A Mathematical Improvement of the Skate Curves

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## Introduction

Styleslalom consists of performing tricks on skates, called steps, while skating around a line of cups or cones, each one placed at the same distance from the others.
In 1995 only about 10 steps had been created and Styleslalom consisted of repeating these steps one after the other from the first to the last cone.
In 1996 one of the authors, E. Perano, introduced a mathematical method to create many more steps from the basic ones that were known in those years and all skaters began to try performing them. Consequently, Styleslalom has expanded exponentially since then, and within a few years it was recognized as a real sporting discipline by all skating federations worldwide.
The method described in this paper can always be used because it may be applied to any step. It is based on three mathematical concepts: the symmetry of functions, their inversion and the principle of duality. In the following section we will present the first two concepts and we will give a description of the curves drawn by each skate in parametric form.

## Some Preliminary Definitions

There are *elemental steps* that cannot be broken up into simpler steps and *composed steps* that contain two or more connected elemental steps [1, Chap. 1].
Every step performance is repeated every $n$ cups ($n \geq 2$).
The *period* $T$ of a step is the lowest $n$ that satisfies this condition and it is always equal to 2 for an elemental step [2].
NOTE 1: Before E. Perano introduced his method, composed steps did not exist.

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OPPOSITE STEP: Given two steps $A$ and $B$, $B$ is defined "the $A$ opposite" and we write $B = (\overline{A})$, if it's obtained performing $A$ in the opposite direction.
SYMMETRIC STEP: a step is defined symmetric when it is equal to its opposite, so that, when performing the opposite, you always obtain the same step.
SWITCHING FIGURE (SWITCH): Trick which is performed with the skates around only one cup and is not repeated when passing the subsequent cups. It is used to change between two different steps.
NOTE 2: as a switch can be executed in a lot of ways, it is possible to perform different switches between two identical steps. The switch used to change from a step $A$ to another step $B$ is also generally different from the one used to switch from step $B$ to step $A$.

## The Mathematical Applications: Symmetry

The features of an elemental step can be described [1, Chap. III] by a mathematical relation, possibly a function, and its corresponding graph.
A first elemental mathematical approach to Styleslalom consists of associating to a step an elemental graph which describes only the periodicity and possible symmetry of the step, without any reference to the trajectories followed by the skates during their performance.
The space covered with the skates is represented on the horizontal axis, using the number of cones which a skater passes while performing the step as unit of measurement, from the first cone to the last one. The function connected to an elemental step must also be periodic with $T = 2$. It means that its graph must repeat itself equally every section of amplitude 2.
The possible symmetry of each step corresponds to the symmetry with respect to the horizontal axis around which the step develops.
As an example, two elemental steps are shown graphically in Figs. 1 and 2.

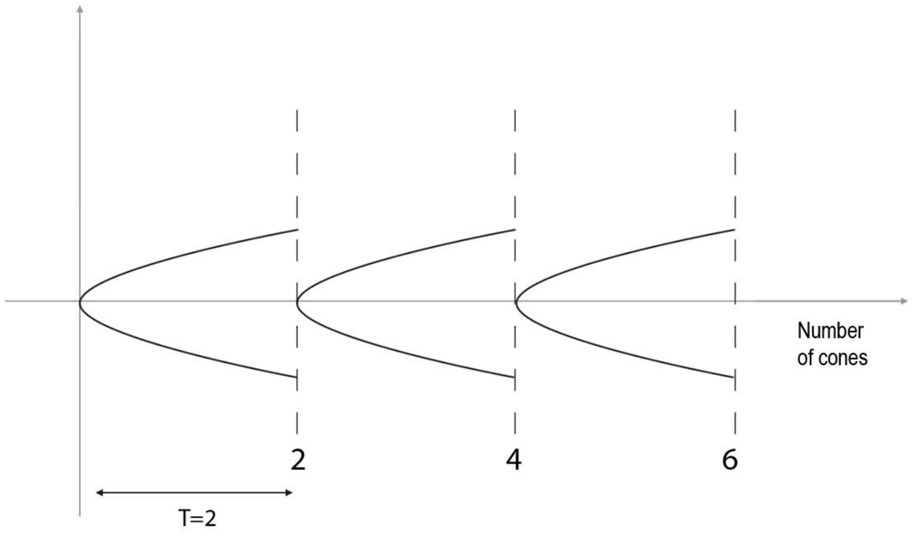


Fig.1

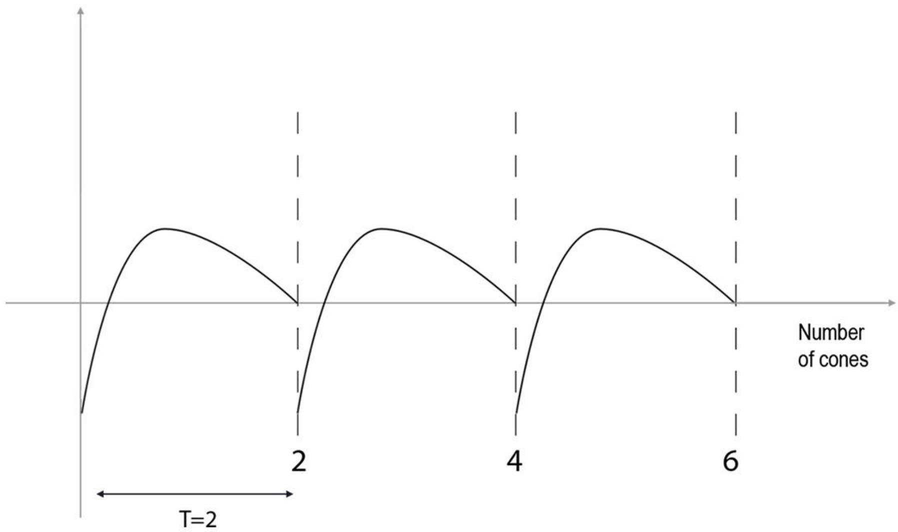


Fig.2

The step of the first example (Fig.1) is symmetric because its graph remains the same after a symmetry about the horizontal axis, whereas the step is not symmetric in the second case (Fig.2). Consequently, it is possible to draw the graph of its opposite, as shown in Fig.3.

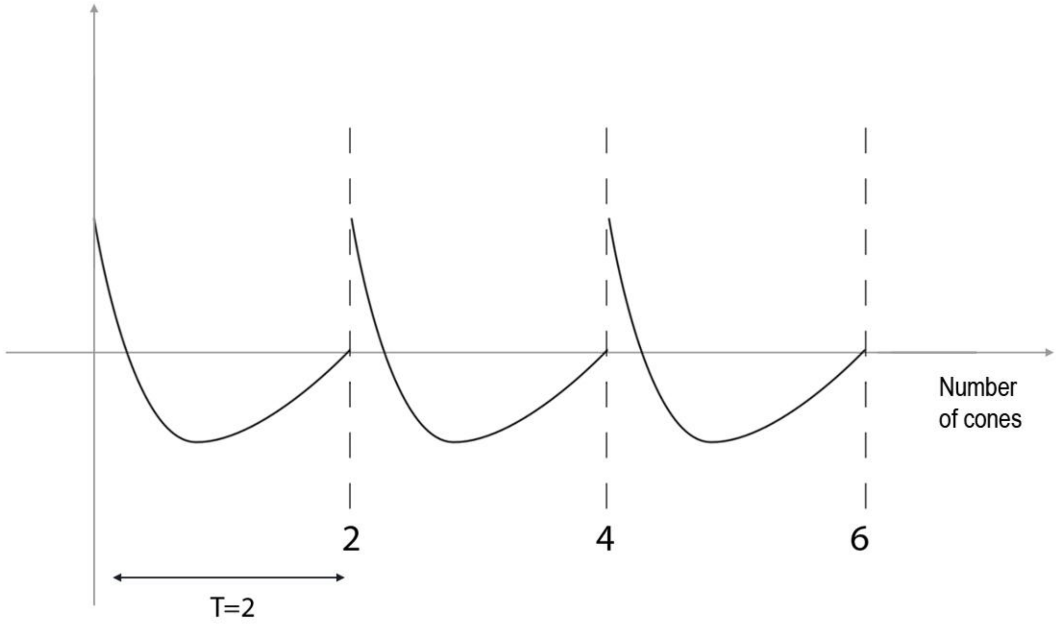


Fig.3

Overall, the graph of a symmetric step, like the one in Fig.1, corresponds to a multi-valued function, that is not a function according to the standard definition.
As the opposites of not symmetric elemental steps are different from the original steps, it is possible to add every step to its opposite in order to create new composed steps.
The equation of a function associated to a not symmetric elemental step performed along a line of 20 cups can be written as follows:

$$f(x) = \sum_{k=0}^{9} f_T\,(x - kT) \qquad kT < x < (k+1)T \qquad k = 0, \dots, 9$$

where $f_T(x)$ describes the step in the fundamental period $T = 2$ , which is repeated identically every two cones.
Similarly, the equation of the function of its opposite becomes:

$$\overline{f(x)} = \sum_{k=0}^{9} [-f_T(x - kT)] \qquad kT < x < (k+1)T \qquad k = 0, \dots, 9$$

We can also write the equation of the $j$-th among $n$ switches which is defined only around one cone and so covers only half a period:

$$g_j(x) = \begin{cases} 0 & x < 0 \\ f_{sj}(x) & 0 < x < \frac{T}{2} \\ 0 & x > \frac{T}{2} \end{cases}$$

or, more simply:

$$g_j(x) = f_{sj}(x)\left[U(x) - U\left(x - \frac{T}{2}\right)\right]$$

where $U$ indicates the Heaviside step function.
Then it is possible to describe the composed symmetric step using this function:

$$f_c(x) = \begin{cases} f_T(x) & 0 < x < T \\ f_{sj}(x - T) & T < x < \frac{3}{2}T \\ -f_T\left(x - \frac{3}{2}T\right) & \frac{3}{2}T < x < \frac{5}{2}T \\ f_{sj}\left(x - \frac{5}{2}T\right) & \frac{5}{2}T < x < 3T \end{cases}$$

For example, consider again the not symmetric step in Fig. 2 (named $B$), and imagine performing it only in the space corresponding to the first period of two cones. Then, skating around the third cone by a SWITCH, execute the opposite step $\overline{B}$ in the next space of width $T = 2$. In this way you obtain a new step that is no longer elemental but is composed of the original step with its opposite, a symmetric step, having a period equal to $3T = 6$. The result is shown in Fig. 4.

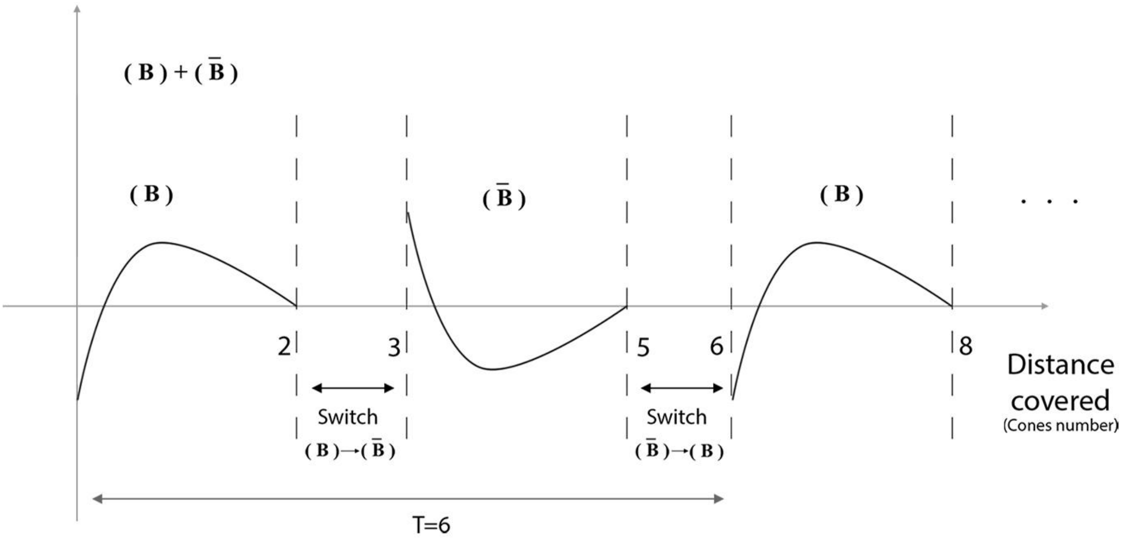


Fig.4

This method allows you to build a symmetric step adding to a not symmetric step its opposite, and it can be summarized in the following RULE 1.

> *Once a not symmetric step is chosen, alternating it with its opposite a composed symmetric step with a longer period* $(T > 2)$ *is obtained.*

The role of the switching figure that allows a step to alternate with its opposite is fundamental for building the resulting composed step.

As a matter of fact, when you alternate the same step with its opposite, you can obtain different choreographies if the switching figure changes.

## Theorem

Given a not symmetric elemental step $A$ and $n$ Switching Figures, it's possible to create $n^2$ composed symmetric steps having a period $T_s = 6$.

## Proof:

Calling $f_A(x)$ the single valued function associated to the elemental step $A$, $\overline{f_A(x)} = -f_A(x)$ the function defined by $\overline{A}$ , the opposite of $A$, and $f_{si}(x)$ the function corresponding to the $i$-th of $n$ Switch Figures, it is possible to create symmetric compositions described by periodical functions, with a period $T_s = 3T = 3 \cdot 2 = 6$, which, in the first period $T_s$, are defined as follows:

$$f_c(x) = \begin{cases} f_T(x) & 0 < x < T \\ f_{si}(x-T) & T < x < \frac{3}{2}T \\ -f_T\left(x - \frac{3}{2}T\right) & \frac{3}{2}T < x < \frac{5}{2}T \\ f_{sj}\left(x - \frac{5}{2}T\right) & \frac{5}{2}T < \ x < 3T \end{cases} \qquad \forall\ i,j : 1 < i,j < n\ .$$

NOTE 3: In a few cases it is possible to alternate an elemental step with its opposite without needing a switching figure. Consequently, the period of the composed step decreases to $2T = 4$.

## The Mathematical Applications: Functions Inversion

The idea of extending a simple step using suitable symmetries in order to build more complex choreographies (step mixtures) can be extended if we use another mathematical tool, the one of inverse function [1, Chap. IV].

In Styleslalom we introduce the following definition:

INVERSE STEP: given two steps $A$ and $B, B$ is defined "the inverse of $A$" and it is indicated by $A^{-1}$, if it is obtained inverting the movements of the skates when step $A$ is performed.

An example of two steps, one the inverse of the other, is skating forward or backward using only one leg and always the same one.

In Mathematical Analysis function inversion is obtained exchanging the variable $x$ with $y$ in the expression of an injective single-valued function; in Styleslalom the inverse of a step is obtained retracing its graph but in the opposite direction, from the point of arrival to the starting point.

For instance, the graph associated to a general step $A$ and the corresponding graph of its inverse, $B = A^{-1}$, are shown in Figs. 5 and 6.

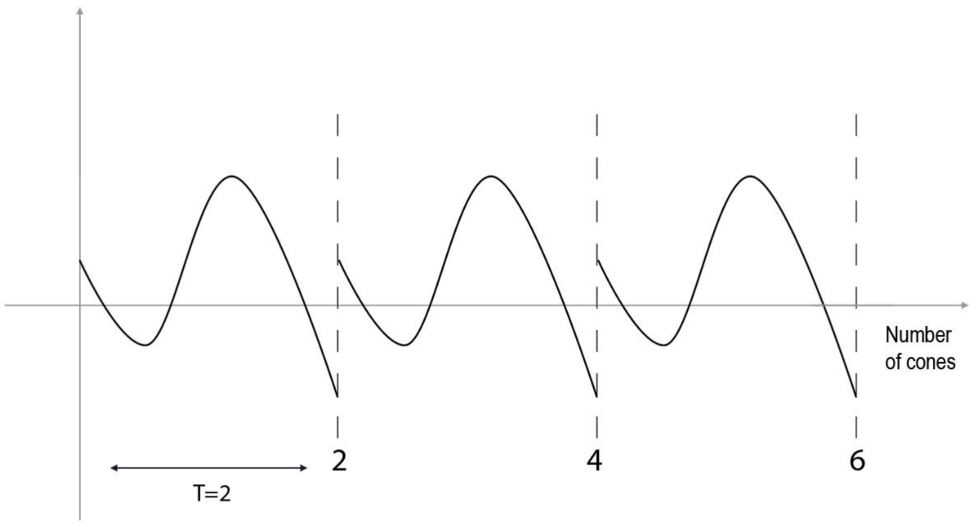


Fig.5

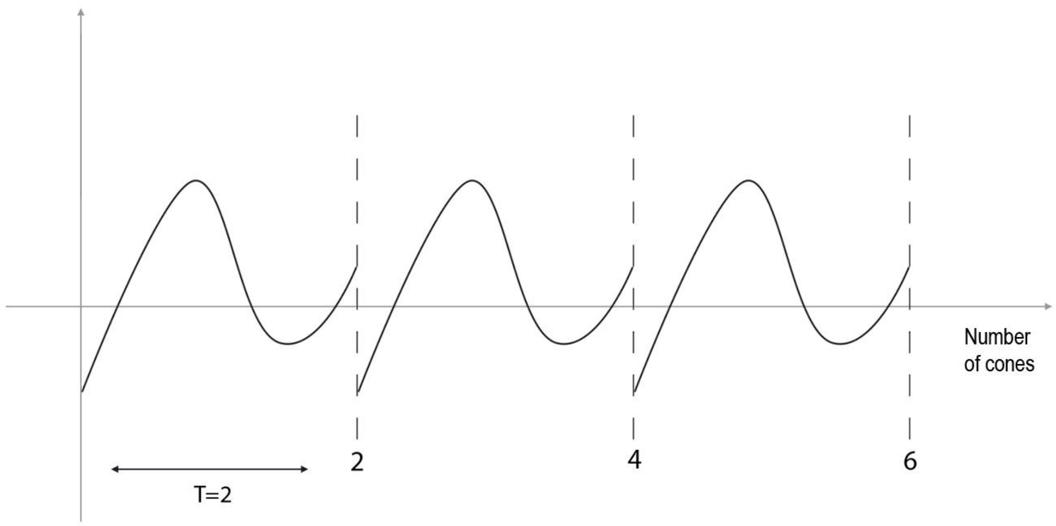


Fig.6

All values assumed by the graph of a step during its performance from the beginning to the end of a period are also assumed by the inverse step but in the reverse order. This means that in a period the inverse step assumes the last position of the original step as its first position and then it goes on taking all intermediate positions in reverse order until it assumes the first position of the original step at the end of the period.
The graph of the inverse step can be traced in a given period reflecting the graph of the original step about the axis of the period (see Fig. 7).

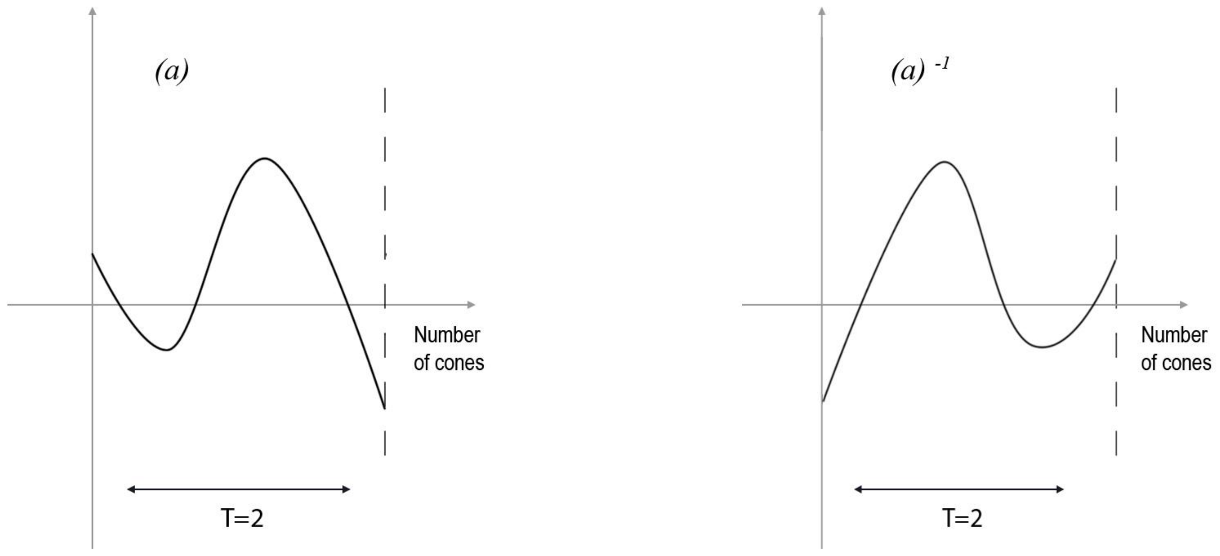


Fig.7

If $f_T(x)$ is the expression of the function associated to an elemental step in the first period $T = 2$, the equation of the function of its inverse in the same period is $f_T(T - x)$.
So the equation of the function associated to the inverse of an elemental step with a period $T = 2$ performed along a line of 20 cups becomes:

$$f_i(x) = \sum_{k=0}^{9} f_T[(k+1)T - x] \qquad kT < x < (k+1)T \qquad k = 0, \dots, 9$$

According to this definition of inverse step, steps coinciding with their inverse would be theoretically possible because it may be the case that the graph of a step does not change after its reflection about the axis of the period. This would mean that the step and its inverse are the same step.

Consequently, we give the following definition:
ANTIMETRIC STEP: a step is named antimetric when it is equal to its inverse, that is, when inverting the movement of the skates the same step is still obtained.
The diagram associated to an antimetric step, in each period, is symmetric with respect to the axis of the period.
Actually an elemental step is never antimetric because it can’t coincide with its inverse, however it is possible to create an antimetric composed steps with longer period adding at to an elemental step its inverse.
So the addition of inverse steps allows the introduction of a second rule to make new more complex compositions, from simpler steps.

*Alternating a not antimetric step with its inverse every cone, a composed antimetric step with a longer period is obtained*

This rule allows you to create an antimetric composed step starting from a simpler step that cannot be antimetric.

NOTE 4: Sometimes it is not possible to perform the inverse of an elemental or not antimetric composed step because it cannot be physically achievable. On the contrary, you can always perform the opposite of a not symmetric step.

NOTE 5: This second rule can also be applied to symmetric compositions resulting from the alternation of an elemental step with its opposite.

Consequently, you can add a symmetric composition to its inverse to obtain a more complex choreography that remains symmetric and, at the same time, has become antimetric.

## A More Detailed Mathematical Approach to Styleslalom

A deeper mathematical application consists of describing the curves drawn by each skate considering it as a material point which moves around the cones.

For example [3], a triangular wave can be traced by both skates as shown in Fig. 8.

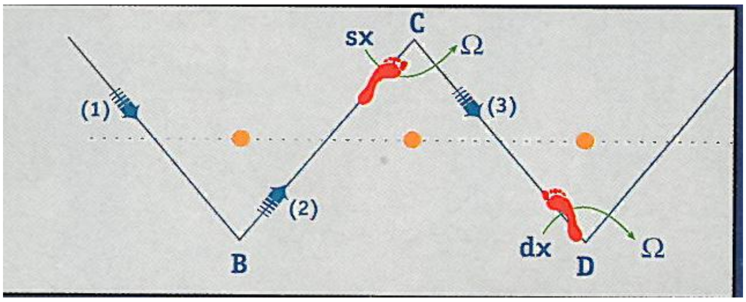


Fig.8

For each triangle, a skate describes one side sliding forward while simultaneously the second skate slides backward on the other side.

Alternatively, the same trend can be described moving on only one skate:

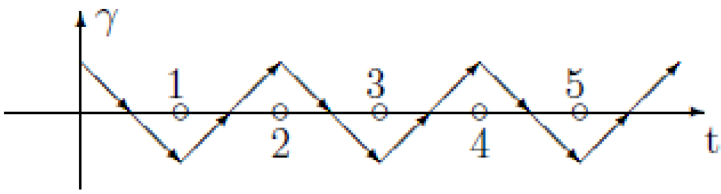


Fig.9

This movement is shown in the next photo sequence (Fig.10).

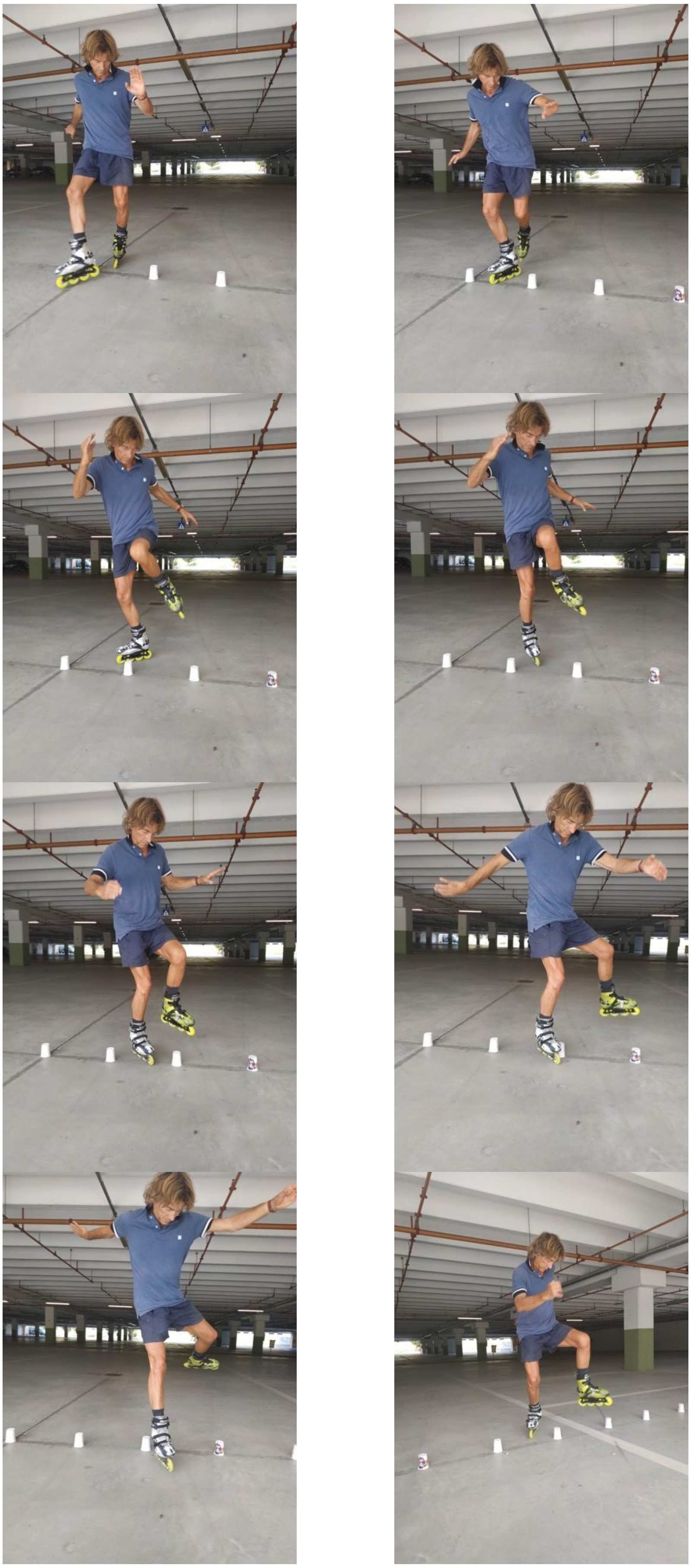

Fig.10

The equation of the triangular wave is the following:

$$\gamma(t) = \sum_{n=1}^{3} \gamma_n(t)$$

with:

$$\gamma_n(t) = \begin{pmatrix} t \\ |t-(2n-1)|-0.5 \end{pmatrix} \qquad 2(n-1) \le t \le 2n$$

In the next example, the curve traced by a skate is a repeated cycloid [4, 5] (Fig.11):

Fig.11

$$\gamma_1(t) = \begin{pmatrix} \dfrac{r[2\pi t - k\sin(2\pi t)]}{2\pi} \\ \\ \dfrac{r[1-k\cos(2\pi t)]}{4} \end{pmatrix} \qquad 0 \le t \le 2 \;\; (k=1)$$

The next photo sequence shows a step where both skates move describing a cycloid, one as in the previous example while the other is a prolate cycloid (Fig.12).

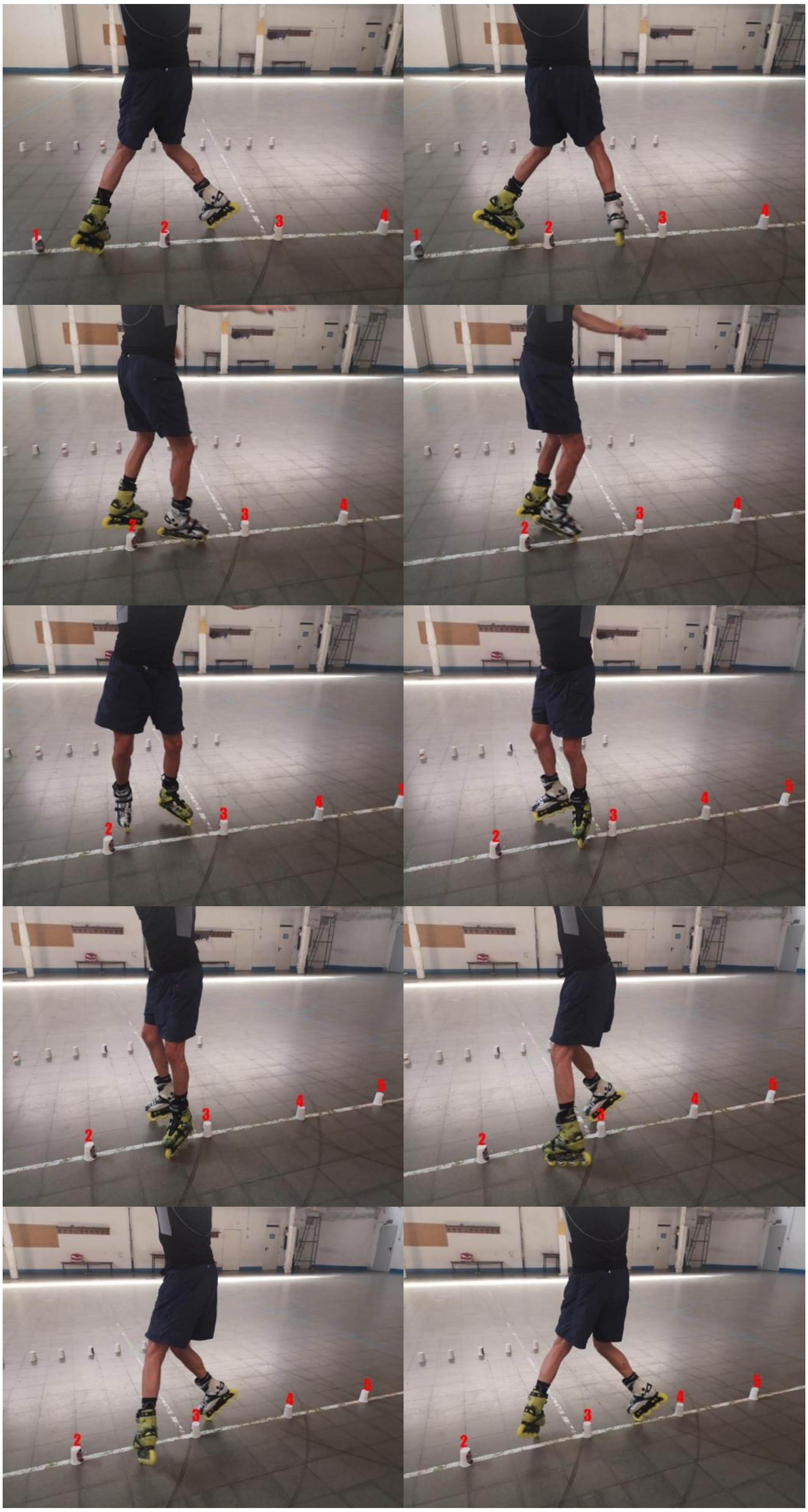

Fig.12

$$\gamma_2(t) = \begin{pmatrix} r[2\pi t - k\sin(2\pi t)] \\ \\ r[1 - k\cos(2\pi t)] \end{pmatrix} \qquad 0 \leq t \leq 2 \;\; (k > 1)$$

NOTE 6: The step shown in the second example is an elemental step with period $T = 2$ because it describes half a cycloid sliding forward and the other half moving backward on the same skate.

On the contrary, the third example contains a step performed on both skates which describe a cycloid sliding forward, so its period is only $T = 1$. The steps whose period is reduced to one are called **pathological steps.**

Finally, the last example shows the movement of the skates while performing the Ala step around only one cone:

Fig.13

In the following figure you can see the drawing of the curve [6] and its equation [4, 5].

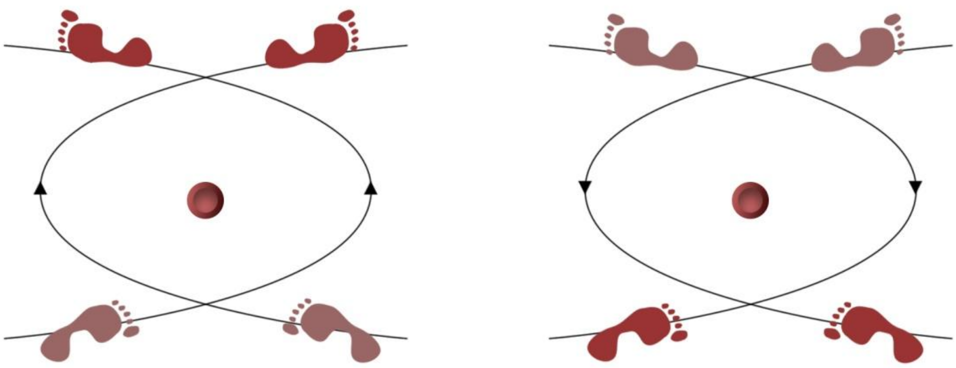

Fig.14

$$\gamma_l(t) = \begin{pmatrix} -\frac{3}{2}a + 2a\cos t \\ a\sin t \end{pmatrix} \qquad -\frac{\pi}{3} \leq t \leq +\frac{\pi}{3}$$

$$\gamma_r(t) = \begin{pmatrix} +\frac{3}{2}a + 2a\cos t \\ a\sin t \end{pmatrix} \qquad +\frac{2\pi}{3} \leq t \leq +\frac{4\pi}{3}$$

letting, for example, $a = 2$.

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