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EXTREMAL KÄHLER METRICS INDUCED BY FINITE OR INFINITE DIMENSIONAL COMPLEX SPACE FORMS

ANDREA LOI, FILIPPO SALIS, AND FABIO ZUDDAS

Abstract. In this paper we address the problem of studying those complex manifolds M equipped with extremal metrics g induced by finite or infinite dimensional complex space forms. We prove that when g is assumed to be radial and the ambient space is finite dimensional then (M, g) is itself a complex space form. We extend this result to the infinite dimensional setting by imposing the strongest assumption that the metric g has constant scalar curvature and is *well-behaved* (see Definition 1 in the Introduction). Finally, we analyze the radial Kähler-Einstein metrics induced by infinite dimensional elliptic complex space forms and we show that if such a metric is assumed to satisfy a stability condition then it is forced to have constant non-positive holomorphic sectional curvature.

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1. Introduction

Extremal Kähler metrics were introduced by Calabi [6] in the compact case as the solution for the variational problem in a Kähler class defined by the square integral of the scalar curvature. Therefore they are a generalization of constant scalar curvature (cscK) and hence of Kähler-Einstein (KE) metrics. Calabi himself constructs nontrivial extremal (namely with nonconstant scalar curvature) metrics on some compact manifolds. In the last thirty years extremal Kähler metrics were rediscovered by several mathematicians due to their link with the stability of complex vector bundles (see e.g. [4], [10], [16], [26] and also the introductory book [29]). The reader is also referred to the recent papers [30], [2], [31] and [32] for the existence of extremal metrics via blowup constructions.

Obviously extremal metrics cannot be defined in the noncompact case as the solutions of a variational problem involving some integral on the manifold. Nevertheless they can be alternatively defined as those metrics such that the (1,0)-part of the Hamiltonian vector field associated to the scalar curvature is holomorphic. In the noncompact case, the existence and uniqueness of such metrics are far from being understood. For example, in [8] (see also [9]), there has been shown the existence of a nontrivial extremal and complete Kähler metric in a complex one-dimensional manifold. More recently M. Abreu [1] inspired by the work of Calabi [6] considered cohomogeneity one examples of extremal metrics on noncompact manifolds.

In this paper we address the issue of classifying those (finite dimensional) complex manifolds M admitting an extremal metric g induced by a finite or infinite dimensional complex space form (S^N, g_c^N) of constant holomorphic sectional curvature c and complex dimension $N \leq \infty$. By the word "induced" we mean that the Kähler manifold (M, g) can be Kähler immersed into (S^N, g_c^N), i.e. there exists a holomorphic map $\varphi : M \to S^N$ such that $\varphi^* g_c^N = g$ (see [5] or the book [24] for an update material on the subject).

If one assumes that (S^N, g_c^N) is complete and simply-connected one has the corresponding three cases, depending on the sign of c:

- for $c = 0$, $S^N = \mathbb{C}^N$ ($S^\infty = \ell^2(\mathbb{C})$) and g_0^N is the flat metric with associated Kähler form

$$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2, \ |z|^2 = \sum_{j=1}^{N} |z_j|^2, \ N \leq \infty; \tag{1}$$

- for $c < 0$, $S^N = \mathbb{C}H^N$ is the N-dimensional complex hyperbolic space, namely the unit ball of $\mathbb{C}^N$ with the metric g_c^N with associated Kähler form

$$\omega_c = \frac{i}{2c}\partial\bar{\partial}\log(1 - |z|^2); \tag{2}$$

- for $c > 0$, $S^N = \mathbb{C}P^N$ is the N-dimensional complex projective space and g_c^N is the metric with associated Kähler form ω_c, given in homogeneous coordinates by:

$$\omega_c = \frac{i}{2c}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2). \tag{3}$$

Notice that when $c = 1$ (resp. $c = -1$) the metric g_c^N is the standard Fubini-Study metric g_{FS} (respectively hyperbolic metric g_{hyp}) of holomorphic sectional curvature 4 (resp. -4). Throughout the paper we will say that a metric g on a complex (connected) manifold is *projectively induced* if (M, g) admits a Kähler immersion into $(\mathbb{C}\mathrm{P}^N, g_{FS})$. We say g is finitely (resp. infinitely) projectively induced if $N < \infty$ (resp. $N = \infty$).

We believe that the extremal metrics induced by a *finite* dimensional complex form are forced to have constant holomorphic sectional curvature as expressed by the following:

Conjecture 1: *Let g be an extremal metric on an n-dimensional complex manifold M induced by a finite dimensional complex space form of constant holomorphic sectional curvature c. Then the following facts hold true:*
(i) if $c \leq 0$ then (M, g) is a complex space form of holomorphic sectional curvature c and the immersion is totally geodesic.
(ii) if $c > 0$, then M an open subset of a flag manifold[1] *(F, h) and $g = h_{|_M}$*

A possible way to attack (i) of Conjecture 1 could be through the following steps: extremal $\rightarrow$ cscK, cscK $\rightarrow$ KE and finally to appeal to a fundamental result of M. Umehara [35] asserting that a Kähler immersion of a KE manifold into a finite dimensional complex space form of non positive holomorphic sectional curvature is totally geodesic. Unfortunately at the moment we are unable to prove any of the two implications. For part (ii) of Conjecture 1 one should try to show the following three facts: extremal $\rightarrow$ cscK, cscK $\rightarrow$ KE and KE $\rightarrow h$. Regarding the step cscK $\rightarrow$ KE a partial result for projectively induced metrics was obtained by S. Kobayashi [17] (see also the work of S.S. Chern [11] for the case of codimension one immersions) which shows that when M is a complete intersection in the complex projective space $\mathbb{C}P^N$ and the restriction of the Fubini–Study metric to M is cscK then it is KE (and hence M is either the quadric of $\mathbb{C}P^N$ or it is totally geodesic by a fundamental result of J. Hano [13]).

The proof of the step KE $\rightarrow h$ represents an important breakthrough in the classification of finite projectively induced KE metrics. The only known facts in this direction are the extension of the above mentioned Chern's result to the codimension 2 case due to K. Tsukada [34] and the proof of the positivity of the Einstein constant

[1] A flag manifold (F, g) is a compact simply-connected Kähler manifold acted upon transitively by its holomorphic isometries group.

of a compact KE submanifold of the complex projective space due to D. Hulin [15] (see also [28] for the case of rotation invariant metrics in codimension 3).

1.1. **Statements of the main results.** In this paper we verify Conjecture 1 under the additional assumption that the metrics involved are *radial* Kähler metrics, i.e. they admit a global Kähler potential $\Phi : M \to \mathbb{R}$ which depends only on the sum $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the local coordinates' moduli. Since M is assumed to be connected this means that there exists a smooth function $f : (r_{\inf}, r_{\sup}) \to \mathbb{R}$, $0 \leq r_{\inf} < r_{\sup} \leq \infty$, such that $\Phi(z) = f(r)$ and

$$\omega = \frac{i}{2}\partial\bar{\partial} f(r),\ r = |z|^2. \tag{4}$$

The prototype of radial Kähler metrics in complex dimension n are the flat metric g_0 on $\mathbb{C}^n$, the hyperbolic metric g_{hyp} on $\mathbb{C}\mathrm{H}^n$ and the Fubini-Study metric g_{FS} on the affine chart $U_0 = \{Z_0 \neq 0\}$ with complex coordinates $z_j = \frac{Z_j}{Z_0}$, $j = 1, \dots n$.

Our first result is then the following:

Theorem 1.1. *Let g be a radial extremal metric on a n-dimensional complex manifold M. Assume that (M, g) can be Kähler immersed into a finite dimensional complex space form (S^N, g_c^N). Then*
(1) If $c \leq 0$ then (M, g) is a complex space form of holomorphic sectional curvature c and the immersion is totally geodesic.
(2) If $c > 0$ then M is an open subset of $\mathbb{C}P^n$, g is an integer multiple of g_c^n, i.e. $g = mg_c^n$, $m \in \mathbb{N}^+$.

Remark 1. The conclusion (2) of the theorem can also be accompained by an explicit description of the Kähler immersion given by a suitable normalization of the Veronese embedding (see [5] for details). Notice also that (2) is a particular case of (ii) of Conjecture 1 since it is not hard to see that a homogeneous Kähler metric h on a flag manifold F which admits a radial potential (on an open subset of F) can exist only when $F = \mathbb{C}P^n$ and g is a multiple of g_{FS}.

It is worth pointing out that Theorem 1.1 is of local nature, i.e. there are no topological assumptions on the manifold M and the Kähler immersions are not required to be injective.

Since an extremal cohomogeneity one toric Kähler metric g on a compact complex manifold T admits a radial Kähler potential on a dense open subset, we get:

Corollary 1.2. *If (T, g) is finitely projectively induced then $(M, g) = (\mathbb{C}P^n, g_{FS})$.*

Unfortunately without any further assumptions Theorem 1.1 does not extend to the infinite dimensional setting. Indeed there exist (see Example 1 in Subsection 2.1 below) extremal (not cscK) radial metrics which can be Kähler immersed into any infinite dimensional complex space form.

Even when dealing with the strongest assumption of cscK metrics one can exhibit examples of infinitely projectively induced cscK (not KE) metrics (see Example 2 and Example 3 in Subsection 2.1).

By analyzing these last examples one discovers two facts: a) such a metric cannot be Kähler immersed into any infinite dimensional complex space form of non positive holomorphic sectional curvature and b) they do not satisfy the following definition of fundamental importance for our analysis.

Definition 1. A radial Kähler metric g with radial potential $f : (r_{\inf}, r_{\sup}) \to \mathbb{R}$ is said to be *well-behaved*[2] if $rf'(r) \to 0$ for $r \to r^+_{\inf}$.

In the following theorem which represents our second result we show that fact a) is true for any cscK metric which is not of constant holomorphic sectional curvature and that well-behaveness is indeed the right condition to impose in order for (2) of Theorem 1.1 to extend to the infinite dimensional setting.

Theorem 1.3. *Let g be a radial cscK metric on a complex manifold M. Assume that (M, g) can be Kähler immersed into an infinite dimensional complex space forms (S^∞, g_c^∞). Then*
(1) If $c \leq 0$ then (M, g) is a complex space form of non positive holomorphic sectional curvature.
(2) If $c > 0$ and g is well-behaved then either (M, g) is a complex space form of non positive holomorphic sectional curvature or M is an open subset of $\mathbb{C}P^n$ and $g = mg_c^n$, $m \in \mathbb{N}^+$.

Remark 2. In (1) of Theorem 1.3 we cannot get to the conclusion that the immersion is totally geodesic as in (1) of Theorem 1.1. Indeed, beside the natural totally geodesic embeddings $(\mathbb{C}^n, g_0) \to (\ell^2(\mathbb{C}), g_0^\infty)$ and $(\mathbb{C}\mathrm{H}^n, g_c^n) \to (\mathbb{C}\mathrm{H}^\infty, g_c^\infty)$ $(c < 0)$ there exist Kähler embeddings of $(\mathbb{C}\mathrm{H}^n, g_c^n)$ into $(\ell^2(\mathbb{C}), g_0)$, for all $c < 0$.

Similar considerations hold true for (2) in Theorem 1.3: $(\mathbb{C}\mathrm{H}^n, g_{c'}^n)$ and $(\mathbb{C}^n, g_0)$ can be Kähler embedded into $(\mathbb{C}\mathrm{P}^\infty, g_c^\infty)$ for all $c' < 0$ and $c > 0$. (The reader is referred to [5] for details).

Finally we ask what happens when cscK is strengthened to the KE condition. In this regards we believe the validity of the following:

Conjecture 2: *A (not well-behaved[3]) radial KE metric induced by $(\mathbb{C}P^\infty, g_c^\infty)$ (for some $c > 0$) is a complex space form.*

In order for the conjecture to make sense we exhibit in Example 5 of Subsection 2.1 a radial KE metric which is not well-behaved.

[2] Clearly if a radial metric g is defined at $r_{\inf} = 0$ then it is well-behaved and in particular, g_0, g_{hyp} and g_{FS} (the latter on the affine charts $U_0 = \{Z_0 \neq 0\}$) are well-behaved and $r_{\inf} = 0$.
[3] Otherwise one can conclude by (2) of Theorem 1.3.

Notice that Conjecture 2 is false without the radiality assumption. Take, for example, any open contractible open subset U of a flag manifold (F, h) different from the complex projective space such that h is projectively induced (see e.g. [33]) then $(U, h_{|U})$ is a KE manifold which admits a global Kähler potential and $h_{|U}$ is projective induced. Another example is given by a bounded symmetric domain equipped with its Bergman metric or any bounded homogeneous domain with a suitable multiple of a homogeneous metric (see [27]). For an example of non radial KE metric which "does not come" from an homogeneous one the reader is referred to [23] where one can find a continuous family of complete and nonhomogeneous KE submanifolds of the infinite dimensional complex projective space (see also [37] for further examples).

Notice also that Conjecture 2 turns out to be true for Ricci flat metrics on complex surfaces (as explained in Example 4 of Subsections 2.1 below).

In Theorem 1.4 we show the validity of Conjecture 2 under a natural stability assumption which the authors of the present paper have already considered in [21].

Definition 2. *Let $c > 0$. A Kähler metric g is said to be* c-stable projectively induced *if there exists $\epsilon > 0$ such that αg is induced by $(\mathbb{C}P^\infty, g_c^\infty)$ for all $\alpha \in (1-\epsilon, 1+\epsilon)$. A Kähler metric g is said to be unstable if it is not c-stable projectively induced for any $c > 0$. When $c = 1$ we simply say that g is* stable-projectively induced.

The reader is referred to [21] for details, examples and further properties of stable projectively induced metrics. Notice that the Fubini-Study metric g_{FS}, and more generally any projectively induced metric on a compact manifold is unstable, while the flat metric g_0 and the hyperbolic metric g_{hyp} are c-stable projectively induced for all $c > 0$ due to the last part of Remark 2.

Theorem 1.4. *Let g be a radial KE metric induced by $(\mathbb{C}P^\infty, g_c^\infty)$ for some $c > 0$. If g is c-stable projectively induced then (M, g) is a complex space form of non-positive holomorphic sectional curvature.*

We point out that Theorem 1.4 extends to arbitrary radial KE metrics [21, Theorem 1.1] valid in the Ricci flat case.

The paper is organized as follows. In Section 2 we summarize some basic facts on extremal Kähler radial metrics. In particular we recall in Lemma 2.1 that these metrics on an n dimensional complex manifold can be described by a rational family $\psi(y)$, $y(r) = rf'(r)$, $r \in (r_{\inf}, r_{\sup})$, depending on four real parameters A, B, C, D (in particular the vanishing of A, B and D is equivalent to the constancy of the holomorphic sectional curvature of the metric involved). In Subsection 2.1 we provide many examples (some already mentioned above) of radial extremal

metrics by fixing some values of the parameters and finding their explicit Kähler potentials. Finally, the last section (Section 3) is dedicated to the proofs of our main results. In Subsection 3.1, after briefly recalling the concept of ϵ-resolvability ($\epsilon = -1, 0, +1$) of rank $N \leq \infty$ of a real analytic Kähler metric g and Calabi's criterium for the inducibility of g into a finite or infinite dimensional complex space form of hyperbolic, flat or elliptic type (depending on the sign of ϵ), we specialize to the case of radial Kähler metrics (Lemma 3.1). In Subsection 3.2 to a given radial extremal metric g we associate a sequence of rational functions $Q_k^\epsilon(y)$, $k \geq 1$, which are the key tools in the proof of our theorems. This is reason why in Lemma 3.3 we deeply analyze these functions and their higher and lower degree coefficients in terms of A, B, C and D of the extremal metric involved. Finally, in Subsection 3.3 one can find the proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.4.

We would like to thank Miguel Abreu for his interest in our work and for stimulating discussions about extremal radial metrics.

2. Radial extremal metrics

Let g be a radial Kähler metric on a complex manifold M, equipped with complex coordinates $z_1, \dots, z_n$. Let $\omega = \frac{i}{2}\partial\bar{\partial} f(r)$ denotes its associated Kähler form where $f : (r_{\inf}, r_{\sup}) \to \mathbb{R}$, $r = |z_1|^2 + \cdots + |z_n|^2$, $0 \leq r_{\inf} < r < r_{sup}$ and $(r_{\inf}, r_{\sup})$ is the maximal domain where the radial potential f is defined.

It is not hard to see that the matrix of the metric g and its inverse read as

$$g_{i\bar{j}} = f''(r)\bar{z}_i z_j + f'(r)\delta_{ij}, \quad g^{i\bar{j}} = \frac{\delta_{ij}}{f'(r)} - \frac{f''(r)}{f'(r)(rf'(r))'}\bar{z}_j z_i. \tag{5}$$

Set

$$y(r) := rf'(r). \tag{6}$$

and

$$\psi(r) := ry'(r). \tag{7}$$

The fact that g is a metric is equivalent to $y(r) > 0$ and $\psi(r) > 0$, $\forall r \in (r_{\inf}, r_{\sup})$. Then

$$\lim_{r \to r_{\inf +}} y(r) = y_{\inf} \tag{8}$$

is a non negative real number. Similarly set

$$\lim_{r \to r_{\sup -}} y(r) = y_{\sup} \in (0, +\infty]. \tag{9}$$

Therefore we can invert the map

$$(r_{\inf}, r_{\sup}) \to (y_{\inf}, y_{\sup}), \ r \mapsto y(r) = rf'(r)$$

on $(r_{\inf}, r_{\sup})$ and think r as a function of y, i.e. $r = r(y)$.

The following lemma provides a classification of radial extremal Kähler metrics. Even if its proof is known (see, for example, [1] and also [36]) we include it here for reader's convenience.

Lemma 2.1. *Set*

$$\psi(y) := \psi(r(y)).$$

A radial Kähler metric g is extremal if and only if

$$\psi(y) = y - \frac{A}{y^{n-1}} - \frac{B}{y^{n-2}} - Cy^2 - Dy^3. \tag{10}$$

for some $A, B, C, D \in \mathbb{R}$. Moreover, the following facts hold true:

(a) *g is a cscK metric*[4] *iff $D = 0$ and the sign of the scalar curvature is equal to the sign of C;*
(b) *g is a KE metric with Einstein constant λ iff $B = D = 0$ and $C = \frac{\lambda}{2(n+1)}$;*
(c) *g has constant holomorphic sectional curvature iff $A = B = D = 0$.*

Proof. From (5), we easily get

$$\det\left(g_{i\bar{j}}\right)(r) = \frac{(y(r))^{n-1}\psi(y)}{r^n}.$$

By straightforward computations (see e.g. [21] for details) we can compute the Ricci tensor's components

$$\mathrm{Ric}_{i\bar{j}}(r) = -\frac{\partial^2 \log\det\left(g_{i\bar{j}}\right)}{\partial z_i \partial \bar{z}_j} = \frac{-\frac{d\sigma}{dy}\psi(y) + \sigma(y) - n}{r^2}\bar{z}_i z_j + \frac{n - \sigma(y)}{r}\delta_{ij}, \tag{11}$$

and the scalar curvature s of g as a function of y

$$s(y) = \sum_{i,\bar{j}=1}^{n} g^{i\bar{j}} Ric_{i\bar{j}} = \frac{n(n-1)}{y} - y^{1-n}\frac{d^2\left[y^{n-1}\psi(y)\right]}{dy^2}, \tag{12}$$

where

$$\sigma(y) := y^{1-n}\frac{d\left[y^{n-1}\psi(y)\right]}{dy} = (n-1)\frac{\psi(y)}{y} + \frac{d\psi}{dy}.$$

Now, by definition a Kähler metric is extremal if and only if the gradient field

$$X = \sum_{i,j=1}^{n} g^{i\bar{j}}\frac{\partial s}{\partial \bar{z}_j}\frac{\partial}{\partial z_i} \tag{13}$$

is holomorphic.

Since the scalar curvature is a radial function, we have $\frac{\partial s}{\partial \bar{z}_j} = s'(r)z_j$: from this and (5) we can rewrite (13) as

$$X = \sum_{i,j=1}^{n}\left(\frac{\delta_{ij}}{f'(r)} - \frac{f''(r)}{f'(r)(rf')'}\bar{z}_j z_i\right)s'(r)z_j\frac{\partial}{\partial z_i} = \sum_{i=1}^{n}\frac{s'(r)}{(rf'(r))'}z_i\frac{\partial}{\partial z_i} \tag{14}$$

[4]with constant scalar curvature equal to $Cn(n+1)$.

It immediately follows that X is holomorphic if and only if $\frac{s'(r)}{(rf'(r))'} = \frac{s'(r)}{y'(r)} = \gamma_1$ for some constant $\gamma_1 \in \mathbb{R}$, i.e.

$$s = \gamma_1 y + \gamma_2 \tag{15}$$

where $\gamma_2 \in \mathbb{R}$. By (12), this means

$$\frac{n(n-1)}{y} - y^{1-n}\frac{d^2\left[y^{n-1}\psi(y)\right]}{dy^2} = \gamma_1 y + \gamma_2$$

which integrated gives

$$\psi(y) = y - \frac{\gamma_1}{(n+1)(n+2)}y^3 - \frac{\gamma_2}{n(n+1)}y^2 + \frac{\gamma_3}{y^{n-2}} + \frac{\gamma_4}{y^{n-1}}$$

which is exactly (10) for

$$A = -\gamma_4, \quad B = -\gamma_3, \quad C = \frac{\gamma_2}{n(n+1)}, \quad D = \frac{\gamma_1}{(n+1)(n+2)}. \tag{16}$$

The proof of (a) follows by (15) and (16) and that of (b) can be easily obtained by using (5) and (11). If g has constant holomorphic sectional curvature then in particular is KE ($B = D = 0$) and the constancy of the norm of the Riemannian tensor $|R|^2$ of g implies $A = 0$ as it follows for example by using the expression of $|R|^2$ in [22]. Finally, if $A = B = D = 0$ then (6), (7) and (10) yield

$$f''(r) + C(f'(r))^2 = 0$$

which integrates explicitly and gives a metric with constant holomorphic sectional curvature. □

Let g be a radial extremal metric as above. By setting $e^t = r$ we deduce by (6) and (7) that the function

$$y(t) := y(e^t)$$

satisfies the ODE

$$\frac{dy}{dt} = \psi(y(t)),$$

where $\psi(y)$ is given by (10).

The following simple lemma will be crucial in the proof of Proposition 3.5 and in Theorem 1.3.

Lemma 2.2. *The following hold true:*

(i) *If* $\lim_{y\to y_{\inf}^+} \psi(y) \neq 0$ *then* $y_{\inf} = 0$.
(ii) *If* $\lim_{y\to y_{\sup}^-} \psi(y) \neq 0$ *then* $y_{\sup} = +\infty$.

Proof. In order to prove (i) assume by contradiction that $y_{\inf} \neq 0$ in (8). Note first that $t_{\inf} := \lim_{r\to r_{\inf}} \log r = -\infty$: otherwise (if $t_{\inf} \in \mathbb{R}$) the function $y(t)$ could be prolonged to an open interval containing $t_{\inf}$ being the solution of the Cauchy

problem

$$\begin{cases} y'(t) = \psi(y(t)) \\ y(t_{\inf}) = y_{\inf} > 0. \end{cases} \tag{17}$$

Thus, by the continuity of $\psi(y)$ at $y_{\inf} \neq 0$,

$$\lim_{y \to y_{\inf}^+} \psi(y) = \lim_{t \to -\infty} \psi(y(t)) = \lim_{t \to -\infty} y'(t) = 0,$$

where the last equality follows by (8) when $t_{\inf} = -\infty$, the desired contradiction. The proof of (ii) is obtained similarly by considering $t_{\sup} = \lim_{r \to r_{\sup}} \log r$. □

Remark 3. In view of the definition of well-behaveness (Definition 1) (i) of Lemma 2.2 can be equivalently expressed by saying that if the a radial metric g is not well-behaved (i.e. $y_{\inf} \neq 0$) then $\psi(y_{\inf}) = 0$.

2.1. **Examples.**

Example 1. Consider the extremal radial metric obtained by taking $A = B = 0$, $C = -3$ and $D = -2$ in (10). In this case we can solve explicitly the ODE equation

$$\psi(y) = \frac{dy}{dt} = y + 3y^2 + 2y^3 = y(y+1)(2y+1)$$

and setting $r = e^t$, we find a unique solution (up to change of complex coordinates) given by the Kähler potential

$$f(r) = \log\left[\frac{1 - \sqrt{1-4r}}{2r}\right], \ 0 < r < \frac{1}{4},$$

which shows that g well-behaved being defined at $r = 0$.

Fix $n \geq 1$ and consider the open unit disk of $\mathbb{C}^n$ of radius $\frac{1}{2}$, namely

$$M = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid 0 < |z|^2 < \frac{1}{4}\}$$

equipped with the Kähler metric g whose associated Kähler form is $\omega = \frac{i}{2}\partial\bar{\partial} f(r)$.

In order to construct a Kähler immersion of (M, g) into $(\mathbb{C}H^\infty, g_{hyp})$ consider the function

$$1 - e^{-f} = \frac{1}{2}\left(1 - \sqrt{1-4r}\right).$$

Now, recall the Taylor expansion

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k,$$

where

$$\binom{1/2}{k} = \frac{1/2(1/2-1)\cdots(1/2-k+1)}{k!}.$$

Therefore

$$1-e^{-f}=\frac{1}{2}\left(1-1-\sum_{k=1}^{\infty}(-4)^k\binom{1/2}{k}r^k\right)=\frac{1}{2}\sum_{k=1}^{\infty}4^k\left|\binom{1/2}{k}\right|r^k.$$

By replacing $r=|z_1|^2+\cdots+|z_n|^2$ then one finds an explicit Kähler immersion via monomials into $(\mathbb{C}\mathrm{H}^\infty, g_{hyp})$:

$$z=(z_1,\dots,z_n)\mapsto\left(\dots,\sqrt{\frac{4^{|j|}}{2}\left|\binom{1/2}{|j|}\right|\frac{|j|!}{j!}}z^j,\dots\right)_{j\in\mathbb{N}^n,|j|\geq 1},$$

where, for $j=(j_1,\dots j_n)\in\mathbb{N}^n$ we set $z^j:=z_1^{j_1}\dots z_n^{j_n}$, $j!:=j_1!\dots j_n!$, $|j|:=j_1+\cdots+j_n$.

By multiplying the metric g by a positive constant one then obtain Kähler immersion of (M,g) into $(\mathbb{C}H^\infty, g_{c'}^\infty)$ for all $c'<0$ and hence into $(\ell^2(\mathbb{C}), g_0)$ and $(\mathbb{C}P^\infty, g_c^\infty)$, for all $c>0$ (cfr. [12, Lemma 5 and Lemma 8]).

It remains an open and interesting problem to classify all extremal radial metrics induced by infinite dimensional complex space form.

Example 2. By taking $n=2$, $A=C=D=0$, $B=1$ in (10) one gets

$$\psi(y)=\frac{dy}{dt}=y-1$$

which can be easily integrated to find the Kähler potential

$$f_{BS}(r)=r+\log r,\ 0<r<+\infty.$$

The scalar (not Ricci) flat Kähler metric g_{BS} corresponding to this potential is the celebrated *Burns-Simanca metric*. Notice that g_{BS} is not well-behaved since $rf'(r)=r+1\to 1$ for $r\to 0^+$. One can show that g_{BS} is projectively induced: an explicit Kähler immersion can be found in [21] (see also [22] and [20]). Moreover, one can easily check that g_{BS} cannot be induced by any complex space form of non positive holomorphic sectional curvature in accordance with (1) of Theorem 1.3.

Example 3. It is not hard to see that the radial Kähler metric corresponding to the Kähler potential

$$f(r)=\log r-\log(1-r^3)$$

provides an example of not well-behaved infinitely projectively induced radial cscK (not KE) metric on the punctured disk of $\mathbb{C}^2$ with *negative* scalar curvature ($s=-24$).

Remark 4. By the previous two examples it is natural to see if there exist projectively induced not well-behaved cscK radial Kähler metrics with *positive* scalar curvature. At the moment we do not know any example of such metrics.

Example 4. In order to describe all the radial Ricci flat metrics one has to solve the ODE ($B = C = D = 0$ in (10)):

$$\frac{dy}{dt} = \psi(y) = \frac{y^n - A}{y^{n-1}}.$$

For either $n = 1$ or $A = 0$ we get the flat metric so we assume $n \geq 2$ and $A \neq 0$. The general solution of (4) is given by $y(t) = (\gamma e^{nt} + A)^{\frac{1}{n}}$ for some $\gamma > 0$. By setting $r = e^t$ we then get

$$y(r) = rf'(r) = (\gamma r^n + A)^{\frac{1}{n}}.$$

By the change of complex variables

$$z_j \mapsto w_j := (\frac{\gamma}{|A|})^{\frac{1}{2n}} z_j, j = 1, \dots, n,$$

and still denoting by $r = |w_1|^2 + \cdots + |w_n|^2$ we deduce that the radial potential $f(r)$ of a Ricci flat (not flat) metric on a n-dimensional complex manifold $n \geq 2$ is, up to the multiplication of a positive constant ($|A|^{-\frac{1}{n}}$), given by:

$$f(r) = \int \left(1 + \frac{A}{|A|} r^{-n}\right)^{\frac{1}{n}}. \tag{18}$$

If $A < 0$ since $f'(r) > 0$ one gets $r_{\inf} = 1$ and thus $y(r) = rf'(r) \to 0$ for $r \to 0^+$ and the corresponding radial Ricci flat metric is well-behaved and not infinitely projectively induced by (2) of Theorem 1.3.

If $A > 0$ then $r_{\inf} = 0$ and $y(r) = rf'(r) \to 1$ and so g is not well-behaved. If we further assume that $n = 2$ one can easily integrate (18) and get

$$f(r) = \sqrt{r^2 + 1} + \ln r - \ln(1 + \sqrt{r^2 + 1})$$

which is the potential of the celebrated Eguchi-Hanson metric g_{EH} on $\mathbb{C}^2 \setminus \{0\}$. It is not hard to see that αg_{EH} is not projectively induced for all $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$ (cfr. the proof of [21, Theorem 1.1]). On the other hand the first and third authors together with M. Zedda have shown in [25, Corollary 1] that αg_{EH} is not infinitely projectively induced for all $\alpha \in \mathbb{Z}^+$. By combining these two facts we deduce that g_{EH} cannot be induced by $(\mathbb{C}P^\infty, g_c^\infty)$ for all $c > 0$. This shows the validity of Conjecture 2 when $n = 2$. The case $n > 2$ and $A > 0$ still remains open.

Example 5. Let $F : (1, +\infty) \to$ be given by

$$F(y) = e^{-\frac{2}{y+2}} \left[\frac{y-1}{y+2}\right]^{\frac{1}{3}} \quad 1 < y < +\infty.$$

Define

$$y(r) = F^{-1}(r), \ 0 < r < 1.$$

One can easily verify that $y(t)$ with $t = \log r$ satisfies the ODE equation

$$\frac{dy}{dt} = \psi(y) = y - \frac{4}{3y} - \frac{1}{3}y^2$$

i.e. we take $n = 2$, $A = \frac{4}{3}$ $C = -\frac{1}{3}$, $B = D = 0$ in (10). By Lemma 2.1 we then get a radial KE metric g with negative Einstein constant ($\lambda = -2$) on a two dimensional complex manifold. Moreover g is not well-behaved since one can easily check that $y \to 1$ as $r \to r_{\inf}^+ = 0$. One can prove (cfr. Remark 6 below) that g is not projectively induced in accordance with Conjecture 2.

Example 6. In this last example we construct a KE radial metric with $r_{\inf} \neq 0$ (and hence $t_{\inf} \neq -\infty$). Let $F : (0, 1) \to \mathbb{R}$ be given by

$$F(y) = \log\left[\frac{\sqrt{2y^2 + y + 1}}{(1-y)^{\frac{1}{4}}}\right],\ 0 < y < 1$$

One can check that

$$y(t) = F^{-1}(y(t)),\ -\frac{3}{\sqrt{7}}\arctan\left(\frac{1}{\sqrt{7}}\right) < t < +\infty,$$

satisfies the ODE

$$\frac{dy}{dt} = \psi(y) = y - \frac{1}{y} - 2y^2,$$

namely we take $n = 2$, $A = -1$, $C = 2$, $B = D = 0$ in (10).

By Lemma 2.1 we then get a radial KE metric g with positive Einstein constant ($\lambda = 12$) on a two dimensional complex manifold. Moreover

$$r_{\inf} = e^{-\frac{3}{\sqrt{7}}\arctan\left(\frac{1}{\sqrt{7}}\right)} \neq 0$$

and g is well-behaved since one can easily check that $y \to 0$ as $r \to r_{\inf}^+$.

Finally, notice that g cannot be induced by any finite or infinite dimensional complex space form as it follows by Theorem 1.1 and Theorem 1.3.

3. The proofs of the main results

3.1. **Radial metrics induced by complex space forms.** Let $\epsilon \in \{-1, 0, 1\}$. Following Calabi [5] we say that a Kähler metric g on a complex manifold M is ϵ–resolvable of rank $N \in \mathbb{N} \cup \{\infty\}$ at $p \in M$ if the matrix (B_{jk}) defined by considering the expansion around the point p of

$$\epsilon(e^{\epsilon D_p(z)} - 1) + (1 - \epsilon^2)D_p(z) = \sum_{m_j, m_k \in \mathbb{N}^n} B_{jk}(z - p)^{m_j}(\bar{z} - \bar{p})^{m_k}, \qquad (19)$$

is positive semidefinite and its rank is N, where $D_p(z)$ is Calabi's diastasis function (cfr. [5], [24]). Here, z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^n z_\alpha^{m_{\alpha,j}}$ and we arrange every n-tuple of nonnegative integers as a sequence $m_j = (m_{1,j}, \ldots, m_{n,j})$ such that $m_0 = (0, \ldots, 0)$, $|m_j| \leq |m_{j+1}|$ for all positive integer j and all the m_j's

with the same $|m_j|$ using lexicographic order. Moreover Calabi's criterium affirms that a Kähler metric g is ϵ–resolvable of rank $N \in \mathbb{N} \cup \{\infty\}$ at p if and only if there exists a neighborhood of p that can be holomorphically and isometrically (Kähler) immersed respectively in $(\mathbb{C}\mathrm{H}^N, g_{hyp})$ for $\epsilon = -1$, $(\mathbb{C}^N, g_0)$ for $\epsilon = 0$ and $(\mathbb{C}\mathrm{P}^N, g_{FS})$ for $\epsilon = 1$. Notice that if M is connected, then the property of resolvabilty does not depend on the choice of the point p.

When the metric g is radial with associate Kähler form $\omega = \frac{i}{2}\partial\bar{\partial} f(r)$, $r \in (r_{\inf}, r_{\sup})$ (as in the previous section) by using Calabi's criterium (see [5]) one can prove the following result which can be obtained by following the same outline of [21, Lemma 2.2] where the authors of the present paper consider the Kähler immersions of radial Kähler metrics into $(\mathbb{C}P^\infty, g_{FS})$ (namely the 1-resolvability of infinite rank).

Lemma 3.1. *Let g be a radial extremal metric on a complex manifold M of complex dimension n. Set*

$$F_\epsilon(r) = \epsilon e^{\epsilon f(r)} + (1-\epsilon^2) f(r), \ r \in (r_{\inf}, r_{\sup}). \tag{20}$$

If g is ϵ-resolvable then the following facts hold true:

- *If $n = 1$,*
$$\det\left(\frac{\partial^{\alpha+\beta} F_\epsilon}{\partial z^\alpha \partial \bar{z}^\beta}\right)_{1\le\alpha,\beta\le I} \ge 0, \qquad \forall I \in \mathbb{Z}^+.$$
- *If $n \ge 2$, $\frac{d^k F_\epsilon}{dr^k} \ge 0$, for every positive index k.*

Moreover if g is ϵ-resolvable of finite rank, there exists an index I such that if $n = 1$ one has $\det\left(\frac{\partial^{\alpha+\beta} F_\epsilon}{\partial z^\alpha \partial \bar{z}^\beta}\right)_{1\le\alpha,\beta\le h} \equiv 0$ and if $n \ge 2$ one has $\frac{d^h F_\epsilon}{dr^h} \equiv 0$, $\forall h > I$.

3.2. **The rational functions $Q_k^\epsilon(y)$.** Given a radial metric g as above, by (20) it is easy to prove by induction that, for $\epsilon = \pm 1$,

$$\frac{d^k F_\epsilon}{dr^k} = \epsilon g_k^\epsilon(r) F_\epsilon(r), \tag{21}$$

where $g_k^\epsilon(r)$ is a function of the derivatives of $f(r)$ determined by the following recursive definition

$$g_1^\epsilon(r) = f'(r); \qquad g_{k+1}^\epsilon(r) = \left(g_k^\epsilon\right)'(r) + \epsilon f'(r) g_k^\epsilon(r). \tag{22}$$

Moreover, we have

$$\frac{d^k F_0}{dr^k} = \frac{d^k f}{dr^k} = g_k^0(r), \tag{23}$$

and (22) holds true for $\epsilon = 0$, i.e.

$$g_1^0(r) = f'(r); \qquad g_{k+1}^0(r) = \left(g_k^0\right)'.$$

By setting as before $r = e^t$, $y(r) = rf'(r)$ and $\psi(y) = r(rf')'$, we can rewrite the recursive formula (22) as

$$g_k^\epsilon(r) = \frac{Q_k^\epsilon(y)}{r^k} \tag{24}$$

where

$$Q_1^\epsilon(y) := y; \qquad Q_{k+1}^\epsilon(y) = (\epsilon y - k)Q_k^\epsilon(y) + \frac{dQ_k^\epsilon}{dy}\psi(y). \tag{25}$$

Remark 5. If g is extremal and $Q_2^\epsilon(y)$ vanishes identically on $(y_{\inf}, y_{\sup})$ then (M, g) is a complex space form. Indeed by (25) with $k = 1$ we get $\psi(y) = y - \epsilon y^2$ and so by Lemma 2.1, g has constant holomorphic sectional curvature.

Notice that for each k, $Q_k^\epsilon(y)$ are functions defined in the open interval $(y_{\inf}, y_{\sup})$ We claim that

$$Q_k^\epsilon(y) = y\prod_{j=1}^{k-1}(\epsilon y - j) + \frac{\psi(y)P_k^\epsilon(y)}{y^{(k-2)n}}, \tag{26}$$

where $P_k^\epsilon(y)$ is a polynomial in y with coefficients depending on A, B, C, D and ϵ and the above product equals 1 when $k = 1$.

We prove our claim by induction. For $k = 1$, we have $Q_1^\epsilon(y) = y$ so that (26) is verified with $P_1^\epsilon(y) = 0$. Now, assume that (26) holds true for some $k > 1$. Then, by using (25) one can easily verify that

$$Q_{k+1}^\epsilon(y) = y\prod_{j=1}^{k}(\epsilon y - j) + \frac{\psi(y)P_{k+1}^\epsilon(y)}{y^{(k-1)n}},$$

where

$$P_{k+1}^\epsilon(y) = y^n(\epsilon y - k)P_k^\epsilon(y) + y^{(k-1)n}\frac{d}{dy}\left(y\prod_{j=1}^{k-1}(\epsilon y - j)\right) + R_k^\epsilon(y), \tag{27}$$

and

$$R_k^\epsilon(y) = y\frac{d\left[y^{n-1}\psi(y)\right]}{dy}P_k^\epsilon(y) + y^n\psi(y)\frac{dP_k^\epsilon}{dy} - \left[(k-1)n - 1\right]y^{n-1}\psi(y)P_k^\epsilon(y).$$

Observe that $R_k^\epsilon(y)$ is a polynomial in y since $y^{n-1}\psi(y)$ is a polynomial by (10). Thus $P_{k+1}^\epsilon(y)$ is a polynomial in y proving our claim.

Notice that $Q_k(y)$ can be written as a finite expression

$$Q_k^\epsilon(y) = \sum_{h=-s}^{t} q_h y^h, \ s, t \in \mathbb{N}, \tag{28}$$

where $q_h := q_h(k, A, B, C, D, \epsilon)$ are real numbers such that $q_h = 0$ for all $h < -s$ and $h > t$. We say that t is the *degree* of $Q_k^\epsilon(y)$, q_t its *leading term*, $-s$ its *lower degree* and q_s its *lower term*, respectively.

The properties of $Q^\epsilon_k(y)$, $y \in (y_{\inf}, y_{\sup})$, needed in the proof of our main results are summarized in the following two lemmata (Lemma 3.2 and Lemma 3.3) and in the two corresponding propositions (Proposition 3.4 and Proposition 3.5).

Lemma 3.2. *Let g be a radial extremal metric on a complex manifold M of complex dimension n. Assume that g is ϵ-resolvable, then we have:*

(a1) *If $n = 1$,*

$$\epsilon^I \det\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}Q^\epsilon_{\alpha+\beta-i}\right)_{1\le\alpha,\beta\le I} \geq 0, \quad \forall I \in \mathbb{Z}^+ \text{if } \epsilon \neq 0,$$

$$\det\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}Q^0_{\alpha+\beta-i}\right)_{1\le\alpha,\beta\le I} \geq 0, \quad \forall I \in \mathbb{Z}^+ \text{if } \epsilon = 0.$$

(a2) *If $n \geq 2$, $Q^\epsilon_k(y) \geq 0$, for every positive index k.*

Moreover if g is ϵ-resolvable of finite rank, then there exists an index I such that:

(b1) *If $n = 1$,*

$$\det\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}Q^\epsilon_{\alpha+\beta-i}\right)_{1\le\alpha,\beta\le h} = 0, \ \forall h \geq I, \tag{29}$$

(b2) *If $n \geq 2$,*

$$Q^\epsilon_k(y) \equiv 0, \ \forall h \geq I. \tag{30}$$

Proof. Notice that for $n = 1$ and $\beta \geq \alpha$ we can write

$$\frac{\partial^{\alpha+\beta}F_\epsilon}{\partial z^\alpha \partial \bar z^\beta} = \frac{\partial^\alpha}{\partial z^\alpha}\left(z^\beta \frac{d^\beta F_\epsilon}{dr^\beta}\right) = \sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}z^{\beta-i}\bar z^{\alpha-i}\frac{d^{\alpha+\beta-i}F_\epsilon}{dr^{\alpha+\beta-i}}.$$

Thus (a1) and (b1) follow by taking into account (21), (23), (24) and Lemma 3.1 for $n = 1$. Similarly (a2) and (b2) follow by Lemma 3.1 for $n \geq 2$. □

Remark 6. Using Lemma 3.2 one can show that some specific radial Kähler metric cannot be induced by a complex space form. For example the KE metric g in Example 5 is not projectively induced since one can check via computer's aid that the associated rational function $Q^1_{11}(y)$ is stricly negative on a right neighborhood of $y = 1$. In order to give further evidence of the validity of Conjecture 2 one could try to show that g cannot be induced by $(\mathbb{C}P^\infty, g^\infty_c)$, for all $c > 0$, or equivalently to show that αg in not projectively induced for any $\alpha > 0$. This does not seem to be an easy task.

Lemma 3.3. *Let g be a radial extremal metric on a complex manifold M of complex dimension $n \geq 2$. Then for $k \geq 2$ we have*[5]*:*

[5]For $k = 1$, one has $Q^\epsilon_1(y) = y$ and hence its the leading and the lower term concide and are equal to 1.

- *the degree of $Q_k^\epsilon(y)$ is equal to $2k-1$ and its leading term is*

$$-D^{k-1}\prod_{j=2}^{k-1}(1-2j); \tag{31}$$

- *the lower degree of $Q_k^\epsilon(y)$ is $n(1-k)+1$ and its lower term is*

$$-A^{k-1}(k-2)!\prod_{j=1}^{k-2}\left(n-\frac{1}{j}\right). \tag{32}$$

In particular

$$Q_2^\epsilon(y) = -\frac{A}{y^{n-1}} - \frac{B}{y^{n-2}} + (\epsilon - C)y^2 - Dy^3 \tag{33}$$

and

$$\begin{aligned} Q_3^\epsilon(y) = \frac{A^2(1-n)}{y^{2n-1}} + \frac{AB(3-2n)}{y^{2n-2}} + \frac{B^2(2-n)}{y^{2n-3}} + \frac{A(n+1)}{y^{n-1}} + \\ + \frac{A\left[C\left(3-n\right)-3\epsilon\right]+Bn}{y^{n-2}} + \frac{(AD+BC)(4-n)-3B\epsilon}{y^{n-3}} + \frac{BD(5-n)}{y^{n-4}} + \\ + (2C^2 - 3C\epsilon - D + \epsilon^2)y^3 + D(5C-3\epsilon)y^4 + 3D^2y^5. \end{aligned} \tag{34}$$

Moreover, the following facts hold true:

(i) *when $D=0$ the degree of $Q_k^\epsilon(y)$ is equal to k and its leading term is*

$$(-1)^{k-1}(k-1)!\prod_{j=1}^{k-1}\left(C-\frac{\epsilon}{j}\right).$$

(ii) *when $A=0$, the lower degree of $Q_k^\epsilon(y)$ is equal to $n+k-nk$ and its lower term is*

$$-B^{k-1}(k-2)!\prod_{j=1}^{k-2}\left(n-\frac{j+1}{j}\right).$$

Proof. It can be obtained by straightforward computations, using (26) and (27) and the induction on k. □

Proposition 3.4. *Let g be a radial extremal metric on a complex manifold M of complex dimension $n \geq 2$. Assume that g is well-behaved and that g is projectively induced. Then then $A = B = 0$ in (10).*

Proof. Since g is well-behaved $y_{\text{inf}} = 0$. Assume by a contradiction that $A \neq 0$. By equation (34) with $\epsilon = 1$ one gets

$$y^{2n-1}Q_3^1(y) \to A^2(1-n) < 0, \quad \text{for} \quad y \to y_{\text{inf}}^+ = 0^+.$$

Then we deduce that $Q_3^1(y)$ would be negative in a right neighborhood of 0 in contrast with Lemma 3.2. Expression (34) with $A = 0$ yields

$$y^{2n-3}Q_3^1(y) \to B^2(2-n), \quad \text{for} \quad y \to 0^+.$$

Then, if $n > 2$, by the same argument just used to show the vanishing of A, one sees that it must be $B = 0$.

It remains to treat the case $n = 2$. On the one hand $Q_3^1(y)$ with $A = 0$ and $n = 2$ becomes a polynomial expression with constant term equals to $2B$ and so

$$Q_3^1(y) \to 2B, \quad \text{for} \quad y \to 0^+.$$

Then by Lemma 3.2 one deduces that $B \geq 0$. On the other hand (33) with $A = 0$ and $n = 2$ rewrites

$$Q_2^1(y) = -B - Dy^3 + (1 - C)y^2.$$

Again by letting $y \to 0^+$ and by Lemma 3.2 one gets $B \leq 0$. So we deduce $B = 0$. The proposition is proved. □

Remark 7. Example 1 in Subsection 2.1 with $n = 1$ shows that the assumption $n \geq 2$ in Lemma 3.4 is necessary. Moreover Example 2 and Example 3 indicate that in the lemma the well-behaveness condition cannot be dropped.

Proposition 3.5. *Let g be a radial extremal metric on a complex manifold M of complex dimension $n \geq 2$. If g is ϵ-resolvable with $\epsilon \leq 0$ then $A = B = 0$ in (10).*

Proof. By the very definition of ϵ-resolvability the Kähler manifold (M, g) can be Kähler immersed into the finite or infinite dimensional flat or complex hyperbolic space. It follows either by Remark 2 in the finite dimensional case or by [12, Lemma 5 and Lemma 8] in the infinite dimensional case that g is infinitely projectively induced. Thus, the proof will be ended if we show that g is well-behaved so to apply Proposition 3.4. Assume by contradiction this is not the case, i.e. $y_{\inf} > 0$. Then by (i) of Lemma 2.2 (cfr. Remark 3) one has $\lim_{y \to y_{\inf}^+} \psi(y) = 0$ which combined with (26) for $k = 2$ and the fact that, by assumption, the metric is ϵ-resolvable with $\epsilon \leq 0$, give $\lim_{y \to y_{\inf}^+} Q_2^\epsilon(y) = -y_{\inf}(|\epsilon| y_{\inf} + 1) < 0$, in contrast with Lemma 3.1. □

3.3. **The proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.4.**

Proof of Theorem 1.1. By multiplying the metric g by $\frac{c}{2}$ (if $c \neq 0$) we can assume the ambient space is one of the following: $(\mathbb{C}H^n, g_{hyp}^n)$, $(\mathbb{C}^n, g_0^n)$, $(\mathbb{C}P^n, g_{FS}^n)$ and so the metric g is ϵ-resolvable with $\epsilon = -1, 0, 1$, respectively. In order to prove (1) and (2) of Theorem 1.1 it is enough to show that g has constant holomorphic sectional curvature and then to appeal to Calabi's classification [5] of Kähler submanifolds of finite dimensional complex space forms.

We consider the cases $n = 1$ and $n \geq 2$ separately.

Case $n = 1$. We are going to show that $D = 0$: this will suffice since by (a) of Lemma 2.1 this would imply g is cscK and hence, since $n = 1$, g has constant

holomorphic sectional curvature. In order to show the vanishing of D observe that by Lemma 3.3 (in the notation of Lemma 3.2) one has

$$\deg\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}Q^{\epsilon}_{\alpha+\beta-i}(y)\right)=\deg Q^{\epsilon}_{\alpha+\beta}(y)=2(\alpha+\beta)-1.$$

Since the metric g is ϵ-resolvable of finite rank we can pick an index I such that (29) holds true, namely

$$\det\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\frac{\beta!}{(\beta-i)!}Q^{\epsilon}_{\alpha+\beta-i}(y)\right)_{1\leq\alpha,\beta\leq I}=0. \tag{35}$$

If σ is an arbitrary chosen permutation on I indices then the degree of $\prod_{\alpha=1}^{I}Q^{\epsilon}_{\alpha+\sigma(\alpha)}(y)$ does not depend on the permutation σ: indeed

$$\deg\left(\prod_{\alpha=1}^{I}Q^{\epsilon}_{\alpha+\sigma(\alpha)}\right)=2\sum_{\alpha=1}^{I}\big(\alpha+\sigma(\alpha)\big)-I=2I^2+I.$$

Therefore the leading term of the left hand side of (35) is given by the determinant of the leading terms of the $Q^{\epsilon}_{\alpha+\beta}$. By (31) of Lemma 3.3 this is given by

$$\det\left(-D^{\alpha+\beta-1}\prod_{j=2}^{\alpha+\beta-1}(1-2j)\right)_{1\leq\alpha,\beta\leq I}$$

and, by a straightforward computation, this is equal to

$$(-1)^I(-2)^{\frac{I}{2}(I-1)}D^{I^2}\prod_{j=2}^{I}(1-2j)^{I-j+1}\prod_{2\leq j<k\leq I+1}(k-j).$$

Hence, by (35) D is forced to be 0.

Case $n\geq 2$. Let $I\in\mathbb{Z}^+$ be the minimal[6] index such that

$$Q^{\epsilon}_{I}\equiv 0, \tag{36}$$

whose existence is guaranteed by (30) in Lemma 3.2. If $I=2$, and hence $Q^{\epsilon}_{2}\equiv 0$, Remark 5 implies that g has constant holomorphic sectional curvature and so the theorem is proved. Hence we can assume $I\geq 3$. We deduce that the leading term and the lower term of $Q^{\epsilon}_{I}(y)$ must vanish. By (31) and (32) of Lemma 3.3 they are given respectively by $-D^{I-1}\prod_{j=2}^{I-1}(1-2j)$ and $-A^{I-1}(I-2)!\prod_{j=1}^{I-2}\left(n-\frac{1}{j}\right)$. Hence we deduce that $A=D=0$. By (c) of Lemma 2.1 the proof of the theorem will be completed if we show that also $B=0$. Notice that for $\epsilon\leq 0$, Proposition 3.5 implies that $B=0$. Therefore it remains to show that $B=0$ when $\epsilon=1$ (and $A=D=0$).

We distiguish two cases: $n\neq 2$ and $n=2$.

[6]The minimality of I will be used only for the cases $n=2$ and $\epsilon=1$ at the end of the proof.

Assume $n \neq 2$. By (i) of Lemma 3.3 the condition $D = 0$ implies that the leading term of $Q^1_I(y)$ is given by

$$- B^{I-1}(I-2)! \prod_{j=1}^{I-2} \left(n - \frac{j+1}{j} \right). \tag{37}$$

Therefore (since $n \neq 2$) it follows by (36) and (37) that $B = 0$ and we are done[7].

Assume $n = 2$. In this case equation (10) with $A = D = 0$ is a polynomial of degree 2 in y, namely

$$\psi(y) = -Cy^2 + y - B. \tag{38}$$

Since $A = 0$ by (ii) of Lemma 3.3 the lower term of $Q^1_I(y)$ is given by

$$(-1)^{I-1}(I-1)! \prod_{j=1}^{I-1} \left(C - \frac{1}{j} \right). \tag{39}$$

By (36) (with $\epsilon = 1$) and (39) it follows that

$$C \in \left\{ 1, \frac{1}{2}, \dots, \frac{1}{I-1} \right\}. \tag{40}$$

In particular $C > 0$[8] and since $\psi(y) > 0$ we deduce that

$$0 \leq y_1 \leq y_{\inf} < y(t) < y_{\sup} \leq y_2 < +\infty, \ \forall t \in (t_{\inf}, t_{\sup}),$$

where y_1 and y_2 are the two distinct roots of (38). Moreover (36) and (26) (with $\epsilon = 1$) yield

$$\psi(y) P^1_I(y) + y^{(I-2)n+1} \prod_{j=1}^{I-1} (y - j) \equiv 0 \tag{41}$$

from which it follows that

$$y_j \in \{0, \dots, I-1\}, \ j = 1, 2. \tag{42}$$

The proof of the theorem will be ended if $y_1 = 0$ since in this case $B = 0$. Let us suppose by contradiction that $y_1 \neq 0$. Thus

$$y_2 \in \{0, \dots, I-2\}, \tag{43}$$

being $y_1 + y_2 = \frac{1}{C}$ at most equal to $I - 1$ by (40). Moreover $y_{\inf} \neq 0$ (since $0 \leq y_1 \leq y_{\inf}$). Thus by (i) (resp. (ii)) of Lemma 2.2 it follows $\psi(y_{\inf}) = 0$ (resp. $\psi(y_{\sup}) = 0$) and hence $y_1 = y_{\inf}$ (resp. $y_2 = y_{\sup}$).

Since $n = 2$ and $I - 1 \geq 2$ we can apply Lemma 3.2 and (25) to obtain

$$Q^1_{I-1}(y) = y \prod_{j=1}^{I-2} (y - j) + \frac{\psi(y) P^1_{I-1}(y)}{y^{(I-3)n}} \geq 0, \ \forall y \in (y_{\inf}, y_{\sup}) = (y_1, y_2). \tag{44}$$

[7] This argument works also for $\epsilon \leq 0$ since (37) does not depend on ϵ.

[8] In accordance to the fact that we will show (M, g) is an elliptic complex space form.

and

$$Q^1_I(y) = (y - I + 1)Q^1_{I-1}(y) + \frac{dQ^1_{I-1}}{dy}\psi(y), \tag{45}$$

which combined with (36) (with $\epsilon = 1$), (43), $\psi(y_2) = 0$ and $\psi(y) > 0$ give $Q^1_{I-1}(y_2) = 0$ and $\frac{dQ^1_{I-1}}{dy} \geq 0$, $\forall y \in (y_{\inf}, y_{\sup})$. Therefore $Q^1_{I-1} \equiv 0$, namely (36) holds true also for $I - 1$ in contrast with the assumption of the minimality of I. This yields the desired contradiction and concludes the proof of the theorem. □

Proof of Theorem 1.3. In order to prove (1) and (2) of Theorem 1.3 we will show that g has constant holomorphic sectional curvature and the proof will follow by Calabi's classification [5] of Kähler submanifolds of infinite dimensional complex space forms.

If $n = 1$ there is nothing to be proved since in this case a cscK metric has constant holomorphic sectional curvature. So assume $n \geq 2$. In order to prove (1) (resp. (2)) of Theorem 1.3 we can assume as before (by multiplying the metric by a suitable constant) that (M, g) admits a Kähler immersion either into $(\mathbb{C}\mathrm{H}^\infty, g_{hyp})$ or $(\ell^2(\mathbb{C}), g_0)$ (resp. $(\mathbb{C}P^\infty, g_{FS})$). By Proposition 3.5 (resp. Proposition 3.4, which can be applied since g is assumed to be well-behaved) we get $A = B = 0$. By combining this with the hypothesis cscK ($D = 0$) and by (a) and (c) of Lemma 2.1 one deduces g has constant holomorphic sectional curvature and we are done. □

The key ingredient in the proof of Theorem 1.4 is the following proposition[9].

Proposition 3.6. *Let g be a radial KE metric on a complex manifold M. Assume that g is not well-behaved and infinitely projectively induced. Then the Einstein constant of g is a rational number. In particular g is unstable.*

Proof. Notice first that the condition that g is not well-behaved implies $n \geq 2$ since a KE metric on a complex 1-dimensional manifold has constant holomoprhic sectional curvature and so it is necessarily well-behaved. We are going to show that

(i) $y_{\inf} \in \mathbb{Z}$,
(ii) $\tilde{n} := n - \frac{\lambda}{2} y_{\inf} \in \mathbb{Z}$

which clearly implies the rationality of λ.

By the very definition of well-behaveness we know that $y_{\inf}$ is a non zero real number. Thus by (i) of Lemma 2.2 one has $\psi(y_{\inf}) = 0$, i.e. $\psi(y) = (y - y_{\inf})\tilde{\psi}(y)$ for some rational function $\tilde{\psi}(y)$. By imposing the KE assumption in (10) and using (b) of Lemma 2.1 one gets

$$\psi(y) = y - \frac{A}{y^{n-1}} - \frac{\lambda}{2(n+1)} y^2,$$

[9]Notice that by the above Conjecture 2 and the fact that any complex space form is well-behaved we believe that the set of metrics satisfying the assumption Proposition 3.6 is empty.

from which one immediately finds $\tilde{\psi}(y_{\inf}) = \frac{d\psi}{dy}(y_{\inf}) = \tilde{n}$. Notice that from this, $\psi(y_{\inf}) = 0$ and $\psi(y) > 0$, for $y > y_{\inf}$, it follows that $\tilde{n}$ must be nonnegative.

Now, combining $\psi(y_{\inf}) = 0$ with (26) with $\epsilon = +1$, one immediately deduces that

$$Q^1_k(y_{\inf}) = y_{\inf}(y_{\inf} - 1)\cdots(y_{\inf} - k + 1) \tag{46}$$

and then, if $y_{\inf} \notin \mathbb{Z}$, one has $Q^1_{[y_{\inf}]+2}(y_{\inf}) < 0$, which by (a2) of Lemma 3.2 contradicts the assumption that the metric is projectively induced. This shows (i).

In order to prove (ii), we show that if $\tilde{n} \notin \mathbb{Z}$ then $Q^1_{y_{\inf}+[\tilde{n}]+2}(y)$ is strictly negative in a right neighbourhood of $y = y_{\inf}$, and the conclusion will follow again by contradiction from (a2) of Lemma 3.2.

By (25), for any positive integer j we easily get

$$Q^1_{j+1}(y_{\inf}) + \tilde{n}\frac{dQ^1_{j+1}}{dy}(y_{\inf}) = \left(Q^1_j(y_{\inf}) + \tilde{n}\frac{dQ^1_j}{dy}(y_{\inf})\right)(y_{\inf} - j + \tilde{n})$$

from which, since $Q^1_1(y_{\inf}) + \tilde{n}\frac{dQ^1_1}{dy}(y_{\inf}) = y_{\inf} + \tilde{n}$ (recall that $Q^1_1(y) = y$), we get

$$Q^1_k(y_{\inf}) + \tilde{n}\frac{dQ^1_k}{dy}(y_{\inf}) = (y_{\inf} + \tilde{n} - k + 1)\cdots(y_{\inf} + \tilde{n} - 1)(y_{\inf} + \tilde{n}), \tag{47}$$

for any integer $k \geq 2$.

By taking in particular $\hat{k} = y_{\inf} + [\tilde{n}] + 2$ and noticing that from (46) it follows that $Q^1_{y_{\inf}+j}(y_{\inf}) = 0$ for any $j \geq 1$, one gets

$$\tilde{n}\frac{dQ^1_{\hat{k}}}{dy}(y_{\inf}) = (\tilde{n} - [\tilde{n}] - 1)(\tilde{n} - [\tilde{n}])\cdots(y_{\inf} + \tilde{n}). \tag{48}$$

Thus, by the assumption $\tilde{n} \notin \mathbb{Z}$ (and $\tilde{n} > 0$), one concludes $\frac{dQ^1_{\hat{k}}}{dy}(y_{\inf}) < 0$, which together with $Q^1_{\hat{k}}(y_{\inf}) = 0$ immediately implies that $Q^1_{\hat{k}}(y)$ is strictly negative in a right neighbourhood of $y = y_{\inf}$, the wished contradiction. The last part of the proposition follows directly by the definition of stable projectively induced metric. □

An interesting consequence of Proposition 3.6 is the following corollary which should be compared to a result of D. Hulin [14] (see also [19, Theorem 1.1] for an alternative proof) on the rationality of the Einstein constant of a finite projectively induced KE metric.

Corollary 3.7. *Let g be a radial KE metric of positive Einstein constant $\lambda > 0$. If g is infinitely projectively induced then λ is a rational number.*

Proof. On the one hand if g is well-behaved then by (2) of Theorem 1.3 (M, g) is a complex space form and the assumption $\lambda > 0$ implies is the complex projective

space, $g = mg_{FS}$ and $\lambda = \frac{2(n+1)}{m} \in \mathbb{Q}$. On the other hand if g is not well-behaved the rationality of λ is guaranteed by Proposition 3.6. □

Proof of Theorem 1.4. By multiplying the metric g by $\frac{c}{2}$ (if $c \neq 0$) we can assume g is infinitely and stable projectively induced. By Proposition 3.6 g is forced to be well-behaved. Thus by (2) of Theorem 1.3 (M, g) is a complex space form of non-positive holomorphic sectional curvature (since g_{FS} is unstable). □

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