then  $A^2 = a^2 - b^2$  and  $\mu = \mu_b$  defines the boundary  $C(x, y) = 0$  of the plasma current to zeroth order in the parameter  $\varepsilon$ . This boundary is an ellipse with major semi-axis  $a$  and minor semi-axis  $b$ .

The equilibrium solution for  $\psi$  is constructed according to the following procedure:

- (i) Using the superposition principle, the solution for the flux function can be written as  $\psi(x, y) = \psi_p(x, y) + \psi_{ext}(x, y)$ , where  $\psi_p \propto I_p$  is the magnetic flux generated by the plasma current flowing inside domain  $D$  and  $\psi_{ext} \propto I_{ext}$  is the magnetic flux generated by the two external current filaments. By construction, both  $\psi_p$  and  $\psi_{ext}$  behave as  $\ln r$  for  $r \rightarrow \infty$ , and so boundary condition (8) is automatically satisfied. Since, in this article, we are concerned only with the leading order solution for  $\psi$  (in powers of  $\varepsilon$ ), it is sufficient to approximate the boundary of  $D$  by the ellipse  $\mu = \mu_b$  when solving for  $\psi_p(x, y)$ .
- (ii) We note that the boundary of  $D$  must be a magnetic flux surface of constant  $\psi$ . However, as we shall see, both  $\psi_p(x, y)$  and

$\psi_{ext}(x, y)$  are not constant over the boundary, only their sum will be. This requirement will impose a relationship between  $I_p$  and  $I_{ext}$  as already anticipated in Eq. (2).

Let us discuss first the flux function generated by the plasma current. Using elementary methods, the solution for  $\psi_p$  inside domain  $D$  (subscript “in”) in Cartesian coordinates is

$$c\psi_{p,in}(x, y) = 2I_p \left[ \frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} \right], \tag{12}$$

and in elliptical coordinates

$$c\psi_{p,in}(\mu, \vartheta) = 2I_p(a^2 - b^2) \left[ \frac{(\cosh \mu \cos \vartheta)^2}{a(a+b)} + \frac{(\sinh \mu \sin \vartheta)^2}{b(a+b)} \right]. \tag{13}$$

The solution for  $\psi_p$  outside  $D$  (subscript “out”) in elliptical coordinates is

$$c\psi_{p,out}(\mu, \vartheta) = I_p \left[ 1 + 2(\mu - \mu_b) + e^{-2\mu} \cos 2\vartheta \right]. \tag{14}$$

On the boundary of  $D$ , i.e., on the ellipse  $\mu = \mu_b$ , we find

$$c\psi_{p,in}(\mu_b, \vartheta) = c\psi_{p,out}(\mu_b, \vartheta) = I_p [1 + e^{-2\mu_b} \cos 2\vartheta]. \tag{15}$$

Thus, boundary conditions (7) are satisfied. However,  $\psi_p(\mu_b, \vartheta)$  is not constant over the elliptical boundary. For  $r \rightarrow \infty$ ,  $c\psi_{p,out} \sim 2I_p \mu \propto \ln r$ , as it should.

Next, the magnetic flux generated by the two, equal external current filaments is

$$c\psi_{ext}(x, y) = I_{ext} \ln \left[ 1 + 2(\hat{y}^2 - \hat{x}^2) + (\hat{x}^2 + \hat{y}^2)^2 \right], \tag{16}$$

where  $\hat{x} = x/l$ ,  $\hat{y} = y/l$ . Clearly,  $\psi_{ext} \propto \ln r$  for  $r \rightarrow \infty$ . In elliptical coordinates

$$c\psi_{ext}(\mu, \vartheta) = I_{ext} \ln \left\{ 1 - \varepsilon \left[ 1 + \cosh(2\mu) \cos(2\vartheta) \right] + \frac{1}{4} \varepsilon^2 \left[ \cosh(2\mu) + \cos(2\vartheta) \right]^2 \right\}. \tag{17}$$

Finally, we impose  $\psi = \text{const}$  on the boundary of  $D$ . Near the boundary, we can expand  $\psi_{\text{ext}}(\mu, \vartheta)$  in powers of  $\varepsilon$ . To leading order, where the boundary can be approximated by the ellipse  $\mu = \mu_b$ , we obtain

$$c\psi(\mu_b, \vartheta) = I_P [1 + e^{-2\mu_b} \cos 2\vartheta] - I_{\text{ext}} \varepsilon [1 + \cosh(2\mu_b) \cos(2\vartheta)] + O(\varepsilon^2). \quad (18)$$

Imposing  $\psi(\mu_b, \vartheta) = \text{const}$ , i.e., independent of  $\vartheta$ , leads to the condition

$$I_P e^{-2\mu_b} = I_{\text{ext}} \varepsilon \cosh(2\mu_b). \quad (19)$$

Using  $\exp(2\mu_b) = (a+b)/(a-b)$  and  $\cosh(2\mu_b) = (a^2 + b^2)/(a^2 - b^2)$ , Eq. (19) can be cast in the form anticipated by Eq. (2). The constant value of  $\psi$  at  $\mu = \mu_b$ , neglecting terms  $O(\varepsilon^2)$ , is

$$\psi(\mu_b, \vartheta) = \frac{2I_P}{c} \frac{ab}{(a^2 + b^2)} = \psi_b. \quad (20)$$

We observe that, using Cartesian coordinates and the leading order approximation

$$c\psi_{\text{ext}}(x, y) \approx 2I_{\text{ext}}(\hat{y}^2 - \hat{x}^2) \quad (21)$$

valid for small  $\varepsilon$  in the proximity of domain  $D$ , the flux function inside  $D$ , neglecting terms of order  $O(\varepsilon^2)$ , becomes

$$\psi(x, y) \approx \psi_b \cdot \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right). \quad (22)$$

We could have expected the result in Eq. (22), which agrees with Gajewski's solution<sup>13</sup> inside  $D$  in the limit  $\varepsilon \rightarrow 0$ . On the other hand, our solution for  $\psi$  outside  $D$  is

$$c\psi(\mu, \vartheta) = I_P \left\{ 1 + 2(\mu - \mu_b) + e^{-2\mu} \cos(2\vartheta) + \frac{l^4}{(a+b)^2(a^2+b^2)} \varepsilon \ln \left\{ 1 - \varepsilon [1 + \cosh(2\mu) \cos(2\vartheta)] + \frac{1}{4} \varepsilon^2 [\cosh(2\mu) + \cos(2\vartheta)]^2 \right\} \right\}. \quad (23)$$

Gajewski's solution is recovered from Eq. (23) by letting  $l \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ . In that limit, the external current contribution (21) holds for small as well as arbitrarily large values of  $x$  and  $y$  and the complete flux function outside domain  $D$  reduces to Eq. (33) of Ref. 13.

A graphic example of the solution in Eq. (23) is given in Fig. 1, where surfaces of constant  $\psi$  are drawn on the plane with normalized coordinates  $x/l$  and  $y/l$ . Because of the up-down and left-right symmetry, only the quadrant with positive  $x$  and  $y$  is shown. The parameters for this figure are  $a/l = 0.5$  and  $b/l = 0.4$ , corresponding to  $\varepsilon = 0.09$ . For such a small but finite value of  $\varepsilon$ , the boundary of domain  $D$  corresponding to the contour line in black going through the points with coordinates  $(x/l = 0.5, y/l = 0)$  and  $(x/l = 0, y/l = 0.4)$  is very closely approximated by the ellipse  $\mu = \mu_b$  represented by the red curve in the figure.

To find the magnetic separatrix, we first solve for the coordinates of the X-points, where  $\nabla\psi = 0$ , and subsequently look for the surface of constant flux passing through such points. In elliptical coordinates

$$\frac{\partial\psi}{\partial\mu} = 0; \quad \frac{\partial\psi}{\partial\vartheta} = 0 \quad \text{at} \quad \mu = \mu_X, \quad \vartheta = \vartheta_X, \quad (24)$$

where  $(\mu_X, \vartheta_X)$  label the X-point coordinates. The corresponding algebra is straightforward, even though somewhat cumbersome, and therefore, we omit the details. Because of up-down symmetry, the X-points are located on the  $x$ -axis, i.e., at values of  $\vartheta_X = 0, \pi$  corresponding to the two solutions of the second of Eq. (24). In Gajewski's limit<sup>13</sup> (subscript "G"), the first of Eq. (24) can be solved explicitly, yielding  $\mu_{XG} = 2\mu_b$ , or equivalently

$$l_{XG} = \frac{a^2 + b^2}{(a^2 - b^2)^{1/2}}, \quad (25)$$

where  $l_X$  is the distance of the X-point from the origin of the Oxy plane. Carrying out the calculation to first order in  $\varepsilon$ , we find  $\mu_X = \mu_{XG} - \varepsilon\mu_{X1}(\delta)$ , where  $\delta = b/a$  and  $\mu_{X1}(\delta) = (1 + \delta^2)^3 / [(1 - \delta^2)(1 + \delta)]^2$ , or equivalently

$$l_X = \left[ 1 - \frac{2\varepsilon_0\mu_{X1}(\delta)\delta}{(1 + \delta^2)} \right] l_{XG}. \quad (26)$$

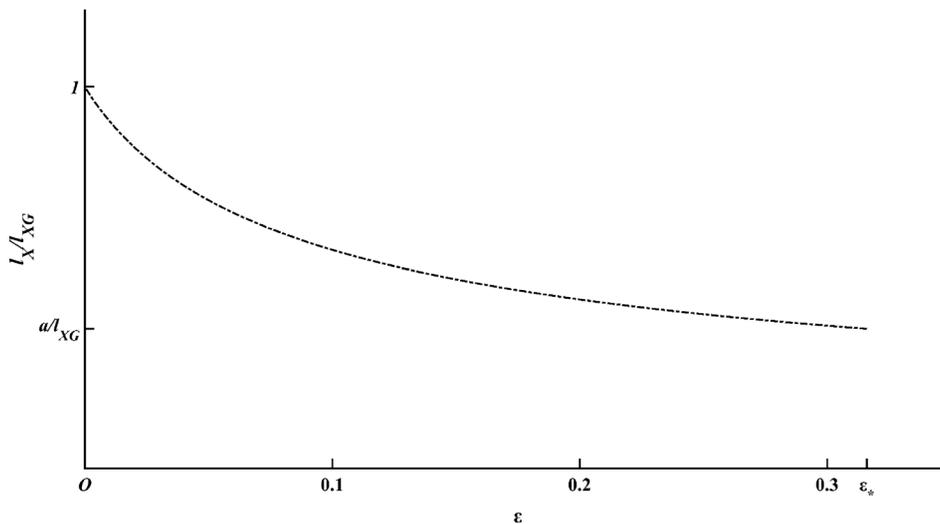
The separatrix is the contour of constant  $\psi$  going through the X-points,  $\psi(\mu, \vartheta) = \psi(\mu_X, 0) \equiv \psi_X$ .

Figure 2 shows a graph of  $l_X/l_{XG}$ , obtained numerically from the full solution in Eq. (23), as a function of  $\varepsilon$  for fixed  $\delta = 0.8$ ; for this value of  $\delta$ ,  $a/l_{XG} = 0.37$ . The curves stop at  $\varepsilon = \varepsilon_*$ , which corresponds to the limit where  $l_X = a$ ; in this case,  $\varepsilon_* = 0.32$ . The equilibrium solution ceases to be valid for  $\varepsilon > \varepsilon_*$ .

In conclusion, we have extended Gajewski's solution<sup>13</sup> for the equilibrium of a plasma column bound by a magnetic separatrix to the case where the external currents are located symmetrically at a finite distance from the boundary of the plasma current density and the latter is distributed uniformly over a domain  $D$  bound by a nearly elliptical magnetic flux surface. Three main results are found in this article: (i) The analysis relies on a small expansion parameter,  $\varepsilon$ , defined in Eq. (1); to leading order in  $\varepsilon$ , the boundary of domain  $D$  can be approximated by an ellipse. (ii) Equilibrium requires that the external currents,  $I_{\text{ext}}$ , are related to the plasma current according to the criterion in Eq. (2). (iii) The geometric structure and topology of the magnetic flux surfaces depend on two parameters only:  $a/l$  and  $\delta = b/a$ , where  $a$  and  $b$  are the major and minor semi-axes of the elliptical boundary, respectively, and  $\pm l$  are the coordinates of the two external current filaments on the  $x$ -axis. Gajewski's equilibrium is recovered in the limit  $l \rightarrow \infty$ .

This analytical equilibrium is expected to be unstable to ideal MHD vertical displacements of the plasma column (in the notations of this article and of Ref. 13, vertical refers to the direction of the  $x$ -axis). Just like in the case of tokamak plasmas with an elongated cross section, we can also expect that modulating in time the external currents can stabilize the vertical instability. In the ideal MHD case, this would mimic the passive feedback stabilization scenario of a tokamak plasma, where time-dependent image currents are induced on the conducting wall containing the plasma.

Of more interest will be to study the case of a resistive plasma extending to the magnetic separatrix. In this respect, we note that the equilibrium solution in this article, where the plasma current density is



**FIG. 2.** X-point coordinate,  $l_X$ , normalized to Gajewski's limiting value  $l_{XG}$ , as a function of  $\varepsilon$  and for fixed  $b/a = 0.8$  and  $\varepsilon_* = 0.32$ .

confined within an elliptical boundary, holds true also in the case where the plasma particle density extends beyond the elliptical boundary and reaches the magnetic separatrix. In this situation, a vertical plasma displacement would be resonant at the magnetic X-points, giving rise to the possibility that current sheets centered at the X-points be driven unstable. In the equivalent tokamak scenario, this type of perturbation is what we refer to as resistive axisymmetric X-point modes.<sup>7</sup> The analytical treatment of these modes will be the subject of a future investigation.

F.P. would like to acknowledge hospitality at the University of Science and Technology of China in Hefei, where part of this research was carried out. This work was sponsored in part by EUROfusion Enabling Research Grant No. AWP17-ENR-MFE-CCFE-01.

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