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Mixed boundary value problems in two-dimensional singularly perturbed domains with the Steklov spectral condition

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Abstract

We investigate the spectrum of the Laplace equation with various, Steklov and Dirichlet, Neumann, boundary conditions in the singularly perturbed two-dimensional domain $\Omega^\varepsilon = \Omega \setminus (\overline{\omega}_1^\varepsilon \cup \ldots \cup \overline{\omega}_2^\varepsilon)$ with small holes ω_j^ε of diameter $O(\varepsilon)$, $\varepsilon > 0$ being a small parameter. We consider all possible combinations of boundary conditions, but the interior boundaries $\gamma_j^\varepsilon = \partial\overline{\omega}_j^\varepsilon$, $j = 1, \ldots, J$, are supplied with conditions of the same type and, of course, the Steklov spectral condition is always put on either $\Gamma = \partial\Omega$, or $\gamma^\varepsilon = \gamma_1^\varepsilon \cup \ldots \cup \gamma_J^\varepsilon$. We construct the asymptotic as $\varepsilon \to +0$ of the eigenvalues λ_n^ε and the corresponding eigenfunctions u_n^ε. However, the main attention is paid to the analysis and error estimates for eigenvalues that present all peculiarities of the asymptotic procedures, while afterwards formulation of theorems on asymptotics of the corresponding eigenfunctions becomes rather standard. Furthermore, the most representative analysis are given in the Steklov–Neumann and pure Steklov problems, so that the Steklov–Dirichlet prolem is discussed condensely. The distinguishing feature of the two-dimensional boundary value problems is the dependence of asymptotic terms on the additional parameter $\zeta = |\ln\varepsilon|^{-1}$, either polynomial and rational, or analytic and homomorphic, and we discard such dependencies for various distributions of three types of the above-mentioned boundary conditions. In general situation only the “logarithmic" asymptotics with remainder of order η^k is available but the perturbation of a simple eigenvalue of the limit problem gets the power-law type with a remainder of order $\varepsilon^m \mathfrak{z}^k$. At the same time, the power-law asymptotics can be derived also in the case of a κ-multiple limit eigenvalue when all κ asymptotic forms for the perturbed eigenvalues have different higher-order terms.

For the pure Steklov problem, we are able to construct and justify the asymptotics of eigenvalues in both, the low-frequency $O(1)O$ and middle-frequency $O(\varepsilon^{-1})$ ranges of the spectrum. However, if the Neumann or Dirichlet conditions enter the problem, only either low-, or middle-frequency range becomes suitable for the asymptotic analysis. With the Neumann or Dirichlet condition on the interior boundary γ^ε, we obtain a limit problem in the intact domain Ω and the low-frequency range is available while these conditions on the exterior boundary Γ give rise to the limit family of exterior Steklov problems in $\Xi_j = \mathbb{R}^2 \setminus \overline{\omega}_j$, $j = 1, \ldots, J$, which describe asymptotics of the perturbed spectrum in the middle-frequency range. The pure Steklov problem in Ω^ε accepts both types of the above-mentioned limit problems

Keywords: Steklov spectral problems, singularly perturbed domains, asymptotics of eigenfunctions and eigenvalues

1 Introduction

?⟨subsec1⟩?
1.1 Formulation of problems

⟨subsec11⟩ We consider the Laplace equation

$$\Delta_x u^\varepsilon(x) = 0,\ x \in \Omega^\varepsilon, \tag{1.1}$$ 1

in the planar domain

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{j=1}^{J} \overline{\omega}_j^\varepsilon, \tag{1.2}$$ 2

where Ω and ω_j, $j = 1, \dots, J$ are domains in the plane $\mathbb{R}^2$ enveloped by simple closed smooth (for simplicity of class C^∞) contours $\Gamma = \partial\Omega$ and $\gamma_j = \partial\omega_j$, respectively. Furthermore,

$$\omega_j^\varepsilon = \{x : \xi^j = \varepsilon^{-1}(x - x^j) \in \omega_j\}, \tag{1.3}$$ 3

where $x^1, \dots, x^J$ are some fixed points in Ω, $x^j \neq x^k$ for $j \neq k$, and $\varepsilon \in (0, \varepsilon_0]$ is a small parameter while the bound $\varepsilon_0 > 0$ is fixed such that

$$\overline{\omega}_j^\varepsilon \subset \Omega,\ \overline{\omega}_j^\varepsilon \cap \overline{\omega}_k^\varepsilon = \varnothing,\ j, k = 1, \dots, J,\ j \neq k \ \text{ for } \varepsilon \leq \varepsilon_0.$$

If necessary, we diminish ε_0 in the sequel but keep the notation. We assume that ω_j contains the coordinate origin.

We supply the equation (1.1) with various boundary conditions on the exterior Γ and interior $\gamma_1^\varepsilon, \dots \gamma_J^\varepsilon$ parts of the boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$ including at least one of the following Steklov spectral conditions

$$\partial_\nu u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \Gamma, \tag{1.4}$$ 4

$$\partial_\nu u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \gamma_j^\varepsilon, j = 1, \dots, J, \tag{1.5}$$ 5

where ∂_ν is the outward normal derivative and λ^ε is the spectral parameter. In addition to the overall Steklov problem (1.1), (1.4),(1.4), we consider all possibl variants of boundary value problems replacing one of the Steklov conditions (1.4) or (1.5) with either the Neumann conditions

$$\partial_\nu u^\varepsilon(x) = 0,\ x \in \Gamma, \tag{1.6}$$ 6

$$\partial_\nu u^\varepsilon(x) = 0,\ x \in \gamma_j^\varepsilon,\ j = 1, \dots, J, \tag{1.7}$$ 7

or the Dirichlet ones

$$u^\varepsilon(x) = 0,\ x \in \Gamma, \tag{1.8}$$ 8

$$u^\varepsilon(x) = 0,\ x \in \gamma_j^\varepsilon,\ j = 1, \dots, J. \tag{1.9}$$ 9

The variational formulation of any introduced problems requires to find an eigenpair $\{\lambda^\varepsilon, u^\varepsilon\} \in \mathbb{R} \times \mathcal{H}^\varepsilon$ verifying the integral identity

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} = \lambda^\varepsilon(\rho^\varepsilon u^\varepsilon, v^\varepsilon)_{\Gamma^\varepsilon} \quad \forall v^\varepsilon \in \mathcal{H}^\varepsilon, \tag{1.10}$$ 10

where $\nabla_x = grad$, $\rho^\varepsilon = 0$ on Γ, resp. on $\gamma^\varepsilon = \gamma_1^\varepsilon \cup \dots \cup \gamma_J^\varepsilon$, in the case when the condition (1.4), resp. (1.5), is excluded, $(,)_\gamma$ is the natural scalar product in the Lebesgue space $L^2(\gamma)$, and $\mathcal{H}^\varepsilon$ is the Sobolev space $H^1(\Omega^\varepsilon)$ if none of the Dirichlet conditions (1.8) and (1.9) is involved into the problem but $\mathcal{H}^\varepsilon$ is the subspace $H_0^1(\Omega^\varepsilon, \Gamma)$, resp. $H_0^1(\Omega^\varepsilon, \gamma^\varepsilon)$, if the conditions (1.8), resp. (1.9), is imposed. Here,

$$H_0^1 = (\Omega^\varepsilon, \upsilon^\varepsilon) = \{u^\varepsilon \in H^1(\Omega^\varepsilon) : u^\varepsilon = 0 \text{ on } \upsilon^\varepsilon\}$$

for any $\upsilon^\varepsilon \subset \Gamma^\varepsilon$.

In any case, problem (1.10) has the eigenvalue sequence

$$0 \leq \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \ldots \leq \lambda_n^\varepsilon \leq \ldots \to +\infty, \quad (1.11)$$ 11

where the eigenvalues' multiplicity is taken into account, and the corresponding eigenfunctions $u_m^\varepsilon \in \mathcal{H}^\varepsilon$ can be subject to the normalization and orthogonality conditions

$$(\rho^\varepsilon u_m^\varepsilon, u_n^\varepsilon)_{\Gamma^\varepsilon} = \delta_{m,n}, \quad m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}, \quad (1.12)$$ 12

where $\delta_{m,n}$ is the Kronecker symbol.

1.2 Existing asymptotic results

⟨subsec12⟩ Asymptotic structures in the spectral Steklov problems having a close relation to the water-wave problems in finite ponds and infinite channels, cf. [1] and [2] and many other monographs and review papers, have been examined in many publications with various formulations and by different approaches. Let us mention some of them.

Infinite asymptotic series for eigenpairs of the Steklov problem (1.10), (1.4), (1.5) in a domain $\Omega^\varepsilon \subset \mathbb{R}^d$, $d \geq 3$, with only one cavity[1] Ω^ε of type (1.3) have been constructed in paper [3] and estimates of asymptotic remainders with any prescribed precision order have been derived. Although asymptotic procedures are based on the well-known methods of compound asymptotic expansions, cf. [4, Ch. 2, 4, 11], the very distinguishing feature of the particular Steklov problem considered in [3] is the existence in the low and middle-frequency ranges of the spectrum of two families of eigenvalues which can be decomposed as infinite asymptotic series[2] in powers in ε. The methods used in [3] do not directly work for the two-dimensional problems in the present paper because of the logarithmic behaviour of the fundamental solution $\Phi(x) = -(2\pi)^{-1} \ln|x|$ of the Laplacian in the plane $\mathbb{R}^2$ — the fact that $\Phi(x)$ in dimension $d \geq 3$ decays as $O\left(|x|^{2-d}\right)$ at infinity was crucially used in [3]. However, in Section **??** we essentially modify the asymptotic method and derive asymptotic expansions f eigenvalues in both the low and middle-frequency ranges of the spectrum (1.11) of the Steklov problem (1.10), (1.4), (1.5) in Ω^ε.

Another type of asymptotic analysis on the basis of a fruitful approach developed in [6], [7], was applied for eigenvalues of the Steklov problem in the domain Ω^ε with a single hole ω_1^ε, i.e., for $J = 1$ in (1.2). It is proved that simple eigenvalues are analytical functions of ε for dimensions $d \geq 3$ and in two variables ε, $|\ln\varepsilon|^{-1}$ in dimension $d = 2$. Meanwhile, this asymptotic analysis applies forcefully for eigenvalues in the low-frequency range of the spectrum. Although results in [6] do not provide explicit formulas for the above-mentioned analytic functions, they demonstrate that some of the formal asymptotic series constructed in [3] do converge. We emphasize tat this type of convergence cannot be verified by means of the asymptotic analysis in the paper [3] as well as by general approaches in the book [4].

1.3 Preliminary description of our further results

?⟨subsec13⟩? A specificity of two-dimensional problems, stationary and spectral, respectively, have been observed in the original papers [8] and [9], see also the monographs [10] and [4], namely, it had been discovered that terms in the asymptotic series for solutions and eigenpairs, respectively, are rational and holomorphic functions of $\mathfrak{z} = |\ln\varepsilon|^{-1}$. In this way, it had become possible to sum up series in powers of $\mathfrak{z}$ obtained in preceding studies of two-dimensional boundary-value problems with singular geometrical perturbations, however, summation of series in powers of ε is not available within the approach in [4] and [10] using the methods of compound and matched asymptotic expansions.

[1]Both the assumptions $J = 1$ and $d \geq 3$ are important for the techniques used in [3]

[2]The authors do not know other singularly perturbed problem where, in addition to eigevalues of order $\varepsilon^0 = 1$, complete asymptotic expansions can be derived for eigenvalues of order ε^{-1}, see discussion in [5] and Section **??**.

The above-mentioned types of dependence on $\mathfrak{z}$ will be proven in Sections 3.4, 5.1, 5.2 and 4.1, 4.3, 5.3, respectively. Note that asymptotic expansions derived in Sections 3.1 and 3.2 do not involve logarithms at all.

All the problems under consideration enjoy an interaction of the holes $\omega_1^\varepsilon, \dots, \omega_J^\varepsilon$, but not in the main asymptotic term. In the low-frequency range of the spectrum of problems with the Steklov condition (1.4) on Γ such interpretation occurs in the low-frequency range at level ε^2, ε and $|\ln\varepsilon|^{-1}$ for the conditions (1.7), (1.5) and (1.9) on γ^ε, respectively. At the same time, the Steklov condition (1.5) on γ^ε provides the interaction in the middle-frequency range in the first correction term of order $\varepsilon^{-1}|\ln\varepsilon|^{-1}$ or $\varepsilon^{-1}|\ln\varepsilon|^{-2}$. In any case the main term of the power-law asymptotics takes this interaction into account. It should be mentioned that our other results about the Laplace equation in $\Omega^\varepsilon \subset \mathbb{R}^d$ with the Steklov and Neumann condition on γ^ε and Γ, respectively, indicate a strong interaction of holes in dimension $d \geq 3$, namely, the exterior problems in Ξ_j are linked by modified Steklov conditions on $\partial\omega_j$, $j = 1, \dots, J$. It is remarkable that the Dirichlet condition on Γ annuls the interaction effect. Similar effects of far-field interaction of small perturbations has been detected in papers [11], [12], [13] but in a quite different spectral problem with local concentrated masses in a three-dimensional domain with the Neumann boudnary condition.

The spectral problem (1.1), (1.4), (1.5) with the Steklov condition on the boundary $\partial\Omega^\varepsilon = \Gamma \cup \gamma^\varepsilon$ possesses some individual and exceptional features. As in the above-mentioned paper [3] for $d \geq 3$, we will detect two series of eigenvalues from the sequence (1.11) in the form

$$\lambda_n^\varepsilon = \lambda_n^0 + O(\varepsilon|\ln\varepsilon|) \qquad (1.13)$$ ser1

$$\lambda_{n(\varepsilon)}^\varepsilon = \varepsilon^{-1}\left(\mu_p(\mathfrak{z}) + O(\varepsilon|\ln\varepsilon|)\right). \qquad (1.14)$$ ser2

The series (1.13) involves the eigenvalue sequence $\{\lambda_n^\varepsilon\}_{n\in\mathbb{N}}$ of the Steklov problems in the intact domain Ω and describes the spectrum and describes the spectrum of the Steklov problem in Ω^ε in its low-frequency range. The series (1.14) is related to the middle-frequency range and is generated by eigenvalues of the family of exterior Steklov problems in $\Xi_1, \dots, \Xi_J$. The first series is also detected by Theorem 6.4 on convergence, while this theorem does nothing with the second series because the eigenvalues in (1.14) rush to infinity when $\varepsilon \to +0$ and, therefore, change their numbers in the ordered eigenvalue sequence (1.11) indefinitely many times.

1.4 Structure of the paper

?⟨subsec14⟩? In Section 2 we sketch the main known information about various limit problems in Ω and Ξ_j, whose eigenpairs and special solutions enter into the asymptotic expansions of the eigenpairs of the problems formulated in Section 1.1.

In 3 and 4 we deal with two variants of the Neumann–Steklov and Dirichlet–Steklov problems in Ω^ε and demonstrate that the behaviour of the spectrum (1.11) is crucially dependent on the position of the spectral boundary condition on either the exterior part of Γ, or the interior part γ_j^ε of the boundary $\partial\Omega^\varepsilon$, while the procedures to construct the asymptotics (1.13) and (1.14) look quite similar to the ones in Section (3.1) and (4.1), respectively.

The convergence theorems for eigenvalues of the various problems under consideration are collected in Section 6. With the help of the classical Lemma on “almost eigenvalues" (a direct consequence of the spectral decomposition of the resolvent) in Section 7 we prove estimates for the asymptotic remainders in the derived asymptotic expansions and conclude with all theorems on asymptotics that are formulated in Sections 3–5.

2 Auxiliary information

⟨sec2⟩
2.1 The interior Steklov problem

?⟨subsec21⟩? It is well-known that the problem

$$-\Delta_x u^0(x) = 0, \quad x \in \Omega^0, \quad \partial_\nu u^0(x) = \lambda^0 u^0(x), \quad x \in \Gamma, \tag{2.1}$$ 21

has the eigenvalue sequence

$$0 \leq \lambda_1^0 < \lambda_2^0 \leq \lambda_3^0 \leq \ldots \leq \lambda_n^0 \leq \ldots \to +\infty, \tag{2.2}$$ 22

and the corresponding eigenfunctions $u_m^0 \in H_0^1(\Omega)$ can be subject to the normalization and orthogonality conditions

$$(u_m^0, u_n^0)_\Gamma = \delta_{m,n}, \quad m, n \in \mathbb{N}. \tag{2.3}$$ 23

The principal eigenfunction $u_1^0(x) = |\Gamma|^{-1/2}$ is constant, and $|\Gamma|$ is the length of the contour Γ.

2.2 The exterior Steklov problem

⟨subsec22⟩ Considering the problem

$$-\Delta_\xi w^j(\xi) = 0, \ \xi \in \Xi_j, \quad \partial_{\nu(\xi)} w^j(\xi) = \mu^j w^j(\xi), \ \xi \in \gamma_j = \partial\omega_j, \tag{2.4}$$ 24

in the exterior domain $\Xi_j = \mathbb{R}^2 \backslash \overline{\omega}_j$, we introduce the space $\mathcal{H}_j$ as the completion of $C_c^\infty(\overline{X}i_j)$ (infinitely differentiable and compactly supported functions) in the norm

$$||w; \mathcal{H}_j|| = \left(||\nabla_\xi w; L^2(\Xi_j)||^2 + ||w; L^2(\gamma_j)||^2\right)^{1/2}. \tag{2.5}$$ 24N

An equivalent weighted norm

$$\left(||\nabla_\xi w; L^2(\Xi_j)||^2 + ||(1+|\xi|)^{-1}(1+|\ln \xi|)^{-1} w; L^2(\Xi_j)||^2\right)^{1/2} \tag{2.6}$$ 25

results from the calssical Hardy inequality with logarithm

$$\int_R^{+\infty} \rho^{-1} \left|\ln \frac{\rho}{R}\right|^{-2} |W(\rho)|^2 \, d\rho \leq 4 \int_R^{+\infty} \rho \left|\frac{dW}{d\rho}(\rho)\right|^2 d\rho, \ \forall W \in \mathcal{C}^\infty[R, +\infty), \ W(R) = 0 \tag{2.7}$$ 25N

together with the Poincare inequality

$$||w; L^2(\mathbb{B}_{2R} \setminus \omega_j)|| \leq c_R \left(||\nabla_\xi w; L^2(\mathbb{B}_{2R} \setminus \omega_j)|| + ||w; L^2(\gamma_j)||\right)$$

where the radius $R > 0$ is chosen such that the disk $\mathbb{B}_R = \{\xi : |\xi| < R\}$ contains $\overline{\omega}_j$. The last condition in (2.7) is achieved by multiplication of w with an appropriate cut-off function.

Constants belong to $\mathcal{H}_j$ and therefore, due to the compact embedding $\mathcal{H}_j \subset L^2(\gamma_j)$, the variational formulation of problem (2.4)

$$(\nabla_\xi w^j, \nabla_\xi v^j)_{\Xi_j} = \mu^j (w^j, v^j)_{\gamma_j} \quad \forall v^j \in \mathcal{H}_j \tag{2.8}$$ 26

possesses the eigenvalue sequence

$$0 = \mu_1^j < \mu_2^j \leq \mu_3^j \leq \ldots \leq \mu_n^j \ldots \to +\infty, \tag{2.9}$$ 27

and the corresponding eienfunctions $w_n^j \in \mathcal{H}_j$ can be subject to the normalization and orthogonality conditions

$$(w_m^j, w_n^j)_{\gamma_j} = \delta_{mn}, \quad m, n \in \mathbb{N}. \tag{2.10}$$ 28

These eigenfunctions admit the asymptotic form

$$w_n^j(\xi) = b_n^j + \widetilde{w}_n^j(\xi) \tag{2.11}$$ 29

where b_n^j is a constant and the remainder $\widetilde{w}_n^j$ satisfies the estimates

$$|\nabla_\xi^k \widetilde{w}_n^j(\xi)| \leq c_{k,n} \, (1+|\xi|)^{-1-k}, \quad k = 0, 1, 2, \ldots, \quad \xi \in \overline{\Xi}_j. \tag{2.12}$$ 30

2.3 The Dirichlet problems

⟨subsec23⟩ We will need the Green function $G(x, y)$ in the domain Ω, which is a distributional solution of the problem

$$\Delta_x G(x,y) = \delta(x-y),\ x \in \Omega, \quad G(x,y) = 0,\ x \in \Gamma, \tag{2.13}$$ D01

where δ is the Dirac mass and $y \in \Omega$ is a parameter. The function $G^j(x) = G(x, x^j)$ with the logarithmic singularity at the fixed point x^j admits the asymptotic form

$$G^j(x) = \delta_{j,k}\frac{1}{2\pi}\ln\frac{1}{r_k} + \mathcal{G}_{jk} + O(r_j), \quad r_j = |x - x^j| \to +0. \tag{2.14}$$ D02

The coefficients $\mathcal{G}_{jk}$, $k = 1, \ldots, J$, compose the $J \times J$-matrix $\mathcal{G}$ which is symmetric.

The exterior homogeneous ($\psi = 0$) Dirichlet problem

$$\Delta_\xi w^j(\xi) = 0,\ \xi \in \Xi_j, \quad w^j(\xi) = \psi(\xi),\ x \in \gamma_j, \tag{2.15}$$ D03

has a solution with logarithmic growth at infinity, namely, the logarithmic capacity potential (see, e.g., [14], [15])

$$E^j(\xi) = \frac{1}{2\pi}\ln\frac{1}{|\xi|} + \frac{1}{2\pi}\ln c_{log}(\omega_j) + \widetilde{E}^j(\xi) \tag{2.16}$$ D04

where $c_{log}(\omega_j) > 0$ is the logarithmic capacity of the set $\omega_j \subset \mathbb{R}^2$ and the remainder $\widetilde{E}^j$ admits the estimate (2.12) (recall thet the coordinate origin $\xi = 0$ belongs to ω^j). For any smooth ψ, problem (2.15) has a unique solution $w^j \in \mathcal{H}_j$ in the form (2.11), where $\widetilde{w}^j$ fulfils the estimates (2.12) and the constant b^j is computed as follows:

$$b^j = \int_{\gamma_j} \psi(\xi)\partial_{\nu(\xi)}E^j(\xi)\,ds_\xi. \tag{2.17}$$?D05?

2.4 The Neumann problems

⟨subsec24⟩ The generalized Green function of the Neumann problem in Ω is defined as the distributional solution of the problem

$$\Delta_x^j G(x,y) = \delta(x-y) - |\Omega|^{-1},\ x \in \Omega, \quad \partial_\nu G(x,y) = 0,\ x \in \partial\Omega, \tag{2.18}$$ N09

where $|\Omega|$ is the area of Ω and the function G is of mean zero over Ω. In this way, the problem (2.18) with the right-hand side

$$a_1\delta(x-x^1) + \ldots + a_J\delta(x-x^J) \tag{2.19}$$ N10A

has a solution in $L^2(\Omega)$ if and only if

$$a_1 + \ldots + a_J = 0. \tag{2.20}$$ NN10

The exterior Neumann problem

$$-\Delta_\xi w^j(\xi) = 0,\ \xi \in \Xi_j, \quad \partial_{\nu(\xi)}w^j(\xi) = \psi(\xi),\ \xi \in \gamma_j, \tag{2.21}$$ N01

with smooth right-hand side ψ of mean zero over γ_j has a solution in $\mathcal{H}_j$ defined up to a constant addendum. Hence, in view of representation (2.11), the decaying ($b_n^j - 0$) solution exists and is unique.

If

$$\int_{\gamma_j} \psi(\xi)\,ds_\xi \neq 0,$$

problem (2.21) has a solution with logarithmic growth at infinity.

Setting $\psi_p = \partial_{\nu(\xi)}\xi_p$ in (2.21), we observe that $\int_{\partial\omega_j} \psi_p(\xi)\, d\xi = 0$ and find a decaying solution $w_p^j \in \mathcal{H}_j$. Furthermore, this harmonics gets the decomposition

$$W_p^j(\xi) = \sum_{q=1}^{2} M_{pq}^j \frac{\partial \Phi}{\partial \xi_q}(\xi) + \widetilde{W}_p^j(\xi) \qquad (2.22)$$ N02

where $\Phi(\xi) = -(2\pi)^{-1} \ln|\xi|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$ and the remainder gets much faster decay than in (2.12)

$$|\nabla_\xi^k \widetilde{W}_p^j(\xi)| \le c_k \, (1+|\xi|)^{-2-k}, \quad k = 0,1,2,\dots, \quad \xi \in \overline{X}i_j. \qquad (2.23)$$?N03?

According to [14, Appendix G], the coefficients M_{pq} in (2.22) form a 2×2-matrix $M^j = M(\overline{\omega}_j)$, which is called the virtual mass matrix of the set $\overline{\omega}_j \subset \mathbb{R}^2$. This matrix is always symmetric and negative definite, since the domain ω_j has positive area $|\omega_j|$. Notice that $M(T)$ is degenerate for a straight crack, e.g., $M_{11}(T) = M_{12}(T) = 0$ for $T = \{\xi : \xi_2 = 0, |\xi_1| < l\}$ because $\psi_1 = 0$.

3 The Steklov-Neumann problems

⟨sec3⟩ 3.1 The Neumann conditions at small holes

⟨subsec31⟩ The asymptotic ansätze for an eigenpair $\{\lambda^\varepsilon, u^\varepsilon\}$ of problem (1.1), (1.4), (1.7) look quite simple

$$\lambda^\varepsilon = \lambda^0 + \varepsilon^2 \lambda' + \overline{\lambda}^\varepsilon, \qquad (3.1)$$ N1

$$u^\varepsilon(x) = u^0(x) + \varepsilon \sum_{j=1}^{J} \chi_j(x) w^j(\xi^j) + \varepsilon^2 u'(x) + \overline{u}^\varepsilon(x), \qquad (3.2)$$ N2

where $\{\lambda^0, u^0\}$ is an eigenpair of the limit Steklov problem, w^j is a boundary layer term localized in the vicinity of the hole ω_j^ε by the cut-off function $\chi_j \in C_c^\infty(\Omega)$,

$$\begin{gathered} \chi(x) = 1 \text{ in } d_j - \text{neighbourhood of } x^j, \quad d_j > 0, \\ \chi_j \chi_k = 0 \text{ for } j \ne k, \ \chi_j = 0 \text{ near } \Gamma. \end{gathered} \qquad (3.3)$$ chi

The correction terms λ' and u' are to be determined, while the remainders $\widetilde{\lambda}^\varepsilon$ and $\widetilde{u}^\varepsilon$ will be estimated in Section ? ????.

Since the harmonics u^0 is smooth near the points $P^1, \dots, P^J$, the leading term in (3.2) leaves the discrepancy

$$\nabla_x u^0(P^j) = \partial_\nu \mathbf{x}^j + O(\varepsilon)$$

in the Neumann condition (1.7) on the contour $\gamma_j^\varepsilon = \partial\omega_j^\varepsilon$, which can be compensated in main by the linear combination

$$w^j(\xi^j) = -\nabla_x u^0(x^j) \cdot \mathbf{w}^j(\xi^j) \qquad (3.4)$$ N3

of the special solutions (2.22) to the exterior Neumann problem (2.21). Here, the central dot stands for the inner product in $\mathbb{R}^2$ and

$$\mathbf{x}^j = (x_1 - x_1^j, x_2 - x_2^j) \quad \text{and} \quad \mathbf{w}^j(\xi^j) = (\mathbf{w}_1^j(\xi^j), \mathbf{w}_2^j(\xi^j))$$

are vector functions in Ω and Xi_j, respectively.

We insert (3.2) into the Laplace equation (1.1) and use the decomposition (2.22) for the boundary layer term (3.4). Performing the coordinate change $\xi^j \mapsto \mathbf{x}^j = \varepsilon \xi^j$, and collecting coefficients of ε^2 yield the Poisson equation

$$-\Delta_x u'(x) = f'(x) := -\sum_{j=1}^{J} [\Delta_x, \chi_j(x)] \nabla_x u^0(P^j) \cdot M^j \nabla_x \Phi(x - x^j), \ x \in \Omega. \tag{3.5}$$
N4

Here, $[\Delta_x, \chi_j] = 2\nabla\chi_j \cdot \nabla_x + \Delta_x \chi_j$ is the commutator of the Laplace operator and the cut-off function χ_j, that is, a first order differential operator, whose coefficients vanish near the point x^j, where $\Phi(x - x^j)$ gets a singularity. Furthermore, according to (3.3), we derive from (1.4) and (3.1), (3.2) the boundary condition

$$\partial_\nu u'(x) - \lambda^0 u'(x) = g'(x) := \lambda' u^0(x), \ x \in \Gamma. \tag{3.6}$$
N5

If λ^0 is a simple eigenvalue in (2.2). then problem (1.4), (1.5) gets only one compatibility condition which, in view of the normalization condition (2.3), reads

$$\int_\Gamma g'(x) u^0(x)\, ds_x + \int_\Omega f'(x) u^0(x)\, dx = 0 \ \Rightarrow \ \lambda' = \lambda' ||u^0; L^2(\Gamma)||^2 = S^0 := -\int_\Omega f'(x) u^0(x)\, dx. \tag{3.7}$$
N6

Integration by parts yield

$$\begin{aligned}
S^0 &= \sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \int_\Omega u^0(x) [\Delta_x, \chi_j(x)] \nabla_x \Phi(x - x^j)\, dx = \\
&= \sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \lim_{\delta \to 0} \int_{\Omega \setminus \mathbb{B}_\delta^j} u^0(x) \Delta_x \left(\chi_j(x) \nabla_x \Phi(x - x^j) \right) dx = \\
&= -\sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \lim_{\delta \to 0} \int_{\partial \mathbb{B}_\delta^j} \left(u^0(x) \frac{\partial}{\partial r_j} \nabla_x \Phi(x - x^j) - \frac{\partial u^0}{\partial r_j}(x) \nabla_x \Phi(x - x^j)\, ds_x \right) ds_x = \\
&= -\sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \frac{1}{2\pi} \int_{\partial \mathbb{B}_\delta^j} \left((\nabla_x u^0(x^j) \cdot \mathbf{x}^j) \frac{\partial}{\partial r_j} \frac{\mathbf{x}^j}{r_j^2} - (\nabla_x u^0(x^j) \cdot \frac{\partial \mathbf{x}^j}{\partial x_j}) \frac{\mathbf{x}^j}{r_j^2} \right) ds_x = \\
&= \sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \nabla_x u^0(x^j).
\end{aligned} \tag{3.8}$$
N7

Here, $\mathbb{B}_\delta^j = \{x : r_j := |x - x^j| < \delta\}$ is a disk, ds_x is the elementary arc length, and M^j is the virtual mass matrix of the set $\overline{\omega}_j$, see Section 2.4.

Thus, we conclude that the correction term in (3.1) takes the form

$$\lambda' = \sum_{j=1}^{J} \nabla_x u^0(x^j) \cdot M^j \nabla_x u^0(x^j) \leq 0, \tag{3.9}$$
?N8?

while a solution of problem (1.4). (1.5) exists and, being defined up to addendum $c' u^0$, becomes unique under the orthogonality condition $(u', u^0)_\Gamma = 0$.

3.2 A multiple eigenvalue

⟨subsec32⟩ Let $\lambda^0 = \lambda_n^0$ be and eigenvalue of problem (2.1) with multiplicity $\kappa > 1$, i.e.,

$$\lambda_{n-1}^0 < \lambda_n^0 = \ldots = \lambda_{n+\kappa-1}^0 < \lambda_{n+\kappa}^0. \tag{3.10}$$
N9

Then we seek for κ numbers $\lambda'_n, \ldots, \lambda'_{n+\kappa-1}$ in the ansatz (3.1) for the eigenvalues $\lambda^\varepsilon_n, \ldots, \lambda^\varepsilon_{n+\kappa-1}$ and set

$$u^{p0}(x) = c^p_n u^0_n(x) + \ldots c^p_{n+\kappa-1} u^0_{n+\kappa-1}(x), \quad p = n, \ldots, n+\kappa-1, \tag{3.11}$$ N10

in the ansatz (3.2) for the corresponding eigenfunctions $u^\varepsilon_n, \ldots, u^\varepsilon_{n+\kappa-1}$ where the coefficient columns $c^p = (c^p_n, \ldots, c^p_{n+\kappa-1})$ satisfy the relations

$$c^p \cdot c^q = \delta_{p,q}, \quad p, q = n, \ldots, n+\kappa-1. \tag{3.12}$$ N11

Repeating the above computations and arguments, we arrive at problem (3.5), (3.6) for u'_p and λ'_p with evident modifications. Due to assumption (3.10), this problem gets κ compatibility conditions, namely its right-hand sides must be orthogonal to the eigenfunctions $u^0_n, \ldots, n^0_{n+\kappa-1}$, cf. (3.7). An obvious modification of calculation (3.8) turns these conditions into the algebraic system

$$\mathcal{M}^n c^p = \lambda'_p c^p \tag{3.13}$$?N110?

where $\mathcal{M}^n$ is a $\kappa \times \kappa$-matrix with entries

$$\mathcal{M}^n_{kl} = \sum_{j=1}^{J} \nabla_x u^0_l(x^j) \cdot M^j \nabla_x u^0_k(x^j). \tag{3.14}$$ N12

Thanks to the general properties of M^j, see Section 2.4, the matric $\mathcal{M}^n$ is symmetric and negative, so that the system (3.12) has the eigenvalues

$$\lambda'_n \leq \lambda'_{n+1} \leq \ldots \leq \lambda'_{n+\kappa-1} \leq 0 \tag{3.15}$$ N13

and the corresponding eigenvectors $c^n, \ldots, c^{n+\kappa-1} \in \mathbb{R}^\kappa$ can be subject to the orthogonality and normalization conditions (3.12). This instantiate the asymptotic ansätze (3.1) and (3.2).

Let us formulate an assertion that will be prove in Section ?????

⟨AS1⟩ **Theorem 3.1.** *For any $N \in \mathbb{N}$, there exist positive ε_N and c_N such that the entries of the eigenvalue sequences* (1.11) *and* (2.2) *of problem* (1.1)*,* (1.4)*,* (1.7) *and* (2.1)*, respectively, are in the relationship*

$$|\lambda^\varepsilon_n - \lambda^0_n - \varepsilon^2 \lambda'_n| \leq c_N \varepsilon^3 \quad \text{for } \varepsilon \in (0, \varepsilon_N], n = 1, \ldots, N, \tag{3.16}$$ N15

where the correction term λ'_n are found by the above described procedure.

3.3 Lower-order terms

?⟨subsec33⟩? As it was mentioned in Section 1.2, the paper [3] provides complete asymptotic expansions of eigenpairs for the Steklov problem in dimension $d \geq 3$. General asymptotic procedure from [4, Ch. 2, 3, 11] allow to construct infinite asymptotic series for eigenpairs in problem (1.1), (1.4), (1.7). However, we will not present adaptation of the procedures and only list some particular features of the analysis.

First of all, in the case $\kappa = 1$ in (3.10), i.e., λ^ε_n is a simple eigenvalue, the procedure becomes quite elementary and routine, because the main terms in the ansätze (3.1) and (3.2) are entirely defined at the above-presented original step, while all further steps repeat the first one in whole. However, for the multiple eigenvalue λ^0_n in (3.10) with $\kappa \geq 2$, the main terms $u^{n0}, \ldots, u^{n+\kappa-10}$ in the eigenfunction ansätz (3.2) are linear combinations (3.11) with certain coefficient columns which are uniquely determined by the normalization and orthogonality conditions (3.12) if and only if the eigenvalues of the $\kappa \times \kappa$-matrix $\mathcal{M}$ with entries (3.14) are simple. On the contrary, for an eigenvalue λ'_p of multiplicity $\tau > 1$,

$$\lambda'_n \leq \lambda'_p = \ldots = \lambda'_{p+\tau-1} \leq 0, \tag{3.17}$$ lamb

the columns $c^p, \dots, c^{p+\tau-1}$ belong to a τ-dimensional subspace in $\mathbb{R}^\kappa$, but remain unfixed. In this way, to specify even the main asymptotic terms of the eigenfunctions $u^\varepsilon_p, \dots, u^\varepsilon_{p+\tau-1}$, it is necessary to find out lower-order terms and to derive an algebraic system of type (3.14) with a $\tau\times\tau$-matrix $\mathcal{M}''$. As a result, one computes the second correction terms $\lambda''_p, \dots, \lambda''_{p+\tau-1}$ in the asymptotic forms

$$\lambda^\varepsilon_q = \lambda^0_n + \varepsilon^2\lambda'_p + \varepsilon^3\lambda''_q + O(\varepsilon^4), \quad q = p, \dots, p+\tau-1, \tag{3.18}$$ qpt

as eigenvalues of the matrix $\mathcal{M}''$. If these eigenvalues are simple, the expansions (3.18) are split at level ε^3 and continuation of the procedure again becomes uncomplicated. At the same time, no tool is created yet to predict if all eigenvalues in the sequence (1.11) can be asymptotically separated in finite number of steps in the procedure. Anyway, a symmetry of the domain (1.2) provides multiple eigenvalues of problem (1.1), (1.4), (1.7).

A distinguishing feature of this problem is that all asymptotic terns of asymptotic expansions of eigenpairs do not depend on $\ln\varepsilon$ — this property can be verified by means of induction, cf. [16], — but we avoid to present necessary cumbersome calculations here. We emphasize that in all other problems investigated in this paper, such dependence occurs in either main, or first correction term.

3.4 The Neumann condition at the exterior boundary

⟨subsec34⟩ For an eigenpair of problem (1.1), (1.5), (1.6), we accept the asymptotic ansätze

$$\lambda^\varepsilon = \varepsilon^{-1}\mu^\varepsilon, \quad \mu^\varepsilon = \mu(\mathfrak{z}) + \widetilde{\mu}^\varepsilon, \tag{3.19}$$ E1

$$u^\varepsilon(x) = a(\mathfrak{z}) + \sum_{j=1}^{J}\big(\mathfrak{z}a^j(\mathfrak{z})G^j(x) + \chi_j(x)w^j(\xi^j,\mathfrak{z})\big) + \widetilde{u}^\varepsilon(x), \tag{3.20}$$ E2

where μ, a and a^j are smooth functions in $\mathfrak{z}\in[0,\mathfrak{z}_0]$, $\mathfrak{z}_0>0$,

$$\mathfrak{z} = |\ln\varepsilon|^{-1}, \tag{3.21}$$ zet

$G^j(x) = G(x,x^j)$ are particular generalized Green functions introduced in Section 2.4, and w^j are boundary layer terms possessing the decay property (2.12). To make the linear combination of $G^1, \dots, G^K$ in (3.20) harmonic, we assume that the coefficient column $\vec{a}(\mathfrak{z}) = \big(a^1(\mathfrak{z}), \dots, a^J(\mathfrak{z})\big)$ fulfils (2.20), that is, $\vec{a}(\mathfrak{z})\in\mathbb{R}^J_\perp$ where

$$\mathbb{R}^J_\perp = \Big\{\mathbf{a}\in\mathbb{R}^J : \mathbf{e}^\top\mathbf{a} = 0, \quad \mathbf{e} = (1,\dots,1)^\top\in\mathbb{R}^J.\Big\} \tag{3.22}$$ E3

Notice that we did not normalize the function (3.20) in $L^2(\gamma^\varepsilon)$. At the same time, the factor ε^{-1} is put into (3.19) in order to have

$$0 = \partial_{\nu(x)}w^j(\varepsilon^{-1}(x-x^j)) - \lambda^\varepsilon w^j(\varepsilon^{-1}(x-x^j)) = \varepsilon^{-1}(\partial_{\nu(x)}w^j(\xi^j) - \mu^\varepsilon w^j(\xi^j)) \tag{3.23}$$ EE0

and $\top$ stands for transposition. Inserting (3.19), (3.20) into the equations (1.1), (1.6) and using the decomposition (2.14) of the generalized Green functions, we arrive at the following family ($j = 1, \dots, J$) of the exterior problems

$$-\Delta_\xi w^j(\xi,\mathfrak{z}) = 0, \ \xi\in\Xi_j,$$

$$\begin{aligned}\partial_{\nu(\xi)}w^j(\xi,\mathfrak{z}) =& \mu(\mathfrak{z})w^j(\xi,\mathfrak{z}) + a(\mathfrak{z}) + \mathfrak{z}\left(a^j(\mathfrak{z})\frac{1}{2\pi}\ln\frac{1}{\varepsilon} + \sum_{k=1}^{J}\mathcal{G}_{jk}a^k(\mathfrak{z})\right) - \\ &-\mathfrak{z}\frac{a^j(\mathfrak{z})}{2\pi}\big(\partial_{\nu(\xi)} - \mu(\mathfrak{z})\big)\ln\frac{1}{|\xi|}, \ \xi\in\partial\omega_j.\end{aligned} \tag{3.24}$$ E4

First of all, setting $\mathfrak{z} = 0$ in (3.24) yields

$$-\Delta_\xi w^j(\xi, 0) = 0,\ \xi \in \Xi_j, \tag{3.25}$$ E5

$$\partial_{\nu(\xi)} w^j(\xi, 0) = \mu(0)\left(w^j(\xi, 0) + a(0) + (2\pi)^{-1} a^j(0)\right),\ \xi \in \Xi_j, \tag{3.26}$$ E6

Regarding (3.26) as an exterior problem with the fixed Neumann datum ψ_j, we write the conditions

$$\int_{\gamma_j} \psi_j(\xi)\, ds_\xi = \mu(0)|\gamma_j| \left(\langle w^j \rangle(0) + a(0) + \frac{a^j(0)}{2\pi} \right) = 0, \quad j = 1, \ldots, J, \tag{3.27}$$ E7

assuring the existence of bounded solutions, see Section 2.4, where

$$\langle w^j \rangle(\mathfrak{z}) = \frac{1}{|\gamma_j|} \int_{\gamma_j} w^j(\xi, \mathfrak{z})\, ds_\xi. \tag{3.28}$$ E8

For any $\mu(0) > 0$ and $\langle w^j \rangle(0) = \left(\langle w^1 \rangle(0), \ldots, \langle w^J \rangle(0)\right)^\top \in \mathbb{R}^J$, the system (3.27) has a unique solution $a(0) \in \mathbb{R}^J_\perp$, see (3.22), so that the boundary condition (3.26) turns into

$$\partial_{\nu(\xi)} w^j(\xi, 0) = \mu(0)\left(w^j(\xi, 0) - \langle w^j \rangle(0)\right) \quad \xi \in \partial\omega_j. \tag{3.29}$$ E9

Finally, since a constant satisfies (3.25), (3.29) for any $\mu(0)$. we impose the orthogonality condition

$$\int_{\partial \mathbb{B}_{R^J}} w^j(\xi)\, ds_\xi = 0 \quad \text{for some } R^j > 0, \quad \mathbb{B}_{R^j} \supset \omega_j. \tag{3.30}$$ N23

The variational formulation of problem (3.25), (3.29), (3.30) reads: to find $\{\mu, w^j\} \in \mathbb{R} \times \mathcal{H}_{j\perp}$ such that

$$(\nabla_\xi w^j, \nabla_\xi v^j)_{\Xi_j} = \mu(w^j - \langle w^j \rangle, v^j - \langle v^j \rangle)_{\gamma_j} \quad \forall v^j \in \mathcal{H}_{j\perp}, \tag{3.31}$$ N25

where, according to Section 2.2,

$$\mathcal{H}_{j\perp} = \{w^j \in \mathcal{H}_j :\ (3.30) \text{ is fulfilled}\}. \tag{3.32}$$ N24

As usual, the integral identity (3.31) is obtained by integration by parts: for any $v^j \in C^\infty_c(\overline{X}i_j)$, we have

$$\begin{aligned} 0 = (-\Delta_\xi w^j, v^j)_{\Xi_j} &= (\nabla_\xi w^j, \nabla_\xi v^j)_{\Xi_j} - (-\partial_{\nu(\xi)} w^j, v^j)_{\gamma_j} = \\ &= (\nabla_\xi w^j, \nabla_\xi v^j)_{\Xi_j} - \mu(w^j - \langle w^j \rangle, v^j)_{\gamma_j} \end{aligned}$$

The substitution $v^j \mapsto v^j - \langle v^j \rangle$ in the last scalar product is possible because $\langle w^j - \langle w^j \rangle \rangle = 0$.

⟨BOT⟩ **Lemma 3.1.** *Problem* (3.31) *gets the eigenvalue sequence*

$$0 < \mu^j_2 \le \mu^j_3 \le \ldots \le \mu^j_n \le \ldots \to +\infty, \tag{3.33}$$ E10

with entries taken form the eigenvalue sequence (2.9) *of the exterior Steklov problem* (2.8). *The corresponding eigenfunctions* $w^j_n \in \mathcal{H}_{j\perp}$ *take the form* $w^{jst}_n - b^{jst}_n$, *where* $w^{jst}_n \in \mathcal{H}_j$ *are the Steklov eigenfunctions satisfying* (2.10) *and* (2.11), *and fulfilling the intrinsic orthogonality and normalization conditions*

$$(w^j_n - \langle w^j_n \rangle, w^j_m - \langle w^j_m \rangle)_{\gamma_j} = \delta_{m,n} \quad m.n \in \mathbb{N}. \tag{3.34}$$ E11

Proof. Since, owing to the orthogonality condition (3.30), the left-hand side of (3.31) is a scalar product in the Hilbert space (3.32), the conclusion on the spectrum (3.33) of problem (3.31) is obtained in a standard way on the basis of the Riesz representation theorem.

The first eigenpair $\{\mu_1^{jst}, w_1^{jst}\}$ of the pure Steklov problem (2.8) is $\{0, \mathrm{const}\}$ and, therefore, according to (2.10), we have

$$\langle w_n^j \rangle = 0,\ n = 2, 3, 4, \dots \tag{3.35}$$ EE1

In view of (3.35) the difference $w_n^j = w_n^{jst} - b_1^{jst} = \widetilde{w}_n^{jst}$, see (2.11), satisfies equation (3.25) and the boundary condition (3.29). Furthermore, $\widetilde{w}_n^{jst} \in \mathcal{H}_{j\perp}$ because

$$\int_{\partial \mathbb{B}_{R^j}} \widetilde{w}_n^{jst}(\xi)\, ds_\xi = - \lim_{R \to +\infty} \int_{\partial \mathbb{B}_R} \Big((\widetilde{w}_n^{jst}(\xi) \partial_\rho \ln \frac{\rho}{R^j} - \ln \frac{\rho}{R^j} \partial_\rho \widetilde{w}_n^{jst}(\xi)) \Big)\, ds_\xi$$

and this limit vanishes due the estimates (2.12). Thus, a Steklov eigenpair gives rise to an eigenpair of problem (3.31). The inverse statement is obvious because the function $w_n^j - \langle w_n^j \rangle$ satisfies the equations (2.4) and falls into the space $\mathcal{H}_j$ containing constants, see (2.5) and (2.6). □

The correspondence between eigenvalues in (2.9) and (3.33) discarded in Lemma 3.1, avoids the null eigenvalues $\mu_1^j = 0$ of the exterior Steklov problems in Ξ_j, $j = 1, \dots, J$, so that the limit problem (3.25), (3.29), (3.30) (or (3.31) in the variational form) does not describe asymptotics of the spectrum of the original problem (1.1), (1.5), (1.6) completely. Clearly, $\lambda_1^\varepsilon = 0$ is a simple eigenvalue in (1.11) with a constant eigenfunction, so that other eigenfunctions u_n^ε with $n > 1$ enjoy the orthogonality condition

$$\sum_{j=1}^{J} \int_{\gamma_j^\varepsilon} u_n^\varepsilon(x)\, ds_x = 0,\ n = 2, 3, 4, \dots \tag{3.36}$$?E0?

To detect other $J - 1$ eigenvalues in (1.11) generated by J eigenvalues $\mu_1^1 = \ldots = \mu_1^J = 0$, we specify the ansatz (3.19) as follows:

$$\mu(\mathfrak{z}) = 0 + \mathfrak{z}\mu'(\mathfrak{z}). \tag{3.37}$$ E12

Then, the limit passage $\mathfrak{z} \to +0$ in (3.24) shows that $w^j(\xi, \mathfrak{z})$ is a decaying solution of the homogeneous ($\psi = 0$) exterior Neumann problem (2.21) and, therefore,

$$w^j(\xi, \mathfrak{z}) = 0 + \mathfrak{z} w^{j\prime}(\xi, \mathfrak{z}). \tag{3.38}$$ E13

In view of (3.37) and (3.38), after multiplying with $\mathfrak{z}^{-1} = |\ln \varepsilon|$, the limit passage in (3.24) leads to the problem

$$\Delta_\xi w^{j\prime}(\xi, 0) = 0,\ \xi \in \Xi_j, \tag{3.39}$$?E113?

$$\partial_{\nu(\xi)} w^{j\prime}(\xi, 0) = \psi^j(\xi) := \mu'(0) \left(a(0) + (2\pi)^{-1} a^j(0) \right) + (2\pi)^{-1} a^j(0) \partial_{\nu(\xi)} \ln |\xi|,\ \xi \in \gamma_j.$$

This is nothing but the exterior Neumann problem which, accordin to Section 2.4, has a unique decaying solution $w^{j\prime}$ provided

$$0 = \int_{\gamma_j} \psi^j(\xi)\, ds_\xi = \mu'(0) |\gamma_j| \left(a(0) + (2\pi)^{-1} a^j(0) \right) - a^j(0). \tag{3.40}$$?E14?

Here, we have applied the trivial equality

$$\frac{1}{2\pi} \int_{\gamma_j} \ln |\xi|\, ds_x i = -\frac{1}{2\pi} \int_{\partial \mathbb{B}_R} \partial_\rho \ln \rho\, ds_\xi = -1. \tag{3.41}$$ EEE

In this way we have arrived at the algebraic system

$$M' \vec{a}(0) = \mu'(0) \left(2\pi a(0) \mathbf{e} + \vec{a}(0) \right) \tag{3.42}$$ E15

with the diagonal $J \times J$-matrix

$$M' = \mathrm{diag}\{M_1, \dots, M_J\}, \ M_j = 2\pi|\gamma_j|^{-1} \tag{3.43}$$ EE15

Recalling that $\vec{a}(0) \in \mathbb{R}^J_\perp$, we denote by $P_\perp = \mathbb{I} - J^{-1}\mathbb{E}$ the orthogonal projector on the subspace (3.22) where $\mathbb{I}$ is the $J \times J$ identity matrix and the matrix $\mathbb{E}$ contains the value 1 at each position.

Projecting (3.42) onto $\mathbb{R}^J_\perp$, we find the positive eigenvalues

$$\mu_2'(0), \dots, \mu_J'(0) > 0 \tag{3.44}$$ E16

of the symmetric matrix $P_\perp M P_\perp$, while the corresponding eigenvectors $\vec{a}_2(0), \dots, \vec{a}_J(0)$ form an orthonormalized basis in $\mathbb{R}^J_\perp$ and the scalars $a_2(0), \dots, a_J(0)$ are computed according to formula

$$a_k(0) = \big(2\pi J \mu_k'(0)\big)^{-1} \mathbf{e}^\top M \vec{a}_k(0), \ k = 2, \dots, J. \tag{3.45}$$?E17?

These deliver main terms in the asymptotic ansätze (3.19) and (3.20) specified by the restrictions (3.37) and (3.38).

3.5 Simple eigenvalues

⟨subsec35⟩ We join the eigenvalue sequences (2.9) with $j = 1, \dots, J$ into the common monotone sequnec

$$0 = \mu_1 = \ldots = \mu_J < \mu_{J+1} \leq \mu_{J+2} \leq \ldots \leq \mu_N \leq \ldots \to +\infty \tag{3.46}$$ E18

of eigenvalues of the exterior Steklov problems (2.4) in the exterior domains Ξ_j, $j = 1, \dots, J$, Let μ_N be a simple eigenvalue in (3.46), that is, μ_N is a simple eigenvalue of one Steklov pronlem, say in Ξ_1, but μ_N does not belong to the spectra of the Steklov problems in $\Xi_2, \dots, \Xi_J$. Surely, $n > J$ and $\mu_N = \mu_n^1 > 0$, so that, by Lemma 3.1, μ_n^1 is simultaneously a simple eigevalue in (3.33), $j = 1$, while the problems (3.31) with $j = 2, \dots, J$ and $\mu = \mu_n^1$ are uniquely solvable in $\mathcal{H}_{j\perp}$.

We recall that, according to our above calculations,

$$\begin{aligned} \mu(\mathfrak{z}) &= \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}), \quad w^1(\xi, \mathfrak{z}) = w_n^1(\xi) + \mathfrak{z} w'(\xi, \mathfrak{z}), \\ a(\mathfrak{z}) &= a(0) + \mathfrak{z} a'(\mathfrak{z}), \quad \vec{a}(\mathfrak{z}) = \vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \\ w^j(\xi, \mathfrak{z}) &= 0 + \mathfrak{z} w^{j\prime}(\xi . \mathfrak{z}), \ j = 2, \dots, J, \end{aligned} \tag{3.47}$$ E19

where $\{\mu_n^1, w_n^1\}$ is an eigenpair of problem (3.31), $j = 1$, and w_n^1 satisfies (3.19), while $a(0) \in \mathbb{R}$ and $\vec{a}(0) \in \mathbb{R}^J_\perp$ are found from system (3.27).

We aim to determine the correction terms in (3.47) as real analytic functions in $\mathfrak{z} \in [0, \mathfrak{z}_0]$, $\mathfrak{z}_0 > 0$. To this end, we consider the whole problems (3.24) involving the dependence on (3.21). regarding them as the exterior Neumann problems, we write the standard compatibility conditions

$$0 = \mu(\mathfrak{z})|\gamma_j| \left(\langle w^j \rangle(\mathfrak{z}) + a(\mathfrak{z}) + \frac{1}{2\pi} a^j(\mathfrak{z}) + \mathfrak{z} \sum_{k=1}^{J} \mathcal{G}_{jk} a^k(\mathfrak{z}) \right) - \mathfrak{z} a^j(\mathfrak{z}) + \mathfrak{z} a^j(\mathfrak{z})\mu(\mathfrak{z}) l_j \tag{3.48}$$ E20

where we use the equality (3.41) as well as the notation (3.28) and

$$l_j = \frac{1}{2\pi} \int_{\gamma_j} \ln \frac{1}{|\xi|} \, ds_\xi. \tag{3.49}$$ E21

These allow us to rewrite the problems as follows:

$$-\Delta_\xi w^j(\xi, \mathfrak{z}) = 0, \ \xi \in \Xi_j$$

$$\partial_{\nu(\xi)} w^j(\xi,\mathfrak{z}) = \mu(\mathfrak{z})\left(w^j(\xi,\mathfrak{z}) - \langle w^j\rangle(\mathfrak{z})\right) - \mathfrak{z} a^j(\mathfrak{z})\varphi^j(\xi,\mu(\mathfrak{z})),\ \xi\in\gamma_j \tag{3.50}$$ E22

where

$$\varphi^j(\xi,\mu) = \frac{1}{2\pi}(\partial_{\nu(\xi)} - \mu)\ln\frac{1}{|\xi|} + 1 - \mu l_j, \quad \int_{\gamma^j}\psi^j(\xi,\mu)\,ds_\xi = 0. \tag{3.51}$$ E23

At the same time, relation (3.48) converts into the algebraic system

$$a(\mathfrak{z})\mathbf{e} + \frac{1}{2\pi}\vec{a}(\mathfrak{z}) + \mathfrak{z}\mathcal{G}\vec{a}(\mathfrak{z}) + \langle\vec{w}\rangle(\mathfrak{z}) = \frac{\mathfrak{z}}{2\pi}M\left(\frac{1}{\mu(\mathfrak{z})}\mathbb{I} + L\right)\vec{a}(\mathfrak{z}) \tag{3.52}$$ E24

where M is the matrix (3.43) and $L = \mathrm{diag}\{l_1,\dots,l_J\}$.

Now, we insert (3.47) into (3.50) and after a simple calculation obtain

$$-\Delta_\xi w^{j\prime}(\xi,\mathfrak{z}) = 0,\ \xi\in\Xi_j,$$

$$\partial_{\nu(\xi)} w^{j\prime}(\xi,\mathfrak{z}) - \mu_n^1\left(w^{j\prime}(\xi,\mathfrak{z}) - \langle w^{j\prime}\rangle(\mathfrak{z})\right) = \mu'(\mathfrak{z})\left(w^j(\xi,0) - \langle w^j\rangle(0)\right) - \\ - (a^j(0) + \mathfrak{z} a^{j\prime}(\mathfrak{z}))\varphi^j(\xi,\mu^1 + \mathfrak{z}\mu'(\mathfrak{z})),\ \xi\in\partial\omega. \tag{3.53}$$ E25

As as mentioned, under the orthogonality condition in (3.51) the problem (3.53) with $j\geq 2$ is uniquely solvable so that denoting $\mathfrak{R}^j(\mu_n^1)$ the inverse operator and recalling $w^j(\xi,0)=0$ yield

$$w^{j\prime}(\cdot,\mathfrak{z}) = -\left(a^j(0) + \mathfrak{z} a^j(\mathfrak{z})\right)\mathfrak{R}^j(\mu_n^1)\varphi^j(\cdot,\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})). \tag{3.54}$$ E26

The problem (3.53) with $j=1$ has μ_n^1 as a simple eigenvalue and, by the Fredholm alternative, requires one additional compatibility condition which, in view of $w^1(\xi,0) = w_n^1(\xi)$ and (3.34), takes the form

$$\mu'(\mathfrak{z}) = \left(a^j(0) + \mathfrak{z} a^{j\prime}(\mathfrak{z})\right)\left(w_n^1 - \langle w_n^1\rangle, \varphi^j(\cdot,\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))\right)_{\gamma_j}. \tag{3.55}$$ E27

Thus, problem (3.54) with $j=1$ becomes as follows:

$$-\Delta_\xi w^{1\prime}(\xi,\mathfrak{z}) = 0,\ \xi\in\Xi_j,$$

$$\partial_{\nu(\xi)} w^{1\prime}(\xi,\mathfrak{z}) - \mu_n^1\left(w^{1\prime}(\xi,\mathfrak{z}) - \langle w^{1\prime}\rangle(\mathfrak{z})\right) = \varphi^{1\prime}(\xi, a'(0) + \mathfrak{z} a^{1\prime}(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})) := \\ (a'(0) + \mathfrak{z} a^{1\prime}(\mathfrak{z}))\left((w_n^1(\xi) - \langle w_n^1\rangle)\left(w_n^1(\xi) - \langle w_n^1\rangle, \varphi^1(\cdot,\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))\right)_{\gamma_1} - \varphi^1(\xi,\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})),\ \xi\in\gamma_1, \tag{3.56}$$?E28?

and its solution $w^{1\prime}(\cdot,\mathfrak{z})\in\mathcal{H}_{1\perp}$ exists and is determined up to an addendum cw_n^1, while the orthogonality conditions

$$\left(w^{1\prime}(\cdot,\mathfrak{z}) - \langle w^{1\prime}\rangle(\mathfrak{z}), w_n^1 - \langle w_n^1\rangle\right)_{\gamma_1} = 0 \tag{3.57}$$ E29

makes the solution unique. Hence, this solution can be written as

$$w^{1\prime}(\cdot,\mathfrak{z}) = (a^1(0) + \mathfrak{z} a^1(\mathfrak{z}))\mathfrak{R}^1(\mu_n^1)\varphi^{1\prime}(\cdot, a^1(0) + \mathfrak{z} a^{1\prime}(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})). \tag{3.58}$$ E30

This solution belongs to the subspace

$$\mathcal{H}_{1\perp}(\mu_n^1) = \{w^{1\prime}\in\mathcal{H}_{1\perp} : (3.57)\text{ is fulfilled}\} \tag{3.59}$$?E32?

In accordance with (3.54) and (3.58), we put

$$\langle w\rangle(\mathfrak{z}) = \mathfrak{r}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})). \tag{3.60}$$?E31?

Finally, we recall the definition of $a(0)$, $\vec{a}(0)$ and transform (3.52) into

$$
\begin{aligned}
a'(\mathfrak{z})\mathbf{e} + \frac{1}{2\pi}\vec{a}'(\mathfrak{z}) = \mathfrak{h}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})) :=& \\
:= \frac{1}{2\pi} M \left(\frac{1}{\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})}\mathbb{I} + L \right) (\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z})) + \mathcal{G}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z})) -& \\
- \mathfrak{r}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))&
\end{aligned}
\qquad \text{E35} \tag{3.61}
$$

Thus,

$$
a'(\mathfrak{z}) = J^{-1}\mathbf{e}^\top \mathfrak{h}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})), \qquad \text{E33} \tag{3.62}
$$

$$
\vec{a}'(\mathfrak{z}) = 2\pi P_\perp \mathfrak{h}(\vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})). \qquad \text{E34} \tag{3.63}
$$

Now we join the unknowns as follows:

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{z}) = (w^{1\prime}(\cdot, \mathfrak{z}), \mu'(\mathfrak{z}), w^{2\prime}(\cdot, \mathfrak{z}), \dots, w^{J\prime}(\cdot, \mathfrak{z}), a'(\mathfrak{z}), \vec{a}'(\mathfrak{z})) \in \mathfrak{H} :=& \\
= \mathcal{H}_{1\perp}(\mu_n^1) \times \mathbb{R} \times \mathcal{H}_{2\perp} \times \dots \mathcal{H}_{J\perp} \times \mathbb{R} \times \mathbb{R}_\perp^J,&
\end{aligned}
\qquad \text{E36} \tag{3.64}
$$

and rewrite formulas (3.58), (3.55), (3.54) with $j = 2, \dots, J$ and (3.62), (3.63) as a nonlinear abstract equation

$$
\mathfrak{u}(\mathfrak{z}) = \mathfrak{T}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})) \quad \text{in } \mathfrak{H}. \qquad \text{E37} \tag{3.65}
$$

Let us list some obvious properties of the operator $\mathfrak{T}$. First, $\mathfrak{T}(0, \mathfrak{u})$ is independent of $\mathfrak{u}$ and we set $\mathfrak{u}^0 = \mathfrak{T}(0, 0)$. Second, the operator is linear in the function unknowns $\vec{w}'(\cdot, \mathfrak{z})$, polynomial in the algebraic unknowns $a'(\mathfrak{z})$, $\vec{a}(\mathfrak{z})$ and rational in $\mu'(\mathfrak{z})$ (recall that $\mu_n^1 > 0$ in $(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))^{-1}$, see (3.61). Third, for a small $\mathfrak{z}$ the mapping $\mathfrak{v} \mapsto \mathfrak{T}(\mathfrak{z}, \mathfrak{v})$ is a contraction in the ball

$$
\mathfrak{B}(\mathfrak{u}^0) = \{\mathfrak{v} \in \mathfrak{H}; ||\mathfrak{v} - \mathfrak{u}^0, \mathfrak{H}|| \leq \rho\}, \quad \rho > 0 \text{ is small}, \qquad \text{ball} \tag{3.66}
$$

namely,

$$
\mathfrak{B}(\mathfrak{u}^0) \ni \mathfrak{v} \mapsto \mathfrak{T}(\mathfrak{z}, \mathfrak{v}) \in \mathfrak{B}(\mathfrak{u}^0),
$$

$$
||\mathfrak{T}(\mathfrak{z}, \mathfrak{v}^1) - \mathfrak{T}(\mathfrak{z}, \mathfrak{v}^2); \mathfrak{H}|| \leq c\mathfrak{z}||\mathfrak{v}^1 - \mathfrak{v}^2; \mathfrak{H}|| \ \forall \mathfrak{v}^1, \mathfrak{v}^2 \in \mathfrak{B}(\mathfrak{u}^0).
$$

Thus, the Banach contraction principle ensures the existence of a unique solution $\mathfrak{u}(\mathfrak{z} \in \mathfrak{B}(\mathfrak{u}^0)$ of the abstract equation (3.65), which additionally admits the estimate

$$
||\mathfrak{u}(\mathfrak{z}) - \mathfrak{u}^0; \mathfrak{H}|| \leq c\mathfrak{z}
$$

and is analytic abstract function in $\mathfrak{z} \in 0, \mathfrak{z}_0$ with some $\mathfrak{z}_0 > 0$.

The solution (3.64) of equation (3.65) implying conditions to solve the family of problems (3.24), $j = 1, \dots, J$, determines all detached terms in the asymptotic ansätze (3.19) and (3.20). Let us formulate an error estimate which will be verifies in Section 7 ?

⟨AS2⟩ **Theorem 3.2.** *Let $\mu_N = \mu_n^1$ be a simple eigenvalue in the united sequence* (3.46) *of the spectra of the exterior Steklov problems* (2.4), $j = 1, \dots, J$*. Then the estimate*

$$
|\lambda_N^\varepsilon - \varepsilon^{-1}\left(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})\right)| \leq c_N(1 + |\ln \varepsilon|) \quad \varepsilon \in (0, \varepsilon_N] \qquad \text{E38} \tag{3.67}
$$

is valid for the corresponding entry of the Neumann–Steklov problem (1.1), (1.5), (1.6), $\mu'(\mathfrak{z})$ *is an analytic function in* $\mathfrak{z} \in [0, \mathfrak{z}_N]$*, and* ε_N*,* c_N *and* $\mathfrak{z}_N$ *are some positive numbers.*

3.6 Eigenvalues of order $\varepsilon^{-1}|\ln\varepsilon|^{-1}$

⟨subsec36⟩ The condition in Theorem 3.2 on the simplicity of μ_N reject from consideration the null eigenvalues of the Steklov problems (2.4) in the case $J > 1$, However, if all eigenvalues (3.44) of the matrix $P_\perp M P_\perp$, see (3.43), are simple, a slight modification of the reduction scheme in Section 3.5 allows us to construct and solve nonlinear equations of type (3.65) in order to find the correction terms $\mu'_j(\mathfrak{z})$ in the ansätze

$$\lambda_j^\varepsilon = \varepsilon^{-1}(0 + \mathfrak{z}\mu'_j(\mathfrak{z}) + \widetilde{\widetilde{\mu}}{}_j^\varepsilon),\ j = 2, \dots, J \tag{3.68}$$ E39

for the initial entries of the sequence (1.11) which starts with $\lambda_1^\varepsilon = 0$.

?⟨HYP⟩? **Remark 3.1.** *A direct calculation shows that*

$$\mu'_2 = \frac{1}{2}(M_1 + M_2) \ \text{ for } J = 2,$$

$$\mu'_{2,3} = \frac{1}{2}\Big(M_1 + M_2 + M_3 \pm \sqrt{M_1^2 + M_2^2 + M_3^2 - M_1M_2 - M_2M_3 - M_1M_3}\Big) \ \text{ for } J = 3.$$

These formulas provoke the hypothesis: all eigenvalues of the matrix $P_\perp M P_\perp$ are simple provided the lengths $|\gamma_1|, \dots, |\gamma_J|$ of the contours $\partial\omega_1, \dots, \partial\omega_J$ are mutually different. We not know how to confirm it.

Let us consider the case

$$|\gamma_1| = \dots = |\gamma_J| \Rightarrow M_1 = \dots = M_J =: \mathbf{m}. \tag{3.69}$$ E40

Then eigenvalues of the matrix $P_\perp M P_\perp = \mathbf{m}P_\perp$ are

$$0 \text{ and } \mu'_2(0) = \dots = \mu'_J(0) = \mathbf{m} > 0 \tag{3.70}$$ E41

while the eigenspace corresponding to $\mathbf{m}$ is nothing but $\mathbb{R}^J_\perp$ and an eigenvector corresponding to null is $\mathbf{e}$, see (3.22). We accept the asymptotic ansätze (3.19) and (3.20) specified as follows:

$$\mu'(\mathfrak{z}) = \mathfrak{z}\mathbf{m} + \mathfrak{z}^2\mu''(\mathfrak{z}), \quad w^j(\xi,\mathfrak{z}) = \mathfrak{z}w^{j\prime}(\xi,0) + \mathfrak{z}^2 w^{j\prime\prime}(\xi,\mathfrak{z}), \tag{3.71}$$ E42

$$a(\mathfrak{z}) = a(0) + \mathfrak{z}a'(\mathfrak{z}) \in \mathbb{R}, \quad \vec{a}(\mathfrak{z}) = \vec{a}(0) + \mathfrak{z}\vec{a}\,'(\mathfrak{z}) \in \mathbb{R}^J_\perp$$

(compare with (3.37), (3.38), and (3.70)). Inserting (3.71) into problem (3.24) extracting term of order $\mathfrak{z}^2$ yield the exterior Neumann problems

$$\Delta_\xi w^{j\prime\prime}(\xi,0) = 0,\ \xi \in \Xi_j,$$

$$\partial_{\nu(\xi)} w^{j\prime\prime}(\xi,0) = \mathbf{m}\Big(w^{j\prime}(\xi,0) + a'(0) + \frac{1}{2\pi}a^{j\prime}(0) + \sum_{j=1}^{J}\mathcal{G}_{jk}a^k(0)\Big) + \tag{3.72}$$?E43?

$$+ \mu''(0)\Big(a(0) + \frac{1}{2\pi}a^j(0)\Big) - a^{j\prime}(0)\frac{1}{2\pi}\partial_{\nu(\xi)}\ln\frac{1}{|\xi|} + \mathbf{m}a^j(0)\frac{1}{2\pi}\ln\frac{1}{|\xi|},\ \xi \in \gamma_j.$$

Note that the terms of order $1 = \mathfrak{z}^0$ and $\mathfrak{z}$ vanish. To assure the existence of the decaying solution, we apply the standard compatibility condition from Section 2.4 and, after using formulas (3.69), (3.41), and (3.49), we obtain the equality

$$2\pi(\langle w^{j\prime}\rangle(0) + a'(0) + \frac{1}{2\pi}a^{j\prime}(0) + \sum_{j=1}^{J}\mathcal{G}_{jk}a^k(0)) + \tag{3.73}$$ E44

$$+ 2\pi\mu''(0)\mathbf{m}^{-1}(a(0) + \frac{1}{2\pi}a^j(0)) - a^{j\prime}(0) + \mathbf{m}a^j(0)l_j = 0$$

We observe that $a^{j\prime}(0)$ disappears from (3.73) and recall that, according to (3.38) and (3.42)

$$w^{j\prime}(\xi,0)=a^j(0)\mathbf{w}^j(\xi) \qquad (3.74)$$?E45?

where $\mathbf{w}^j$ is a decaying harmonics in Ξ_j satisfying the condition

$$\partial_{\nu(\xi)}\mathbf{w}^j(\xi)=1+(2\pi)^{-1}\partial_{\nu(\xi)}\ln|\xi|,\ \xi\in\partial\omega_j. \qquad (3.75)$$ E45N

We compose a linear system from equations (3.73), $j=1,\dots,J$, and project it onto $\mathbb{R}^J_\perp$ while noting that the component $c\mathbf{e}$ in the system can be annulled by fixing the number $2\pi a'(0)$ on the left of (3.73). As a result, we obtain the following eigenvalue problem in $\mathbb{R}^J_\perp$

$$\mu''(0)\vec{a}(0)=M''\vec{a}(0) \qquad (3.76)$$ E46

where

$$M''=-\mathbf{m}P_\perp(2\pi\mathcal{G}+2\pi\mathcal{W}+\mathbf{m}L)P_\perp, \qquad (3.77)$$ E47

$$L=\operatorname{diag}\{l_1,\dots,l_J\},\quad \mathcal{W}=\operatorname{diag}\{\langle\mathbf{w}^1\rangle,\dots,\langle\mathbf{w}^J\rangle\}. \qquad (3.78)$$ EE47

We emphasize that the matrix $P_\perp\mathcal{G}P_\perp$ is symmetric, cf. a comment to (2.14), and the definition (2.18) of G. The matrix (3.77) depends on the shape of $\omega^1,\dots,\omega^J$ through the values l_j and $\langle\mathbf{w}^1\rangle$, $j=1,\dots,J$ as well as on the shape of Ω and the position of the points $x^1,\dots x^j\in\Omega$. through the matrix g.

The limit problem (3.76) in the $(J-1)$-dimensional subspace $\mathbb{R}^J_\perp$ gives us the real eigenalues

$$\mu''_2(0),\dots,,\mu''_J(0) \qquad (3.79)$$ E48

and the corresponding orthonormalized eigenvectors $\vec{a}_{(2)}(0),\dots,\vec{a}_{(J-1)}(0)$, which together with the scalars $a_{(2)}(0),\dots,a_{(J-1)}(0)$, computed from (3.73) concretize the main terms in representation (3.71) and the ansätze (3.19), (3.20).

If all the eigenvalues (3.79) are simple, a slight modification of the procedure applied in Section 3.5 allows us to contruct the correction term $\mathfrak{z}^2\mu''(\mathfrak{z})$ in (3.71) as an analytic function in $\mathfrak{z}$. In this way, we formulate the following assertions, see Section 7 ?

⟨AS3⟩ **Theorem 3.3.** *The initial positive terms in the eigenvalue sequence* (1.11) *of problem* (1.1), (1.5), (1.6) *satisfy the formulas*

$$|\lambda^\varepsilon_j-\varepsilon^{-1}\mathfrak{z}\mu'_j(0)|\le c_j\varepsilon^{-1}\mathfrak{z}^2\quad\text{for }\mathfrak{z}\in(0,\mathfrak{z}_J],\ j=2,\dots,J, \qquad (3.80)$$ E49

where $\mu'_j(0)$ are eigenvalues (3.44) *of the matrix $P_\perp M'P_\perp$ and $c_J,\mathfrak{z}_J$ are positive numbers. If additionally, the relation* (3.69) *is valid and the eigenvalues* (3.79) *of the matrix* (3.77) *are simple, then*

$$|\lambda^\varepsilon_j-\varepsilon^{-1}\mu'_j(0)|\le c'_j|\ln\varepsilon|^2\quad\text{for }\mathfrak{z}\in(0,\mathfrak{z}'_J],\ j=2,\dots,J, \qquad (3.81)$$?EE50?

where $c'_J,\mathfrak{z}'_J>0$ and μ'_j is an analytic function in $\mathfrak{z}\in[0,\mathfrak{z}'_J]$ such that $\mu'_j(\mathfrak{z})=\mathbf{m}+\mathfrak{z}\mu''_j(0)+O(\mathfrak{z}^2)$ with coefficients from (3.69) *and* (3.79).

3.7 General situation

⟨subsec37⟩ Without assumption on simplicity of eigenvalues in the united spectrum (3.46) of the exterior Steklov problems (2.4), $j=1,\dots,J$, we were able to construct the "logarithmic" asymptotics of eigenvalues of problem (1.1), (1.5), (1.6), however with precision of arbitrary order $\mathfrak{z}^N$. In the case of a simple eigenvalue μ_N the result of Theorem 3.2 demonstrates that the formal asymptotic series in power of $\mathfrak{z}$ converges. The same can be said about the result of Theorem 3.3. For an eigenvalue μ_N of multiplicity $\kappa_N>1$, we can establish that the series converges only in the case when, at q-th step of the asymptotic procedure with a certain $q\in\mathbb{N}$, we obtain just κ_N different terms $\mu^q_N,\dots,\mu^q_{N+\kappa_N-1}$

and simultaneously fix the main terms in the expansion (3.20) of the eigenfunctions $u^{\varepsilon}_N, \dots, u^{\varepsilon}_{N+\kappa_N-1}$. At the same time, no tool is known yet to conclude in advance that such "asymptotic splitting" of the eigenvalues $\lambda^{\varepsilon}_N, \dots, \lambda^{\varepsilon}_{N+\kappa_N-1}$ can be predicted to occur in finite number of steps.

We formulate a simplest assertion which relates the sequences (1.11) and (3.46) and reflects the traditional principle of the first non-trivial asymptotic term.

⟨AS4⟩ **Theorem 3.4.** *For any $N \geq J$, there exist positive numbers c_N and $\mathfrak{z}_N$ such that the first N eigenvalues of problem* (1.1), (1.5), (1.6) *satisfy the estimates*

$$|\lambda^{\varepsilon}_j - \varepsilon^{-1}\mathfrak{z}\mu'_j(0)| \leq c_N \varepsilon^{-1}\mathfrak{z}^2 \quad \text{for } \mathfrak{z} \in [0, \mathfrak{z}_N], j = 2, \dots, J,$$

$$|\lambda^{\varepsilon}_n - \varepsilon^{-1}\mu'_n| \leq c_N \varepsilon^{-1}\mathfrak{z} \quad \text{for } \mathfrak{z} \in (0, \mathfrak{z}_N], j = J+1, \dots, N,$$

where $\mu'_2(0), \dots, \mu'_J(0)$ are positive eigenvalues of the matrix $P_{\perp}M'P_{\perp}$, see (3.43), *and $\mu_{J+1}, \dots, \mu_N$ are positive entries of the united sequence* (3.46) *of the exterior Steklov problems.*

4 The Steklov-Dirichlet problem

⟨sec4⟩ 4.1 The Dirichlet conditions at small holes

⟨subsec41⟩ To construct asymptotics of eigenpairs in the problem, we employ an algorithm from [9], see also [4, Ch. 9, §1 and §2], which originally served for other perturbed problems. We accept the asymptotic ansätze

$$\lambda^{\varepsilon} = \lambda^0 + \Lambda(\mathfrak{z}) + \widetilde{\lambda}^{\varepsilon}, \tag{4.1}$$ D1

$$u^{\varepsilon}(x) = u^0(x) + U(x,\mathfrak{z}) + \sum_{j=1}^{J} \big(a^j(\mathfrak{z})G^j(x) + \chi^j(x)w^j(\xi^j,\mathfrak{z})\big) + \widetilde{u}^{\varepsilon}(x). \tag{4.2}$$ D2

First of all, we assume that λ^0 is a simple eigenvalue of the limit Steklov problem (2.1) in Ω, for example, $\lambda^0 = 0$, and the corresponding eigenfunction u^0 is normalized in $L^2(\Gamma)$, see (2.3). We need to find the main correction terms $\{\Lambda(\mathfrak{z}), U(x,\mathfrak{z})\}$ depending on the small parameter $\mathfrak{z}$, the dacaying boundary-layer terms $w^j(\cdot,\mathfrak{z})$ in Ξ_j, $j = 1, \dots, J$, as well as the coefficients $\vec{a}(\mathfrak{z}) = (a^1(\mathfrak{z}), \dots, a^J(\mathfrak{z}))^{\top}$ of a linear combination of the generalized Green functions of the Steklov problem (2.1) with the spectral parameter $\lambda = \lambda^0$ defined similarly to (2.22) as the distributional solution to the problem

$$-\Delta_x G^j(x) = \delta(x - x^j), \ x \in \Omega, \quad \partial_\nu G^j(x) - \lambda^0 G^j(x) = u^0(x^j)u^0(x), \ x \in \Gamma, \tag{4.3}$$?D3?

$$\int_{\Gamma} u^0(x)G^j(x)\, ds_x = 0. \tag{4.4}$$ D3N

These solutions obey the decomposition (2.14) with a symmetric coefficient matrix $\mathcal{G} = (\mathcal{G}_{jk})_{j,k=1}^{J}$.

We insert (4.2) into the Dirichlet condition on γ^{ε}_j and obtain

$$w^j(\xi,\mathfrak{z}) = -u^0(x^j) - U(x^j,\mathfrak{z}) + a^j(\mathfrak{z})\frac{1}{2\pi}\ln(\varepsilon|\xi|) - \sum_{k=1}^{J}\mathcal{G}_{jk}a^k(\mathfrak{z}), \quad \xi \in \gamma_j. \tag{4.5}$$ D5

According to Section 2.3, a decaying harmonics w^j with the Dirichlet datum (4.5) exists if and only if

$$a_j(\mathfrak{z})\frac{1}{2\pi}(|\ln\varepsilon| + \ln c_{\log}(\omega_j)) + \sum_{k=1}^{J}\mathcal{G}_{jk}a^k(\mathfrak{z}) = -u^0(x^j) - U(x^j,\mathfrak{z}). \tag{4.6}$$ D7

Moreover, it is unique and takes the form

$$w^j(\xi,\mathfrak{z}) = a^j(\mathfrak{z})\widetilde{E}^j(\xi), \tag{4.7}$$?D6?

see the logarithmic capacity potential (2.16). Setting

$$\vec{u}^0 = (u^0(x^1), \dots, u^0(x^J)), \quad \vec{U}(\mathfrak{z}) = (U(x^1, \mathfrak{z}), \dots, U(x^J, \mathfrak{z})), \tag{4.8}$$?D7N?

$$\mathcal{N} = \mathcal{N} + (2\pi)^{-1} \mathrm{diag}\,\{\ln c_{\log}(\omega^1), \dots, \ln c_{\log}(\omega^J)\}, \tag{4.9}$$ D9

the system of equations (4.6) gives us the coefficient column

$$\vec{a}(\mathfrak{z}) = -\mathfrak{z}\left((2\pi)^{-1}\mathbb{I} + \mathfrak{z}\mathcal{N}\right)^{-1}\left(\vec{u}^0 + \vec{U}(\mathfrak{z})\right). \tag{4.10}$$ D8

Clearly, (4.10) is an analytic function in $\mathfrak{z} \in [0, \mathfrak{z}_0]$ if $\vec{U}(\mathfrak{z})$ possesses the same property, and moreover $\vec{a}(0) = 0$.

We find the pair $\{U, \Lambda\}$ from the Steklov problem

$$-\Delta U(x, \mathfrak{z}) = 0, \ x \in \Omega,$$

$$\partial_\nu U(x, \mathfrak{z}) - \lambda^0 U(x, \mathfrak{z}) = F(x, \mathfrak{z}) :=$$

$$:= \Lambda(\mathfrak{z})\left(u^0(x) + \sum_{j=1}^J a^j(\mathfrak{z}) G^j(x) + U(x, \mathfrak{z})\right) + u^0(x) \sum_{j=1}^J a^j(\mathfrak{z}) u^0(x^j), \ x \in \Gamma, \tag{4.11}$$ D10

and impose the orthogonality condition

$$\int_\Gamma u^0(x) U(x, \mathfrak{z})\, ds_x = 0 \tag{4.12}$$ D10N

to make the solution U unique. Since λ^0 is simple, problem (4.11) gets the only compatibility condition $(F, u^0)_\Gamma = 0$ which, in view of (4.4) and (4.12), converts into

$$\Lambda(\mathfrak{z}) = -\vec{a}(\mathfrak{z})^\top \vec{u}^0 = \mathfrak{z}(\vec{u}^0)^\top((2\pi)^{-1}\mathbb{I} + \mathfrak{z}\mathcal{N})^{-1}(\vec{u}^0 + \vec{U}(\mathfrak{z})) \tag{4.13}$$ D11

where (2.3) and (4.10) were used.

We regard (4.11)–(4.13) as a non-linear system for $\{U(\cdot, \mathfrak{z}), \Lambda(\mathfrak{z})\} \in H^2_\perp(\Omega; \lambda_0) \times \mathbb{R}$, where $H^2_\perp(\Omega; \lambda_0) = \{v \in H^2(\Omega) : (v, u^0)_\Gamma = 0\}$. The non-linearity is quadratic. The mappings $H^{1/2}_\perp(\Gamma) \ni F \mapsto U \in H^2_\perp(\Omega; \lambda_0)$ and $H^2_\perp(\Omega) \ni U(\cdot, \mathfrak{z}) \mapsto \{U(\cdot, \mathfrak{z})|_\Gamma, \vec{U}(\mathfrak{z})\} \in H^{1/2}_\perp(\Gamma) \times \mathbb{R}^J$ are an isomorphism and a compact operator, respectively; here $H^{1/2}_\perp(\Gamma)$ is the Sobolev–Slobodetski space under the orthogonality condition $(v, u^0)_\Gamma = 0$. Finally, in view of (4.13) and (4.9), we have

$$\Lambda(0) = 0 \in \mathbb{R}, \ \vec{a}(0) = 0 \in \mathbb{R}^J \ \Rightarrow \ U(x, 0) = 0.$$

Summing up the above-listed properties of the system (**??**)–(4.13), we apply the Banach contraction principle and find a unique small solution which satisfies the estimate

$$|\Lambda(\mathfrak{z})| = ||U(\cdot, \mathfrak{z}); H^2_\perp(\Omega; \lambda_0)|| \le c\mathfrak{z}$$

and depends analytically on $\mathfrak{z} \in [0, \mathfrak{z}_0]$, $\mathfrak{z}_0 > 0$.

The following assertion will be proved in Section 7 ?

⟨AS5⟩ **Theorem 4.1.** *Let $\lambda^0 = \lambda^0_n$ be a simple eigenvalue of the Steklov problem* (2.1)*. There exist positive ε_n and c_n such that the entry λ^ε_n of the eigenvalue sequence* (1.11) *of problem* (1.1), (1.4), (1.9) *satisfies the inequality*

$$|\lambda^\varepsilon_n - \lambda^0_n - \Lambda_n(|\ln\varepsilon|^{-1})| \le c_n \varepsilon \quad \text{for } \varepsilon \in (0, \varepsilon_n] \tag{4.14}$$ D13

where the analytic function $\mathfrak{z} \mapsto \Lambda_N(\mathfrak{z})$ is found from problem (4.11)–(4.13) *and enjoys estimate* (**??**).

?⟨DirNull⟩? **Remark 4.1.** *If $\lambda^0 = \lambda^0_n = 0$ and therefore $u^0(x) = |\Gamma|^{1/2}$, then, according to* (4.13) *we obtain*

$$\Lambda_1 |\ln\varepsilon|^{-1} = \frac{2\pi}{|\ln\varepsilon|}|\vec{u}^0|^2 + O\left(|\ln\varepsilon|^{-2}\right) = \frac{2\pi J}{|\ln\varepsilon||\Gamma|} + O\left(|\ln\varepsilon|^{-2}\right).$$

4.2 A multiple eigenvalue

?⟨subsec42⟩? Assuming that $\lambda^0 = \lambda_n^0$ is an eigenvalue of multiplicity $\kappa > 1$, see (3.2), we keep the ansätze (4.1) for $\lambda_n^\varepsilon, \dots, \lambda_{n+\kappa-1}^\varepsilon$ and (4.2) for $u_n^\varepsilon, \dots, u_{n+\kappa-1}^\varepsilon$, but accept the representations (3.11) of the main regular terms u^{p0}, $p = n, \dots, n+\kappa-1$, with the coefficient columns $c^p(\mathfrak{z}) = (c_n^p(\mathfrak{z}), \dots, c_{n+\kappa-1}^p(\mathfrak{z}))^\top$ and the eigenfunctions $u_n^0, \dots, u_{n+\kappa-1}^0$ of problem (2.1) which correspond to λ_n^0 and fulfil the orthogonality and normalization condition (2.3).

Let us list changes in the asymptotic procedure of Section 4.1 while seeking the main terms in the following ingredients of the ansätze:

$$\Lambda_p(\mathfrak{z}) = \mathfrak{z}\Lambda'(\mathfrak{z}), \quad U_p(x,\mathfrak{z}) = \mathfrak{z}U_p'(x,\mathfrak{z}), \quad a_{(p)}^j(\mathfrak{z}) = \mathfrak{z}a_{(p)}^{j\prime}(\mathfrak{z}), \quad w_{(p)}^j(\xi,\mathfrak{z}) = \mathfrak{z}w_{(p)}^{j\prime}(\xi,\mathfrak{z}), \tag{4.15}$$ D99

First of all, the particular generalized Green functions are found from problem

$$-\Delta G^j(x) = \delta(x - x^j), \ x \in \Omega,$$
$$\partial_\nu G^j(x) - \lambda_n G^j(x) = \sum_{q=n}^{n+\kappa-1} u_q^0(x^j) u_q^0(x), \ x \in \Gamma,$$
$$\int_\Gamma u_q^0(x) G^j(x)\, ds_s = 0, \ q = n, \dots, n+\kappa-1.$$

Second, we compute the coefficients

$$a_{(p)}^{j\prime}(0) = -2\pi \vec{u}^{p0} = -2\pi \sum_{m=n}^{n+\kappa-1} c_m^p u_m^0(x^j) \tag{4.16}$$ D15

from the compatibility condition (**??**) in the exterior Dirichlet problem, see (4.5). Then, we compose the right-hand side of problem (4.11) for $U_p'(x,\mathfrak{z})$ and $\Lambda'(\mathfrak{z})$:

$$F_p'(x,\mathfrak{z}) = \Lambda'(\mathfrak{z})\Big(u^{p0}(x) + \sum_{j=1}^J a^{j\prime}(\mathfrak{z}) G^j(x) + U_p'(x,\mathfrak{z}) + \sum_{q=n}^{n+\kappa-1} u_q^0(x) \sum_{j=1}^J a^j(\mathfrak{z}) u_q^0(x^j)\Big). \tag{4.17}$$ D16

Finally, the compatibility conditions

$$\int_\Gamma u_m^0(x) F_p'(x,\mathfrak{z})\, ds_x = 0, \ m = n, \dots, n+\kappa-1 \tag{4.18}$$ D17

in the Steklov problem Ω lead us to the linear algebraic system

$$\Lambda_p'(0) c_m^p = 2\pi \sum_{q=n}^{n+\kappa-1} c_q^p \sum_{j=1}^J u_q^0(x^j) u_m^0(x^j), \ m = n, \dots, n+\kappa-1. \tag{4.19}$$ D18

We have set $\mathfrak{z} = 0$ in (4.17), (4.18) and have applied (4.16), (2.3). The $\kappa\times\kappa$-matrix $\mathcal{U}$ of system (4.19) is a sum of the symmetric positive matrices

$$\mathcal{U}^j = (\mathcal{U}_{qm}^j)_{q,m=n}^{n+\kappa-1}, \quad \mathcal{U}_{qm}^j = 2\pi u_q^0(x^j) u_m^0(x^j) \tag{4.20}$$?D19?

and possesses κ eigenvalues

$$0 \le \Lambda_n'(0) \le \ldots \le \Lambda_{n+\kappa-1}'(0). \tag{4.21}$$ D20

The corresponding eigenvectors can be subject to the orthogonality and normalization condition (3.12),

The main correction terms in the asymptotic ansätze (4.1), (4.2) specified by (4.15), have been constructed. If all eigenvalues (4.21) are simple, repeating of arguments in Section 3.5 helps to find the ingredients (4.15) as analytic functions in $\mathfrak{z} \in [0, \mathfrak{z}_n]$, $\mathfrak{z}_n > 0$. However, we do not know an elementary geometric condition to provide this simplicity. In any case, the procedure to construct formal series in power of $\mathfrak{z} = |\ln\varepsilon|^{-1}$ for the eigenpairs $\{\lambda_p^\varepsilon, u_p^\varepsilon\}$, $p = n, \dots, n+\kappa-1$, can be continued. Nevertheless, in Section 7 ? we will prove error estimates for the detected correction terms only.

⟨AS6⟩ **Theorem 4.2.** *For any $N \in \mathbb{N}$, there exist positive $\mathfrak{z}_N$ and c_N such that the entries of the eigenvalue sequences* (1.11) *and* (2.2) *for problems* (1.1), (1.4), (1.9) *and* (2.1) *are in the relationship*

$$|\lambda_n^\varepsilon - \lambda_n^0 - \mathfrak{z}\Lambda_n'(0)| \leq c_N \mathfrak{z}^2 \quad \text{for } \mathfrak{z} \in (0, \mathfrak{z}_N],\ n = 1, \dots, N, \tag{4.22}$$ D21

where $\Lambda_1'(0), \dots, \Lambda_N'(0)$ are determined by the above-descried procedure.

4.3 The Dirichlet condition the exterior boundary

⟨subsec43⟩ As in Sections 3.4–3.6, the asymptotic ansätze for the eigenpairs $\{\varepsilon_n^\varepsilon, u_n^\varepsilon\}$ of problem (1.1), (1.5), (1.8), with indexes $n = 1, \dots, J$ and indexes $n > J$ are different. We proceed with the construction of asymptotics, cf. (3.68),

$$\lambda_p^\varepsilon = \varepsilon^{-1}(0 + \mathfrak{z}\mu_p(\mathfrak{z}) + \widetilde{\mu}_p^\varepsilon),\ p = 1, \dots, J, \tag{4.23}$$ D22

of the initial terms in the sequence (1.11). Inserting (4.23) and the ansätz

$$u_p^\varepsilon(x) = \sum_{j=1}^J (a_p^j(\mathfrak{z}) G^j(x) + \chi_j(x) w_p^j(\xi^j, \mathfrak{z})) \tag{4.24}$$ D23

for eigenfunctions into the Laplace equation (1.1) and the Steklov condition (1.5) on γ_j^ε, we obtain the problem for the boundary-layer term

$$\Delta_\xi w_p^j(\xi, \mathfrak{z}) = 0,\ \xi \in \Xi_j,$$

$$\partial_{\nu(\xi)} w_p^j(\xi, \mathfrak{z}) = \mathfrak{z}\mu_p(\mathfrak{z})(w_p^j(\xi^j, \mathfrak{z}) + a_p^j(\mathfrak{z})\frac{1}{2\pi}\ln\frac{1}{\varepsilon|\xi|} + \sum_{k=1}^J \mathcal{G}_{jk} a_p^k(\mathfrak{z}) - \\ - \frac{1}{2\pi} a_p^j(\mathfrak{z}) \partial_{\nu(\xi)} \ln \frac{1}{\varepsilon|\xi|},\ \xi \in \gamma_j. \tag{4.25}$$ D24

We set $\mathfrak{z} = 0$ in (4.25) and write the standard compatibility condition in the exterior Neumann problem which, owing to (3.41), turns into the equations

$$\mu_p(0) a_p^j(0) = 2\pi a_p^j(0) \tag{4.26}$$ D25

so that we arrive at the formula $\mu_1(0) = \ldots = \mu_J(0) = 2\pi$ which is not very informative because cannot provide the splitting of the eigenvalues $\lambda_1^\varepsilon, \dots, \lambda_J^\varepsilon$. Thus we continue and write

$$\mu_p(\mathfrak{z}) = 2\pi + \mathfrak{z}\mu_p'(\mathfrak{z}), \quad a_p^j(\mathfrak{z}) = a_p^j(0) + \mathfrak{z} a_p^{j\prime}(\mathfrak{z}), \quad w_p^j(\xi, \mathfrak{z}) = w_p^j(\xi, 0) + \mathfrak{z} w_p^{j\prime}(\xi, \mathfrak{z}) \tag{4.27}$$ D26

where $\vec{a}_{(p)}(0) \in \mathbb{R}^J$ is still arbitrary and $w_p^j(\xi, 0) = a_p^j(0)\mathbf{w}^j(\xi)$, where $\mathbf{w}^j$ is a decaying harmonics in Ξ_j with the Neumann datum (3.75). Inserting (4.27) into (4.25) and collecting terms of order $\mathfrak{z}$, we obtain the exterior Neumann problem

$$= \Delta_\xi w_p^{j\prime}(\xi, \mathfrak{z}) = 0,\ \xi \in \Xi_j,$$

$$\partial_{\nu(\xi)} w_p^{j\prime}(\xi, \mathfrak{z}) = \mu_p(0)\Big(w_p^j(\xi^j, 0) + \frac{1}{2\pi} a_p^{j\prime}(0) + \frac{1}{2\pi} a_p^j(0) \ln \frac{1}{|\xi|} + \sum_{k=1}^J \mathcal{G}_{jk} a_p^k(0) \Big) + \\ + \mu_p'(0) \frac{1}{2\pi} a_p^j(0) - \frac{1}{2\pi} a_p^{j\prime}(0) \partial_\nu \ln \frac{1}{|\xi|},\ \xi \in \gamma_j.$$

The same compatibility condition as above gives us the relation

$$2\pi|\gamma_j|(a_p^j(0)(\langle \mathbf{w}^j\rangle + l_j) + \sum_{k=1}^{J} \mathcal{G}_{jk} a_p^k(0)) + \mu_p'(0)\frac{1}{2\pi}|\gamma_j| a_p^j(0) = 0. \qquad (4.28)$$ D27

Notice that $a_p^{j\prime}(0)$ disappeared due to (4.26) and the quantity (3.49) was involved. Employing the matrices (3.78), we rewrite the relations (4.28), $j = 1, \ldots, J$, in the form

$$\mu_p'(0)\vec{a}_{(p)}(0) = M'\vec{a}_{(p)}(0), \quad M' = -4\pi^2(\mathcal{W} + L + \mathcal{W}). \qquad (4.29)$$ D28

Hence, in (4.27),

$$\mu_1'(0) \le \ldots \le \mu_J'(0) \ \text{ and } \ \vec{a}_{(1)}(0), \ldots, \vec{a}_{(J)}(0) \in \mathbb{R}^J \qquad (4.30)$$ D29

are eigenvalues of the symmetric matrix M' in (4.30) and the corresponding orthonormalized eigenvectors.

This asymptotic procedure can be continued. If all eigenvalues in (4.30) are simple and therefore the eigenvectors are fixed uniquely, a slight modification of the approach in Section 3.6 helps to determine the ingredients (4.27) in the ansätze (4.23), (4.24) as analytic functions in $\mathfrak{z} \in [0, \mathfrak{z}_J]$, $\mathfrak{z}_J > 0$.

The eigenvalue asymptotics (4.23) are generated by the null eigenvalues of the exterior Steklov problems (2.4) in $\Xi_1, \ldots, \Xi_J$. As for positive eigenvalues in the united sequence (3.46) of the exterior Steklov eigenvalues, we may repeat the consideration in Section 3.5 on simple eigenvalues and the representation (3.19) with an analytic function μ in the variable (3.21). However, we present error estimate only for the main asymptotic terms in the logarithmic decompositions which is in accord with the concept of the first nontrivial correction term, cf. Theorem 3.4.

⟨AS65⟩ **Theorem 4.3.** *For any $N \in \mathbb{N}$, $N \ge J$, there exist positive $\mathfrak{z}_n$ and c_N such that the entries in the eigenvalue sequences* (1.11) *and* (3.46) *of problem* (1.1), (1.3), (1.8) *and the exterior Steklov problems* (2.4) *with $j = 1, \ldots, J$, respectively, are in the relationship*

$$|\lambda_n^\varepsilon - \varepsilon^{-1}\mathfrak{z}(2\pi + \mathfrak{z}\mu_n'(0))| \le c_N \varepsilon^{-1}\mathfrak{z}^3 \text{ for } \mathfrak{z} \in (0, \mathfrak{z}_N], \ n = 1, \ldots, J, \qquad (4.31)$$ D30

$$|\lambda_n^\varepsilon - \varepsilon^{-1}\mu_n| \le c_N \varepsilon^{-1}\mathfrak{z} \text{ for } \mathfrak{z} \in (0, \mathfrak{z}_N], \ n = J+1, \ldots, N, \qquad (4.32)$$?D31?

where $\mu_1'(0), \ldots, \mu_J'(0)$ are eigenvalues of the matrix M', see (4.30) *and* (4.29).

5 The pure Steklov problem

⟨sec5⟩ 5.1 Preliminary discussion

⟨subsec51⟩ In this section we consider problem (1.1), (1.4),(1.5) with the spectral Steklov condition on the whole boundary $\partial\Omega^\varepsilon = \Gamma \cup \gamma^\varepsilon$. Asymptotic procedures remain quite simial to the above-described ones, but, in contrast to our analysis in sections 3 and 4 we will construct two types of eigenvalue asymptotics, namely, we will employ the ansätz (3.1) in the low-frequency range of the spectrum (1.11). The possibility to accept different ansätze discovered in the paper [3] and clarified in dimension $d \ge 3$ only, is supported by the following observation.s. First of all, for $\lambda^\varepsilon = \lambda^0 + o(1)$, the Steklov condition (1.5) in the stretched coordinates reads:

$$\varepsilon^{-1}\partial_{\nu(\xi)} w^\varepsilon(\xi) = \lambda^\varepsilon w^\varepsilon(\xi), \ \xi \in \partial\omega_j, \quad \Rightarrow \quad \partial_{\nu(\xi)} w^\varepsilon(\xi) = \ldots, \ \xi \in \partial\omega_j. \qquad (5.1)$$ S0

Thus, neglecting the small right-hand side forms the Neumann boundary condition as in Section 2.4. At the same time, for $\lambda^\varepsilon = \varepsilon^{-1}\mu^\varepsilon$, $\mu^\varepsilon \ge c > 0$, the Steklov condition (1.4) on the exterior boundary Γ turns into the Dirichlet condition (1.8) in the following way:

$$\partial_\nu u^\varepsilon(x) = \varepsilon^{-1}\mu^\varepsilon u^\varepsilon(x), \ x \in \Gamma, \quad \Rightarrow \quad u^\varepsilon(x) = \varepsilon(\mu^\varepsilon)^{-1}\partial_n u^\varepsilon(x), \ x \in \Gamma. \qquad (5.2)$$?S00?

In Section ? we will continue to discuss two asymptotic series of eigenvalues in the sequence (1.11).

5.2 The low-frequency range

⟨subsec52⟩ Let λ^0 be a simple eigenvalue in (2.2). For an eigenvalue of problem (1.1), (1.4), (1.5), we accept the ansätze

$$\lambda^\varepsilon = \lambda^0 + \lambda\lambda' + \widetilde{\lambda}^\varepsilon, \tag{5.3}$$
S1

$$u^\varepsilon(x) = u^0(x) + \varepsilon \sum_{j=1}^{J} \chi_j(x) w^j(\xi^j) + \varepsilon u'(x, \mathfrak{z}) + \widetilde{u}^\varepsilon(x) \tag{5.4}$$
S2

which look quite similar to (3.1) and (3.2) but have a different order of the correction terms and allow for the dependence on $\ln\varepsilon$. This dependence is caused by our observation (5.1), namely, the term $\varepsilon\lambda^\varepsilon u^\varepsilon(x^j + \varepsilon\xi^j)$ in the boundary condition on $\gamma_j = \partial\omega_j$ furnishes the following Neumann datum for the boundary-layer term

$$\psi^j(\xi^j) = -\lambda^0 u^0(x^j) - \nabla_x u^0(x^j) \cdot_{\nu(\xi)} \xi^j.$$

Hence, a solution to the exterior Neumann problem in Ξ_j takes the form

$$w^j(\xi^j) = -\lambda^0 u^0(x^j)|\gamma_j| \mathbf{w}_0^j(\xi^j) - \nabla_x u^0(x^j) \cdot \mathbf{w}^j(\xi^j) \tag{5.5}$$
S3

where in addition to (3.4) the special solution $\mathbf{w}_0^j$ of problem (2.21) with $\psi(\xi^j) = |\gamma^j|^{-1}$ appears. According to Section 2.4 such problem does not have a bounded solution but the solution with the logarithmic growth at infinity

$$\mathbf{w}_0^j(\xi^j) = \frac{1}{2\pi} \ln \frac{1}{|\xi^j|} + \widetilde{\mathbf{w}}_0^j(\xi^j) \tag{5.6}$$
S4

where the remainder satisfies (2.12) and the coefficient of $\ln|\xi^j|$ is found by the calculation

$$\int_{\partial\omega_j} \partial_{\nu(\xi)} \mathbf{w}_0^j(\xi)\, ds_\xi = -\lim_{R\to+\infty} \int_{\partial\mathbb{B}_R} \frac{\partial \mathbf{w}_0^j}{\partial|\xi|}(\xi)\, ds_\xi = 1.$$

In view of (5.5) and (5.6), (2.22) we obtain the following problem for the correction terms:

$$-\Delta_x u'(x, \mathfrak{z}) = f'(x, \mathfrak{z}) := \lambda^0 \sum_{j=1}^{J} u^0(x^j) \frac{|\gamma_j|}{2\pi} [\Delta, \chi_j(x)] \left(\ln \frac{1}{|x - x^j|} - \ln\varepsilon \right), \ x \in \Omega,$$

$$\partial_\nu u'(x, \mathfrak{z}) - \lambda^0 u'(x, \mathfrak{z}) = \lambda' u^0(x), \ x \in \Gamma. \tag{5.7}$$
S5

Recalling that λ^0 is a simple eigenvalue of the Steklov prolem (2.1) and the corresponding eigenfunction u^0 is normalized in $L^2(\Gamma)$, we write the only compatibility condition in problem (5.7) as follows:

$$\lambda' = \int_\Gamma |u^0(x)|^2\, ds_x = -\int_\Omega u^2(x) f'(x, \mathfrak{z})\, dx =$$

$$= -\lambda^0 \sum_{j=1}^{J} u^0(x^j) \frac{|\gamma_j|}{2\pi} \int_\Omega u^0(x) [\Delta, \chi_j(x)] \left(\ln \frac{1}{|x - x^j|} - \ln\varepsilon \right) dx =$$

$$= -\lambda^0 \sum_{j=1}^{J} u^0(x^j)|\gamma_j| \lim_{\delta\to+0} \int_{\Omega\setminus\mathbb{B}_\delta(x^j)} u^0(x) \Delta_x \left(\chi_j(x)(\Phi(x - x^j) - \frac{\ln\varepsilon}{2\pi} \right) dx = \tag{5.8}$$
S6

$$= -\lambda^0 \sum_{j=1}^{J} u^0(x^j)|\gamma_j| \lim_{\delta\to+0} \int_{\partial\mathbb{B}_\delta^j} \left(u^0(x) \frac{\partial}{\partial x_j} \Phi(x) - \left(\Phi(x) - \frac{\ln\varepsilon}{2\pi} \right) \frac{\partial u^0}{\partial x_j}(x) \right) ds_x =$$

$$= -\lambda^0 \sum_{j=1}^{J} |u^0(x^j)|^2 |\gamma_j|.$$

Thus, the correction term in the ansätz (5.3) for a simple eigenvalue is obtained. Notice that, although $u^0(x,\mathfrak{z})$ depends on $\ln\varepsilon$ through the right-hand side in (5.7), the number λ' is independent of $\ln\varepsilon$.

For a multiple eigenvalue λ_n^0 as in (3.10), arguments and computation are very similar. We assume the representation (3.11) of the main term in the ansätze (**??**) for the eigenfunctions $u_n^\varepsilon,\dots,u_{n+\kappa-1}^\varepsilon$ and, with the help of an evident modification of calculation (5.8), we conclude that the numbers $\lambda_n',\dots,\lambda_{n+\kappa-1}'$ in (3.15) and the coefficient columns $c^n,\dots,c^{n+\kappa-1}$ subject to (3.12), are found from the algebraic system (**??**) where $\mathcal{M}^n$ is a symmetric negative matrix of size $\kappa\times\kappa$ with entries

$$\mathcal{M}_{kl}^n = -\lambda^0\sum_{j=1}^{J} u_l^0(x^j)|\gamma^j|u_k^0(x^j). \qquad (5.9)\ \boxed{\text{S7}}$$

Let us formulate an assertion that will be verifies in Section ? .

⟨AS7⟩ **Theorem 5.1.** *For any $N\in\mathbb{N}$, there exist positive ε_N and c_N such that the entries of the eigenvalue sequences* (1.11) *and* (2.2) *of problems* (1.1), (1.4), (1.5), *and* (2.1), *respectively, are in the relationship*

$$|\lambda_n^\varepsilon-\lambda_n^0-\varepsilon\lambda_n'|\le c_N\varepsilon^2|\ln\varepsilon|^2 \text{ for } \varepsilon\in(0,\varepsilon_N],\ n=1,\dots,N, \qquad (5.10)\ \boxed{\text{S8}}$$

where the correction terms λ_n' are found by the above-described procedure.

It should be mentioned that both the problems in Theorem 5.1 have the null eigenvalue, for which the estimate (5.10) is of no need.

5.3 The middle-frequency range

⟨subsec53⟩ As it was mentioned in Section 5.1, the formal procedure to construct main terms in the asymptotics of the eigenvalues (3.19) of the Steklov problem (1.1), (1.4), (1.5) is the same as for the Steklov–Dirichlet problem (1.1), (1.5), (1.8). However, the final assertion differs slightly from Theorem 4.3. We will formulate this theorem and comment on it, see also Section 7?, as well as investigate in detail the case of a simple eigenvalue skept in Section 4.3.

⟨AS8⟩ **Theorem 5.2.** *For any $N\in\mathbb{N}$, $N\ge J$, there exist positive $\mathfrak{z}_N$, and c_N such that, for the entries $\mu_1,\dots,\mu_N$ of the united eigenvalue sequences* (3.46) *of the exterior Steklov problem* (2.4) *with $j=1,\dots,J$ fulfil the inequalities*

$$|\lambda_{p_n(\varepsilon)}^\varepsilon-\varepsilon^{-1}\mathfrak{z}(2\pi+\mathfrak{z}\mu_n'(0))|\le c_N\varepsilon^{-1}\mathfrak{z}^3 \text{ for } \mathfrak{z}\in(0,\mathfrak{z}_N],\ n=1,\dots,J, \qquad (5.11)\ \boxed{\text{zz1}}$$

$$|\lambda_{p_n(\varepsilon)}^\varepsilon-\varepsilon^{-1}\mu_n|\le c_N\varepsilon^{-1}\mathfrak{z} \text{ for } \mathfrak{z}\in(0,\mathfrak{z}_N],\ n=J+1,\dots,N, \qquad (5.12)\ \boxed{\text{zz2}}$$

where $\lambda_{p_1(\varepsilon)}^\varepsilon,\dots,\lambda_{p_N(\varepsilon)}^\varepsilon$ are eigenvalues of the Steklov problem (1.1), (1.4), (1.5) *in Ω^ε, $p_n(\varepsilon)\ne p_m(\varepsilon)$ for $n\ne m$, and $\mu_1'(0),\dots,\mu_J'(0)$ are eigenvalues of the matrix M' composed in Section 4.3.*

Notice that the estimates (4.31), (4.31) of Theorem 4.3 involve eienvalues λ_n^ε from (1.11) and μ_n from (3.46) with the same indexes $n=1,\dots,N$, while in Theorem 5.2 the index $p_n(\varepsilon)$ of the Steklov eigenvalue in Ω^ε differs from n, depends on the small parameter ε and grows unboundedly as $\varepsilon\to+0$. The latter property is supported by the following observation based on Theorem 5.1: the number $\mathcal{N}^\varepsilon$ of eienvalues $\lambda_n^\varepsilon\in(0,2\pi\varepsilon^{-1})$ subject to the estimate (5.10) tends to infinity when $\varepsilon\to+0$ while $p_n(\varepsilon)\ge\mathcal{N}^\varepsilon$ for $n=1,\dots,N$.

As in Section 3.5, we now consider a simple eigenvalue μ_N in the united sequence (3.46), namely $\mu_N=\mu_n^1$ is a simple eigenvalue of the exterior Steklov problem (2.4) with $j=1$ (in necessary, we relabel the holes $\Omega_1^\varepsilon,\dots,\Omega_J^\varepsilon$) while μ_N does not belong to the spectrum of other problems with $j=2,\dots,J$. In particular, $N>J$ and $\mu_n^1>0$. The corresponding eigenfunction w_n^1 enjoys the normalization (2.10) and the representation (2.11). We accept the ansätze

$$\lambda^\varepsilon=\varepsilon^{-1}(\mu_n^1+\mathfrak{z}\mu'(\mathfrak{z})+\widetilde{\mu}^\varepsilon, \qquad (5.13)\ \boxed{\text{se1}}$$

$$u^\varepsilon(x) = \sum_{j=1}^{J} (\mathfrak{z} a^j(\mathfrak{z}) G^j(x) + w^j(\xi^j, \mathfrak{z})) + \widetilde{u}^\varepsilon(x), \qquad (5.14) \text{ se2}$$

where $G^1, \ldots, G^J$ are particulat Green functions, see (2.13) and (**??**) and

$$w^j(\xi, \mathfrak{z}) = w^j(\xi, 0) + \mathfrak{z} w^{j\prime}(\xi, \mathfrak{z}), \qquad (5.15) \text{ se3}$$

$$w^1(\xi, 0) = \{\widetilde{w}^1_n(\xi), \quad w^2(\xi, 0) = 0, \ldots, w^J(\xi, 0) = 0, \qquad (5.16) \text{ se4}$$

are boundary-layer terms with the decay property (2.12). Aiming to determine $\mu\prime(\mathfrak{z})$, $\vec{a}(\mathfrak{z}) = (a^1(\mathfrak{z}), \ldots, a^J(\mathfrak{z}))$ and $\vec{w}\,\prime(\mathfrak{z}) = (w^{1\prime}(\cdot, \mathfrak{z}), \ldots, w^{J\prime}(\cdot, \mathfrak{z}))$ as analytic functions in (3.21), we insert (5.13)–(5.15) into the Laplace equation (1.1) and the Steklov conditions (1.5); notice that the Dirichlet condition (1.8) is fulfilled completely. Since the boundary layers (5.15) decay as $O(|\xi|^{-1})$, the discrepancy in (1.1) is small, of order ε, and we obtain the problems

$$-\Delta_\xi w^j(\xi, \mathfrak{z}) = 0, \ \xi \in \Xi_j,$$

$$\partial_{\nu(\xi)} w^j(\xi, \mathfrak{z}) = \psi^j(\xi, \mathfrak{z}) := (\mu^1_n + \mathfrak{z}\mu'(\mathfrak{z}))(w^j(\xi, \mathfrak{z}) + \mathfrak{z} a^j(\mathfrak{z}) \frac{1}{2\pi} \ln \frac{1}{\varepsilon|\xi|} + \qquad (5.17) \text{ se5}$$
$$+ \mathfrak{z} \sum_{k=1}^{J} g_{jk} a^k(\mathfrak{z})) + \mathfrak{z} \frac{a^j(\mathfrak{z})}{2\pi} \partial_{\nu(\xi)} \ln |\xi|, \quad \xi \in \gamma^j,$$

which again are considered as exterior Neumann problems with fixed right-hand side $\psi^j(\xi, \mathfrak{z})$. To have $\psi^j(\xi, 0) = 0$, we put

$$\vec{a}(\mathfrak{z}) = \vec{a}(0) + \mathfrak{z}\vec{a}'(\mathfrak{z}), \quad \vec{a}(0) = (2\pi b^1_n, 0, \ldots, 0)^\top \in \mathbb{R}^J. \qquad (5.18) \text{ se6}$$

We emphasize that $a^1(0) = 2\pi b^1_n$ with b^1_n taken from (2.11) provides the equality $w^1_n(\xi) = \widetilde{w}^1_n(\xi) + (2\pi)^{-1} a^1(0)$ on the right of (5.17) after setting $\mathfrak{z} = 0$.

Applying the standard compatibility condition for the existence of a unique decaying solution of problem (2.21) and taking (5.15), (5.16), and (5.18) into account, we arrive at

$$0 = \frac{1}{\mathfrak{z}} \int_{\gamma_j} \psi^j(\xi, \mathfrak{z})\, ds_\xi = (\mu^1_n + \mathfrak{z}\mu'(\mathfrak{z}))|\gamma_j|(\langle w^{j\prime}\rangle(\mathfrak{z}) + \frac{1}{2\pi} a^{j\prime}(\mathfrak{z}) +$$
$$+ (a^j(0) + \mathfrak{z} a^{j\prime}(\mathfrak{z})) l_j + \sum_{k=1}^{J} \mathcal{G}_{jk}(a^k(0) + \mathfrak{z} a^{k\prime}(\mathfrak{z}))) - (a^j(0) + \mathfrak{z} a^{j\prime}(\mathfrak{z})), \qquad (5.19) \text{ se7}$$

where (3.41) and (3.49) were used.

First, we deal with $j > 1$ when $a^j(0) = 0$ according to (5.18). By (5.19) and (5.16), we convert problem (5.17) into

$$-\Delta_\xi w^{j\prime}(\xi, \prime) = 0, \xi \in \Xi_j,$$

$$\partial_{\nu(\xi)} w^{j\prime}(\xi^j, \mathfrak{z}) - \mu^1_n (w^{j\prime}(\xi, \mathfrak{z}) - \langle w^{j\prime}\rangle(\prime)) = \mathfrak{z}\varphi^j(\xi, \mathfrak{z}) := \qquad (5.20) \text{ se8}$$
$$= \mathfrak{z}\mu'(\mathfrak{z})(w^j(\xi, \mathfrak{z}) - \langle w^{j\prime}\rangle(\mathfrak{z})) - \mathfrak{z}(\mu^1_n + \mathfrak{z}\mu'(\mathfrak{z}) a^{j\prime}(\mathfrak{z}) \left(\frac{1}{2\pi} \ln \frac{1}{|\xi|} - \frac{1}{|\gamma_j|} l_j \right) +$$
$$+ \mathfrak{z} a^{j\prime}(\mathfrak{z}) \left(\frac{1}{2\pi} \partial_{\nu(\xi)} \ln |\xi| + \frac{1}{|\gamma_j|} \right), \ \xi \in \gamma^j. \qquad (5.21) \{?\}$$

In view of Lemma 3.1 the positive spectra of problems (2.4) and (3.25), (3.29) coincide with each other, and thus, problem (5.20) with $j = 2, \ldots, J$ has the parameter $\mu^j = \mu^1_n$ outside the spectrum (3.33) and has a unique decaying solution because the right-hand $\mathfrak{z}\varphi^j(\xi,\mathfrak{z})$ (fixed, at the moment) is of zero mean over γ_j. At the same time, the relation (5.19) with $j>1$ can be rewritten as follows:

$$a^j(\mathfrak{z}) = -\langle w^{j\prime}\rangle(\mathfrak{z}) - g_{j1}a^1(0) - \mathfrak{z}l_j a^{j\prime}(\mathfrak{z}) - \mathfrak{z}\sum_{k=1}^{J}\mathcal{G}_{jk}a^{k\prime}(\mathfrak{z}) - \frac{\mathfrak{z}a^{j\prime}(\mathfrak{z})}{(\mu_n^1+\mathfrak{z}\mu'(\mathfrak{z}))|\gamma_j|}. \tag{5.22}$$ se9

Let $j=1$. Since μ_n^1 is a simple eigenvalue of problem (5.17) with $j=1$, we have to take into account the additional compatibility condition while rewriting the boundary condition as follows:

$$\partial_{\nu(\xi)}w^1(\xi,\mathfrak{z}) - \mu_n^1 w^1(\xi,\mathfrak{z}) = \psi^1(\xi,\mathfrak{z}) := \psi^1(\xi,\mathfrak{z}) - \mu_n^1 w^1(\xi,\mathfrak{z}), \quad \xi\in\gamma_1. \tag{5.23}$$?smu1?

The right-hand side ψ_0^1 must be orthogonal in $L^2(\gamma_1)$ to the eigenfunction w_n^1 of the exterior Steklov problem in Ξ_1 and this requirement

$$\mathfrak{z}^{-1}\int_{\gamma^1} w_n^1(\xi)\psi_0^1(\xi,\mathfrak{z})\,ds_\xi = 0$$

converts into the following equation for the correction term $\mu'(\mathfrak{z})$ in (5.13):

$$\mu'(\mathfrak{z}) = \mu'(\mathfrak{z})\int_{\gamma_1}|w_n^1(\xi)|^2\,ds_\xi = \int_{\gamma_1} w_n^1(\xi)\Big((\mu_n^1+\mathfrak{z}\mu'(\mathfrak{z}))(w^{1\prime}(\xi,\mathfrak{z}) + \frac{1}{2\pi}a^{1\prime}(\mathfrak{z}) +$$

$$+\,(a^1(0)+\mathfrak{z}a^{1\prime}(\mathfrak{z}))\,\frac{1}{2\pi}\ln\frac{1}{|\xi|} + \mathcal{G}_{11}a^1(0) + \sum_{k=1}^{J}\mathcal{G}_{1k}a^{k\prime}(\mathfrak{z})) + \tag{5.24}$$ ba1

$$+\,\frac{1}{2\pi}\,(a^1(0)+\mathfrak{z}a^{1\prime}(\mathfrak{z})))\,\partial_{\nu(\xi)}\ln|\xi|\Big)\,ds_\xi. \tag{5.25}$$ {?}

Finally, we transform the Neumann compatibility condition in problem (5.17) with $j=1$ and transform the relation

$$\mathfrak{z}^{-1}\int_{\gamma_1}\psi^1(\xi,\mathfrak{z})\,ds_\xi = 0$$

into the following equation for the last scalar unknown $a^{1\prime}(\mathfrak{z})$, see (5.18),

$$a^{1\prime}(\mathfrak{z}) = \frac{a^{1\prime}(0)+\mathfrak{z}a^{1\prime}(\mathfrak{z})}{|\gamma^1|(\mu_n^1+\mathfrak{z}\mu'(\mathfrak{z}))} - \langle w^{1\prime}\rangle(\mathfrak{z}) - (a^1(0)+\mathfrak{z}a^{1\prime}(\mathfrak{z}))l_j - \tag{5.26}$$ ba2

$$- \mathcal{G}_{11}a^1(0) - \mathfrak{z}\sum_{k=1}^{J}\mathcal{G}_{1k}a^{k\prime}(\mathfrak{z}).$$

Now problem (5.17), $j=1$, takes the form (5.20), $j=1$, while formula (5.24) for $\mu'(\mathfrak{z})$ assures its compatibility condition

$$\int_{\gamma^1}(w_n^1(\xi) - \langle w^1\rangle)\varphi(\xi,\mathfrak{z})\,ds_\xi = 0.$$

Thus, collecting the derived relations (5.20), (5.24), (5.22), (5.26) gives us an abstract equation of type (3.65) with a contraction operator $\mathfrak{T}(\mathfrak{z},\cdot)$ in a small ball (3.66) for the vector

$$\mathfrak{u}(\mathfrak{z}) = (w^{1\prime}(\cdot,\mathfrak{z}), \mu'(\mathfrak{z}), w^{2\prime}(\cdot,\mathfrak{z}), \dots, w^{J\prime}(\cdot,\mathfrak{z}), \vec{a}'(\mathfrak{z})). \tag{5.27}$$ ba3

Arguing in the same way as in Section (3.5), we find a solution (5.27) and concretize the asymptotic ansätze (5.13) and (5.14) specified in (5.15), (5.16), and (5.18).

?⟨AS9⟩? **Theorem 5.3.** *Let $\mu_N = \mu_n^1$ be a simple eigenvalue in the united sequence* (3.46) *of the spectra* (2.9) *of the exterior Steklov problems* (2.4), $j=1,\dots,J$. *Then there exist ε_N, $c_N>0$ and $p_n^1(\varepsilon)\in\mathbb{N}$ such that*

$$|\lambda^\varepsilon_{p_n^1(\varepsilon)} - \varepsilon^{-1}\left(\mu_n^1+\mathfrak{z}\mu'(\mathfrak{z})\right)| \le c_N \quad \text{for } \varepsilon\in(0,\varepsilon_N]. \tag{5.28}$$ fin

Here $\varepsilon^\varepsilon_{p_n^1(\varepsilon)}$ is an eigenvalue of the pure Steklov problem (1.1), (1.4), (1.5), *and μ' is an analytic function in $\mathfrak{z} = |\ln\varepsilon|^{-1}\in(0,|\ln\varepsilon_N|^{-1}]$ determined by the above procedure.*

6 The convergence theorems

⟨sec6⟩
6.1 The Steklov problem

⟨subsec61⟩ We fix a number $n \in \mathbb{N}$ and consider the eigenpair $\{\lambda_n^\varepsilon, u_n^\varepsilon\}$ of problem (1.1), (1.4), (1.5). In Remark **??** we show that

$$\lambda_n^\varepsilon \le c_n \quad \text{for } \varepsilon \in (0, \varepsilon_n] \tag{6.1}$$ T1

with some $\varepsilon_n > 0$. From (6.1) and (1.10),(1.1) with $\rho^\varepsilon = 1$ on Γ^ε, it follows that

$$||\nabla_x u_n^\varepsilon; L^2(\Omega^\varepsilon)||^2 = \lambda_n^\varepsilon ||u_n^\varepsilon; L^2(\Gamma^\varepsilon)||^2 \le c_n. \tag{6.2}$$?T2?

We now construct an extension $\widehat{u}_n^\varepsilon \in H^1(\Omega)$ of u_n^ε such that

$$||\nabla_x \widehat{u}_n^\varepsilon; L^2(\Omega)||^2 \le c||\nabla_x u_n^\varepsilon; L^2(\Omega^\varepsilon)||^2. \tag{6.3}$$ T3

To this end, we introduce the function

$$\mathbb{B}_R \setminus \omega_j \ni \xi \mapsto U_{nj}^\varepsilon(\xi) = u_n^\varepsilon(x^j + \varepsilon\xi) \tag{6.4}$$ T4

and its mean-value

$$\overline{U}_{nj}^\varepsilon = |\mathbb{B}_R \setminus \omega_j|^{-1} \int_{\mathbb{B}_R \setminus \omega_j} U_{nj}^\varepsilon(\xi)\, d\xi.$$

The Poincaré inequality

$$||U_{nj}^\varepsilon - \overline{U}_{nj}^\varepsilon; L^2(\mathbb{B}_R \setminus \omega_j)|| \le c_j ||\nabla_\xi (U_{nj}^\varepsilon - \overline{U}_{nj}^\varepsilon); L^2(\mathbb{B}_R \setminus \omega_j)|| = c_j ||\nabla_\xi U_{nj}^\varepsilon; L^2(\mathbb{B}_R \setminus \omega_j)||$$

assures that the difference $U_{nj}^{\varepsilon\perp} = U_{nj}^\varepsilon - \overline{U}_{nj}^\varepsilon$ satisfies

$$||U_{nj}^{\varepsilon\perp}; H^1(\mathbb{B}_R \setminus \omega_j)|| \le c_j ||\nabla_\xi U_{nj}^\varepsilon; L^2(\mathbb{B}_R \setminus \omega_j)||$$

and has an extension $\widetilde{U}_{nj}^\varepsilon \in H^1(\mathbb{B}_R)$ such that

$$||\widehat{U}_{nj}^\varepsilon; H^1(\mathbb{B}_R \setminus \omega_j)|| \le c_j ||\nabla_\xi U_{nj}^\varepsilon : L^2(\mathbb{B}_R \setminus \omega_j)||. \tag{6.5}$$ T5

Setting

$$\widehat{u}_n^\varepsilon(x) = u_n^\varepsilon(x),\ x \in \Omega^\varepsilon, \quad \widehat{u}_n^\varepsilon(x) = \widehat{U}_{nj}^\varepsilon(\varepsilon^{-1}(x - x^j)) + \overline{U}_{nj}^\varepsilon,\ x \in \omega_j^\varepsilon,\ j = 1, \dots, J, \tag{6.6}$$ T6

provides the desired extension and proved (6.3) on the base of (6.5) and (6.4).

In view of (6.1) and (6.3) we find a positive infinitesimal sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that, as $\varepsilon = \varepsilon_k \to +0$,

$$\lambda_n^\varepsilon \to \widehat{\lambda}_n^0, \tag{6.7}$$ T7

$$\widehat{u}_n^\varepsilon \rightharpoonup \widehat{u}_n^0 \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Gamma). \tag{6.8}$$ T8

Hence, performing the limit passage $\varepsilon = \varepsilon_k \to +0$ in the integral identity (1.10) with a test function $v \in C_c^\infty(\overline{\Omega} \setminus \{x^1, \dots, x^J\})$, we observe that, for a small $\varepsilon = \varepsilon_j$, $\lambda_n^\varepsilon(u_n^\varepsilon, v)_{\gamma_j^\varepsilon} = 0$ and obtain

$$(\nabla_x \widehat{u}_n^0, \nabla_x v)_\Omega = \widehat{\lambda}_n^0 (\widehat{u}_n^0, v)_\Gamma, \tag{6.9}$$ T9

while any test function $v \in H^1(\Omega)$ is available in (6.9) due to a density argument. To conclude that $\{\widehat{\lambda}_n^0, \widehat{u}_n^0\}$ is an eigenpair of problem (2.1), it suffices to verify that

$$||\widehat{u}_n^0; L^2(\Gamma)|| = 1. \tag{6.10}$$ T10

To this end, we write the estimates

$$||r_j^{-1}(1+|\ln r_j|)^{-1}u; L^2(\Omega)||^2 \le c||u; H^1(\Omega)||^2, \ j=1,\dots,J, \tag{6.11}$$?T11?

which are supported by the one-dimensional Hardy inequality with logarithm

$$\int_0^1 \frac{1}{r}|\ln r|^{-2}|U(r)|^2\,dr \le 4\int_0^1 r\left|\frac{dU}{dr}(r)\right|^2 dr \ \forall U\in H^1(0,1), \ U(1)=0. \tag{6.12}$$ HAR

Furthermore, applying to $\widehat{u}_n^\varepsilon$ the trace inequality

$$||u; L^2(\gamma_j^\varepsilon)||^2 \le c\varepsilon(1+|\ln\varepsilon|)^2!!u; H^1(\Omega)||^2 \tag{6.13}$$ T12

which can be derived by the coordinate dilation $x\mapsto \xi^j=\varepsilon^{-1}(x-x^j)$ and using (6.13), we see that

$$1=||u_n^\varepsilon; L^2(\Gamma^\varepsilon)||^2 = ||\widehat{u}_n^\varepsilon; L^2(\Gamma)||^2 + ||\hat{u}_n^\varepsilon; L^2(\gamma^\varepsilon)||^2 \to ||\widehat{u}_n^0; L^2(\Gamma)||^2 \tag{6.14}$$ T13

and (6.10) is true, indeed.

Unfortunately, the above arguments do not help us to verify that $\widehat{\lambda}_n^0=\lambda_n^0$ and $\widehat{u}_n^0=u_n^0$. We however formulate the final assertion whose proof will be completed in Section ?????.

⟨COS⟩ **Theorem 6.1.** *The eigenvalue sequence* (1.11) *and* (2.2) *of problem* (1.1), (1.4), (1.5), *and* (2.1), *respectively, are in the relationship*

$$\lambda_n^0 = \lim_{\varepsilon\to+0}\lambda_n^\varepsilon. \tag{6.15}$$?T14?

Moreover, the eigenfunctions u_n^0, $n\in\mathbb{N}$, satisfying the normalization and orthogonality conditions (2.3), *can be obtained by the limit passage* (6.8) *from the eigenfunctions u_n^ε, $n\in\mathbb{N}$, satisfying condition* (1.12).

6.2 The Steklov condition on Γ only

⟨subsec62⟩ The above considerations apply to the Neumann–Steklov and Dirichlet–Steklov problems (1.1), (1.4), (1.6), and (1.1), (1.4), (1.9). Moreover, in the Dirichlet case the extension of $u_n^\varepsilon \in H_0^1(\Omega^\varepsilon,\gamma^\varepsilon)$ by zero over the holes is available while in both cases the conclusion (6.14) simplifies, too.

The next assertion will be finalized in Section ????

?⟨COND⟩? **Theorem 6.2.** *Theorem 6.1 remains valid under the change* (1.5) $\mapsto$ (1.7) *and* (1.5) $\mapsto$ (1.9).

6.3 The Dirichlet condition on Γ

⟨subsec63⟩ Let $\{\lambda_n^\varepsilon, u_n^\varepsilon\}$ be an eigenpair of problem (1.1), (1.5), (1.8), while $||u^\varepsilon; L^2(\gamma^\varepsilon)||=1$ and, as will be explained in Section 7.1,

$$\lambda_n^\varepsilon =\le c_n\varepsilon^{-1}. \tag{6.16}$$ K1

We denote $\mathbf{u}_n^\varepsilon=\sqrt{\varepsilon}u_n^\varepsilon$ and derive from (1.10) and (6.16) that

$$||\nabla_x\mathbf{u}_n^\varepsilon; L^2(\Omega^\varepsilon)||^2 = \varepsilon||\nabla_x u_n^\varepsilon; L^2(\Omega^\varepsilon)||^2 = \varepsilon\lambda_n^\varepsilon||u_n^\varepsilon; L^2(\gamma^\varepsilon)||^2 \le c_n. \tag{6.17}$$?K2?

We construct the extension $\widehat{\mathbf{u}}_n^\varepsilon\in H^1(\Omega)$ of $\mathbf{u}_n^\varepsilon$ such that

$$||\nabla_x\mathbf{u}_n^\varepsilon; L^2(\Omega)|| \le c||\nabla_x\mathbf{u}_n^\varepsilon; L^2(\Omega^\varepsilon)|| \le c_n, \tag{6.18}$$ K0

see (6.6) and (6.3), and, owing to (1.8), apply the Friedrichs inequality

$$||\mathbf{u}_n^\varepsilon; L^2(\Omega)|| \le c||\nabla_x\mathbf{u}_n^\varepsilon; L^2(\Omega)|| \le c_n. \tag{6.19}$$ K3

We set

$$w_n^{j\varepsilon}(\xi^j) = \chi_j(x^j + \varepsilon\xi^j)\mathbf{u}_n^\varepsilon(x^j + \varepsilon\xi^j) \tag{6.20}$$ K4

and observe that, according to (1.12), with $\rho^\varepsilon = 1$ on γ^ε and definitions (6.20), (3.3),

$$1 = \sum_{j=1}^{J}\int_{\gamma_j^\varepsilon}|u_n^\varepsilon(x)|^2\,ds_x = \varepsilon^{-1}\sum_{j=1}^{J}\int_{\gamma_j^\varepsilon}|\widehat{\mathbf{u}}_n^\varepsilon(x)|^2\,ds_x = \sum_{j=1}^{J}\int_{\gamma_j}|w_n^{j\varepsilon}(\xi)|^2\,ds_\xi,$$

$$||\nabla_\xi w_n^{j\varepsilon}; L^2(\mathbb{R}^2\setminus\omega_j)||^2 = \int_{\mathbb{B}_{2R/\varepsilon}\setminus\omega_j}|\chi_j\nabla_\xi\widehat{\mathbf{u}}_n^\varepsilon + \widehat{\mathbf{u}}_n^\varepsilon\nabla_\xi\chi_j|^2\,d\xi = \tag{6.21}$$ K5

$$= \int_{\Omega^\varepsilon}|\chi_j\nabla_\xi\widehat{\mathbf{u}}_n^\varepsilon + \widehat{\mathbf{u}}_n^\varepsilon\nabla_\xi\chi_j|^2\,dx \le c_j||\widehat{\mathbf{u}}_n^\varepsilon; H^1(\Omega)|| \le c_{nj}.$$

The Poincaré inequality also demonstrates that

$$||w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R/\varepsilon}\setminus\omega_j)||^2 \le c\left(||\nabla_\xi w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R/\varepsilon}\setminus\omega_j)||^2 + ||w_n^{j\varepsilon}; L^2(\gamma_j)||^2\right)^2 \le c. \tag{6.22}$$ K6

Hence, $w_n^{j\varepsilon} \in \mathcal{H}_j$ has a norm uniformly bounded in ε, compare (2.6) and (6.21),(6.22). We recall (6.16) and find a positive infinitesimal sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that, as $\varepsilon = \varepsilon_k \to +0$,

$$\mu_n^\varepsilon = \varepsilon\lambda_n^\varepsilon \to \widehat{\mu}_n^0, \tag{6.23}$$ K7

$$w_n^{j\varepsilon} \to w_n^{j0} \text{ weakly in } \mathcal{H}_j \text{ and strongly in } L^2(\gamma_j), L^2(\mathbb{B}_{2R}\setminus\omega_j). \tag{6.24}$$ K8

For any $\widehat{v}_j \in C_c^\infty(\mathbb{R}^2\setminus\omega_j)$, we insert the test function $v^\varepsilon(x) = \varepsilon^{-1/2}\widehat{v}_j(\varepsilon^{-1}(x - x^j))$ into the integral identity (1.10), observe that $v^\varepsilon \in H_0^1(\Omega^\varepsilon, \Gamma)$ for a small $\varepsilon > 0$, and perform the limit passage $\varepsilon + 0$. Since $w_n^{j\zeta}(\varepsilon^{-1}(x - x^j)) = \varepsilon^{1/2}u_n^\varepsilon(x)$ for $x \in \operatorname{supp} v^\varepsilon$ while $w_n^{j\varepsilon} = 0$ on γ_l^ε with $l \neq j$, we have

$$0 = (\nabla_x w_n^{j\varepsilon}, \nabla_x v^\varepsilon)_{\Omega^\varepsilon} - \lambda^\varepsilon(w_n^{j\varepsilon}, v^\varepsilon)_{\gamma_j^\varepsilon} \to$$

$$\to (\nabla_x i\widehat{w}_n^{j0}, \nabla_\xi\widehat{v}_j)_{\mathbb{R}^2\setminus\omega_j} - \widehat{\mu}_n^0(\widehat{w}_n^{j0}, \widehat{v}_j)_{\gamma_j} = 0. \tag{6.25}$$?K8N?

Now, we can take any $\widehat{v}_j \in \mathcal{H}_j$ because $C_c^\infty(\mathbb{R}^2\setminus\omega_j)$ is dense in $\mathcal{H}_j$. Hence, $\{\widehat{\mu}_n^0, \widehat{w}_n^{j\varepsilon}\}$ is an eigenpair of the exterior Steklov problem (2.8) provided $\widehat{w}_n^{j0} \not\equiv 0$. On the base of (6.20) and (6.21) we obtain that

$$1 = \sum_{j=1}^{J}||u_n^\varepsilon; L^2(\gamma_j^\varepsilon)||^2 = \sum_{j=1}^{J}||w_n^{j0}; L^2(\gamma_j)||^2 \to \sum_{j=1}^{J}||w_n^{j0}; L^2(\gamma_j)||^2 = 1.$$

Thus, at least for one index $j = 1, \ldots, J$ the limit passages (6.23), (6.24) give us an eigenpair.

To formulate the final assertion whose proof will be concluded in Section 7 ? ????, we join the eigenvalue sequences (2.9), $j = 1, \ldots, J$, into

$$0 = \mu_1^0 = \ldots = \mu_J^0 < \mu_{J+1}^0 \le \mu_{J+2}^0 \le \ldots \le \mu_n^0 \le \ldots \to +\infty. \tag{6.26}$$ K9

In other words, (6.26) consists of all eigenvalues of the exterior Steklov problems (2.8) in the domains $\Xi_1, \ldots, \Xi_J$.

⟨COD⟩ **Theorem 6.3.** *The eigenvalue sequences* (1.11) *and* (6.26) *of problems* (1.1), (1.5), (1.8) *and* (2.8), *respectively, are in the relationship*

$$\mu_n^0 = \lim_{\varepsilon\to+0}\varepsilon\lambda_n^\varepsilon,\ n \in \mathbb{N}. \tag{6.27}$$ K10

The vector eigenfunctions $\mathcal{W}_n = \{w_n^1, \ldots, w_n^J\} \in \mathcal{H}_1 \times \ldots \mathcal{H}_J$ of the family of exterior Steklov problems can be obtained through the limit passage (6.24) *and fulfill the normalization and orthogonality conditions*

$$\sum_{j=1}^{J}(w_n^j, w_m^j)_{\gamma_j} = \delta_{n,m}\ n, m \in \mathbb{N}. \tag{6.28}$$?K11?

6.4 The Neumann condition on Γ

?⟨subsec64⟩? We again take an eigenpair $\{\lambda_n^\varepsilon, u_n^\varepsilon\}$ with the normalized eigenfunction in $L^2(\gamma^\varepsilon)$ and a positive eigenvalue satisfying (6.16) (see Remark **??**). We denote by $\widehat{\mathbf{u}}_n^\varepsilon \in H^1(\Omega)$ the extension of $\mathbf{u}_n^\varepsilon = \sqrt{\varepsilon} u_n^\varepsilon$ fulfilling (6.18). However, the Neumann condition (1.6) at the exterior boundary Γ rejects the Friedrichs inequality (6.19) but, as its substituter, we apply the inequality

$$||\widehat{\mathbf{u}}_n^\varepsilon; L^2(\Omega)||^2 \leq c \left(||\nabla_x \widehat{\mathbf{u}}_n^\varepsilon; L^2(\Omega)||^2 + \left| \int_{\partial\mathbb{B}_R(x^1)} \widehat{\mathbf{u}}_n^\varepsilon(x)\, ds_x \right|^2 \right). \quad (6.29)$$ K12

Here, we took into account that the Dirichlet semi-norm degenerates for constant functions only and used lemma about equivalent norm. Note that the circle $\partial\mathbb{B}_R(x^1)$ around the hole ω_1^ε is involved and $\overline{\mathbb{B}_R(x^1)} \cap \omega_j^\varepsilon = \varnothing$ for $j = 1, \dots, J$.

To estimate the last term in (6.29), we first observe that the functions (6.20) are harmonic in $\mathbb{B}_{2R} \setminus \omega_j$, they satisfy condition (3.23) and are subject to the estimate

$$||\nabla_\xi w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R} \setminus \omega_j)||^2 + ||w_n^{j\varepsilon}; L^2(\partial\omega_j)||^2 \leq \varepsilon\lambda_n^\varepsilon + 1 \leq c_n. \quad (6.30)$$?K13?

Hence, using local estimates [17] of solutions to elliptic problems yields

$$\begin{aligned} &||w_n^{j\varepsilon}; L^2(\partial\mathbb{B}_R)|| \leq ||\partial_\rho w_n^{j\varepsilon}; L^2(\partial\mathbb{B}_R)|| \leq c||w_n^{j\varepsilon}; H^2(\mathbb{B}_R \setminus \omega_j)|| \leq c \big(||\Delta w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R} \setminus \omega_j)|| + \\ &+ ||\partial_{\nu(\xi)} w_n^{j\varepsilon} - \mu_n^\varepsilon w_n^{j\varepsilon}; H^{1/2}(\partial\omega_j)|| + ||w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R} \setminus \omega_j)|| \big) \leq \\ &\leq c \big(0 + 0 + ||\nabla w_n^{j\varepsilon}; L^2(\mathbb{B}_{2R} \setminus \omega_j)|| + ||w_n^{j\varepsilon}; L^2(\partial\omega_j)||\big) \leq c_n. \end{aligned} \quad (6.31)$$ K14

Owing to equations (1.1) and (1.6), we have

$$\sum_{j=1}^J \int_{\partial\mathbb{B}_R} \partial_\rho w_n^\varepsilon(\xi)\, d\xi = \sum_{j=1}^J \int_{\partial\mathbb{B}_{\varepsilon R}(x^j)} \partial_r \mathbf{u}_n^\varepsilon(x)\, ds_x = 0. \quad (6.32)$$ K15

We set $a_1 = J - 1$ and $a_2 = \dots = a_J = -1$ in (2.19), cf. (2.20), and insert the harmonic linear combination

$$\mathbf{G}(x) = (J-1)G^1(x) - G^2(x) - \dots - G^J(x)$$

of the particular generalized Green functions $G^j(x) = G(x, x^j)$, see (2.18), into the Green formula on $\Omega \setminus (\mathbb{B}_{\varepsilon R}(x^1) \cup \dots \cup \mathbb{B}_{\varepsilon R}(x^1))$ together with the function $\mathbf{u}_n^\varepsilon$. Sice, according to the decomposition (2.14) of G^j,

$$\mathbf{G}(x) = (2\pi)^{-1} b_j |\ln\varepsilon| + O(1), \quad \partial_{r_j} \mathbf{G}(x) = O(\varepsilon^{-1}),\ x \in \partial\mathbb{B}_{\varepsilon R}(x^j),$$

we detect that, in view of (6.31),

$$\begin{aligned} &\frac{1}{2\pi} |\ln\varepsilon| \left| \sum_{j=1}^J b_j \int_{\partial\mathbb{B}_{\varepsilon R}} \partial_\rho w_n^{j\varepsilon}(\xi)\, d\xi \right| \leq c \sum_{j=1}^J \big(||\partial_\rho w_n^{j\varepsilon}; L^2(\partial\mathbb{B}_R)|| + \\ &+ ||\partial_\rho w_n^{j\varepsilon}; L^2(\partial\mathbb{B}_R)|| \big) \leq c_n \end{aligned} \quad (6.33)$$ K16

Combining (6.32) and (6.33) leads us to

$$\left| \int_{\partial\mathbb{B}_{\varepsilon R}(x^1)} \partial_{r_1} \mathbf{u}_n^\varepsilon(x)\, ds_x \right| = \left| \int_{\partial\mathbb{B}_R} \partial_\rho w_n^{1\varepsilon}(\xi)\, ds_\xi \right| \leq \frac{c_n}{|\ln\varepsilon|}. \quad (6.34)$$ K17

Now, we apply the Green formula for $\mathbf{u}_n^\varepsilon$ and $\ln(R^{-1}r_j)$ in the annulus $\mathbb{B}_R(x^1)\setminus\mathbb{B}_{\varepsilon R}(x^1)$ to conclude that

$$\frac{1}{R}\left|\int_{\partial\mathbb{B}_R(x^1)}\mathbf{u}_n^\varepsilon(x)\,ds_x\right| = \left|\int_{\partial\mathbb{B}_{\varepsilon R}(x^1)}\big(\mathbf{u}_n^\varepsilon(x)\frac{\partial}{\partial r_j}\ln\frac{r_j}{R}-$$

$$-\ln\varepsilon\partial_{r_j}\mathbf{u}_n^\varepsilon(x)\big)\,ds_x\right| \leq \left|\int_{\partial\mathbb{B}_R}\big(|w_n^{1\varepsilon}(\xi)|+|\ln\varepsilon||\partial_\rho w_n^{1\varepsilon}(\xi)|\big)\,ds_\xi\right|\leq c_n.$$

Here, we used (6.31) and especially (6.34) with the infinitesimal bound $c_n|\ln\varepsilon|^{-1}$.

Thus, the right-hand side of (6.29) is uniformly bounded in $\varepsilon\in(0,\varepsilon_n]$ with some $\varepsilon_n >$ and we have the desired estimates

$$||w_n^{j\varepsilon};\mathcal{H}_j||\leq c_n,\ \ j=1,\dots,J,$$

for the functions (6.20) and (6.24). Now we repeat word-by-word the arguments from Section 6.3 and conclude with the convergence theorem on the eigenvalue problem with the Neumann conditions on Γ and the Steklov conditions on $\gamma_1^\varepsilon,\dots,\gamma_J^\varepsilon$.

⟨CON⟩ **Theorem 6.4.** *The assertions of Theorem 6.3 remain valid for the eigenpairs of problem* (1.1)*,* (1.5)*,* (1.6).

7 Justification of asymptotics

⟨sec7⟩ 7.1 The middle-frequency range of the pure Steklov problem

⟨subsec71⟩ The Sobolev space $H^1(\Omega^\varepsilon)$ with the scalar product

$$\langle u^\varepsilon, v^\varepsilon\rangle_\varepsilon = (\nabla_\xi u^\varepsilon,\nabla_\xi v^\varepsilon)_{\Omega^\varepsilon}+(u^\varepsilon,v^\varepsilon)_{\partial\Omega^\varepsilon} \tag{7.1}$$ J1

is denoted by $\mathcal{H}^\varepsilon$. we introduce the positive continuous symmetric, and therefore self-adjoint, operator $\mathcal{T}^\varepsilon$ as follows

$$\langle \mathcal{T}^\varepsilon u^\varepsilon, v^\varepsilon\rangle_\varepsilon = (u^\varepsilon,v^\varepsilon)_{\partial\Omega^\varepsilon}\quad \forall u^\varepsilon,v^\varepsilon\in\mathcal{H}^\varepsilon. \tag{7.2}$$ J2

Since the embedding $H^1(\Omega^\varepsilon)\subset L^2(\partial\Omega^\varepsilon)$ is compact, this property is attributed to $\mathcal{T}^\varepsilon$ as well. Hemce, by [18, Theorems 10.1.5 and 10.2.2], the essential spectrum of $\mathcal{T}^\varepsilon$ consists of the only point $\tau=0$ while the discrete spectrum constitutes the positive infinitesimal sequence

$$\tau_1^\varepsilon\geq\tau_2^\varepsilon\geq\ldots\geq\tau_n^\varepsilon\geq\ldots\to+0 \tag{7.3}$$ J3

where the eigenvalues multiplicity is taken into account. Comparing (7.1), (7.2) and (1.10) with $\rho^\varepsilon=1$, we see that the variational formulation of problem (1.1), (1.4), (1.5) is equivalent to the abstract equation

$$\mathcal{T}^\varepsilon u^\varepsilon=\tau^\varepsilon u^\varepsilon\quad\text{in}\ \ \mathcal{H}^\varepsilon. \tag{7.4}$$ J4

with the spectral parameter τ^ε,

$$\tau^\varepsilon=(1+\lambda^\varepsilon)^{-1}\Leftrightarrow\lambda^\varepsilon=(\tau^\varepsilon)^{-1}-1. \tag{7.5}$$ J5

The relationship (7.5) transforms (7.3) into the unbounded sequence (1.11); in particular $\tau_2^\varepsilon>\tau_1^\varepsilon=1$.

The next assertion is known [19] as Lemma on "almost eigenvalues and eigenvectors" and follows directly from the spectral decomposition od resolvent, cf. [18, Ch. 6, Section 2].

⟨NEAR⟩ **Lemma 7.1.** *Let $\mathcal{U}^\varepsilon \in \mathcal{H}^\varepsilon$ and $t^\varepsilon \in \mathbb{R}_+$ be such that*

$$||\mathcal{U}^\varepsilon; \mathcal{H}^\varepsilon|| = 1, \ ||\mathcal{T}^\varepsilon \mathcal{U}^\varepsilon - t^\varepsilon \mathcal{T}^\varepsilon; \mathcal{H}^\varepsilon|| == \delta^\varepsilon \in (0, t^\varepsilon). \tag{7.6}$$ J6

Then the closed segment $[t^\varepsilon - \delta^\varepsilon, t^\varepsilon + \delta^\varepsilon]$ contains at least one eigenvalue of the operator $\mathcal{T}^\varepsilon$. Moreover, for any $\alpha > 1$, there exists a coefficient columns $c^\varepsilon = (c^\varepsilon_{K(\varepsilon)}, \dots, c^\varepsilon_{K(\varepsilon)+X(\varepsilon)-1})^\top \in \mathbb{R}^{X(\varepsilon)}$ such that

$$||\mathcal{U}^\varepsilon - \sum_{k=K(\varepsilon)}^{K(\varepsilon)+X(\varepsilon)-1} c^\varepsilon_k u^\varepsilon_k; \mathcal{H}^\varepsilon|| \le \frac{2}{\alpha}, \qquad \sum_{k=K(\varepsilon)}^{K(\varepsilon)+X(\varepsilon)-1} |c^\varepsilon_k|^2 = 1 \tag{7.7}$$ J7

where $\tau^\varepsilon_{K(\varepsilon)}, \dots, \tau^\varepsilon_{K(\varepsilon)+X(\varepsilon)-1}$ is the list of all eigenvalues in (7.3) which belong to $[t^\varepsilon - \delta^\varepsilon\alpha, t^\varepsilon + \delta^\varepsilon\alpha]$ and the corresponding eigenvectos $u^\varepsilon_{K(\varepsilon)}, \dots, u^\varepsilon_{K(\varepsilon)+X(\varepsilon)-1}$ of $\mathcal{T}^\varepsilon$ are orthonormalized in $\mathcal{H}^\varepsilon$.

Based on the analysis in Section 5.2, we propose the following approximations for eigenpairs;

$$t^\varepsilon_p = (1 + \lambda^0_n + \varepsilon\lambda'_p)^{-1}, \ \mathcal{U}^\varepsilon_p = ||\mathcal{V}^\varepsilon_p; \mathcal{H}^\varepsilon||^{-1}\mathcal{V}^\varepsilon_p, \ p = n, \dots, n + +\kappa - 1. \tag{7.8}$$ J8

$$\mathcal{V}^\varepsilon_p(x) = u^{p0}(x) + \varepsilon u^{p\prime}(x, \mathfrak{z}) + \varepsilon \sum_{j=1}^{J} \chi_j(x) w^{pj}(\xi^j), \tag{7.9}$$?J9?

where $\lambda^0_n > 0$ is an eigenvalue of the Steklov problem (2.1) with multiplicity $\kappa \ge 1$, see (3.10), u^{p0} are linear combinations (3.11) of the corresponding eigenfunctions $u^0_n, \dots, u^0_{n+\kappa-1}$, orthonormalized in $L^2(\Gamma)$, (2.3), with coefficient columns $c^n, \dots, c^{n+\kappa-1}$ which are eigenvectors of the matris $\mathcal{M}$ with entries (5.9), while $\lambda'_n, \dots, \lambda'_{n+\kappa-1}$ are the corresponding eigenvalues, (3.15). Finally, w^{pj} are the usual boundary-layer terms but the solutions (5.5) of the exterior Neumann problem in Ξ_j with the logarithmic growth as $|\xi| \to +\infty$.

Let us mention that, by construction,

$$\begin{aligned}
&||\nabla_x u^{p0}; L^2(\Omega \setminus \Omega^\varepsilon)|| \le c\varepsilon, \quad \varepsilon||\nabla_x u^{p\prime}; L^2(\Omega^\varepsilon)|| \le c\varepsilon, \quad \varepsilon||u^{p\prime}; L^2(\Gamma)|| \le c\varepsilon|\ln\varepsilon| \\
&||\mathcal{V}^\varepsilon_p; L^2(\gamma^\varepsilon_j)|| \le c\varepsilon^{1/2}, \\
&\varepsilon||\nabla_x(\chi_j w^{pj}); L^2(\Omega^\varepsilon)|| \le c\varepsilon \sum_{j=1}^{J} \left(\int_{\Omega^\varepsilon} r_j^{-2} \, dx \right)^{1/2} \le c\varepsilon|\ln\varepsilon|^{1/2}
\end{aligned}$$

and hence, according to (2.1), (2.3), and (**??**),

$$\begin{aligned}
&|\langle \mathcal{V}^\varepsilon_p, \mathcal{V}^\varepsilon_q \rangle_\varepsilon - \delta_{p,q}(1 + \lambda^0_n)| \le |(\nabla_x u^{p0}, \nabla_x u^{q0})_\Omega + (u^{p0}, u^{q0})_\Gamma - \delta_{p,q}(1 + \lambda^0_n)| + \\
&+ c\varepsilon|\ln\varepsilon| = \sum_{k,l=n}^{n+\kappa-1} c^p_k c^q_l ((\nabla_x u^0_k, \nabla_x u^0_l)_\Omega + (u^0_k, u^0_l)_\Gamma) - \delta_{p,q}(1 + \lambda^0_n)| + \\
&+ c\varepsilon|\ln\varepsilon| = 0 + c\varepsilon|\ln\varepsilon|.
\end{aligned} \tag{7.10}$$ J10

To compute the value δ^ε_p in Lemma 7.1, we write

$$\begin{aligned}
&||\mathcal{T}^\varepsilon \mathcal{U}^\varepsilon_p - t^\varepsilon_p \mathcal{U}^\varepsilon_p; \mathcal{H}^\varepsilon|| = \sup |\langle \mathcal{T}^\varepsilon \mathcal{U}^\varepsilon_p - t^\varepsilon_p \mathcal{U}^\varepsilon_p, \mathcal{W}^\varepsilon \rangle_\varepsilon = \\
&= ||\mathcal{V}^\varepsilon_p; \mathcal{H}^\varepsilon||^{-1} t^\varepsilon_p \sup |(\nabla_x \mathcal{V}^\varepsilon_p, \nabla_x \mathcal{W}^\varepsilon)_{\Omega^\varepsilon} - (\lambda^0_n + \varepsilon\lambda'_p)(\mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)_{\partial\Omega^\varepsilon}| = \\
&= ||\mathcal{V}^\varepsilon_p; \mathcal{H}^\varepsilon||^{-1} t^\varepsilon_p \sup |(\Delta_x \mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)_{\Omega^\varepsilon} + (\partial_\nu \mathcal{V}^\varepsilon_p - (\lambda^0_n + \varepsilon\lambda'_p)\mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)_{\partial\Omega^\varepsilon}|
\end{aligned} \tag{7.11}$$ J11

Here the supremum is computed over the unit sphere in $\mathcal{H}^\varepsilon$, i.e., $||\mathcal{W}^\varepsilon; \mathcal{H}^\varepsilon|| = 1$, that, in particular, means:

$$||\nabla_x \mathcal{W}^\varepsilon; H^1(\Omega^\varepsilon)|| + ||r_j^{-1}(1 + |\ln r_j|)^{-1} \mathcal{W}^\varepsilon; L^2(\Omega^\varepsilon)|| + \varepsilon^{-1/2}|\ln\varepsilon|^{-1}||\mathcal{W}^\varepsilon; L^2(\gamma^\varepsilon_j)|| \le c. \tag{7.12}$$ J12

In view of (7.8) and (7.10), the factor of the last supremum in (7.11) is uniformly bounded in ε. By construction in Section 5.2 we have

$$-\Delta_x \mathcal{V}_p^\varepsilon = \mathcal{F}^\varepsilon := -\varepsilon\Delta_x u^{p\prime} - \varepsilon\sum_{j=1}^{J}[\Delta_x, \chi_j] w^{pj} = -\varepsilon\sum_{j=1}^{J}[\Delta_x, \chi_j]\widehat{w}^{pj}$$

and

$$|(\mathcal{F}_p^\varepsilon, \mathcal{W}^\varepsilon)_{\Omega^\varepsilon}| \le c\varepsilon^2 ||\mathcal{W}^\varepsilon; L^2(\Omega^\varepsilon)|| \le c\varepsilon^2$$

because $|\widetilde{w}(\xi^j)|+|\nabla_x\widetilde{w}(\xi^j)| \le c\varepsilon$ for $x\in supp|\nabla_x\chi_j|$, see (2.12) and (3.3). Furthermore, on the exterior boundary we obtain

$$g_p^{0\varepsilon} = \partial_\nu u^{p0} + \varepsilon\partial_\nu u^{p\prime} - (\lambda_n^0 + \varepsilon\lambda_p')(u^{p0} + \varepsilon u^{p\prime}) = -\varepsilon^2\lambda_p' u^{p\prime},$$

$$|(g_p^{0\varepsilon}, \mathcal{W}_p^\varepsilon)_\Gamma| \le c\varepsilon^2 ||\mathcal{W}_p^\varepsilon; L^2(\Gamma)|| \le c\varepsilon^2.$$

Finally, on the boundaries γ_j^ε of the small holes we have

$$\begin{aligned} g_p^{j\varepsilon}(x) &= \partial_{\nu(x)}(u^{p0}(x) + \varepsilon u^{p\prime}(x,\mathfrak{z}) + \varepsilon w^{pj}(\xi^j)) - (\lambda_n^0 + \varepsilon\lambda_p')(u^{p0}(x) + \varepsilon u^{p\prime}(x,\mathfrak{z}) + \varepsilon w^{pj}(\xi^j)) = \\ &= (\partial_{\nu(\xi)} w^{pj}(\xi^j) + \partial_{\nu(\xi)}\xi^j\cdot\nabla_x u^{p0}(x^j) - \lambda_n^0 u^{p0}(x^j)) + \widetilde{g}_p^{j\varepsilon}(x), \quad |\widetilde{g}_p^{j\varepsilon}(x)| \le c\varepsilon|\ln\varepsilon|, \end{aligned}$$

and, owing to (7.12),

$$|(g_p^{j\varepsilon}, \mathcal{W}^\varepsilon)_{\gamma_j^\varepsilon}| \le c\varepsilon|\ln\varepsilon||\gamma_j^\varepsilon|^{1/2}||\mathcal{W}^\varepsilon; L^2(\gamma_j^\varepsilon)|| \le c\varepsilon^2|\ln\varepsilon|^2.$$

Collecting these estimates, we see that

$$\delta_p^\varepsilon = ||\mathcal{T}^\varepsilon\mathcal{U}_p^\varepsilon - t_p^\varepsilon\mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon|| \le c_n\varepsilon^2|\ln\varepsilon|^2 \tag{7.13}$$?38?

and, therefore, one finds at least one eigenvalue of the operator $\mathcal{T}^\varepsilon$ such that

$$c_n\varepsilon^2|\ln\varepsilon|^2 \ge |\tau_{m^\varepsilon(p)}^\varepsilon - t_p^\varepsilon| = |(1+\lambda_{m^\varepsilon(p)}^\varepsilon)^{-1} - (1+\lambda_n^0+\varepsilon\lambda_p')^{-1}|. \tag{7.14}$$ J13

A simple calculation derives from (7.14) estimate (5.10) for $\lambda_{m^\varepsilon(p)}^\varepsilon$ and $\lambda_n^0+\varepsilon\lambda_p'$. We however cannot conclude that $m^\varepsilon(p_1)\ne m^\varepsilon(p_2)$ in the case $p_1\ne p_2$ and
$lambda_{p_1}' \ne \lambda_{p_2}'$.

In order to prove that a $c\varepsilon^2|\ln\varepsilon|^2$-neighbourhood of the point $\lambda_n^0+\varepsilon\lambda_p'$ contains at least τ eigenvalues $\lambda\lambda_{m^\varepsilon(p)}^\varepsilon,\dots,\lambda_{m^\varepsilon(p)+\tau-1}^\varepsilon$ for an eigenvalue λ_p' of the matrix $\mathcal{M}^n$ with multiplicity $\tau>1$, see (3.17), we employ the second part of Lemma 7.1 and find orthonormalized columns $c_{(p)}^\varepsilon,\dots,c_{(p+\tau-1)}^\varepsilon\in\mathbb{R}^{X(\varepsilon)}$ such that

$$||\mathcal{U}_q^\varepsilon - S_q^\varepsilon; \mathcal{H}^\varepsilon|| \le 2\alpha^{-1}, \quad q=p,\dots,p+\tau-1. \tag{7.15}$$ J14

Here, S_q^ε is the linear combination indicated in (7.7) for $\mathcal{U}_q^\varepsilon$ and $X(\varepsilon)$ is the number of eigenvalues of $\mathcal{T}^\varepsilon$ in the segment

$$[t_p^\varepsilon - \alpha\delta_\bullet^\varepsilon, t_p^\varepsilon + \alpha\delta_\bullet^\varepsilon], \quad \delta_\bullet^\varepsilon = \max\{\delta_p^\varepsilon,\dots\delta_{p+\tau-1}^\varepsilon\} \le c_\bullet\varepsilon^2|\ln\varepsilon|^2.$$

We have

$$\begin{aligned} &|(c_{(k)}^\varepsilon)^\top c_{(l)}^\varepsilon - \delta_{k,l}| = |\langle S_k^\varepsilon, S_l^\varepsilon\rangle_\varepsilon - \delta_{k,l}| = \\ &= |\langle S_k^\varepsilon, S_l^\varepsilon - \mathcal{U}_l^\varepsilon\rangle_\varepsilon + \langle S_k^\varepsilon - \mathcal{U}_k^\varepsilon, \mathcal{U}_l^\varepsilon\rangle_\varepsilon + \langle\mathcal{U}_k^\varepsilon,\mathcal{U}_l^\varepsilon\rangle_\varepsilon - \delta_{k,l}| \le \\ &= 2\alpha^{-1} + 2\alpha^{-1} + c\varepsilon|\ln\varepsilon|. \end{aligned} \tag{7.16}$$ JJJ

Here, we used the equality $||S_k^\varepsilon;\mathcal{H}^\varepsilon||=1$ according to (7.7) and the estimate $|\langle \mathcal{U}_k^\varepsilon,\mathcal{U}_l^\varepsilon\rangle_\varepsilon-\delta_{k,l}|\le c\varepsilon|\ln\varepsilon|$ supported by (7.10). Thus, for a small $\varepsilon>0$ and a big $\alpha>1$, the columns $c^\varepsilon_{(p)},\dots,c^\varepsilon_{(p+\tau-1)}\in\mathbb{R}^{X(\varepsilon)}$ are almost "almost orthonormalized" so that $X(\varepsilon)\ge\tau$ indeed and the segment (7.15) contains at least τ eigenvalues $\tau^\varepsilon_{m^\varepsilon(p)},\dots,\tau^\varepsilon_{m^\varepsilon(p)+\tau-1}$ of the operator $\mathcal{T}^\varepsilon$, while the relationship (7.5) delivers the desired eigenvalues $\lambda^\varepsilon_{m^\varepsilon(p)},\dots,\lambda^\varepsilon_{m^\varepsilon(p)+\tau-1}$ of the Steklov problem (1.1), (1.4), (1.5).

In a small neighbourhood of each eigenvalue λ_n^0 of the Steklov problem in Ω with multiplicity κ_n we have found eigenvalues

$$\lambda^\varepsilon_{m^\varepsilon(p)},\dots,\lambda^\varepsilon_{m^\varepsilon(p)+\kappa_n-1}\in[\lambda_n^0-C_n\varepsilon,\lambda_n^0+C_n\varepsilon]\quad\text{for}\ \ \varepsilon\in(0,\varepsilon_n]\tag{7.17}$$ J15

of the Steklov problem in Ω^ε. This fact confirms the relation (6.1) required in Section 6.1 to prove the convergences (6.7) and (6.8). Moreover, $m(\varepsilon)\ge n$ in (7.17). Recalling all the above estimates and the bounds $c_n\varepsilon^2|\ln\varepsilon|^2$, in order conclude the proof of Theorem 5.1, we need to verify the equality $m(\varepsilon)=n$. To this end, arguing by contradiction, we may assume that $m(\varepsilon)>n$, and for an infinitesimal positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, we detect eigenvalues $\lambda_1^\varepsilon,\dots,\lambda^\varepsilon_{n+\kappa_n}\in[0,\lambda_n^0+C_n\varepsilon]$ and the corresponding eigenfunctions $u_1^\varepsilon,\dots,u^\varepsilon_{n+\kappa_n}\in H^1(\Omega^\varepsilon)$, verifying condition (1.12) with $\rho^\varepsilon=1$. Thus, in view of formulas (6.7), (6.8) and $||u_p^\varepsilon;L^2(\gamma_j^\varepsilon)||\le c_p\varepsilon^{1/2}|\ln\varepsilon|$, the limits $\widetilde{u}_1^\varepsilon,\dots,\widetilde{u}^\varepsilon_{n+\kappa_n}\in H^1(\Omega)$ indicated in Section 6.1 satisfy the orthogonality and normalization conditions (2.3), but correspond to eigenvalues $\widetilde{\lambda}_1^\varepsilon,\dots,\widetilde{\lambda}^\varepsilon_{n+\kappa_n}$ in the segment $[0,\lambda_n^0]$ which contains just $n+\kappa_n-1$ eigenvalues in the sequence (2.2). This contradiction concludes the proof of Theorem 5.1.

7.2 The Dirichlet and Neumann conditions at small holes

?⟨subsec72⟩? Theorem 3.1 about asymptotics of the spectrum (1.11) of problem (1.1),(1.4), (1.6) considered in Sections 3.1, 3.2 and 6.2 can be proved just along the same lines as in the previous section. The only deviation appears in the improved bound in estimate (3.16), due to the visible differences in the asymptotic ansätze (3.1),(3.2), and (5.3), (5.4), which in turn provide different orders of discrepancies in abstract equations of type (7.4), see the last formula in (7.6).

Much more significant deviation of the bounds $O(\varepsilon)$ and $O(|\ln\varepsilon|^{-2})$ is observed in the estimates (4.14) and (4.22) in Theorems 4.1 and 4.2 serving the Stekhlov–Dirichlet problem (1.1), (1.5), (1.8). The difference originates in the use of "logarithmic" and "power-low" asymptotic expansions while the latter, in some sense, is obtained by summation of infinite series in powers of $\mathfrak{z}=|\ln\varepsilon|^{-1}$. We will further comment of these kinds of expansions in the next section.

7.3 The Neumann condition at the exterior boundary

⟨subsec73⟩ We again use Lemma 7.1 on "almost eigenvalues and eigenvectors" but in a bit different framework. Namely, instead of the scalar product 7.1 in the Sobolev space $\mathcal{H}^\varepsilon=H^1(\Omega^\varepsilon)$, we employ the following one:

$$\langle u^\varepsilon,v^\varepsilon\rangle_\varepsilon=(\nabla u^\varepsilon,\nabla v^\varepsilon)_{\Omega^\varepsilon}+\frac{1}{\varepsilon}(u^\varepsilon,v^\varepsilon)_{\gamma_\varepsilon}.\tag{7.18}$$ V1

The new factor $1/\varepsilon$ of the last term in (7.18) and its restrictrion on the interior boundary $\gamma^\varepsilon=\gamma_1^\varepsilon\cup\ldots\cup\gamma_J^\varepsilon$ is due to the replacement of the spectral parameter $\lambda^\varepsilon\mapsto\mu^\varepsilon$ and its disappearance from the boundary condition on Γ, cf. (3.19) and (1.6). At the same time, the operator $\widetilde{T}^\varepsilon$ is still defined by (7.2) and the variational formulation (1.10) of problem (1.1), (1.5), (1.6) is equivalent to the abstract equation (7.4) with the new spectral parameter

$$\tau^\varepsilon=(\varepsilon^{-1}+\lambda^\varepsilon)^{-1}=\varepsilon^{-1}(1+\mu^\varepsilon)^{-1}\iff\lambda^\varepsilon=(\tau^\varepsilon)^{-1}-\varepsilon^{-1}.\tag{7.19}$$?V2?

Notice that $\tau_1^\varepsilon=\varepsilon$ and $\lambda_1^\varepsilon=0$ in (7.3) and (1.11), respectively. To prove Theorem 3.2 about perturbation of a simple eigenvalue $\mu_N=\mu_n^1$ in (3.46), see 3.5, we choose the folllowing approximation of an eigenpair $\{\lambda_N^\varepsilon, u_N^\varepsilon\}$:

$$t_N^\varepsilon = (\varepsilon^{-1} + \varepsilon^{-1}(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z})))^{-1}, \quad \mathcal{U}_N^\varepsilon = ||\mathcal{V}_N^\varepsilon; \mathcal{H}^\varepsilon||^{-1}\mathcal{V}_n^\varepsilon,$$

$$\mathcal{V}_n^\varepsilon(x) = a(\mathfrak{z}) + \sum_{j=1}^{J}(\mathfrak{z}a^j(\mathfrak{z})G^j(x) + \chi_j(x)w^j(\xi^j;\mathfrak{z})) \tag{7.20}$$ V3

extracting the ingredients (3.47) of the asymptotic ansätze (3.19) and (3.20). First of all, we observe that formulas (7.20) and (7.18) assure the inequalities

$$|t_N^\varepsilon| \geq c\varepsilon, \quad C \geq ||\mathcal{V}_N^\varepsilon; \mathcal{H}^\varepsilon|| \geq c > 0. \tag{7.21}$$?V4?

Then, the discrepancy δ_n^ε in (7.6) is equal to

$$\begin{aligned}\sup|\langle\mathcal{T}^\varepsilon\mathcal{U}_N^\varepsilon - t_N^\varepsilon\mathcal{U}_N^\varepsilon, \mathcal{W}_N^\varepsilon\rangle_\varepsilon| = ||\mathcal{V}_N^\varepsilon; \mathcal{H}^\varepsilon||^{-1}t_N^\varepsilon \sup|(\nabla\mathcal{V}_N^\varepsilon, \nabla\mathcal{W}^\varepsilon)_{\Omega^\varepsilon} - \varepsilon^{-1}(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))(\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon)_{\gamma_\varepsilon}| \leq \\ \leq c\varepsilon \sup|(\Delta\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon)_{\Omega^\varepsilon} - (\partial_\nu\mathcal{V}_N^\varepsilon - \varepsilon^{-1}((\mu_n^1) + \mathfrak{z}\mu'(\mathfrak{z}))\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon)_{\gamma_\varepsilon}|\end{aligned} \tag{7.22}$$ V5

where the supremum is computed over the unit sphere in $\mathcal{H}^\varepsilon$, i.e., $||\mathcal{W}^\varepsilon; \mathcal{H}^\varepsilon|| = 1$.

?⟨LL⟩? **Lemma 7.2.** *The inequality*

$$||u^\varepsilon; L^2(\Omega^\varepsilon)|| \leq c(1 + |\ln\varepsilon|)||u^\varepsilon; \mathcal{H}^\varepsilon|| \tag{7.23}$$ V6

is valid where $j = 1, \dots, J$, $r_j = |x - x_j|$ and the factor c is independent of $u^\varepsilon \in H^1(\Omega^\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$.

Proof. For the extension $\widehat{u}^\varepsilon \in H^1(\Omega)$ of $u^\varepsilon \in \mathcal{H}^\varepsilon$, see (6.3), we set

$$\widehat{u}^\varepsilon(x) = \widehat{u}_0^\varepsilon(x) + \widehat{u}_\perp^\varepsilon(x), \qquad \int_\Omega \widehat{u}_\perp^\varepsilon(x)\,dx = 0.$$

Owing to the Poincaré inequality supported by the last orthogonality condition we have

$$||\widehat{u}_\perp^\varepsilon; L^2(\Omega)||^2 \leq c_\Omega||\nabla\widehat{u}_\perp^\varepsilon; L^2(\Omega)||^2 = c_\Omega||\nabla\widehat{u}^\varepsilon; L^2(\Omega)||^2 \leq c||u^\varepsilon; \mathcal{H}^\varepsilon||^2.$$

Applying the one-dimensional Hardy inequality (6.12) in polar coordinates and the trace inequality (6.13) we obtain

$$||u^\varepsilon; \mathcal{H}^\varepsilon||^2 \geq \frac{1}{\varepsilon}\sum_{j=1}^{J}||u^\varepsilon; L^2(\gamma_j^\varepsilon)||^2 \geq \frac{1}{\varepsilon}\sum_{j=1}^{J}\left(\tfrac{1}{2}||\widehat{u}_0^\varepsilon; L^2(\gamma_j^\varepsilon)||^2 - ||\widehat{u}_\perp^\varepsilon; L^2(\gamma_j^\varepsilon)||^2\right)$$

hence

$$|\widehat{u}_0^\varepsilon|^2 \leq 2|\gamma|\left(||u^\varepsilon; \mathcal{H}^\varepsilon||^2 + \frac{1}{\varepsilon}\sum_{j=1}^{J}||\widehat{u}_\perp^\varepsilon; L^2(\gamma_j^\varepsilon)||^2\right) \leq c(1 + |\ln\varepsilon|^2)||u^\varepsilon; \mathcal{H}^\varepsilon||^2.$$

□

According to (3.47) and (2.12), the harmonics w^j gets the decay rate

$$|w^j(\xi,\mathfrak{z})| + (1 + |\xi|)|\nabla_\xi w^j(\xi,\mathfrak{z})| \leq c_j(1 + |\xi|)^{-1}. \tag{7.24}$$ V0

Hence, thanks to (7.23), the term $|(\Delta\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon)_{\Omega^\varepsilon}|$ in (7.22) satisfies

$$|(\Delta\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon)_{\Omega^\varepsilon}| \leq c||\mathcal{W}^\varepsilon; L^2(\Omega^\varepsilon)||\sum_{j=1}^{J}\max_{x\in\upsilon_j}\left(|\nabla_x w^j(\xi^j,\mathfrak{z})| + |w^j(\xi^j,\mathfrak{z})|\right) \leq c\varepsilon(1 + |\ln\varepsilon|) \tag{7.25}$$ V7

because $|\xi^j| \geq c_j \varepsilon^{-1}$, $c_j > 0$ for $x \in \upsilon_j = \text{supp}|\nabla \chi_j|$. Furthermore, w^j solves problem (3.24), and therefore

$$\partial_{\nabla} u \mathcal{V}_N^\varepsilon - \varepsilon^{-1}(\mu_n^1 + \mu'(\mathfrak{z}))\mathcal{V}_N^\varepsilon = \sum_{k=1}^J a^k(\mathfrak{z}) \left(\partial_\nu \widetilde{G}^k(x) - \varepsilon^{-1}(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))\widetilde{G}^k(x) \right), \quad x \in \gamma_j^\varepsilon$$

where

$$\widetilde{G}^j(x) = G^j(x) + \chi_j(x) \frac{1}{2\pi} \ln \frac{1}{r_j} - \sum_{k=1}^J \chi_k(x) \mathcal{G}_{jk}, \quad \widetilde{G}^k \in C^\infty(\overline{\Omega}),$$

is the regular part of the generalized Green function, see (2.18) and (2.14). As a result, we have

$$| \left(\partial_\nu \mathcal{V}_N^\varepsilon - \varepsilon^{-1}(\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))\mathcal{V}_N^\varepsilon, \mathcal{W}^\varepsilon \right)_{\gamma^\varepsilon} | \leq c \sum_{j=1}^J |\gamma_j^\varepsilon|^{-1/2} ||\mathcal{W}^\varepsilon; L^2(\gamma_j^\varepsilon)|| \leq$$

$$\leq c\varepsilon^{1/2}\varepsilon^{1/2} ||\mathcal{W}^\varepsilon; \mathcal{H}^\varepsilon|| = c\varepsilon. \tag{7.26} \boxed{\texttt{V8}}$$

In view of (7.25) and (7.26) the discrepancy (7.22) does not exceed $c\varepsilon(1+|\ln\varepsilon|)$ and Lemma 7.1 delivers an eigenvalue $\tau_{m(\varepsilon)}^\varepsilon$ of the operator $\mathcal{T}^\varepsilon$ defined in (7.2), such that

$$|\tau_{m(\varepsilon)}^\varepsilon - t_N^\varepsilon| \leq c\varepsilon^2(1 + |\ln\varepsilon|)$$

yields

$$|\mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}) - \varepsilon\lambda_{m(\varepsilon)}^\varepsilon| \leq c_N \mathcal{V}^\varepsilon (1 + |\ln\varepsilon|)(1 + \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))(1 + \varepsilon\lambda_{m(\varepsilon)}^\varepsilon). \tag{7.27} \boxed{\texttt{V9}}$$

If $\varepsilon \leq \varepsilon_N$ and $\varepsilon_N > 0$ is fixed to fulfil

$$c_N \varepsilon_N (1 + |\ln\varepsilon_N|)(1 + \mu_n^1 + |\ln\varepsilon_N|^{-1}\mu'(|\ln\varepsilon_N|^{-1})) \leq \frac{1}{2}, \tag{7.28} ?\underline{\texttt{V10}}?$$

we derive $1 + \varepsilon\lambda_{m(\varepsilon)}^\varepsilon \leq 2(1 + \mu_n^1 + \mathfrak{z}\mu'(\mathfrak{z}))$ from (7.27) and conclude the estimate (3.67) for $\lambda_{m(\varepsilon)}^\varepsilon$. The coincidence $m(\varepsilon) = N$ follows from Theorem 6.4 on the convergence (6.27) of eigenvalues of the Steklov–Neumann problem (1.1), (1.5), (1.6); see the end of this section.

Let s now confirm assertions of Theorems 3.4 and 3.3. First of all we consider a positive eigenvalue μ_N of multiplicity $X \geq 1$ in the sequence (3.46), i.e., similarly to (3.17) we have

$$0 \leq \mu_{N-1} < \mu_N = \ldots = \mu_{N+X-1} < \mu_{N+X}. \tag{7.29} ?\underline{\texttt{U1}}?$$

By Lemma 3.1, μ_N appears just X times in the spectra (3.33) of problems (3.25), (3.29), (3.30) (or (3.31) in the differential form) with $j = 1, \ldots, J$ and we form the vectors $\vec{w}_{(p)} = (w_{(p)}^1, \ldots, w_{(p)}^J)$, $p = N, \ldots, N * X - 1$, as follows:

$$\mu_N = \mu_{m^j}^j \quad \Longrightarrow \quad w_{(p)}^j = w_{m^j}^j \text{ and } w_{(p)}^k = 0 \text{ for } k \neq j.$$

The eigenfunctions $w_{m^j}^j$ obey the normalization and orthogonality conditions (3.34) so that

$$(\vec{w}_{(p)}, \vec{w}_{(q)})_\gamma := \sum_{j=1}^J (w_{(p)}^j, w_{(q)}^j)_{\gamma_j} = \delta_{p,q}, \quad p, q = N, \ldots, N + X - 1. \tag{7.30} ?\underline{\texttt{U2}}?$$

In Lemma 7.1 we set

$$t_p^\varepsilon = (\varepsilon^{-1} + \varepsilon^{-1}\mu_N)^{-1} = \varepsilon(1 + \mu_N)^{-1}, \quad p = N, \ldots, N + X - 1. \tag{7.31} ?\underline{\texttt{U3}}?$$

$$\mathcal{U}_p^\varepsilon = ||\mathcal{V}_p^\varepsilon; \mathcal{H}^\varepsilon||^{-1}\mathcal{V}_p^\varepsilon, \quad \mathcal{V}_p^\varepsilon(x) = a_{(p)}^0 + \sum_{j=1}^J (\mathfrak{z} a_{(p)}^j G^j(x) + \chi_j(x) w_{(p)}^j(\xi^j)),$$

where the scalar $a^0_{(p)}$ and the column $\vec{a_{(p)}} = (a^0_{(p)}, \ldots, a^0_{(p)}) \in \mathbb{R}^J_\perp$ are found from the system

$$a^0_{(p)} + (2\pi)^{-1} a^j_{(p)} = -\langle w^j_{(p)} \rangle, \quad j = 1, \ldots, J. \tag{7.32}$$ U4

The discrepancies

$$\delta^\varepsilon_p = ||\mathcal{V}^\varepsilon_p; \mathcal{H}^\varepsilon||^{-1} t^\varepsilon_p \sup |(\Delta \mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)^\varepsilon_\Omega - (\partial_\nu \mathcal{V}^\varepsilon_p - \varepsilon^{-1}\mu_N \mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)_{\gamma^\varepsilon}| \tag{7.33}$$ U5

with $p = N, \ldots, N + X - 1$ and $||\mathcal{W}^\varepsilon; \mathcal{H}^\varepsilon|| = 1$ are evaluated quite similar to the above calculation. Indeed, the scalar product in $L^2(\Omega^\varepsilon)$ from (7.33) satisfies (7.25) and, owing to (7.33) and (2.14), we have

$$\partial_\nu \mathcal{V}^\varepsilon_p(x) - \varepsilon^{-1}\mu_N \mathcal{V}^\varepsilon_p(x) = g^\varepsilon_{jp}(x) := \tag{7.34}$$ U6

$$= \varepsilon^{-1}\Big(\partial_{\nu(\xi)} w^j_{(p)}(\xi^j) - \mu_N(w^j_{(p)}(\xi^j) - \langle w^j_{(p)}(\xi^j)\rangle)\Big) +$$

$$+ \varepsilon^{-1}\mathfrak{z}\frac{a^j_{(p)}}{2\pi}\left(\partial_{\nu(\xi)} - \mu_N\right) \ln\frac{1}{|\xi^j|} - \mathfrak{z}(\partial_{\nu(x)} - \varepsilon^{-1}\mu_p)\sum_{k=1}^J a^k_{(p)}(G^j(x) - \frac{\delta_{jk}}{2\pi}\ln\frac{1}{R_J}, \quad x \in \gamma_j.$$

The first expression on the right-hand side of (7.34) vanishes due to the definition of $w^j_{(p)}$ and, therefore, $|g^\varepsilon_{jp}(x)| \le c\varepsilon^{-1}\mathfrak{z}$ and

$$|(\partial_\nu \mathcal{V}^\varepsilon_p - \varepsilon^{-1}\mu_N \mathcal{V}^\varepsilon_p, \mathcal{W}^\varepsilon)_{\gamma^\varepsilon}| \le c\sum_{j=1}^J ||\mathcal{W}^\varepsilon; L^2(\gamma^\varepsilon_j)|||\gamma^\varepsilon_j|^{1/2} \max_{x\in\gamma^\varepsilon_j} |g^\varepsilon_{jp}(x)| \le \tag{7.35}$$ U7

$$\le c\varepsilon^{1/2}||\mathcal{W}^\varepsilon; \mathcal{H}^\varepsilon||\varepsilon^{1/2}\varepsilon^{-1}\mathfrak{z} = c\mathfrak{z} = c|\ln\varepsilon|^{-1}.$$

From formula (7.39) below, it follows that $||\mathcal{V}^\varepsilon_p; \mathcal{H}^\varepsilon|| \ge c > 0$ and, hence, (7.33) and (7.25), (7.35) give us:

$$\delta^\varepsilon_p \le c_N \varepsilon |\ln\varepsilon|^{-1}$$

while Lemma 7.1 delivers an eigenvalue $\tau^\varepsilon_{m^p(\varepsilon)}$ of $\mathcal{T}^\varepsilon$ enjoying the inequality

$$|\tau^\varepsilon_{m^p(\varepsilon)} - \varepsilon(1 + \mu_N)^{-1}| \le c_N \varepsilon |\ln\varepsilon|^{-1}. \tag{7.36}$$ U8

Similarly to (7.37), (7.24) we impose a proper restriction $\varepsilon < \varepsilon_N$ and derive from (7.36) that

$$|\lambda^\varepsilon_{m^p(\varepsilon)} - \mu_N| \le 2c_N |\ln\varepsilon|^{-1}(1 + \mu_N)^2. \tag{7.37}$$ U9

It suffices to prove that indexes of entries in the sequence (1.11) involved into (7.37) are nothing but

$$m^N(\varepsilon) = N, \ldots, m^{N+X-1}(\varepsilon) = N + X - 1. \tag{7.38}$$?U0?

First of all we prove that there exists at least X different eigenvalues λ^ε_M verifying the inequality (7.37), maybe, with a bigger bound $C_N\varepsilon|\ln\varepsilon|^{-1}$. To this end, we compute the following scalar products (7.18) of $\mathcal{V}^\varepsilon_p$ and $\mathcal{V}^\varepsilon_q$.

Recalling that $x^j \in \omega^\varepsilon_j$ and $w^j_{(p)}$ satisfies (2.12) yields

$$\mathfrak{z}^2 \int_{\Omega^\varepsilon} |\nabla_x G^j(x)|^2\, dx \le c\mathfrak{z}^2\left(1 + \int_{\varepsilon\rho}^R \left(\frac{1}{r}\right)^2 r\, dr\right) \le c\mathfrak{z} \text{ with } R, \rho > 0,$$

$$\left|\int_{\Omega^\varepsilon} \nabla_x(\chi_j(x) w^j_{(p)}(\xi^j)) \cdot \nabla_x(\chi_j(x) w^j_{(q)}(\xi^j))\, dx - \int_{\Xi_j} \nabla_\xi w^j_{(p)}(\xi^j) \cdot \nabla_\xi w^j_{(q)}(\xi^j)\, d\xi\right| \le c\varepsilon.$$

Furthermore, the relation (7.32) yields

$$|\mathcal{V}_p^\varepsilon(x) - (w^j_{(p)}(\xi^j) - \langle w^j_{(p)}(\xi^j)\rangle)| \le c\mathfrak{z} \text{ for } \ x \in \gamma_j^\varepsilon.$$

As a result, we obtain the inequalities

$$|\langle \mathcal{V}_p^\varepsilon, \mathcal{V}_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}(1+\mu_N)| \le c\mathfrak{z}. \tag{7.39}$$ UUU

Notice that the subtrahend $\delta_{p,q}(1+\mu_N)$ is due to the normalization and orthogonality conditions (3.34) and the integral identity (3.31). From (7.39) we derive that $||\mathcal{V}_p^\varepsilon; \mathcal{H}^\varepsilon|| = \sqrt{1+\mu_N} + O(\sqrt{\mathfrak{z}})$ and

$$|\langle \mathcal{U}_p^\varepsilon, \mathcal{U}_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}| \le c\mathfrak{z}, \quad p,q = N, \dots, N+X-1. \tag{7.40}$$?U11?

Now we denote by $S_N^\varepsilon, \dots, S_{N+X-1}^\varepsilon$ the linear combination of the orthonormalized eigenvectors $u^\varepsilon_{\kappa(\varepsilon)}, \dots, u^\varepsilon_{\kappa(\varepsilon)+X(\varepsilon)-1}$ $\mathcal{H}^\varepsilon$ of the operator $\mathcal{T}^\varepsilon$ which correspond to eigenvalues in the segment $[t^\varepsilon - \delta^\varepsilon\alpha, t^\varepsilon + \delta^\varepsilon\alpha]$ with $t^\varepsilon = t_p^\varepsilon$, $\delta^\varepsilon = \max\{\delta_N^\varepsilon, \dots, \delta_{N+X-1}^\varepsilon\}$ and $\alpha > 1$. For the coefficient columns $c_N^\varepsilon, \dots, c_{N+X-1}^\varepsilon$ of these combinations, we repeat the calculation (7.16) to derive the estimates

$$|(c^\varepsilon_{(p)})^\top (c^\varepsilon_{(q)} - \delta_{p,q}| \le 4\alpha^{-1} + c\mathfrak{z}, \quad p,q = N, \dots, N+X-1,$$

and to conclude the inequality $X(\varepsilon) \ge X$ for small $\mathfrak{z}$ and α^{-1} because $c_N^\varepsilon, \dots, c_{N+X-1}^\varepsilon \in \mathbb{R}^{X(\varepsilon)}$ are "almost orthonormalized" vectors. Similar arguments apply to the eigenvalues $\lambda_1^\varepsilon, \dots, \lambda_J^\varepsilon$ obtained as perturbation of the null Steklov eigenvalues $\mu_1^1, \dots, \mu_1^J = 0$ in exterior domains, indicated in (3.80) but excluded from our previous consideration. However, we have to take into account correction terms of order $\varepsilon^{-1}\mathfrak{z}$, so that the factor $\mathfrak{z}$ in all above derived estimates are replaced by $\mathfrak{z}^2$.

the above listed inferences allow us to conclude with proofs of Theorems 3.4 and 3.3 with the help of Theorem 6.4 when hypothesis (6.16) is confirmed. This is to be done by means of traditional arguments, cf. the end of Section 7.1 with an obvious and slight modification.

7.4 The Dirichlet and Steklov conditions at the exterior boundary

?⟨subsec74⟩? For the Dirichlet–Steklov (1.1), (1.5), (1.8), the asymptotic procedure and the justfication approach are quite the same as for the Neumann–Steklov problem (1.1), (1.5), (1.6) presented in Section 3.4–3.7 and 7.3. Howevere, a simplification occurs due to the Dirichlet condition on Γ, which assures the Friedrichs inequality

$$||\widehat{u}^\varepsilon; L^2(\Omega)|| \le c_\Omega ||\nabla \widehat{u}^\varepsilon; L^2(\Omega)|| \tag{7.41}$$ U12

which replaces the inequality (7.23) with the big factor $1 + |\ln\varepsilon|$ in the case of the Neumann condition on Γ.

A Poincaré inequality, similar to (7.23)

$$||\widehat{u}^\varepsilon; L^2(\Omega)|| \le c_\Omega \left(||\nabla \widehat{u}^\varepsilon, L^2(\Omega)|| + ||\nabla \widehat{u}^\varepsilon; L^2(\Gamma)||\right) \le C||u^\varepsilon; \mathcal{H}^\varepsilon|| \tag{7.42}$$ U13

is valid in the case of the pure Steklov problem (1.1), (1.4), (1.5), because the norm $||u^\varepsilon; \mathcal{H}^\varepsilon||$ generated by the scalar product (7.1), contains the trace norm $||u^\varepsilon : L^2(\Omega)||$. The inequality (7.23), as well as (7.41) and (7.42), is used to estimate discrepancies left in the Laplace equation (1.1) by the approximate eigenfunctions due to the multiplication. of the boundary layer terms w^j with the cut-off functions χ_j. Since $w^j(\xi)$ decays as $O(|\xi|^{-1})$ at infinity, these discrepancies are of order ε and, therefore, the big factor $1 + |\ln\varepsilon|$ does not play any role in the error estimation for the "logarithmic" asymptotics, cf. Theorem 3.4, 4.3 and 5.2. However, in the case of the powrer-law asymptotics the disappearance of $1 + |\ln\varepsilon|$ from (7.42) makes the bound of the error estimate (5.28) for the Steklov problem less than the bound of (3.67) for the Neumann–Steklov problem.

One important point which distinguishes Theorem 5.2 from Theorem 3.3 and 4.3, is but the different indexes $p_n(\varepsilon)$ and n of the eigenvalues in the error estimates (5.11), (5.12) has been discussed in Section 5.3. The very reason of this disparity, namely the accumulation of the spectrum of problem (1.1), (1.4), (1.5), in the low-frequency range, is reflected in the justification scheme too: all the rescaled eigenvalues $\varepsilon\lambda_n^\varepsilon$ converge to $+0$, so that an important step in the scheme displayed in the end of Section 7.1 and 7.3 cannot be performed in the middle-frequency range of the spectrum of the pure Steklov problem.

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