ON THE ARITHMETICALLY COHEN-MACAULAY PROPERTY FOR SETS OF POINTS IN MULTIPROJECTIVE SPACES

GIUSEPPE FAVACCHIO, ELENA GUARDO, AND JUAN MIGLIORE

Abstract. We study the arithmetically Cohen-Macaulay (ACM) property for finite sets of points in multiprojective spaces, especially $(\mathbb{P}^1)^n$. A combinatorial characterization, the $(\star)$-property, is known in $\mathbb{P}^1 \times \mathbb{P}^1$. We propose a combinatorial property, $(\star_s)$ with $2 \le s \le n$, that directly generalizes the $(\star)$-property to $(\mathbb{P}^1)^n$ for larger n. We show that X is ACM if and only if it satisfies the $(\star_n)$-property. The main tool for several of our results is an extension to the multiprojective setting of certain liaison methods in projective space.

1. Introduction

A motivating problem in algebraic geometry and commutative algebra concerns multiprojective spaces. Given a finite collection of points $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ it is interesting to describe the homological invariants of the quotient ring of X. An important property is whether the collection is arithmetically Cohen-Macaulay (ACM) or not, i.e. whether the quotient ring is a Cohen-Macaulay ring. It is no longer the case (as it is in projective space) that a finite set of points is automatically ACM. It is of interest to understand which finite sets of points are ACM.

A characterization of finite sets with the ACM property is only known in $\mathbb{P}^1 \times \mathbb{P}^1$. We know several classifications of ACM sets of reduced and fat points in $\mathbb{P}^1 \times \mathbb{P}^1$, in terms of the Hilbert function, separators, and combinatorial properties (for example see [1, 3, 4, 5, 6, 7, 8, 9]). Unfortunately, Examples 3.4, 4.10 in [6] and Examples 3.4, 3.12, 5.10 in [7] show that these characterizations cannot be generalized to other ambient spaces such as $\mathbb{P}^n \times \mathbb{P}^m$ or $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$, but there remains the hope that they can be generalized to $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^n$.

Our focus in this paper is to better understand ACM sets of points in $(\mathbb{P}^1)^n$ by extending some standard tools in the homogeneous setting to the multihomogeneous setting. These include *basic double G-linkage, liaison addition, and liaison.* Although our focus in this paper is not on Hilbert functions, we do give multigraded Hilbert function formulas for these generalized constructions.

For a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$, it is known (cf. for instance [10] Theorem 4.11) that X is ACM if and only if it satisfies the so-called $(\star)$-property (see page 5 of this paper for the definition). We first give a new proof of this result using liaison theory (Corollary 2.9). In $\mathbb{P}^1 \times \mathbb{P}^1$, the $(\star)$-property is equivalent to the *inclusion property* (Definition 2.5).

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However, we use liaison addition to show in Example 2.12 that the inclusion property does not characterize ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (or $(\mathbb{P}^1)^n$ for larger n).

On the other hand, we show in Proposition 2.7 that the inclusion property does imply the ACM property in $(\mathbb{P}^1)^n$. If $X \subset (\mathbb{P}^1)^n$ is an ACM set of points and π_i is any projection to a copy of $(\mathbb{P}^1)^{n-1}$ then X decomposes in a natural way to a disjoint union of level subsets (see Definition 2.5). We show in Theorem 3.2 that these level subsets are all ACM, as are their complements.

We introduce for sets of points in $(\mathbb{P}^1)^n$ the $(\star_s)$-property for $2 \leq s \leq n$ (Definition 3.6), a generalization of the $(\star)$-property, and we show in Theorem 3.16 that for $s = n$ this characterizes the ACM property.

In [11] the authors began the study of the Hilbert function of *any* finite set of points, X, in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We hope that the characterization in this paper will help in the future classification of Hilbert functions of ACM finite sets of points in $(\mathbb{P}^1)^n$.

2. Some constructions and a new proof

We work over a field of characteristic zero.

Definition 2.1. For $V = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ we define

$$\pi_i : V \to \mathbb{P}^{a_1} \times \cdots \times \widehat{\mathbb{P}^{a_i}} \times \cdots \times \mathbb{P}^{a_n}$$

to be the projection omitting the i-th component and

$$\eta_i : V \to \mathbb{P}^{a_i}$$

to be the projection to the i-th component.

Let $\underline{e}_1, \ldots, \underline{e}_n$ be the standard basis of $\mathbb{N}^n$. Let $x_{i,j}$, with $1 \leq i \leq n$ and $0 \leq j \leq a_i$ for all i, j, be the variables for the different $\mathbb{P}^{a_i}$. Let

$$R = K[x_{1,0}, \ldots, x_{1,a_1}, \ldots, x_{n,0}, \ldots, x_{n,a_n}],$$

where the degree of $x_{i,j}$ is $\underline{e}_i$.

A subscheme X of V is defined by a saturated ideal, I_X, generated by a system of multihomogeneous polynomials in R in the obvious way. We say that X is *arithmetically Cohen-Macaulay (ACM)* if R/I_X is a Cohen-Macaulay ring.

Let $N = a_1 + \cdots + a_n + n$. Given a subscheme X of V together with its homogeneous ideal I_X, we can also consider the subscheme $\bar{X}$ of $\mathbb{P}^{N-1}$ defined by I_X. Notice that if X is a zero-dimensional subscheme of V, I_X almost never defines a zero-dimensional subscheme of $\mathbb{P}^{N-1}$. For example, if $n = 2, a_1 = a_2 = 1$, then a finite subset, X, of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a finite union of lines, $\bar{X}$, in $\mathbb{P}^3$ (of a certain type). The subscheme $X \subset V$ is ACM if and only if the subscheme $\bar{X} \subset \mathbb{P}^{N-1}$ is ACM.

The following construction is a special case of so-called *Basic Double G-Linkage* – cf. [16] Lemma 3.4 for a more general version, or [10] Theorem 4.9 for what is used here.

Proposition 2.2 ([16] Corollary 3.5)**.** *Let $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{P}^n$ be ACM of the same dimension ≥ 1. Let $H_1, \ldots, H_r$ be hypersurfaces, defined by forms $F_1, \ldots, F_r$, such that for each i, H_i contains no component of V_j for any $j \leq i$. Let $W_0 \subset V_1$ be an ACM subscheme, and for each $i \geq 1$ let W_i be the ACM scheme defined by the corresponding* *hypersurface sections: $I_{W_i} = I_{V_i} + (F_i)$. Let Z be the sum of the W_i, viewed as divisors on V_r. Then*

(i) Z is ACM.

(ii) As ideals we have

$$I_Z = I_{V_r} + F_r \cdot I_{V_{r-1}} + F_r F_{r-1} I_{V_{r-2}} + \cdots + F_r F_{r-1} \cdots F_2 I_{V_1} + F_r F_{r-1} \cdots F_1 I_{W_0}.$$

(iii) Let $d_i = \deg F_i$. The Hilbert functions are related by the formula

$$\begin{aligned} h_Z(t) &= h_{W_r}(t) + h_{W_{r-1}}(t-d_r) + h_{W_{r-2}}(t-d_r-d_{r-1}) + \dots \\ &\quad + h_{W_1}(t-d_r-d_{r-1}-\cdots-d_2) + h_{W_0}(t-d_r-d_{r-1}-\cdots-d_1). \end{aligned}$$

A multihomogeneous version of the above proposition, taking W_0 to be empty (since for the purposes of this paper it is not needed), is given in the next proposition. Recall that the ideal of a zero-dimensional subscheme of the multiprojective space defines a subscheme of higher dimension in projective space.

Proposition 2.3. *Let $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ be ACM of the same dimension ≥ 1. Fix $h \in \{1, \dots, n\}$. Let $H_1, \dots, H_r$ be hypersurfaces, defined by multihomogeneous forms $F_1, \dots, F_r$ with degree* $\deg(F_i) = D_i \underline{e}_h := \underline{d_i} \in \mathbb{N}^n$, *for some $D_i \in \mathbb{N}$, such that for each i, H_i contains no component of V_j for any $j \leq i$. Let W_i be the ACM schemes defined by the corresponding hypersurface sections: $I_{W_i} = I_{V_i} + (F_i)$. Let Z be the sum of the W_i, viewed as divisors on V_r. Then*

(i) Z is arithmetically Cohen-Macaulay.

(ii) As ideals we have

$$I_Z = I_{V_r} + F_r \cdot I_{V_{r-1}} + F_r F_{r-1} I_{V_{r-2}} + \cdots + F_r F_{r-1} \cdots F_2 I_{V_1} + (F_r F_{r-1} \cdots F_1).$$

(iii) The Hilbert functions are related by the formula

$$\begin{aligned} h_Z(\underline{t}) &= h_{W_r}(\underline{t}) + h_{W_{r-1}}(\underline{t} - \underline{d_r}) + h_{W_{r-2}}(\underline{t} - \underline{d_r} - \underline{d_{r-1}}) + \dots \\ &\quad + h_{W_1}(\underline{t} - \underline{d_r} - \underline{d_{r-1}} - \cdots - \underline{d_2}). \end{aligned}$$

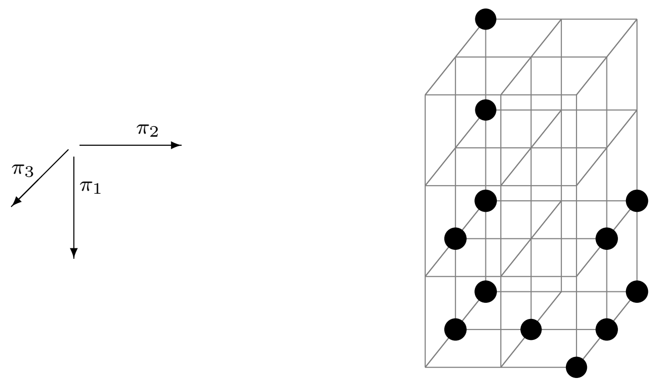
Remark 2.4. As the name suggests, basic double G-linkage actually does something stronger: it preserves the even Gorenstein liaison class (originally [15], but see [16] Lemma 3.4 (iv)). This means that in Proposition 2.2 something stronger is actually true: without initially assuming that W_0 is ACM we have that Z is ACM if and only if W_0 is, since the ACM property is preserved under liaison. This latter is a standard fact whose roots go back at least to Gaeta in the 40s and 50s, to Hartshorne in the 60s, to Rao in the 70s, and to Schenzel [18] in 1982 (the first time Gorenstein liaison was considered rather than only complete intersection liaison), and is based on the fact that the Hartshorne-Rao modules are invariant up to shifts and duals in the Gorenstein liaison class, and are all zero if and only if the scheme is ACM. These facts are collected in [14] Lemma 1.2.3 and Theorem 5.3.1.

This has the following consequence, which we will use in Corollary 2.9. We follow the notation of Proposition 2.2. Let $W_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set. Let V_1 be a union of hyperplanes of multidegree $(0,1)$ containing W_0 and let F_1 be a hyperplane of multidegree $(1,0)$ not containing any point of X. Let W_1 be the complete intersection of V_1 and F_1, and let $Z = W_0 \cup W_1$. Then Z is ACM if and only if W_0 is ACM.

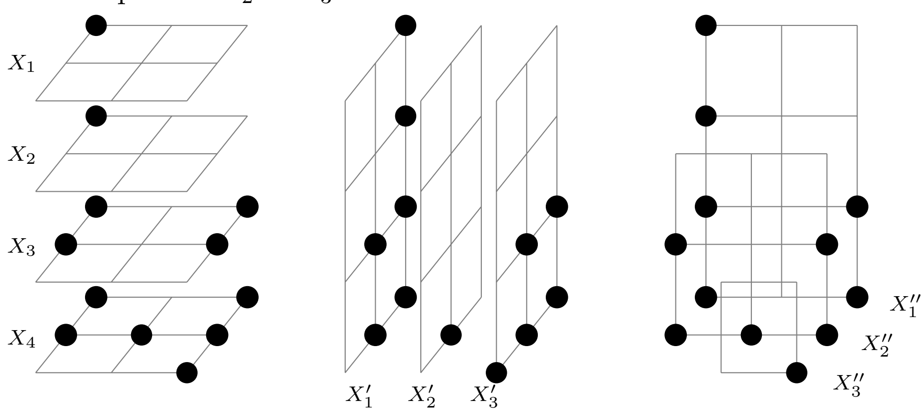
Definition 2.5. Let $X \subset (\mathbb{P}^1)^n$ be a finite, reduced subscheme and fix a value of i, $1 \le i \le n$. Let $\{[k_1, \ell_1], \dots, [k_t, \ell_t]\} = \eta_i(X)$. For $1 \le j \le t$, let $\mathbb{H}_j$ be the hyperplane defined by $\ell_j x_{i,0} - k_j x_{i,1}$ and let $X_j = X \cap \mathbb{H}_j$. We call the X_j the *i-level sets* of X. We say that X has the *inclusion property with respect to* π_i if the subsets $\pi_i(X_j)$ of $(\mathbb{P}^1)^{n-1}$, for $1 \le j \le t$, admit a total ordering by inclusion and are all ACM.

Note that the i-level sets are a natural stratification of the points of X obtained by taking all points with prescribed i-th coordinate.

Example 2.6. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the following set of points.



The next pictures show the decompositions of X as unions of 1-level sets, 2-level sets and 3-level sets respectively. Note that X has the inclusion property with respect to π_1 but not with respect to π_2 or π_3.



Indeed, we have $\pi_1(X_4) \supseteq \pi_1(X_3) \supseteq \pi_1(X_2) \supseteq \pi_1(X_1)$, but no such chain of inclusions holds for $\pi_2(X'_1), \pi_2(X'_2)$ and $\pi_2(X'_3)$, or for $\pi_3(X''_1), \pi_3(X''_2)$ and $\pi_3(X''_3)$.

Proposition 2.7. *Let $X \subset (\mathbb{P}^1)^n$ be a finite set. Assume that for some $1 \le i \le n$, X has the inclusion property with respect to π_i. Then X is ACM.*

Proof. Recall that the inclusion property includes the assumption that the i-level sets are all ACM. We first note that if W is a finite subset of $(\mathbb{P}^1)^{n-1}$ then $\pi_i^{-1}(W)$ is a finite union of lines (copies of the i-th $\mathbb{P}^1$) in V sitting over W. We have that the finite set W is ACM if and only if the curve $\pi_i^{-1}(W)$ is ACM, since they are defined by the same equations. Furthermore, in the notation of Definition 2.5, $X_j = \pi_i^{-1}(\pi_i(X_j)) \cap \mathbb{H}_j$.

Notice that

$$X = [(\pi_i^{-1}(\pi_i(X_1)) \cap \mathbb{H}_1] \cup \cdots \cup [(\pi_i^{-1}(\pi_i(X_t)) \cap \mathbb{H}_t].$$

The inclusion property then implies an analogous inclusion property for the curves $\pi_i^{-1}(\pi_i(X_j))$. Since all these curves are ACM, the result follows from Proposition 2.2. □

Remark 2.8. Note that for $\mathbb{P}^1 \times \mathbb{P}^1$, X satisfies the inclusion property with respect to π_i (for $i = 1, 2$) if and only if it satisfies the so-called ($\star$)-*property*, namely that even after re-indexing, X contains no subset of type (a) in Remark 3.10 below (cf. [10] Definition 3.19), where it is understood that the intersection points that are non-bullets do not lie in X. In Definition 3.6 we will extend the ($\star$)-property to higher dimension.

The following is part of the known classification of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (again see [10] Theorem 4.11). We now give a short new proof of this result.

Corollary 2.9. *Let $V = \mathbb{P}^1 \times \mathbb{P}^1$. Let $X \subset V$ be a finite set of points. Then X is ACM if and only if X satisfies the inclusion property with respect to either π_1 or π_2.*

Proof. Notice that in $\mathbb{P}^1$ all finite subsets are ACM, so i-level sets in $\mathbb{P}^1 \times \mathbb{P}^1$ are automatically ACM. Then the fact that if X satisfies the inclusion property with respect to one of the projections then X is ACM follows immediately from Proposition 2.7.

We now prove the converse. Suppose that X is ACM but does not satisfy the inclusion property with respect to either projection. Without loss of generality, we may assume that there is no value j such that

$$|\pi_1^{-1}(\pi_1(X)) \cap \mathbb{H}_j| = |X_j| \quad \text{or} \quad |\pi_2^{-1}(\pi_2(X)) \cap \mathbb{H}_j| = |X_j|.$$

In other words, thinking of X as a subset of the intersection points of a grid of "horizontal" lines and "vertical" lines, we may assume that no row or column contains the maximum possible number of points of X. Indeed, if this were the case then we can remove such a row or column of points, and what remains is still ACM thanks to Remark 2.4.

We now consider X as a union of lines in $\mathbb{P}^3$. The "vertical" lines of our grid correspond to a union of planes in $\mathbb{P}^3$ containing X, as do the "horizontal" lines, and these unions have no plane in common. Hence they provide a geometric link of X to some union of lines Y in $\mathbb{P}^3$, which is again ACM by standard results in liaison theory (cf. [14]). Notice that the product of the minimal number of "vertical" lines in the grid containing X is a minimal generator of I_X, as is the product of the minimal number of "horizontal" lines in the grid containing X (Theorem 1.2 in [3]).

The key observation is that if we view Y as part of our grid, it is simply the points of the grid that do not belong to X. By our observation, the minimal set of "vertical" and "horizontal" lines containing Y are identical to those containing X. If we link back using the same complete intersection, we re-obtain X. But now in both links we have used minimal generators for the ideal. Since X has codimension two (a crucial ingredient!), if X were ACM, a sequence of two links using minimal generators in both cases would result in a set of points whose number of minimal generators is two less than that of X. (This is due primarily to Apery and Gaeta; see for instance [14] Theorem 6.1.3, applied twice.) Since we instead have exactly X again, X is not ACM so we have our contradiction. □

It is natural to wonder if the analogue of Corollary 2.9 holds for a larger product of $\mathbb{P}^1$'s. We now show that this is not the case. For simplicity we will give our example for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, but the same idea and construction works for any number. Of course one direction of a supposed analogue of Corollary 2.9 is given by Proposition 2.7, so we need to exhibit an ACM set of points not satisfying the inclusion property with respect to any of the three projections. We will use the following result, which generalizes an unpublished result of P. Schwartau, and is a multihomogeneous version of [2] Corollary 1.6, Theorem 1.3 and Corollary 1.5.

Theorem 2.10. *Let $V_1, \dots, V_r$ be subschemes of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$, with $2 \le r \le n$. Assume that V_i are all equidimensional of codimension r. Choose multihomogeneous polynomials $F_1, \dots, F_r$ with $\underline{d_i} = \deg F_i = D_i \cdot \underline{e_i} \in \mathbb{N}^n$ so that*

$$F_i \in \bigcap_{\substack{1 \le j \le r \\ j \ne i}} I_{V_j}.$$

and $(F_1, \dots, F_r)$ is a regular sequence. Let V be the complete intersection scheme defined by $(F_1, \dots, F_r)$. Let $I = F_1 I_{V_1} + \cdots + F_r I_{V_r}$ and let Z be the scheme defined by I. Then

(i) As sets, $Z = V_1 \cup \cdots \cup V_r \cup V$.
(ii) I is a saturated ideal.
(iii) If $h_X(\underline{t})$ denotes the Hilbert function of a scheme X then we have

$$h_Z(\underline{t}) = h_V(\underline{t}) + h_{V_1}(\underline{t} - \underline{d_1}) + \cdots + h_{V_r}(\underline{t} - \underline{d_r}).$$

(iv) Z is ACM if and only if $V_1, \dots, V_r$ are all ACM.

Remark 2.11. If $\Delta h_X(\underline{t})$ denotes the first difference of the Hilbert function of a scheme X (see [7], Definition 2.8) then we note that item (iii) is equivalent to

$$\Delta h_Z(\underline{t}) = \Delta h_V(\underline{t}) + \Delta h_{V_1}(\underline{t} - \underline{d_1}) + \cdots + \Delta h_{V_r}(\underline{t} - \underline{d_r}). \tag{2.1}$$

With this construction we now show that the analogue of Corollary 2.9 does not hold for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.12. In $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ consider the points

$$V_1 = ([1,1],[1,1],[1,1]) \ , \ V_2 = ([2,1],[2,1],[2,1]) \ , \ V_3 = ([3,1],[3,1],[3,1]).$$

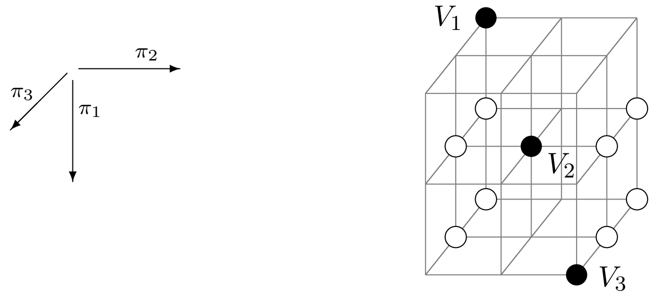
Note that $I_{V_i} = (x_{1,0} - i x_{1,1}, x_{2,0} - i x_{2,1}, x_{3,0} - i x_{3,1})$, and all three are ACM. Let

$$\begin{array}{rcl} F_1 & = & (x_{1,0} - 2x_{1,1})(x_{1,0} - 3x_{1,1}) \\ F_2 & = & (x_{2,0} - \ \ x_{2,1})(x_{2,0} - 3x_{2,1}) \\ F_3 & = & (x_{3,0} - \ \ x_{3,1})(x_{3,0} - 2x_{3,1}) \end{array}$$

Let

$$I = F_1 I_{V_1} + F_2 I_{V_2} + F_3 I_{V_3}.$$

Theorem 2.10 shows that I is the saturated ideal of the union of 11 points, namely V_1, V_2, V_3 and the 8 points of intersection of F_1, F_3 and F_3, and that X is ACM. One checks, however, that X fails to have the inclusion property with respect to any direction.



The same idea can be used to construct an example for a product of any number of copies of $\mathbb{P}^1$.

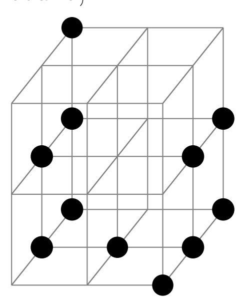
Remark 2.13. Using formula (2.1) we can write the first difference of the Hilbert function of X in Example 2.12. We get

$$\Delta h_X(0,j,k) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & 0 & \cdots \end{array} \qquad \Delta h_X(1,j,k) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

$\Delta h_X(2,0,0) = 1$ and $\Delta h_X(i,j,k) = 0$ otherwise. One can check that there is no hyperplane containing 6 points, but $\sum_{j,k} \Delta h_X(0,j,k) = 6$. So we cannot generalize Theorem 3.1 in [11] i.e. we are not able to count the number of points on a plane directly from the Hilbert function even in the ACM case.

Having the preceding example, it becomes of great interest to find a characterization of those sets of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and, more generally, in $(\mathbb{P}^1)^n$ that are ACM. We will do this in the next section.

Remark 2.14. It is not hard to show that if we move the point $V_2 = ([2,1],[2,1],[2,1])$ to the bottom plane in the above picture,



the Hilbert function remains the same but now X does have the inclusion property. This shows that from the Hilbert function one cannot determine whether or not an ACM set of points has the inclusion property.

3. ACM sets of points in $(\mathbb{P}^1)^n$

Let $R = K[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, \ldots, x_{n,0}, x_{n,1}]$ be the coordinate ring for $(\mathbb{P}^1)^n$, which we shall also view as the coordinate ring for $\mathbb{P}^{2n-1}$. Let $X \subset (\mathbb{P}^1)^n$ be a finite set of points.

Since I_X defines both a set of points in $(\mathbb{P}^1)^n$ and a union of linear varieties in $\mathbb{P}^{2n-1}$, we will abuse notation and denote by X also the subvariety of $\mathbb{P}^{2n-1}$ defined by this ideal. As above, we can view the i-level sets of X with respect to any direction, and we often refer to them simply as level sets.

Notation 3.1. Let X be as above. We write $X = X_1 \cup X_2 \cup \cdots \cup X_r$ to represent the decomposition of X into level sets with respect to some direction; without loss of generality we assume it is the first. Viewed in $\mathbb{P}^{2n-1}$, any given X_i is the intersection of a hyperplane defined by a linear form ℓ_i in the variables $x_{1,0}$ and $x_{1,1}$ and a variety $\overline{X}_i$ in $\mathbb{P}^{2n-1}$ defined by the variables $x_{2,0}, x_{2,1}, \dots, x_{n,0}, x_{n,1}$. $\overline{X}_i$ is ACM in $\mathbb{P}^{2n-1}$ if and only if X_i is ACM in the i-th copy of $(\mathbb{P}^1)^{n-1}$, which in turn is equivalent to X_i being ACM as a subscheme of $\mathbb{P}^{2n-1}$ (with ideal $(\ell_i) + I_{\overline{X}_i}$). We denote by $I_{\overline{X}_i}$ the corresponding ideal in R. For convenience we will denote by $A_{1,i}$ the hyperplanes in the variables $x_{1,0}$ and $x_{1,1}$ containing at least one point of X, by $A_{2,i}$ the hyperplanes in the variables $x_{2,0}$ and $x_{2,1}$ containing at least one point of X, etc. We will abuse notation and use the same notation for the corresponding linear forms.

Theorem 3.2. *Let $X \subset (\mathbb{P}^1)^n$ be a finite set. Choose any of the n projections; without loss of generality assume it is π_1. Let $X_1, \dots, X_r$ be the level sets with respect to this projection. If X is ACM then for each i, both X_i and $X \backslash X_i = X_1 \cup \cdots \cup \widehat{X_i} \cup \cdots \cup X_r$ are ACM.*

Proof. Notice that the first assertion follows immediately from the second, by removing level sets one at a time.

We now prove that $X \backslash X_i$ is ACM. We have n families of linear forms, namely the $A_{1,i}$ that are linear combinations of $x_{1,0}$ and $x_{1,1}$, the $A_{2,i}$ that are linear combinations of $x_{2,0}$ and $x_{2,1}$, etc. We replace each of these linear forms by a new variable. Supposing that

$$\begin{gathered} |\{A_{1,i} \mid A_{1,i} \cap X \neq \emptyset\}| = r_1, \\ |\{A_{2,i} \mid A_{2,i} \cap X \neq \emptyset\}| = r_2, \\ \vdots \\ |\{A_{n,i} \mid A_{n,i} \cap X \neq \emptyset\}| = r_n, \end{gathered}$$

let us call the new variables $a_{1,1}, \dots, a_{1,r_1}, a_{2,1}, \dots, a_{2,r_2}, \dots, a_{n,1}, \dots, a_{n,r_n}$. Let S be the polynomial ring in these $r_1 + \cdots + r_n$ variables. We form the monomial ideal in S given by the intersection of ideals of the form $(a_{1,i}, a_{2,j}, \dots, a_{n,k})$ corresponding to the components of X. This intersection defines a height n monomial ideal, $J \subset S$.

Consider J as an ideal, say $\overline{J}$, in the ring $T = S[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, \dots, x_{n,0}, x_{n,1}]$, where S is defined in the previous paragraph. Being a cone, $\overline{J}$ continues to be a height n monomial ideal. Consider the linear forms $a_{1,i_1} - A_{1,i_1},\ a_{2,i_2} - A_{2,i_2}, \dots, a_{n,i_n} - A_{n,i_n}$, where $1 \leq i_1 \leq r_1, \dots, 1 \leq i_n \leq r_n$. Let L be the ideal generated by all these linear forms. We have that

$$R/I_X \cong T/(\overline{J}, L),$$

the former of which is ACM. Since the ideals I_X of R and $\overline{J}$ of T both have height n, we can view the addition of each linear form in L as a proper hyperplane section, giving that $T/\overline{J}$ is also Cohen-Macaulay.

Now let x be any of the $r_1 + r_2 + \cdots + r_n$ variables of S (viewed in T). Corollary 3.2(a) of [12] (see also Theorem 1.5 in [13] and Proposition 1.2 in [17]) shows that the following inequality holds for the projective dimension

$$\mathrm{pd}(T/(\overline{J}, x)) \leq \mathrm{pd}(T/\overline{J}) + 1.$$

From the exact sequence

$$0 \to T/(\overline{J} : x)(-1) \to T/\overline{J} \to T/(\overline{J}, x) \to 0$$

it then follows, by the Depth Lemma (see [19] Lemma 1.3.9 or [20] Lemma 3.1.4), that $T/(\overline{J} : x)$ is also Cohen-Macaulay. Then again passing to the hyperplane sections, we see that $X \backslash X_i$ is ACM. □

The following corollary is immediate.

Corollary 3.3. *If $X \subset (\mathbb{P}^1)^n$ is ACM then the union of any number of level sets of X in any given direction is ACM.*

Corollary 3.4. *If $X \subset (\mathbb{P}^1)^n$ is a finite ACM set of points then its multihomogeneous ideal (hence also its homogeneous ideal) is minimally generated by products of linear forms of type $A_{i,j}$.*

Proof. This follows from the argument in Theorem 3.2, since the monomial ideal passes to an ideal generated by products of linear forms, and the ACM property means that the Betti diagram (in particular the minimal generators) is preserved under proper hyperplane sections. □

Remark 3.5. Corollary 3.4 is not true without the ACM assumption. Indeed, the ideal of three general points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is easily seen to have minimal generators of degree $(1,1,0), (1,0,1), (0,1,1)$ that are not products of linear forms.

Let $P, Q \in (\mathbb{P}^1)^n$. We denote by $Y_{P,Q}$ a height n multihomogeneous complete intersection of least degree containing P and Q, with each minimal generator being a product of at most two hyperplanes in the same family $\{A_{j,1}, \ldots, A_{j,r_j}\}$ (see Notation 3.1 and Theorem 3.2 for the notation).

Definition 3.6. Let X be a finite set of points in $(\mathbb{P}^1)^n$ and s be an integer such that $2 \leq s \leq n$. Then X has the $(\star_s)$ *property* if, for any integer s', such that $2 \leq s' \leq s$, there do not exist two points $P, Q \in (\mathbb{P}^1)^n$ with either of the following properties:

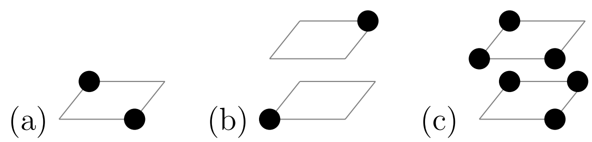
- (i) $P, Q \in X$ such that the ideal defining $Y_{P,Q}$ has exactly s' minimal generators of degree 2 and $X \cap Y_{P,Q} = \{P, Q\}$;
- (ii) $P, Q \notin X$ such that the ideal defining $Y_{P,Q}$ has exactly s' minimal generators of degree 2 and $Y_{P,Q} \cap X = Y_{P,Q} \setminus \{P, Q\}$.

Remark 3.7. If P, Q have the property that $Y_{P,Q}$ has only one minimal generator of degree 2, this does not violate the $(\star_s)$ property for any s because of the condition that $2 \leq s'$. Thus a set X with the $(\star_s)$-property may have two such points.

Example 3.8. Let $X := \{Q_{112}, Q_{121}, Q_{122}, Q_{211}, Q_{212}, Q_{221}\}$ be the set of 6 points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where $Q_{ijk} := \{[i:1], [j:1], [k:1]\}$. One can check that X has the $(\star_2)$ property and it does not have the $(\star_3)$ property. Indeed, for instance the smallest complete intersection containing Q_{112}, Q_{121} is defined by (A_1, B_1B_2, C_1C_2), which contains a third point of X, where for the convenience of the reader we have denoted $A_i := x_{10} - ix_{11}$, $B_j := x_{20} - jx_{21}$ and $C_k := x_{30} - kx_{31}$. However, X fails the $(\star_3)$ property. Indeed, the smallest complete intersection Y containing the points $Q_{111}, Q_{222} \notin X$ is defined by (A_1A_2, B_1B_2, C_1C_2), and $Y \cap X = Y \setminus \{Q_{111}, Q_{222}\}$ (in this case this is actually equal to X).

Remark 3.9. It is natural for us, considering the ambient space we are studying, to define the $(\star_s)$ property by using a geometric interpretation. Moreover, we will describe this property from a combinatorial point of view in Lemma 3.14 and Lemma 3.15.

Remark 3.10. For points in $\mathbb{P}^1 \times \mathbb{P}^1$, the $(\star_2)$-property coincides with the $(\star)$-property of [10] Definition 3.19 (see also Remark 2.8 above). For points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we can rewrite Definition 3.6 as follows. Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X has the $(\star_3)$-*property* if there is no complete intersection $Y_{P,Q}$ whose intersection with X (after possibly reindexing) has any of the following three forms:



If $X \subset (\mathbb{P}^1)^n$ satisfies the $(\star_n)$-property then (for example) in particular the behavior of type (a), (b) or (c) does not occur.

Corollary 3.11. *Let $X \subset (\mathbb{P}^1)^n$ be a finite set and assume that X is ACM. Then X satisfies the $(\star_n)$-property.*

Proof. We will view X as lying in $\mathbb{P}^{2n-1}$. We first note that it is enough to prove that X contains no subset of type (i) in Definition 3.6. Indeed, the fact that X does not contain a subset of the form (ii) follows from (i) by liaison. To see this, consider a complete intersection containing X, of the form $(\prod A_{1,i}, \prod A_{2,i}, \dots, \prod A_{n,i})$. It links X to a union of $(n-1)$-planes X' (still viewed in $\mathbb{P}^{2n-1}$), and X contains a subset of the form (i) if and only if X' contains one of the form (ii). But by standard facts in liaison theory (cf. [14]), X is ACM if and only if X' is ACM, so we are done.

Thus it remains only to prove that if X is ACM then it does not contain a subset of the form given in (i). Suppose to the contrary that such a subset does occur in X. Then we can selectively remove level sets with respect to different projections, until we remain only with the *non-degenerate* set $\{P, Q\}$ in a suitable copy of $(\mathbb{P}^1)^s$ for some $s \le n$. By a repeated application of Theorem 3.2, we obtain the assertion that $\{P, Q\}$ is ACM in $(\mathbb{P}^1)^s$. But this is clearly impossible since the ideals of P and of Q do not share any linear forms, so viewed as subschemes of $\mathbb{P}^{2s-1}$, P and Q are disjoint. Contradiction. □

Proposition 3.12. *Let $X \subset (P^1)^n$ be a finite set. Choose any of the n projections; without loss of generality assume that it is π_1. Let $Y_1 = X_1$ be any level set (after possibly re-indexing) with respect to this projection and let $Y_2 = X_2 \cup \cdots \cup X_r$ be the union of the remaining level sets, with $r \ge 3$. Then X is ACM if and only if the following conditions hold.*

(a) Both Y_1 and Y_2 are ACM.

(b) $I_{Y_1} + I_{Y_2}$ *is the saturated ideal of a dimension* $(n-2)$ *union of linear spaces.*
(c) *The scheme defined by* $I_{Y_1} + I_{Y_2}$ *is ACM of dimension* $(n-2)$.

Proof. In both directions we will use the exact sequence

$$0 \to I_{Y_1} \cap I_{Y_2} \to I_{Y_1} \oplus I_{Y_2} \to I_{Y_1} + I_{Y_2} \to 0. \tag{3.1}$$

Observe that $I_X = I_{Y_1} \cap I_{Y_2}$. Note that two components of X, which are all $(n-1)$-planes in $\mathbb{P}^{2n-1}$, meet in dimension $(n-2)$ if and only if $(n-1)$ of their n defining multihomogeneous hyperplanes coincide. They meet in lower dimension if and only if fewer than $(n-1)$ of their defining multihomogeneous hyperplanes coincide.

Assume first that (a), (b) and (c) hold. Then the ACM property for X comes immediately from a consideration of cohomology of the long exact sequence coming from the sheafification of (3.1).

Now assume that X is ACM. Part (a) is Theorem 3.2. Part (b) is immediate from the cohomology sequence, since X is ACM so its first cohomology is zero. Again considering the long exact sequence in cohomology coming from the sheafification of (3.1), the fact that X is ACM and both Y_1 and Y_2 are ACM of dimension $(n-1)$ immediately gives that the $(n-2)$-dimensional scheme defined by $I_{Y_1} + I_{Y_2}$ is ACM, giving (c). □

Remark 3.13. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set (a priori not necessarily ACM). Choose any of the 3 projections; without loss of generality assume it is π_1. Let $X_1, \dots, X_r$ be the level sets with respect to this projection. If X has the $(\star_3)$-property (so in particular X does not contain any set in configuration (a) in Remark 3.10) then, for each j, X_j is ACM thanks to Corollary 2.9 and Remark 2.8.

Our next goal is to show that $X \subseteq (P^1)^n$ is ACM if and only if it has the $(\star_n)$-property. We first make a small modification of the notation introduced at the beginning of this section. Given $\underline{u} \in \mathbb{N}^n$, we denote by $P_{\underline{u}}$ the point whose ideal is generated by

$$(A_{1u_1}, A_{2,u_2}, \dots, A_{n,u_n}).$$

Define $d(\underline{v}, \underline{w}) := |\{i \mid v_i \neq w_i\}|$.

Lemma 3.14. *Let* $X \subset (\mathbb{P}^1)^n$ *be a finite set with the* $(\star_s)$*-property, for some* $2 \leq s \leq n$. *Moreover, suppose that* $\underline{v}, \underline{w} \in \mathbb{N}^n$ *are such that* $d(\underline{v}, \underline{w}) = r \leq s$ *and* $P_{\underline{v}}, P_{\underline{w}} \in X$. *Then there exist* $\underline{u}_0, \dots \underline{u}_r \in \mathbb{N}^n$ *such that*

- $\underline{u}_0 = \underline{v}$, $\underline{u}_r = \underline{w}$;
- $P_{\underline{u}_i} \in X$ *for* $i = 1, \dots, r$;
- $d(\underline{u}_i, \underline{u}_{i-1}) = 1$ *for* $i = 1, \dots, r$.

Proof. Note that the $(\star_s)$-property implies the $(\star_i)$-property for $i \leq s$ by definition. We proceed by induction on r. If $r = 1$ the result is trivial, and for $r = 2$ the result follows immediately from the $(\star_2)$-property. Now take $\underline{v}, \underline{w} \in \mathbb{N}^n$ such that $d(\underline{v}, \underline{w}) = r > 2$ and $P_{\underline{v}}, P_{\underline{w}} \in X$. Since X has the $(\star_s)$-property, there exists $P_{\underline{u}} \in X$ such that $u_j \in \{v_j, w_j\}$, for each component of $\underline{u}$. Then apply the inductive hypothesis on the vectors $\underline{v}, \underline{u}$ and $\underline{u}, \underline{w}$. □

Lemma 3.15. *Let* $X \subset (\mathbb{P}^1)^n$ *be a finite set with the* $(\star_s)$*-property. Moreover, suppose that* $\underline{v}, \underline{w} \in \mathbb{N}^n$ *are such that* $d(\underline{v}, \underline{w}) = r \leq s$, $v_1 \neq w_1$, *and* $P_{\underline{v}}, P_{\underline{w}} \in X$. *Then there exist* $\underline{a}, \underline{b} \in \mathbb{N}^n$ *such that*

- $P_{\underline{a}}, P_{\underline{b}} \in X$,
- $a_1 \neq b_1$,
- $d(\underline{a}, \underline{b}) = 1$
- $a_i, b_i \in \{v_i, w_i\}$ *for* $i = 1 \dots n$.

Proof. It follows from Lemma 3.14. □

Theorem 3.16. *Let $X \subset (\mathbb{P}^1)^n$ be a finite set. Then X has the $(\star_n)$-property if and only if X is ACM.*

Proof. If X is ACM then it was shown in Corollary 3.8 that X satisfies the $(\star_n)$-property, so we only have to prove the converse. We proceed by simultaneous induction on n and on t, the number of level sets with respect to some projection (say π_1). We have already shown the case $n = 2$, so we can assume $n \geq 3$.

If t is equal to 1, the result follows from the inductive hypothesis on n. Let $t > 1$ and $X = X_1 \cup X_2 \cup \cdots \cup X_t$. Let $Y_1 = X_1$ be any level set (after possibly re-indexing) with respect to this projection and let $Y_2 = X_2 \cup \cdots \cup X_t$ be the union of the remaining level sets. Given $\underline{v} \in \mathbb{N}^{n-1}$, we denote by $L_{\underline{v}}$ the line in $(\mathbb{P}^1)^n$ through $P_{(1,\underline{v})}$ whose ideal is generated by $(A_{2v_1}, A_{3,v_3}, \dots, A_{n,v_{n-1}})$ for some $A_{i,j}$.

We assume $A_{1,1} \in R_{\underline{e}_1}$ is the linear form defining the hyperplane containing Y_1. We denote by $\hat{Y}_1$ the set of lines $L_{\underline{v}}$ passing through a point of Y_1, i.e. $\hat{Y}_1 = \pi_1^{-1}(\pi_1(X_1))$. (Viewed in $\mathbb{P}^{2n-1}$, $\hat{Y}_1$ is a union of codimension $n-1$ linear spaces.)

By induction on n and on t, we know that Y_1 and Y_2 are ACM. In particular, this means that $\hat{Y}_1$ is also ACM. Hence we have an equality of saturated ideals $I_{Y_1} = (A_{1,1}) + I_{\hat{Y}_1}$. Then it follows from the following exact sequence

$$0 \to I_{Y_1} \cap I_{Y_2} \to I_{Y_1} \oplus I_{Y_2} \to (A_{1,1}) + I_{\hat{Y}_1} + I_{Y_2} \to 0$$

that it is enough to show that $I_{\hat{Y}_1} + I_{Y_2}$ is an ACM ideal (clearly of height n) and $A_{1,1}$ is a regular form in $R/(I_{\hat{Y}_1} + I_{Y_2})$. We proceed by steps.

(σ_1) *We show that $\hat{Y}_1 \cap Y_2$ is an ACM set of points.*

By the inductive hypothesis on t, it suffices to show that it has the $(\star_n)$-property. More precisely, we prove that $\hat{Y}_1 \cap Y_2$ has the $(\star_s)$-property for every s such that $2 \leq s \leq n$.

- $\hat{Y}_1 \cap Y_2$ *contains no subset of type (i).* Let $\underline{v}, \underline{w} \in \mathbb{N}^n$ be such that $d(\underline{v}, \underline{w}) = s$ and $P_{\underline{v}}, P_{\underline{w}} \in \hat{Y}_1 \cap Y_2$. Note that both points are also in X but not in Y_1. However, by construction, $P_{(1,v_2,\dots,v_n)}$, $P_{(1,w_2,\dots,w_n)} \in Y_1$ (so in particular, in X).

  Assume, by contradiction, that $P_{\underline{v}}$ and $P_{\underline{w}}$ define a subset of $\hat{Y}_1 \cap Y_2$ of type (i), i.e. that no other point in the smallest complete intersection containing $P_{\underline{v}}, P_{\underline{w}}$ belongs to $\hat{Y}_1 \cap Y_2$. Since X has the $(\star_n)$-property, by Lemma 3.14 there exist $\underline{u}_0, \dots, \underline{u}_s$ as in the statement of lemma, "joining" $P_{\underline{v}}$ to $P_{\underline{w}}$.

  By our assumption, in particular $P_{\underline{u}_{s-1}}$ is not in $\hat{Y}_1 \cap Y_2$, but it is in X. Notice that the "path" from $P_{\underline{v}}$ to $P_{\underline{w}}$ is obtained by changing one coordinate at a time from $\underline{v}$ to $\underline{w}$ in the shortest possible way. Since neither $\underline{v}$ nor $\underline{w}$ has a 1 as first coordinate, $\underline{u}_{s-1}$ is not of the form $(1, z_2, \dots, z_n)$. Then by applying Lemma 3.15

to the points $P_{(1,v_2,\dots,v_n)}$ and $P_{\underline{u}_{s-1}}$, we get a contradiction by forcing a point of $\hat{Y}_1 \cap Y_2$ to lie in the complete intersection.

- $\hat{Y}_1 \cap Y_2$ *contains no subset of type (ii).* Indeed, if it did then this subset is contained in X, contradicting the $(\star_n)$-property of X.

(σ_2) *We make a technical observation concerning the "outlier" points.*

We denote by Y_1' the set of points $P_{(1,\underline{v})} \in Y_1$ (where now $\underline{v} \in \mathbb{N}^{n-1}$) such that the line $L_{\underline{v}}$ has empty intersection with Y_2, and we denote by $Y_2' := Y_2 \setminus (\hat{Y}_1 \cap Y_2)$. Let $F \in I_{\hat{Y}_1 \cap Y_2}$ and assume that F is a product of linear forms of type $A_{i,j}$. Taking the ideal of the empty set to be R, *we claim that*

$$F \in (I_{Y_2'}) \cup (I_{Y_1'}).$$

We assume that both Y_1' and Y_2' are non-empty; otherwise the statement is trivial. Assume by contradiction that $F \notin (I_{Y_2'}) \cup (I_{Y_1'})$. Then there exist $P := P_{(1,\underline{u})} \in Y_1'$ and $Q := P_{(2,\underline{v})} \in Y_2'$, such that $F \notin (A_{1,1}, A_{2,u_1}, \dots A_{n,u_{n-1}})$ and $F \notin (A_{1,2}, A_{2,v_1}, \dots A_{n,v_{n-1}})$. Now, $P_{(1,\underline{u})} \in Y_1'$ implies $P_{(2,\underline{u})} \notin Y_2$; moreover, since $P_{(2,\underline{v})} \in Y_2'$ we have $P_{(1,\underline{v})} \notin Y_1$. But X has the $(\star_n)$-property, so by Lemma 3.15 there exist $\underline{w} \in \mathbb{N}^{n-1}$ such that $P_{(1,\underline{w})}, P_{(2,\underline{w})} \in X$. Thus $P_{(2,\underline{w})} \in \hat{Y}_1 \cap Y_2$. Since, for any index i, $w_i \in \{u_i, v_i\}$ and $F \in (A_{1,2}, A_{2,w_1}, \dots A_{n,w_{n-1}})$ is a product of linear forms $A_{i,j}$ we get either $F \in I_P$ or $F \in I_Q$, which contradicts the assumption.

(σ_3) *We show that* $I_{\hat{Y}_1 \cap Y_2} \subseteq I_{\hat{Y}_1} + I_{Y_2}$.

From (σ_1) we know that $\hat{Y}_1 \cap Y_2$ is ACM, so $I_{\hat{Y}_1 \cap Y_2}$ is minimally generated by products of linear forms of type $A_{i,j}$ (by Corollary 3.4). Let $F \in I_{\hat{Y}_1 \cap Y_2}$ be such a generator. From the minimality of F we note that $F \notin (A_{1,1})$. From (σ_2) we have $F \in I_{Y_1'} \cup I_{Y_2'}$.

Assume first that $F \in I_{Y_2'}$. Then trivially $F \in I_{Y_2} \subseteq I_{\hat{Y}_1} + I_{Y_2}$.

Assume now that $F \notin I_{Y_2'}$; in particular, there exists a point, say $P_{(2,\underline{u})} \in Y_2'$, such that $F \notin (A_{1,2}, A_{2,u_1}, \dots, A_{n,u_{n-1}})$. We collect the relevant facts:

(f_1) $P_{(1,\underline{u})} \notin Y_1$ by definition of Y_2';

(f_2) $F \in I_{Y_1'}$ by (σ_2);

(f_3) $F \in I_{\hat{Y}_1 \cap Y_2}$, $F \notin (A_{1,2}, A_{2,u_1}, \dots, A_{n,u_{n-1}})$ and $F \notin (A_{1,1})$.

We want to show that $F \in I_{\hat{Y}_1}$. Choose any point $P := P_{(1,\underline{v})} \in Y_1$. We consider two cases.

- If $P = P_{(1,\underline{v})} \in Y_1'$ then from (f_2) and (f_3) we get $F \in (A_{1,1}, A_{2,v_1}, \dots, A_{n,v_{n-1}})$ so $F \in (A_{2,v_1}, \dots, A_{n,v_{n-1}})$ since $F \notin (A_{1,1})$ and F is a product of linear forms of type $A_{i,j}$.
- Assume $P = P_{(1,\underline{v})} \in Y_1 \setminus Y_1'$.
  - We first use the $(\star_n)$-property of X. We have $P_{(1,\underline{v})}, P_{(2,\underline{u})} \in X$ but $P_{(1,\underline{u})} \notin X$ by definition of Y_2'. It follows by Lemma 3.15 applied to the points $P_{(1,\underline{v})}$ and $P_{(2,\underline{u})}$ of X that there is some point $P_{(2,\underline{z})} \in \hat{Y}_1 \cap Y_2$ with $z_h \in \{u_h, v_h\}$ (here we have used $P_{(1,\underline{u})} \notin X$). Importantly, since $P \notin Y_1'$, at least one of the coordinates z_h must be v_h.

- Now we have: F vanishes at $P_{(2,\underline{z})}$, F does not vanish at $P_{(2,\underline{u})}$, and the components of $\underline{z}$ are all from $\underline{v}$ or $\underline{u}$. Therefore, since $\underline{u} \neq \underline{z}$ and F is a product of linear forms of the form $A_{i,j}$, at least one linear factor of F comes from $\underline{v}$. Therefore $F \in I_{\hat{Y}_1}$ as desired.

This concludes the proof of (σ_3).

To complete the proof of our theorem, note that in R we always have

$$I_{\hat{Y}_1 \cap Y_2} \supseteq \sqrt{I_{\hat{Y}_1} + I_{Y_2}} \supseteq I_{\hat{Y}_1} + I_{Y_2}.$$

Thus, $I_{\hat{Y}_1} + I_{Y_2}$ is the ideal of an ACM set of reduced points in $(\mathbb{P}^1)^n$, as desired. Moreover, this implies that $A_{1,1}$ is a regular form in $R/(I_{\hat{Y}_1} + I_{Y_2})$ since no point of $\hat{Y}_1 \cap Y_2$ belongs to the hyperplane defined by $A_{1,1}$.

□

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