Intersecting Architectural Surfaces Between Graphic and Analytic Representations

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Abstract. Representing an architectural shape, mediating design/formal/semantic needs, means respecting its specificity according to the purposes with which one operates; therefore, teaching how to represent an architectural shape is a complex operation, especially if this happens in the first year of the degree course in Architecture where the heterogeneity of students' background requires a preliminary definition of a common language. Students are firstly introduced to theoretical geometries which underlie architectural shapes. So, they have to know the basis of Geometry (both Descriptive and Analytical) in order to proceed within these issues. This process requires to underline the two 'souls' of architectural shapes: the theoretical and the build one. Moreover, it also leads to investigate two different types of theoretical shapes: the one that lies behind the design idea and the other one which underlies the built. We propose teaching examples focused on reading architectural shapes as a result of intersections of surfaces.

Keywords: Intersecting Surfaces, Descriptive Geometry, DGS, CAAD, Visual Thinking, Theoretical Form.

1 Introduction

One of the very first issue that students of Architecture have to face deals with analyzing built architectures in order to understand the geometrical properties of their volumes and to recognize their constituent geometries. If we think at this issue from an architectural point of view, the use of Geometry and its elements (such as points, lines, planes, then surfaces and solids) has its roots in the ancient past, and there might be a connection between the idea of mathematical mental models [1] and the architectural design process, when we think of architectural artefacts as results of the creative application of Geometry and its basic elements. On the other hand, applications of Architecture are, somehow, materialization of «abstract mathematical statements» [2]. These statements could then act as mental models for the first architectural composition/shapes recognition exercises, as important architects from the past, such as Leon Battista Alberti, Francesco di Giorgio Martini, Andrea Palladio, Vincenzo Scamozzi and Guarino Guarini, suggest in their treatises [3]. So, nowadays, students of Architecture must acquire an interdisciplinary knowledge to study the built form, both contemporary and historical, while developing basic skills for its analysis [4]. In fact, the knowledge that 'flows' into an architectural design is not only related to its specific disciplines but includes a variety of methodologies and interpretations derived from other areas. In this contribution, we focus on the formalization of complex architectural structures, in particular of roofing systems. The relationship between Representation and precision is normally linked to 'measurement' [5]. We treat it from the point of view of rigor in the geometric sequence for graphic tracing of surfaces intersections and in parallel with an analytical description [6].

We focus on particular architectural/geometric realities generated by intersecting surfaces and on the analysis of a variety of approaches and tools for representing them. We present, through some examples, an integrative teaching process, aimed for the first year Bachelor students of Architecture, experimented for years in courses of Architectural Drawing and Survey Laboratory (ADSLab from here on). Our approach links tools of shape visualization, from Descriptive Geometry to the first steps of 3D modelling, with basis of Analytical Geometry and its formalisms to show these mathematical topics as necessary tools in an architect's professional toolbox. We want our students to face both, so we provide them with a set of interdisciplinary tools, in order to develop this kind of critical reading of Architecture, by enhancing their spatial visualization abilities in recognizing 3D geometry and by promoting critical shape-reading activities to foster spatial prefiguration, in the sense of Leopold [7] and Nagy-Kondor [8].

2 Academic Context

University newcomers hardly recognize usefulness and applicability of abstract Mathematics proposed in the course of Calculus; on the other hand, it is often left to students' own initiative to make connections between disciplines: for example, the relations between geometric theory and its graphical representation. In view of the use of highly specialized digital methods, with which students will be trained in more advanced courses, we try to make them aware of the mathematical representation underlying graphic constructions and we invite them to use a language that makes use of parametric equations. It is a matter of analytically describing and intersecting geometric entities, in a concatenation of steps, as they learned in the first semester Calculus course for the case of planes and straight lines in space. In this way, students can gradually acquire a sense of the objects on which they will operate.

Following Sfard's theory [9], [10], we see the definition of a mathematical object as the recursive tree of its manifold visual realizations, knowing that to understand it, it is not enough to be able to write an equation, but it is necessary to re-create the equation from a network of inter-related realizations.

There are possible learning difficulties of epistemological nature related to the recognition of geometric objects in space, which involve understanding of concepts, symbols, procedures and different types of representations. Even only from a mathematical point of view, indeed, a geometric object can be described by Cartesian equations, or by parametric equations, or even considering it as a set of points that verify the same property; therefore, it is important to provide students with adequate flexibility to switch from one register to another, in the sense of Duval [11]. With this regard, we chose to use Computer-aided architectural design software (CAAD, such as AutoCAD) and Dynamical Geometric Software (DGS, such as GeoGebra) because our students are at their very first year of academic studies, thus they are not comfortable with highly specialized software. In other words, we want them to use simple tools they already know, or they get easy management within their first weeks in academic path. The same is for the choice of objects to be analyzed: our students starts university without specialist knowledge of architectural issues, thus we need them to experiment with simple tasks, avoiding difficulties in spatial prefiguration which in general is not developed during high school studies, at least in our Italian context. Here an interdisciplinary approach, based on the shared language of Geometry, becomes useful to set up the critical approach that characterizes architect's professional figure. We propose the use of a DGS to provide a suitable environment where students can freely manipulate mathematical objects, and where they can learn how the dependent objects will be affected, by investigating mathematical relations dynamically and through their multiple representations. In fact, DGS is a tool which today is part of the educational path of secondary schools in mathematical disciplines, whose interdisciplinary value in support of the architect's training has not yet been studied. Moreover, there are nexuses with the program of the Calculus course taught in parallel to ADSLab, where the use of the DGS has a strong communicative impact (e.g. about analytical representation of planes and straight lines in the Euclidean space). Also, a DGS can be used by teachers to introduce the foundations of parametric culture applied to Architecture through small exercises. For more complex geometries, first year students do not have sufficient mathematical notions, however they can benefit from seeing it in use, as a training tool for spatial prefiguration supported by analytical descriptions, to understand what is at the basis of this elaboration process, just as children start to use new adult words before they fully understand the meaning of the words, see Vygotsky [12]. Summing up, the possibility of seeing theoretical geometry described also through a DGS type software allows to integrate languages, also intercepting users less predisposed to spatial prefiguration and more anchored to analytical representation. As for the teaching of Representation, students have heterogeneous training and come from secondary schools that do not always contemplate drawing among their subjects. So, it is necessary to start from the concept of projection by introducing it right from the definition of projective source and projecting rays, defining projection as intersection of the projecting ray with the reference surface. In the same way, we remember the need to recognize the characterizing and descriptive elements of a surface so that one can represented it completely and unambiguously and can therefore operate on it to identify points or sections. It is interesting to compare the analytical description, its representation with a DGS software, the representation in orthographic projection and a view of the 3D model created with CAAD (Fig. 1). Each tool favors aspects of the constituent geometry through its own specialized language: their interpolation gives a wider picture of the problem and reveals the importance of

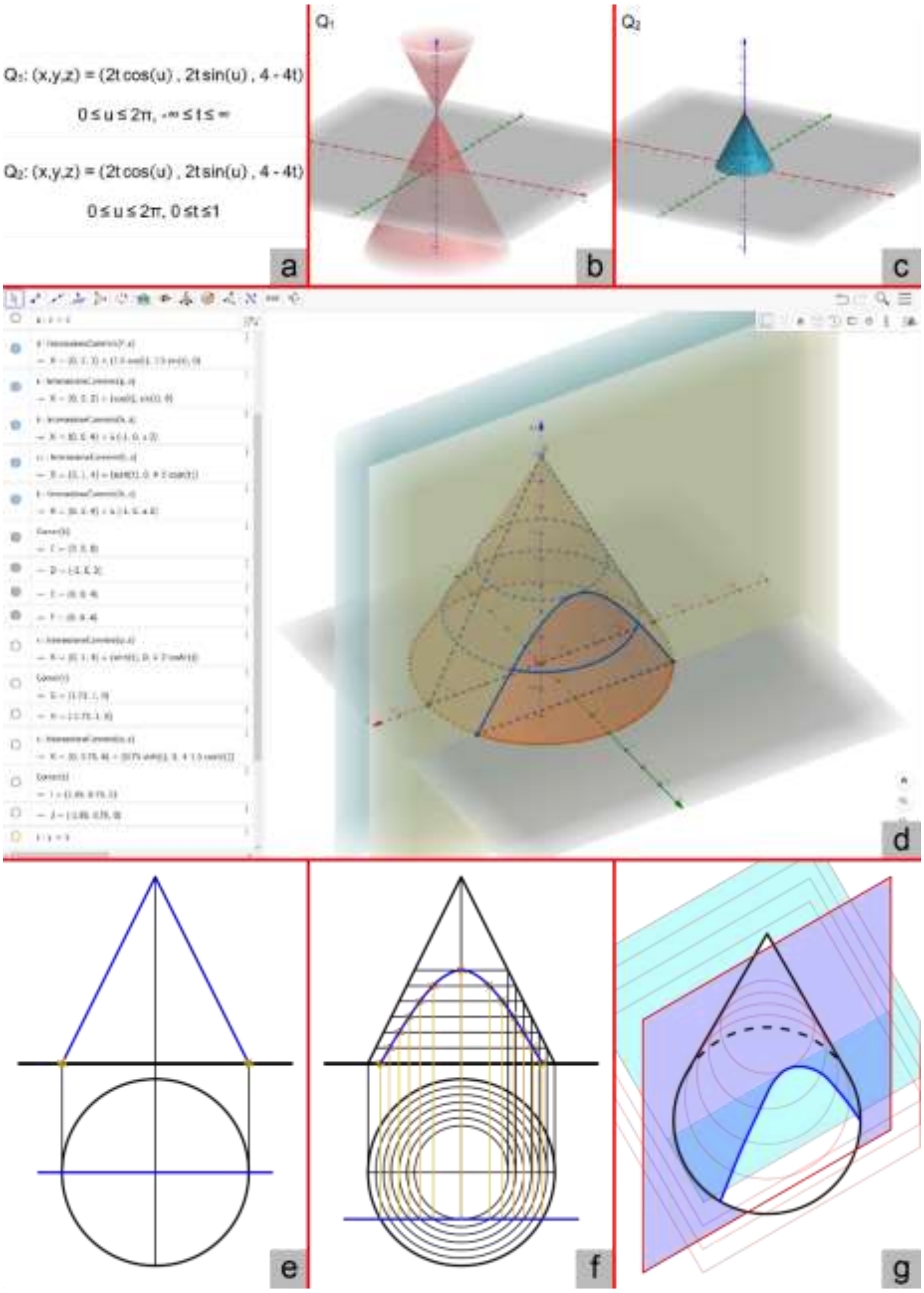

Fig. 1. Cone descriptions and sections. a) analytical representations. b-d) DGS: b) Q1; c) Q2; d) independent visualizations of horizontal sectioning planes identifying circumferences and vertical sectioning plane identifying a hyperbola. e-g) graphic representations: e) vertical sectioning plane trough the axis; f-g) vertical sectioning plane identifying a hyperbola defined by the use of horizontal sectioning planes.

investigating images with a critical approach to avoid the risk of not grasping their specificities and limits.

This theoretical introduction allows a formalization of the ADSLab teaching process and of the consequent operating sequence shared in the classroom in the teacher/student relationship, between tools used for teaching and skills acquired by students at the end of the course. Spatial representation can be faced, graphically, applying foundation topics of Descriptive Geometry such as belonging conditions, parallelism and orthogonality of simple elements in succession, in a ‘crescendo’ of complexity that can be managed through a structuring process: this experience is supported by effective interdisciplinary clarifications to show useful connections to architectural practice.

It is fundamental to distinguish between what are the purposes of the ADSLab, that is, making students capable of using certain tools autonomously and instead what are the means used to make this happen. In the context of this contribution (see examples) we will therefore analyze a use of representation tools, such as correlated projection planes and digital/physical modeling, and simple elements which are introduced from the beginning of the course through textual, graphic and plastic descriptions with the use of physical models that can be reproduced independently by the students. We work in a bidirectional path that leads from the real object to its synthetic representation and from this again to the real object, becoming aware of the fact that any representation is an interpretation and result of reasoned selection of the elements deemed as significant [14].

3 Methodology

The solution of many representative geometric questions can be traced back to the same operating mode used to introduce the concept of projection, that is, the determination of the straight line/surface intersection with the use of a chosen auxiliary plane that contains the straight line under analysis and that, between the infinite possible, sections in the most ‘simple’ and therefore also ‘rigorous’ way the surface in question; this process makes use of the geometric properties already at the basis of the selective critical process in the projective phase and therefore can be considered the basis for dealing with the most complex intersections between surfaces.

The more elementary problem is the search for the straight line/plane intersection. The solution follows from the introduction of a plane that contains the straight line and that cuts the plane in question along a straight line belonging to the same plane as the first, so the consequent intersection gives the intersection of two coplanar lines graphically and in a rigorous way. Note that it is not always possible to obtain an equally rigorous result by relating other entities.

To analyze the intersection between surfaces in the graphical way, one has to operate on auxiliary planes which section both surfaces, by identifying a section on each; therefore, the coplanar sections offer a ‘picture’ of what is happening on that plane and indicate whether there are points of contact between the surfaces, identified by the intersections of the sections. These points must then be identified on all the projection planes as belonging to the auxiliary section plane as well as to the surfaces under analysis.

Therefore, their position becomes rigorous only if the identified sections are rigorously graphically and geometrically traced, all other intersections are affected by graphic approximation as any curve constructed by points. To obtain the intersections, the points found must be connected taking into account the surface they belong to and the operating methods are those that necessarily influence the outcome of the study. Finally, we emphasize that the choice of how many and which sectioning planes are useful for the intersection construction depends on the geometries being analyzed, on their mutual position and on the precision required by the exercise.

4 Discussion

ADSLab is structured through lectures (normally in the classroom, but today converted in on-line lessons due to the COVID-19 related emergency), always followed by moments of guided application and further possibilities for individual exercise, case studies, where multiple approaches converge and test the ability to build relationships between knowledge, and moments of complementary activity, such as targeted contributions from external teachers and/or laboratory activities [14].

Within the course, examples are presented both abstract and applied to the built, in order to highlight the need for a critical approach also in the study of a theoretical intersection.

4.1 Example 1

The representation of a plane becomes an opportunity to verify languages and tools integration. The concept is inherently complex, because the plane is infinite, but it is always drawn to have limited size: only portions of it can be represented.

Via orthographic projections, the way to describe the plane is through its traces, i.e. intersections with the reference system.

From an analytical point of view, the plane is represented by an equation (Fig. 2). Points, lines and planes allow to represent volumes of buildings, vertical distribution elements and pitched roofing systems: going from the representation of a plane to that of a volume of a simple building or a small house, means recognizing a portion of a straight line in each edge and a portion of a plane in each wall.

4.2 Example 2

We analyze the study of shadows of a small tea filter house shaded by a light placed at infinite distance. Drawing its shadow on the reference planes lies in making pass planes parallel to the ray of light through the segments of the shadow dividing line and constructing the intersection between those planes and the surface which receives the shadow (Fig. 3). Also, students are asked for both a physical paper modeling and a digital 3DS modeling.

The preparation of a model requires that properties and geometric characteristics of the shapes to be described, remain unchanged; otherwise, the results lose the geometric code that allows the construction [15] (Fig. 4). It is a matter of analytically describing and intersecting planes and straight lines in space, as they learned in the Calculus Course, in a concatenation of steps. Moreover, we propose the use of GeoGebra to make students aware of analytical geometrical language and its potentialities (see Fig. 4a).

In view of the use of highly specialized digital methods, we invite them to use a language that makes use of parametric equations, to describe what happens for example when moving the light source (Fig. 3a).

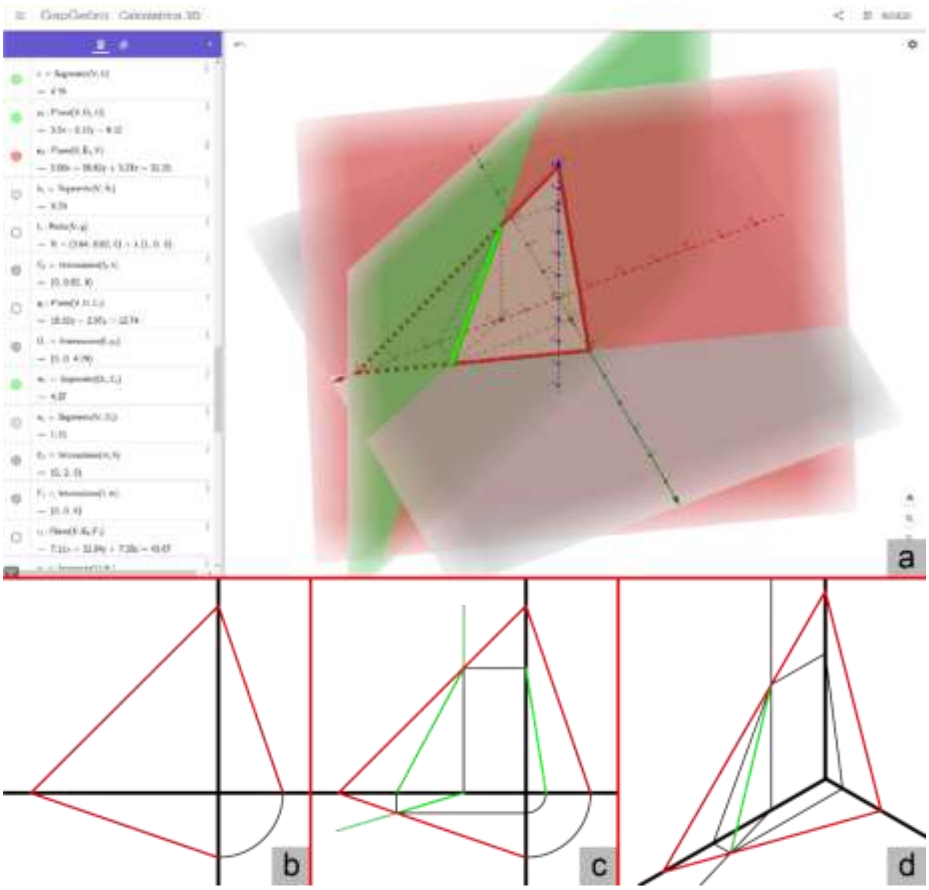

Fig. 2. Generic plane and intersection between planes. a) DGS; b-c) orthographic views; d) isometric axonometry view.

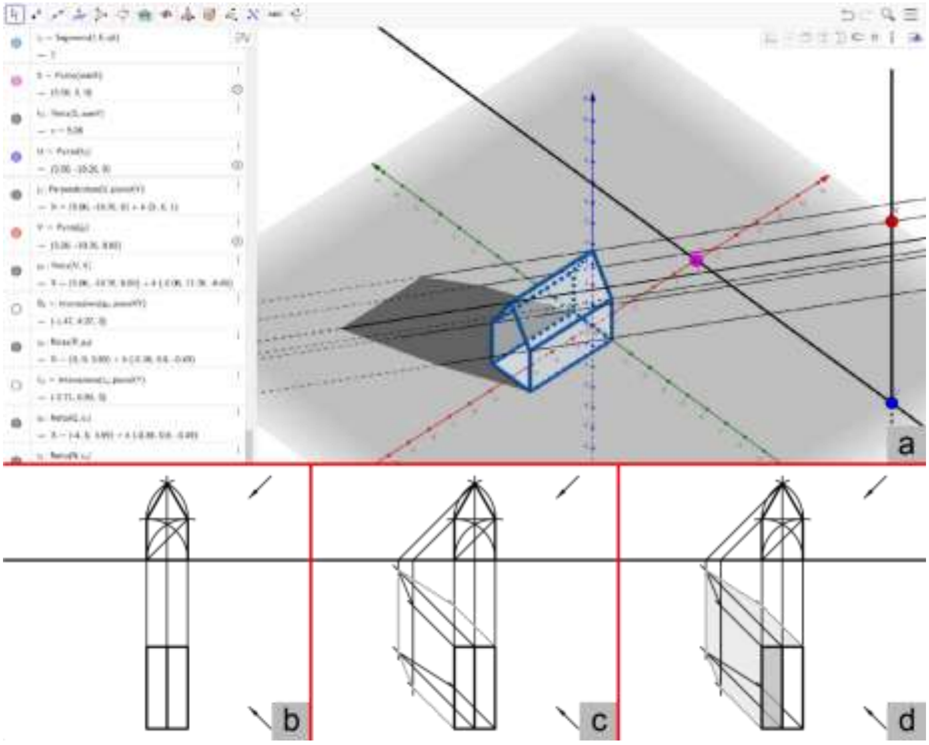

Fig. 3. Study of the tea filter house (light source at infinite distance). a) DGS; parametrizing the direction of the rays allows to verify the variety of possible solutions of the shadow brought on the reference planes (an analogous construction is required for its shade); b-d) orthographic projections, shade and shadows.

4.3 Example 3

In order to analyze a groin vault with our students, we consider intersections of ruled surfaces such as cylinders [14] or cones: for our purpose, we use intersecting circular surfaces with coplanar and orthogonal axes (Fig. 5).

In the case of two cylinders (Fig. 5a, d, g), the graphic way, based on planes parallel to the translation axis of one of the two surfaces (and orthogonal to the other), gives a sequence of rigorously defined points. Similarly, the use of planes parallel to both axes also offers geometrically rigorous intersections (Fig. 5g). The result obtained by joining them in an arbitrary manner is therefore subject to errors directly related to the distribution and number of planes. In the same hypotheses, we consider intersection of a conical and a cylindrical surface; if we choose sectioning planes parallel to the base of the cone and therefore parallel to the cylinder translation axis (Fig. 5b), we observe that each plane identifies on both surfaces geometrically rigorous sections (a circumference and a straight line), whose intersection gives geometrically identified points; however, the result obtained by joining them, in CAAD as sequence of segments or polyline or spline, similarly to the previous case, is subject to errors according to number and

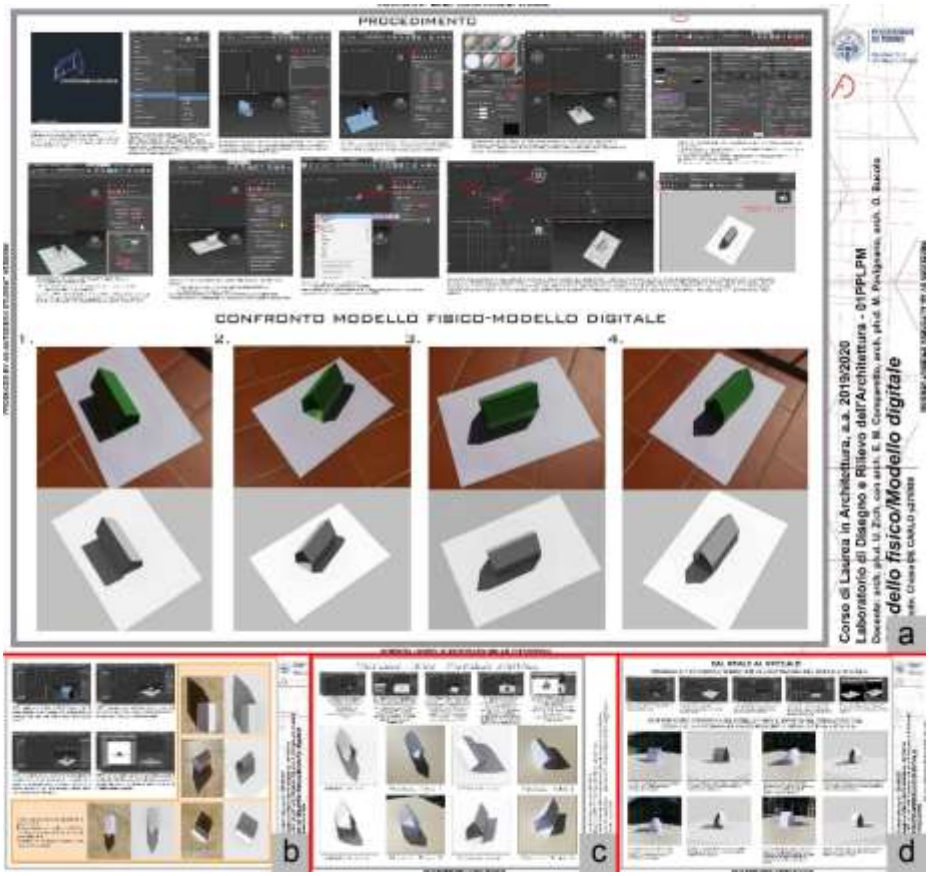

Fig. 4. ADSLab, students' exercises for the comparison between physical and digital models of the tea filter house shade/shadows/light source/point of view: a) Chiara De Carlo; b) Elisa Cusumano; c) Filippo Crovella; d) Jacopo Di Franco.

distribution of planes. Any other choice of planes would induce on the cone traced sections, bringing approximation errors, that would invalidate the intersection outcome.

About intersecting cones (Fig. 6), as outcome of a critical process of data discretization, we compare graphic solutions, obtained via Descriptive Geometry by CAAD – in Fig. 7 the hypotheses of similar circular cones with orthogonal and coplanar axes imply that the plane section is an ellipse, increasing the possible ways of its tracing – and their different levels of approximation with a mathematical formalization (obtained via Analytical Geometry) and the use of a DGS, reasoning about subjectivity and objectivity of the respective representations.

Here students do not have sufficient mathematical tools to give an analytical description of mathematical objects that come into play; moreover, the same DGS is not able to provide intersection curves between cones with a simple command, however it proves to be a useful spatial prefiguration tool of analytical descriptions, when it is forced to show those curves with an input of their parametric equations (Fig. 6b).

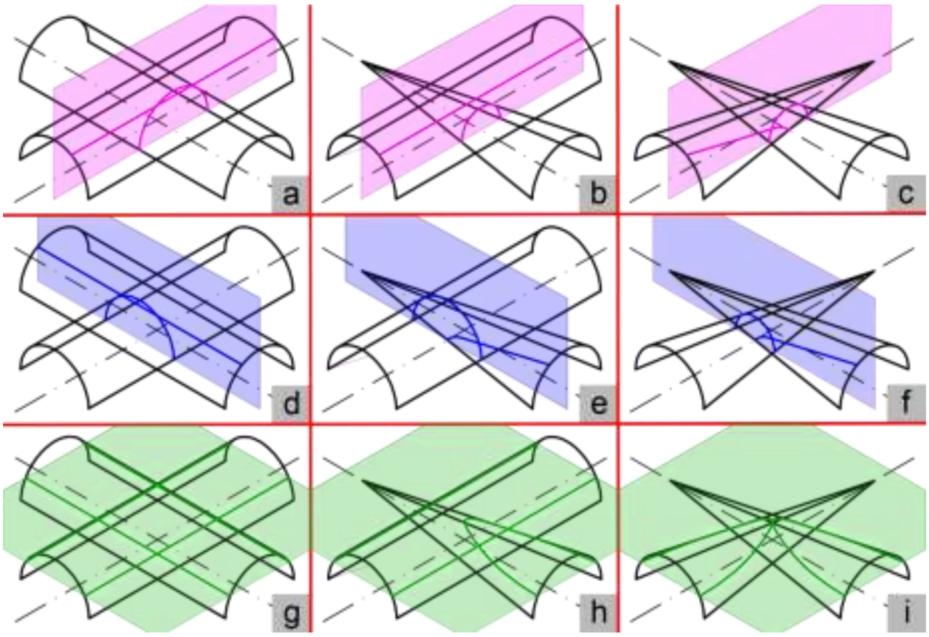

Fig. 5. Studying intersections between ruled surfaces: evaluation of the position and distribution of useful sectioning planes. a/d/g) cylinder/cylinder; b/e/h) cylinder/cone; c/f/i) cone/cone. a-f) sectioning planes parallel to an axis and perpendicular the other one; g-i) sectioning planes parallel to both axes.

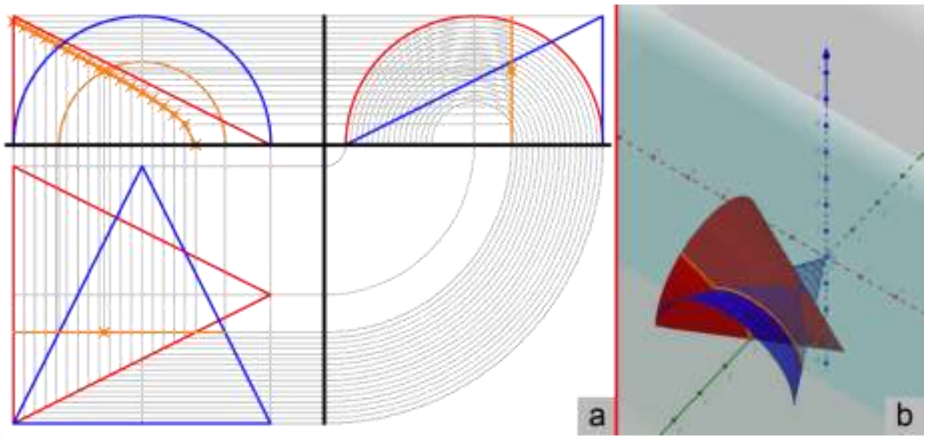

Fig. 6. Using an auxiliary plane to study intersection of surfaces. a) orthographic projection; b) DGS 3D view of the intersection curves.

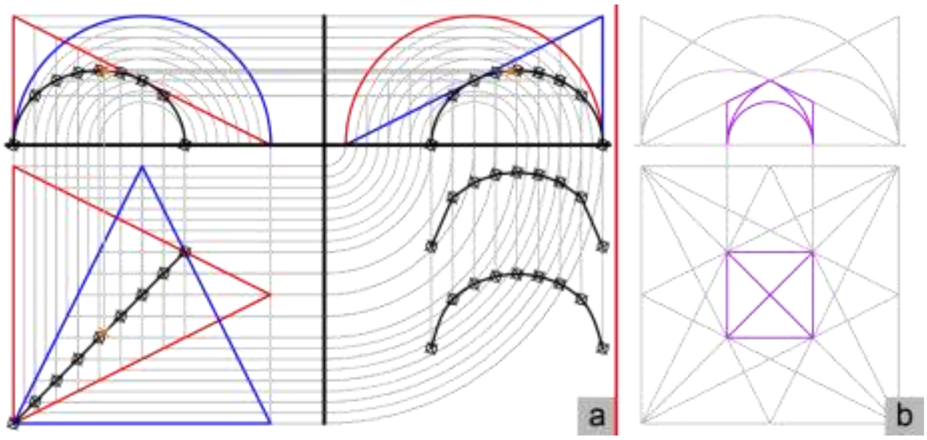

Fig. 7. a) Graphic intersection between two cones and critical evaluation of the union of the points. b) mutual position of four cones as the geometry constituting a groin vault.

5 Conclusions and possible outlooks

We treated the relationship between Representation and precision from the point of view of rigor in the geometric sequence for graphic analysis of surfaces intersections. This was done in comparison with the analytical description provided using a DGS tool. Following the narrative thread of our experiences, we observe a degree of increasing complexity in the study of intersections between surfaces, directly connected with the highlighted limits of their representation. When using the parameter of geometric rigor as a discriminant to critically read the results, we can conclude that the graphic way allows to rigorously represent only plane/straight line and plane/plane intersections. In some cases of intersection between ruled surfaces, the particular arrangement between the parts and the choice of appropriate section planes allow to rigorously solve the search for the intersection points, but do not allow to obtain their union in an equally objective way, while the last cone/cone case explicitly shows how, although they belong to the ruled surfaces family, the study of their intersection in a graphic way is always affected by errors due to the tracing of section plane curves identified on each surface in order to build intersections.

Our experience has shown that, even if the analytical description of these intersections is absolutely rigorous, their DGS visualization requires a direct user's intervention, otherwise intersection curves are not expressly visible. Graphic language reveals its semantic efficacy if contextualized by Discipline and proves to be an interesting subject of discussion for the formation of critical ability in the use of basic CAAD and DGS tools, especially if integrated with the mathematical discourse of analytic geometry and its formalism to solve space problems, in the sense of Sfard [10]. Our idea is to broaden the interdisciplinarity between Mathematics and Drawing in the first year of the bachelor's degree in Architecture, in order to provide students with skills that will be a must requirements in the future applications of parametric design, by fostering their consciousness of the geometric 'soul' of architectural shapes. We want to continue with our project, by creating new interactions between Mathematics and Drawing for the study of more complex kind of surfaces and their intersections.

Acknowledgements For M. Pavignano and U. Zich the contribution was developed within the research project M.A.G.I.A. 2019. C. Cumino is member of the National Group for Algebraic and Geometric Structures (GNSAGA-INDAM).

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