The Hilbert function of bigraded algebras in k[P1×P1]

THE HILBERT FUNCTION OF BIGRADED ALGEBRAS IN $k[\mathbb{P}^1 \times \mathbb{P}^1]$

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Abstract. We classify the Hilbert functions of bigraded algebras in $k[x_1, x_2, y_1, y_2]$ by introducing a numerical function called a *Ferrers* function.

1. Introduction

Let $S := k[x_1, \dots, x_n]$ be the standard graded polynomial ring and let $I \subseteq S$ be a homogeneous ideal. The quotient ring S/I is called a *standard graded k-algebra*. The Hilbert function of S/I is defined as $H_{S/I} : \mathbb{N} \to \mathbb{N}$ such that

$$H_{S/I}(t) := \dim_k (S/I)_t = \dim_k S_t - \dim_k I_t.$$

A famous theorem, due to Macaulay (cf. [7]) and pointed out by Stanley (cf. [10]), characterizes the numerical functions that are Hilbert functions of a standard graded k-algebra, i.e. the functions H such that $H = H_{S/I}$ for some homogeneous ideal $I \subseteq S$. Macaulay's theorem is expressed in the language of O-sequence; for a modern treatment of this result see [3].

It is of interest to find an extension of the above theorem to the multi-graded case. Multi-graded Hilbert functions arise in many contexts. Properties related to the Hilbert function of multi-graded algebras are studied for instance in [1, 2, 8, 9, 11]. A generalization of Macaulay's theorem to multi-graded rings is an open problem. A partial result is Theorem 4.14 in [1]. It gives non-sharp bounds on the growth of the Hilbert function of a bigraded algebra.

The goal of this work is to generalize the Macaulay's Theorem in the first significant case of bigraded algebras. The main result of this paper is Theorem 4.10, where we classify the numerical functions $H : \mathbb{N}^2 \to \mathbb{N}$ which are Hilbert functions of a bigraded algebra in $k[x_1, x_2, y_1, y_2]$ where $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$.

The paper is structured as follows. In Section 2 we give the necessary background and notation. In Section 3 we introduce a set of partitions of a number, and we define a numerical function called a *Ferrers* function. Finally, in Section 4 we investigate a connection between partitions and set of monomials, and we prove that Ferrers functions characterize the Hilbert functions of bigraded algebras.

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2. Maximal growths for the Hilbert function of a bigraded algebra

Let k be an infinite field, and let $R := k[x_1, x_2, y_1, y_2] = k[\mathbb{P}^1 \times \mathbb{P}^1]$ be the polynomial ring in 4 indeterminates with the grading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$. Then $R = \oplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$

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where $R_{(i,j)}$ denotes the set of all homogeneous elements in R of degree (i,j). $R_{(i,j)}$ is generated, as a k-vector space, by the monomials $x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2}$ such that $i_1+i_2=i$ and $j_1+j_2=j$. An ideal $I\subseteq R$ is called a *bigraded* ideal if it is generated by homogeneous elements with respect to this grading. A bigraded algebra R/I is the quotient of R with a bigraded ideal I. The Hilbert function of a bigraded algebra R/I is defined such that $H_{R/I}:\mathbb{N}^2\to\mathbb{N}$ and $H_{R/I}(i,j):=\dim_k(R/I)_{(i,j)}=\dim_k R_{(i,j)}-\dim_k I_{(i,j)}$ where $I_{(i,j)}=I\cap R_{(i,j)}$ is the set of the bihomogeneous elements of degree (i,j) in I.

Throughout this notes we will work with the degree lexicographical order on R induced by $x_1>x_2>y_1>y_2$. With this ordering we recall the definition of bilex ideal, introduced and studied in [1]. We refer to [1] for all preliminaries and for further results on bilex ideals.

Definition 2.1 ([1], Definition 4.4)**.** A set of monomials $L\subseteq R_{(i,j)}$ is called *bilex* if for every monomial $uv\in L$, where $u\in R_{(i,0)}$ and $v\in R_{(0,j)}$, the following conditions are satisfied:

- if $u'\in R_{(i,0)}$ and $u'>u$, then $u'v\in L$;
- if $v'\in R_{(0,j)}$ and $v'>v$, then $uv'\in L$.

A monomial ideal $I\subseteq R$ is called a *bilex ideal* if $I_{(i,j)}$ is generated as k-vector space by a bilex set of monomials, for every $i,j\geq 0$.

Bilex ideals play a crucial role in the study of the Hilbert function of bigraded algebras.

Theorem 2.2 ([1],Theorem 4.14)**.** *Let $J\subseteq R$ be a bigraded ideal. Then there exists a bilex ideal I such that $H_{R/I}=H_{R/J}$.*

The next theorem gives an upper bound for the growth of the Hilbert function of bigraded algebras. It is an reformulation of [1], Theorem 4.18.

Theorem 2.3. *Let $I\subseteq R$ be a bigraded ideal. For all $(i,j)\in\mathbb{N}^2$, if $p:=\left\lfloor\frac{H_{R/I}(i,j)}{(i+1)}\right\rfloor$ and $q:=\left\lfloor\frac{H_{R/I}(i,j)}{(j+1)}\right\rfloor$, then*

$$\begin{cases} H_{R/I}(i+1,j)\leq H_{R/I}(i,j)+p\\ H_{R/I}(i,j+1)\leq H_{R/I}(i,j)+q\end{cases}$$

Remark 2.4. The bound in Theorem 2.3 is not sharp even if $\dim_k(I_{(i,j)})=2$. Take, for instance, $I':=(x_1y_1,x_1y_2)+(x_1,x_2,y_1,y_2)^4$ and $I'':=(x_1y_1,x_2y_1)+(x_1,x_2,y_1,y_2)^4$. Then the Hilbert functions of the associated bigraded algebras are

$H_{R/I'}:=$

	0	1	2	3	4	...
0	1	2	3	4	0	...
1	2	2	3	0	0	...
2	3	2	0	0	0	...
3	4	0	0	0	0	...
4	0	0	0	0	0	...

$H_{R/I''}:=$

	0	1	2	3	4	...
0	1	2	3	4	0	...
1	2	2	2	0	0	...
2	3	3	0	0	0	...
3	4	0	0	0	0	...
4	0	0	0	0	0	...

.

By Theorem 2.3 we have $\begin{cases} H_{R/I}(2,1)\leq 3\\ H_{R/I}(1,2)\leq 3\end{cases}$ but the numeric function

$H:=$

	0	1	2	3	4	...
0	1	2	3	4	0	...
1	2	2	3	0	0	...
2	3	3	0	0	0	...
3	4	0	0	0	0	...
4	0	0	0	0	0	...

is not the Hilbert function of any bigraded algebra. Indeed, for $t \geq 0$, $\sum_{i+j=t} H(i,j)$ gives rise to the sequence $(1,4,8,14,0,\ldots)$ which fails to be an O-sequence. Thus, this example also shows two different maximal growths of the Hilbert function from the degree $(1,1)$ to $(1,2)$ and $(2,1)$.

To formalize the idea of maximal growths we need to introduce a partial order in $\mathbb{N}^2$. For $(a,b),(c,d) \in \mathbb{N}^2$ we say that $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$. Moreover we say that $(a,b) < (c,d)$ iff $(a,b) \leq (c,d)$ and $a < c$ or $b < d$.

Definition 2.5. Let $I \subseteq R$ be a bigraded ideal. We say that $H_{R/I}$, the Hilbert function of R/I, has a maximal growth in degree (i,j) if $(H_{R/I}(i+1,j), H_{R/I}(i,j+1))$ is a maximal element in the set $\{(H_{R/J}(i+1,j), H_{R/J}(i,j+1)) \mid R/J$ a bigraded algebra with $H_{R/J}(i,j) = H_{R/I}(i,j)\}$.

Note that the above definition does not require that R/I and R/J have the same Hilbert function in the degrees less than (i,j).

3. Partitions of a number and Ferrers functions

In this section we introduce a partition of a number which slightly generalizes Definition 3.12 in [6] that allows the authors to characterize arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 3.1. Given $h, \ell_1, \ell_2 \in \mathbb{N}$ we say that $\alpha := (p_1, \ldots, p_t) \in \mathbb{N}^t$ is a partition of h of *sides* (ℓ_1, ℓ_2) if $h = p_1 + \cdots + p_t$, $\ell_1 \geq p_1 \geq \ldots \geq p_t \geq 0$ and $t = \ell_2$. We denote by $\lambda_1(\alpha)$ the number of entries in α equal to ℓ_1, i.e. $\lambda_1(\alpha) := |\{j \mid p_j = \ell_1\}|$ and by $\lambda_2(\alpha) := p_{\ell_2}$. Moreover we call $\lambda(\alpha) := (\lambda_1(\alpha), \lambda_2(\alpha)) \in \mathbb{N}^2$ the *size* of α.

For example, $\alpha = (5,5,5,4,1,1)$ is a partition of 21 of sides $(5,6)$ with size $\lambda(\alpha) = (3,1)$.

We denote by $\mathcal{S}(h)^{(\ell_1,\ell_2)}$ the set of all the partitions of h of sides (ℓ_1,ℓ_2), and by $\mathcal{L}(h)^{(\ell_1,\ell_2)}$ the set of the sizes of the elements in $\mathcal{S}(h)^{(\ell_1,\ell_2)}$.

Example 3.2. $\mathcal{S}(4)^{(3,3)} := \{(3,1,0),(2,2,0),(2,1,1)\}$ and $\mathcal{L}(4)^{(3,3)} := \{(1,0),(0,0),(0,1)\}$.

Furthermore, we set $\mathcal{S}(\cdot)^{(\ell_1,\ell_2)} := \cup_{h \in \mathbb{N}} \mathcal{S}(h)^{(\ell_1,\ell_2)}$. We introduce in $\mathcal{S}(\cdot)^{(\ell_1,\ell_2)}$ an inner operation. Take $\alpha := (p_1, \ldots, p_t)$ and $\alpha' := (p'_1, \ldots, p'_t)$ elements in $\mathcal{S}(\cdot)^{(\ell_1,\ell_2)}$. Then we define

$$\alpha \cap \alpha' := (\min\{p_1, p'_1\}, \ldots, \min\{p_t, p'_t\}) \in \mathcal{S}(\cdot)^{(\ell_1,\ell_2)}.$$

Let $\alpha, \alpha' \in \mathcal{S}(\cdot)^{(\ell_1,\ell_2)}$. We say that $\alpha' \leq \alpha$ iff $\alpha' \cap \alpha = \alpha'$, i.e. the entries in α' are less than or equal to the entries in α componentwise.

Let $\alpha := (p_1, p_2, \ldots, p_t) \in \mathcal{S}(\cdot)^{(\ell_1,\ell_2)}$ be a partition of size (λ_1, λ_2). Then we associate to α partitions of sides (ℓ_1+1, ℓ_2) and (ℓ_1, ℓ_2+1) as follows. We denote by $\alpha^{(1)} := (p'_1, p'_2, \ldots, p'_t) \in \mathcal{S}(\cdot)^{(\ell_1+1,\ell_2)}$ where

$$p'_j := \begin{cases} p_j + 1 & \text{if } j \leq \lambda_1 \\ p_j & \text{if } j > \lambda_1 \end{cases},$$

and we denote by $\alpha^{(2)} := (p_1, p_2, \ldots, p_{t-1}, p_t, p_t) \in \mathcal{S}(\cdot)^{(\ell_1,\ell_2+1)}$.

We are ready to introduce the Ferrers functions.

Definition 3.3. Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a numerical function. We say that H is a Ferrers function if $H(0,0) = 1$ and, for any $(i,j) \in \mathbb{N}^2$, there exists a partition of $H(i,j)$ of sides $(i+1, j+1)$, namely $\alpha_{ij} \in \mathcal{S}(H(i,j))^{(i+1,j+1)}$, such that all the partitions satisfy the condition

$$\begin{cases} \alpha_{ij} \leq \alpha^{(1)}_{i-1j} & \text{if } i > 0 \\ \alpha_{ij} \leq \alpha^{(2)}_{ij-1} & \text{if } j > 0 \end{cases}$$

Example 3.4. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be the numerical function $H(i,j) = (i+1)(j+1)$. For any $i, j \in \mathbb{N}$, the only partition of sides $(i+1, j+1)$ of the integer $(i+1)(j+1)$ is $\alpha_{ij} := (\underbrace{i+1, i+1, \cdots, i+1}_{j+1})$. For any $i, j \in \mathbb{N}$, we have $\alpha^{(1)}_{i-1j} = \alpha_{ij}$ and $\alpha^{(2)}_{ij-1} = \alpha_{ij}$. Then the conditions in Definition 3.3 are satisfied and therefore H is a Ferrers function.

Remark 3.5. Note that if H is a Ferrers function then we have a bound on the growth of H since

$$H(i,j) \le \min\{H(i-1,j) + \lambda_1(\alpha_{i-1j}), H(i,j-1) + \lambda_2(\alpha_{ij-1})\}.$$

Proposition 3.6. *Let H be a Ferrers function, set $g_1 := H(i+1,j) - H(i,j)$ and $g_2 := H(i,j+1) - H(i,j)$. Then $(g_1, g_2) \le (\lambda_1, \lambda_2)$, for some $(\lambda_1, \lambda_2) \in \mathcal{L}(H(i,j))^{(i+1,j+1)}$.*

Proof. Since H is a Ferrers function then, from Remark 3.5, there exists $\alpha_{ij} \in \mathcal{S}(H(i,j))^{(i+1,j+1)}$, such that $(H(i+1,j), H(i,j+1)) \le (H(i,j)+\lambda_1(\alpha_{ij}), H(i,j)+\lambda_2(\alpha_{ij}))$. Therefore $(g_1, g_2) \le (\lambda_1(\alpha_{ij}), \lambda_2(\alpha_{ij}))$. □

Example 3.7. The numeric function

$$H := \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \dots \\ \hline 0 & 1 & 2 & 3 & 0 & \dots \\ 1 & 2 & 2 & 3 & 0 & \dots \\ 2 & 3 & 3 & 0 & 0 & \dots \\ 3 & 0 & 0 & 0 & 0 & \dots \end{array}$$

fails to be a Ferrers function since $\mathcal{S}(2)^{(2,2)} = \{(2,0), (1,1)\}$ and $(1,1) \notin \mathcal{L}(2)^{(2,2)} = \{(1,0), (0,1)\}$.

4. The Hilbert function of a bigraded algebra

In order to relate Ferrers functions to bigraded algebras we introduce a correspondence between partitions and sets of monomials.

Definition 4.1. We denote by $T(p,q)_{(a,b)} \in R_{(a,b)}$, where $(0,0) \le (p,q) \le (a+1, b+1) \in \mathbb{N}^2$, the monomial

$$T(p,q)_{(a,b)} := \begin{cases} x_1^{p-1} x_2^{a-p+1} y_1^{q-1} y_2^{b-q+1} & \text{if } (a,b), (p,q) \ge (1,1) \\ y_1^{q-1} y_2^{b-q+1} & \text{if } a = 0, b > 0, (p,q) \ge (1,1) \\ x_1^{p-1} x_2^{a-p+1} & \text{if } b = 0, a > 0, (p,q) \ge (1,1) \\ 1 & \text{if } a = b = 0, (p,q) = (1,1) \\ 0 & \text{if } p = 0 \text{ or } q = 0 \end{cases}$$

Given $\alpha := (p_1, \dots, p_t) \in \mathcal{S}(h)^{(a+1,b+1)}$, a partition of an integer h of sides $(a+1, b+1)$, we denote by $\mathcal{M}(\alpha) \subseteq R_{(a,b)}$ the set of monomials $T(p', q')_{(a,b)}$ where $(p', q') \le (p_i, i)$ for some $i = 1, \dots, t$.

Note that for any $(1,1) \le (p,q), (p',q') \le (a+1, b+1)$ then $(p,q) \ne (p',q')$ iff $T(p,q)_{(a,b)} \ne T(p',q')_{(a,b)}$.

Example 4.2. Let be $\alpha := (3,3,2,1,0) \in \mathcal{S}(9)^{(2,4)}$, then we can draw α as a Ferrers diagram (see [6], Definition 3.13) with rows and columns labeled

$$\alpha := \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \bullet & \bullet & \bullet & \bullet & \\ 1 & \bullet & \bullet & \bullet & & \\ 2 & \bullet & \bullet & & & \end{array}$$

Then $\mathcal{M}(\alpha) \subseteq R_{(2,4)}$ is the set of all the monomials $x_1^i x_2^{2-i} y_1^j y_2^{4-j}$ where (i,j) is a non-empty entry in the above diagram.

Remark 4.3. For any $\alpha := (p_1,\dots,p_t) \in \mathcal{S}(\cdot)^{(a+1,b+1)}$ the set of monomials of degree (a,b) not in $\mathcal{M}(\alpha)$ is a bilex set. Indeed, let $u := x_1^{a_1}x_2^{a_2} \in R_{(a,0)}$ and $v := y_1^{b_1}y_2^{b_2} \in R_{(0,b)}$ be monomials such that $uv \notin M(\alpha)$, i.e. $(a_1+1, b_1+1) \not\leq (p_i, i)$ for any $i = 1,\dots,t$. Given a monomial $u' := x_1^{c_1}x_2^{c_2} \in R_{(a,0)}$ with $u' > u$, then $c_1 > a_1$ and $(c_1+1, b_1+1) > (a_1+1, b_1+1)$. Therefore $T(c_1+1, b_1+1)_{(a,b)} = u'v \notin \mathcal{M}(\alpha)$.

One can also check that $M(\alpha)$ is a bilex set of monomials with respect to the order $x_2 > x_1 > y_2 > y_1$.

Definition 4.4. Let $I \subseteq R$ be a monomial bilex ideal and $(a,b) \in \mathbb{N}^2$. We denote by $\mathcal{M}_{ab}(I)$ the set of monomial of degree (a,b) not in $I_{(a,b)}$.

Moreover we denote by

$$p_i(\mathcal{M}_{ab}(I)) := \max\left(\{p' \in \mathbb{N} \mid T(p',i)_{(a,b)} \in \mathcal{M}_{(a,b)}(I)\} \cup \{0\}\right)$$

and by

$$\alpha_{\mathcal{M}_{ab}(I)} := \left(p_1(\mathcal{M}_{ab}(I)),\dots,p_{b+1}(\mathcal{M}_{ab}(I))\right).$$

Remark 4.5. From the definition of monomial bigraded ideal, it is immediate to check that $\alpha_{\mathcal{M}_{ab}(I)} \in \mathcal{S}(\cdot)^{(a+1,b+1)}$. Indeed

$$a+1 \geq p_1(\mathcal{M}_{ab}(I)) \geq p_2(\mathcal{M}_{ab}(I)) \geq \cdots \geq p_{b+1}(\mathcal{M}_{ab}(I)).$$

Example 4.6. Take the bilex ideal minimally generated only in degree $(2,3)$

$$I = (x_1x_2y_1^2y_2, x_1x_2y_1^3, x_1^2y_1^2y_2, x_1^2y_1^3).$$

Using the notation introduced in Definition 4.1 we write

$$I = \left(T(2,3)_{(2,3)}, T(2,4)_{(2,3)}, T(3,3)_{(2,3)}, T(3,4)_{(2,3)}\right).$$

Then the set $\mathcal{M}_{(2,3)}(I)$ introduced in Definition 4.4 is

$$\mathcal{M}_{(2,3)}(I) = \left\{\begin{array}{llll} T(1,1)_{(2,3)}, & T(1,2)_{(2,3)}, & T(1,3)_{(2,3)}, & T(1,4)_{(2,3)}, \\ T(2,1)_{(2,3)}, & T(2,2)_{(2,3)}, & & \\ T(3,1)_{(2,3)}, & T(3,2)_{(2,3)} & & \end{array}\right\}.$$

Thus we have $\alpha_{\mathcal{M}_{(2,3)}(I)} = (3,3,1,1) \in \mathcal{S}(\cdot)^{(3,4)}$.

The following result holds.

Lemma 4.7. *Let $\alpha_1, \alpha_2 \in \mathcal{S}(\cdot)^{(\ell_1,\ell_2)}$ be such that $\alpha_1 \leq \alpha_2$. Then $\mathcal{M}(\alpha_1) \subseteq \mathcal{M}(\alpha_2)$.*

Proof. It is trivial. □

Lemma 4.8. *Let L be a bilex set of monomials of degree (a,b) and let $I := (L) \subseteq R$ be the ideal generated by the elements in L. Then*

i) $\alpha_{\mathcal{M}_{ab}(I)} \in \mathcal{S}(H_{R/I}(a,b))^{(a+1,b+1)}$;
ii) $\alpha_{\mathcal{M}_{a+1b}(I)} = \alpha^{(1)}_{\mathcal{M}_{ab}(I)}$ *and* $|\mathcal{M}_{a+1b}(I)| = |\mathcal{M}_{ab}(I)| + \lambda_1(\alpha_{\mathcal{M}_{ab}(I)})$;
iii) $\alpha_{\mathcal{M}_{ab+1}(I)} = \alpha^{(2)}_{\mathcal{M}_{ab}(I)}$ *and* $|\mathcal{M}_{ab+1}(I)| = |\mathcal{M}_{ab}(I)| + \lambda_2(\alpha_{\mathcal{M}_{ab}(I)})$.

Proof. Item $i)$ follows from $\mathcal{M}_{ab}(I) \subseteq R_{(a,b)}$ and $H_{R/I}(a,b) = \dim_k R_{(a,b)} - \dim_k I_{(a,b)} = |\mathcal{M}_{ab}(I)|$. Let $\alpha_{\mathcal{M}_{ab}(I)} = (p_1,\dots,p_t) \in \mathcal{S}(\cdot)^{(a+1,b+1)}$ and $\alpha_{\mathcal{M}_{a+1b}(I)} = (p'_1,\dots,p'_t) \in \mathcal{S}(\cdot)^{(a+1,b+1)}$. Assume $i \in \{1,\dots,t\}$ such that $p'_i < a+2$, then $p'_i = \max\{q \mid T(q,i)_{(a+1,b)} \in \mathcal{M}_{a+1b}\} = \max\{q \mid T(q,i)_{(a+1,b)} \notin I_{(a+1,b)}\} \leq$

$\max\{q \mid T(q,i)_{(a,b)} \notin I_{(a,b)}\} = p_i$. Since I is only generated in degree (a,b), we get $p_i = p'_i$. If $p'_i = a+2$, then $T(a+2,i)_{(a+1,b)} \notin I_{(a+1,b)}$ and then $T(a+1,i)_{(a,b)} \notin I_{(a,b)}$. Analogously we get item $iii)$.

□

Example 4.9. Let $\alpha := (3,3,2,1,0) \in \mathcal{S}(\cdot)^{(2,4)}$, then $\alpha^{(1)} = (4,4,2,1,0) \in \mathcal{S}(\cdot)^{(3,4)}$ and $\alpha^{(2)} = (3,3,2,1,0,0) \in \mathcal{S}(\cdot)^{(2,5)}$. Using Ferrers diagrams as in Example 4.2 we have

$\alpha^{(1)} :=$

	0	1	2	3	4
0	•	•	•	•	
1	•	•	•		
2	•	•			
3	•	•			

$\alpha^{(2)} :=$

	0	1	2	3	4	5
0	•	•	•	•		
1	•	•	•			
2	•	•				

We are ready to prove the main result of this paper.

Theorem 4.10. *Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a numerical function. Then the following are equivalent*

1) *H is a Ferrers function;*
2) *$H = H_{R/I}$ for some bigraded ideal $I \subseteq R$.*

Proof. (1) → (2) Let H be a Ferrers function and let $\{\alpha_{ab}\} \in \mathcal{S}(H(a,b))^{(a+1,b+1)}$ be the set of partitions as required in Definition 3.3. For each $(a,b) \in \mathbb{N}^2$, let $I_{(a,b)}$ be the k-vector space generated by monomials of degree (a,b) not in $\mathcal{M}(\alpha_{ab})$. Then we claim that $I := \oplus_{(a,b)\in\mathbb{N}^2} I_{(a,b)}$ is an ideal of R. Note that, in order to prove the claim, it is enough to show that $(x_1,x_2,y_1,y_2)T \in I$ for any monomial $T \in I$. Thus, let $T(p,q)_{(a,b)} \in I_{(a,b)}$ be monomial of degree (a,b). Say $\alpha_{(a,b)} = (c_1,\dots,c_b)$, $\alpha_{(a+1,b)} = (c'_1,\dots,c'_b)$ and $\alpha_{(a,b+1)} = (c''_1,\dots,c''_{b+1})$. We collect the relevant facts

i) $c_q < p$ by $T(p,q)_{(a,b)} \notin \mathcal{M}(\alpha_{ab})$;
ii) $c'_q \le c_q$ and $c''_q \le c_q$. This follows from $\alpha_{a+1b} \le \alpha^{(1)}_{ab}$, $\alpha_{ab+1} \le \alpha^{(2)}_{ab}$ and *i)*.
iii) $x_1 \cdot T(p,q)_{(a,b)} = T(p+1,q)_{(a+1,b)}$ and $x_2 \cdot T(p,q)_{(a,b)} = T(p,q)_{(a+1,b)}$ by Definition 4.1;
iv) $y_1 \cdot T(p,q)_{(a,b)} = T(p,q+1)_{(a,b+1)}$ and $y_2 \cdot T(p,q)_{(a,b)} = T(p,q)_{(a,b+1)}$ by Definition 4.1.

Assume by contradiction $T(p+1,q)_{(a+1,b)} \notin I$, i.e., $T(p+1,q)_{(a+1,b)} \in \mathcal{M}(\alpha_{a+1,b})$. Thus, by Definition 4.1 we have $(c'_q,q) \ge (p+1,q)$, and then $c'_q \ge p+1 > c_q$. This contradicts *ii)*. Analogously, if $T(p,q)_{(a+1,b)} \in \mathcal{M}(\alpha_{a+1,b})$ then $c'_q \ge p > c_q$, contradicting *ii)*. In a similar way, by using the inequalities $c_q \ge c''_q \ge c''_{q+1}$, one can show that $T(p,q+1)_{(a,b+1)} \in I$ and $T(p,q)_{(a,b+1)} \in I$.

(2) → (1) Let $I \subseteq R$ be a bilex ideal such that $H_{R/I} = H$. Then, for any $(a,b) \in \mathbb{N}^2$, we set $\alpha_{ab} := \alpha_{\mathcal{M}_{ab}(I)}$. By item *i)* in Lemma 4.8, we have $\alpha_{ab} \in \mathcal{S}(H_{R/I}(a,b))^{(a+1,b+1)}$. Moreover, by items *ii)* and *iii)* in Lemma 4.8, the condition in Definition 3.3 holds since $(x_1,x_2)I_{(a-1,b)} \subseteq I_{(a,b)}$ and $(y_1,y_2)I_{(a,b-1)} \subseteq I_{(a,b)}$.

□

Remark 4.11. Lemma 4.8 gives the maximal growths for an Hilbert function H of a bigraded algebra R/I. A maximal growth for H in degree (a,b) is $(H(a,b)+\lambda_1, H(a,b)+\lambda_2)$ with (λ_1,λ_2) a maximal element in $\mathcal{L}(H(a,b))^{(a+1,b+1)}$. The set of maximal growths is not enough to describe the behavior of an

Hilbert function. E.g. let H be such that

$$H := \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 0 & \cdots \\ 1 & 2 & 4 & 6 & 8 & 10 & 0 & \cdots \\ 2 & 3 & 6 & 9 & 8 & 9 & 0 & \cdots \\ 3 & 4 & 8 & 8 & 10 & 0 & 0 & \cdots \\ 4 & 5 & 10 & 9 & 0 & 0 & 0 & \cdots \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

To ensure the maximal growth in degree $(2,3)$ and $(3,2)$ we have to take $\alpha_{32} = (4,2,2)$ and $\alpha_{23} = (3,3,1,1)$. Then $\lambda_1(\alpha_{23}) = \lambda_2(\alpha_{32}) = 2$ so $H(3,3) \leq \min\{H(2,3)+2, H(3,2)+2\} = 10$ but $\alpha_{23}^{(1)} \cap \alpha_{32}^{(2)} = (4,2,1,1)$ and then H fails to be a Ferrers function.

We end this paper with an application of Ferrers functions. Admissible functions were introduced in [5] in order to study the Hilbert functions of reduced 0-dimensional schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 4.12 ([5], Definition 2.2)**.** Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function and denote by $c_{ij} := \Delta H(i,j) = H(i,j) + H(i-1,j-1) - H(i-1,j) - H(i,j-1)$. Then we say that H is an admissible function if

(1) $c_{ij} \leq 1$, and $c_{ij} = 0$ for $i \gg 0, j \gg 0$;
(2) if $c_{ij} \leq 0$ then $c_{uv} \leq 0$, for all $(u,v) \geq (i,j)$;
(3) $0 \leq \sum_{t=0}^{j} c_{it} \leq \sum_{t=0}^{j} c_{i-1t}$ and $0 \leq \sum_{t=0}^{i} c_{tj} \leq \sum_{t=0}^{i} c_{tj-1}$.

Theorem 2.12 in [5] shows that the Hilbert function of a 0-dimensional scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ is an admissible function. However the converse fails to be true ([5] Example 2.14). Even if, in this paper, we do not worry about the geometrical point of view, it is still interesting to ask if an admissible function is a Ferrers function. Theorem 4.13 gives a positive answer to this question.

Theorem 4.13. *If H is an admissible function, then H is a Ferrers function.*

Proof. Let $(a,b) \in \mathbb{N}$ and $r \in \{0,\dots,b\}$, we set $p_{r+1}^{(ab)} := \sum_{i=0}^{a} c_{ir}$ and $\alpha_{ab} := (p_1^{(ab)}, \dots, p_{r+1}^{(ab)}, \dots p_{b+1}^{(ab)})$. From the definition of admissible function, we have $\alpha_{ab} \in \mathcal{S}(H(a,b))^{(a+1,b+1)}$ and also $\alpha_{a-1b}^{(1)} \geq \alpha_{ab}$ and $\alpha_{ab-1}^{(2)} \geq \alpha_{ab}$. □

Example 4.14. Example 2.14 in [5] shows an admissible function that fails to be the Hilbert function of a reduced set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

$$H := \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 5 & \cdots \\ 1 & 2 & 4 & 6 & 8 & 10 & 10 & \cdots \\ 2 & 3 & 6 & 8 & 9 & 10 & 10 & \cdots \\ 3 & 4 & 8 & 10 & 10 & 10 & 10 & \cdots \\ 4 & 5 & 10 & 10 & 10 & 10 & 10 & \cdots \\ 5 & 5 & 10 & 10 & 10 & 10 & 10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Meanwhile, as Theorem 4.13, H is a Ferrers function since one can write

$\alpha_{ij} :=$

	0	1	2	3	4	$\cdots$
0	(1)	$(1,1)$	$(1,1,1)$	$(1,1,1,1)$	$(1,1,1,1,1)$	$\cdots$
1	(2)	$(2,2)$	$(2,2,2)$	$(2,2,2,2)$	$(2,2,2,2,2)$	$\cdots$
2	(3)	$(3,3)$	$(3,3,2)$	$(3,3,2,1)$	$(3,3,2,1,1)$	$\cdots$
3	(4)	$(4,4)$	$(4,4,2)$	$(4,4,2,0)$	$(4,4,2,0,0)$	$\cdots$
4	(5)	$(5,5)$	$(5,5,0)$	$(5,5,0,0)$	$(5,5,0,0,0)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Thus, the associated ideal $I := (x_1^2y_1^2, x_1x_2y_1^3, x_2^3y_1^3, x_1^4y_2^2, x_1^4y_1y_2, x_1^5, y_1^5) \subseteq R$, has Hilbert function $H_{R/I} = H$.

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