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TIME-FREQUENCY ANALYSIS OF THE DIRAC EQUATION

S. IVAN TRAPASSO

Abstract. The purpose of this paper is to investigate several issues concerning the Dirac equation from a time-frequency analysis perspective. More precisely, we provide estimates in weighted modulation and Wiener amalgam spaces for the solutions of the Dirac equation with rough potentials. We focus in particular on bounded perturbations, arising as the Weyl quantization of suitable time-dependent symbols, as well as on quadratic and sub-quadratic non-smooth functions, hence generalizing the results in [40]. We then prove local well-posedness on the same function spaces for the nonlinear Dirac equation with a general nonlinearity, including power-type terms and the Thirring model. For this study we adopt the unifying framework of vector-valued time-frequency analysis [57]; most of the preliminary results are stated under general assumptions and hence they may be of independent interest.

1. Introduction

In this note we consider the Cauchy problem for the n-dimensional Dirac equation with a potential V:

$$\begin{cases} i\partial_t\psi(t,x) = (\mathcal{D}_m + V)\psi(t,x), \\ \psi(0,x) = \psi_0(x), \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d. \tag{1}$$

Here $\psi(t,x) = (\psi_1(t,x),\dots,\psi_n(t,x)) \in \mathbb{C}^n$ is a vector-valued complex wavefunction and the Dirac operator $\mathcal{D}_m$ is defined by

$$\mathcal{D}_m = 2\pi m\alpha_0 - i\sum_{j=1}^{d}\alpha_j\partial_j, \tag{2}$$

where $m \geq 0$ (mass) and $\alpha_0,\alpha_1,\dots,\alpha_d \in \mathbb{C}^{n\times n}$ is a set of *Dirac matrices*, i.e. $n\times n$ Hermitian matrices satisfying the identities

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I_n, \quad \forall\, 0 \leq i,j \leq d, \tag{3}$$

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(I_n is the $n \times n$ identity matrix). For $d=3$ and $n=4$ the standard choice for such matrices is the so-called Dirac's representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i=1,2,3, \qquad \alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{4}$$

where we introduced the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

In general, for any d there exist several iterative schemes to obtain a set of Dirac matrices and in general the dependence of the (even) dimension $n=n(d)$ on d is a consequence of the chosen construction [37].

The study of the Dirac equation, like other dispersive equations, may certainly take advantage from the techniques of modern harmonic analysis. In the last decades we have witnessed an increasing interest in the application to PDEs of strategies and function spaces arising in time-frequency analysis. Even if it is impossible to offer a comprehensive list of results, we suggest the papers [5, 6, 9, 10, 11, 12, 13, 38, 39, 58, 61] and the monographs [30, 59] as examples of the manifold aspects one can handle from this perspective.

The optimal environment for this approach is provided by modulation spaces, which were introduced by Feichtinger in the '80s [20, 21]. In the first instance they can be thought of as Besov spaces with isometric boxes in the frequency domain instead of dyadic annuli. In fact, a much more insightful definition is given in terms of the global decay of the phase-space concentration of a function or a distribution. To be precise, given a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a non-zero Schwartz window function $g \in \mathcal{S}(\mathbb{R}^d)$, the short-time Fourier transform $V_g f$ is defined as

$$V_g f(x,\xi) = \mathcal{F}[f g(\cdot - x)](\xi), \qquad (x,\xi) \in \mathbb{R}^{2d},$$

where $\mathcal{F}$ denotes the Fourier transform. The modulation space $M^{p,q}(\mathbb{R}^d)$, $1 \le p,q \le \infty$ is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|V_g f(x,\xi)\|_{L^q(\mathbb{R}^d_\xi;L^p(\mathbb{R}^d_x))} < \infty$. A better control on the regularity is achieved by introducing weights of polynomial type: for $r,s \in \mathbb{R}$ the $M^{p,q}_{r,s}(\mathbb{R}^d)$-norm of f is given by $\|V_g f(x,\xi)\|_{L^q_s(\mathbb{R}^d_\xi;L^p_r(\mathbb{R}^d_x))}$, where

$$u \in L^q_s(\mathbb{R}^d) \Leftrightarrow (1+|\cdot|^2)^{s/2} u \in L^q(\mathbb{R}^d), \tag{6}$$

and similarly for $L^p_r(\mathbb{R}^d)$. In particular, the parameter $s \ge 0$ can be interpreted as the degree of fractional differentiability of $f \in M^{p,q}_{r,s}$. A strictly related family of spaces is obtained by reversing the order of integration in the mixed-Lebesgue norm. The space $W^{p,q}_{r,s}(\mathbb{R}^d)$, traditionally called Wiener amalgam space, contains distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $\|V_g f(x,\xi)\|_{L^q_s(\mathbb{R}^d_x;L^p_r(\mathbb{R}^d_\xi))} < \infty$. There is in fact a deeper connection among these spaces, since it turns out that the elements of $W^{p,q}_{r,s}(\mathbb{R}^d)$ are Fourier transforms of functions in $M^{p,q}_{r,s}(\mathbb{R}^d)$; see Section 3 for more details.

The relevance of these function spaces to the study of dispersive PDEs is closely related to the evolution of the phase-space concentration under the corresponding propagators. As an example, while the Schrödinger propagator $e^{it\triangle}$ is not bounded on $L^p(\mathbb{R}^d)$ except for $p = 2$, it is a bounded unimodular Fourier multiplier on any modulation space $M^{p,q}(\mathbb{R}^d)$ [6]. Many results of this type, including improved dispersive and Strichartz estimates, are also known for the wave equation and the Klein-Gordon equation (see the list of papers above).

To the best of our knowledge, the only contribution in this spirit concerning the Dirac equation is the recent paper [40] by Kato and Naumkin. The authors proved estimates for the solutions of the Dirac equation (1) in the free case (Theorem 1.1) and also for quadratic and subquadratic time-dependent smooth potentials (Theorem 1.2); the latter setting also includes an electromagnetic potential with linear growth. Broadly speaking, the main difficulty in dealing with (1) lies in that it is a system of coupled equations, hence a strategy for disentangling the components is needed. For instance, this can be done approximately at the level of phase space (see [40, Eq. 3.17]) or by projection onto the spectrum of the Dirac operators (see the proof of the dispersive estimate [40, Eq. 1.8]). Another standard procedure consists of exploiting the connection with the wave and Klein-Gordon equations when $m = 0$ and $m > 0$ respectively. Nevertheless, when a non-zero potential V is taken into account most of these procedures loose their usefulness and new ideas are required (cf. for instance [7, 8, 15, 19]).

The first aim of this paper is to offer a different point of view that does not require an explicit decoupling technique nor any preliminary knowledge about the Klein-Gordon equation. A naive look at (1) would suggest to treat it like a Schrödinger-type equation with matrix-valued Hamiltonian $\mathcal{H} = \mathcal{D}_m + V$. For the free case ($V = 0$) the corresponding propagator $U(t) = e^{-it\mathcal{D}_m}$ can be formally viewed as a Fourier multiplier with matrix symbol

$$\mu_t(\xi) = \exp\left[-2\pi it\left(m\alpha_0 + \sum_{j=1}^{d}\alpha_j\xi_j\right)\right]. \tag{7}$$

This perspective naturally leads to consider estimates on vector-valued modulation and Wiener amalgam spaces by studying the regularity of μ_t and extending the ordinary boundedness results for Fourier multipliers and more general pseudodifferential operators. Roughly speaking, the definition of the modulation space $M^{p,q}(\mathbb{R}^d, E)$, E being a complex Banach space in general, coincides with the one given above with $|\cdot|$ replaced by the norm on E; such spaces were first considered by Toft [56] and then extensively studied by Wahlberg [57]. The study of the Dirac equation would only require to consider finite-dimensional vector spaces such as $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$, so that the subtleties connected with infinite-dimensional target spaces are not relevant here and most of the proofs reduce to componentwise computation. Nevertheless, we decided to embrace this wider perspective and thus the first part of the paper is devoted to extend some results of scalar-valued time-frequency analysis to the vector-valued context. In our opinion, the price of developing these tools in full generality is repaid by a unifying and powerful framework which provides very natural and compact proofs for the main results on the Dirac equation. In passing, we remark that the core of results concerning vector-valued time-frequency analysis is in fact of independent interest and falls into the larger area of infinite-dimensional harmonic analysis [28, 36, 60], with possible applications to abstract evolution equations [1, 3] and generalized stochastic processes [26].

In that spirit, we are then able to prove the following estimates for the free Dirac propagator.

Theorem 1.1. *Let $1 \le p,q \le \infty$ and $r,s \in \mathbb{R}$; denote by X any of the spaces $M^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C}^n)$ or $W^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C}^n)$. Let $\psi(t,x)$ be the solution of* (1) *with $V\equiv 0$. For any $t\in\mathbb{R}$ there exists a constant $C_X(t)>0$ such that*

$$\|\psi(t,\cdot)\|_X \le C_X(t)\,\|\psi_0\|_X\,.$$

In particular, if $X = M^{p,q}_{0,s}(\mathbb{R}^d,\mathbb{C}^n)$ there exists a constant $C'>0$ such that

$$C_X(t) \le C'(1+|t|)^{d|1/2-1/p|}. \tag{8}$$

While the results are not unexpected in themselves in view of the discussion above on the connection with the Klein-Gordon propagator, we remark that our method improves the known estimates in two aspects. First, we are able to cover weighted modulation and Wiener amalgam spaces with no extra effort, resulting in a more precise description of the action of the propagator (no loss of derivatives in Theorem 1.1 or asymptotic smoothing in Theorem 3.1 below). On the other hand, at least for modulation spaces we are able to explicitly characterize the time-dependence of the constant $C(t)$ in (8) in a straightforward way, essentially by inspecting the symbol (7).

The second purpose of this note is to provide boundedness results on modulation and Wiener amalgam spaces for suitable potentials V in (1). We relax the regularity assumptions in [40] in two aspects. First, we replace the multiplication operator by V with a genuine matrix pseudodifferential operator σ^{w} in the Weyl form, where the matrix symbol σ belongs to the so-called Sjöstrand class [49]. In the ordinary scalar-valued framework this is a prime example of an exotic symbol class still yielding bounded Weyl operators on $L^2(\mathbb{R}^d)$. A closer inspection reveals that this function space is nothing but the modulation space $M^{\infty,1}$ and it is well known that symbols in this space associate with bounded operators on any (unweighted) modulation space

[29]. This characterization also extends to operator-valued symbols on Hilbert-valued modulation spaces [57]. In addition, while the dependence on the time of the potential V is assumed to be smooth in [40], we require here a milder condition, namely continuity for the narrow convergence; see Definition 2.16 for a precise characterization. In the following claim we use the spaces $\mathcal{M}^{p,q}_{r,s}$ and $\mathcal{W}^{p,q}_{r,s}$ defined as the closure of the Schwartz class in the corresponding modulation and Wiener amalgam spaces respectively.

Theorem 1.2. *Let $1 \leq p,q \leq \infty$, $\gamma \geq 0$ and $r,s \in \mathbb{R}$ be such that $|r|+|s| \leq \gamma$; denote by X any of the spaces $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C}^n)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C}^n)$. Let $T>0$ be fixed and assume the map $[0,T] \ni t \mapsto \sigma(t,\cdot) \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^d,\mathbb{C}^{n\times n})$, to be continuous for the narrow convergence. For any $\psi_0 \in X$ there exists a unique solution $\psi \in C([0,T],X)$ to (1) with $V=\sigma(t,\cdot)^{\mathrm{w}}$. The corresponding propagator is bounded on x.*

We then consider the case of potentials with quadratic and sub-quadratic growth as in [40]. As a consequence of a useful splitting lemma, namely Proposition 3.2 below, we are able to prove a generalized rough counterpart of the smooth scenario considered in [40, Thm. 1.2]. In particular, the potential contains non-smooth functions with a certain number of derivatives in the Sjöstrand class plus a perturbation in the Weyl form.

Theorem 1.3. *Let $1 \leq p \leq \infty$ and $\psi_0 \in \mathcal{M}^p(\mathbb{C}^n)$. Consider the Cauchy problem (1) with potential*

$$V = QI_n + L + \sigma^{\mathrm{w}}, \tag{9}$$

where

- *$Q:\mathbb{R}^d \to \mathbb{C}$ is such that $\partial^\alpha Q \in M^{\infty,1}(\mathbb{R}^d)$ for $\alpha \in \mathbb{N}^d$, $|\alpha|=2$,*
- *$L:\mathbb{R}^d \to \mathbb{C}^{n\times n}$ is such that $\partial^\alpha L \in M^{\infty,1}(\mathbb{R}^d,\mathbb{C}^{n\times n})$ for $\alpha \in \mathbb{N}^d$, $|\alpha|=1$, and*
- *$\sigma \in M^{\infty,1}(\mathbb{R}^d,\mathbb{C}^{n\times n})$.*

For any $t \in \mathbb{R}$ there exists a constant $C(t)>0$ such that the solution ψ of (1) satisfies

$$\left\|\psi(t,\cdot)\right\|_{\mathcal{M}^p} \leq C(t)\left\|\psi_0\right\|_{\mathcal{M}^p}.$$

Furthermore, if V is as in (9) and $Q=0$, then for any $1 \leq p,q \leq \infty$ and $t \in \mathbb{R}$ there exists a constant $C(t)>0$ such that the solution ψ of (1) satisfies

$$\left\|\psi(t,\cdot)\right\|_{\mathcal{M}^{p,q}} \leq C(t)\left\|\psi_0\right\|_{\mathcal{M}^{p,q}}.$$

In the last part of the paper we study the local well-posedness for the nonlinear setting, namely

$$\begin{cases} i\partial_t\psi\,(t,x) = \mathcal{D}_m\psi(t,x) + F(\psi(t,x)), \\ \psi\,(0,x) = \psi_0\,(x)\,, \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{10}$$

where the nonlinear term F considered below comes in the form of a vector-valued real-analytic entire function $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(0) = 0$, i.e.

$$F_j(z) = \sum_{\alpha,\beta\in\mathbb{N}^n} c^j_{\alpha,\beta} z^\alpha \bar{z}^\beta, \qquad j = 1, \dots, n. \tag{11}$$

We remark that this general choice includes nonlinearities of power type, such as

$$F(\psi) = |\psi|^{2k}\psi, \qquad k \in \mathbb{N}; \tag{12}$$

and the cubic nonlinearity known as the Thirring model, namely

$$F(\psi) = (\alpha_0\psi, \psi)\alpha_0\psi; \tag{13}$$

The choice of even powers in (12) and entire functions as in (11) are standard in the context of modulation and amalgam spaces, because of the Banach algebra property enjoyed by certain spaces of these families [51]. On the other hand, the nonlinear spinor field appearing in the Thirring model has been largely investigated; cf. for instance [4, 34, 43, 44], also in view of its physical relevance - it is a model for self-interacting Dirac fermions in quantum field theory [50, 54].

The main result in this respect reads as follows.

Theorem 1.4. *Let $1 \le p \le \infty$ and $r, s \ge 0$; denote by X any of the spaces $M^{p,1}_{0,s}(\mathbb{R}^d, \mathbb{C}^n)$ or $W^{1,p}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$. If $\psi_0 \in X$ then there exists $T = T(\|\psi_0\|_X)$ such that the Cauchy problem (10) with F as in (11) has a unique solution $\psi \in C^0([0, T], X)$.*

We conclude this introduction by emphasizing a few aspects that may be further developed in the context of modulation spaces, such as Strichartz estimates and perturbations due to a magnetic field, i.e. the Dirac operator in (2) becomes $\mathcal{D}_{m,A} = 2\pi m\alpha_0 - i\sum_{j=1}^d \alpha_j(\partial_j - iA_j)$, where $A(x) = (A_1(x), \dots, A_d(x))$, $x \in \mathbb{R}^d$, is a static magnetic potential. We also point out that more general nonlinear terms could be considered, for instance as in the Soler model [50] and other interactions arising in condensed matter; cf. [47] for the state of the art in 1+1 dimensions.

2. Preliminaries

2.1. **Notation.** We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $x \cdot y$ is the scalar product on $\mathbb{R}^d$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of temperate distributions by $\mathcal{S}'(\mathbb{R}^d)$. The brackets $\langle f, g\rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$.

The characteristic function on a set $A \subseteq E$ is denoted with χ_A. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $|x|_\infty = \max\{|x_1|, \dots, |x_d|\}$.

The conjugate exponent p' of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$. The symbol $\lesssim$ means that the underlying inequality holds up to a positive constant factor $C > 0$:

$$f \lesssim g \quad \Rightarrow \quad \exists C > 0 : f \le Cg.$$

We write $f \asymp g$ to say that both $f \lesssim g$ and $g \lesssim f$ hold.

We choose the following normalization for the Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx, \qquad \xi \in \mathbb{R}^d.$$

We define the involution * as $f^*(t) = \overline{f(-t)}$. For any $x, \xi \in \mathbb{R}^d$, the modulation M_ξ and translation T_x operators are defined as

$$M_\xi f(t) = e^{2\pi i t\cdot\xi} f(t), \qquad T_x f(t) = f(t-x).$$

For $m > 0$ and $t \in \mathbb{R}^d$ we set $\langle\xi\rangle_m \coloneqq \sqrt{m^2+\xi^2}$. We omit the subscript for $m = 1$, namely $\langle\xi\rangle$ stands for $\langle\xi\rangle_1$. Denote by J the canonical symplectic matrix in $\mathbb{R}^{2d}$:

$$J = \begin{pmatrix} 0_d & I_d \\ -I_d & 0_d \end{pmatrix}.$$

In what follows we always denote by E a complex Banach space with norm $|\cdot|_E$, whereas the symbol H is reserved for a complex separable Hilbert space. The topological dual space of E is denoted by E'. The brackets $(\cdot,\cdot)$ are used for the duality between E' and E and in particular for the inner product in H - we assume $(\cdot,\cdot)$ to be conjugate-linear in the second argument. Given two normed spaces X and Y, the space of continuous linear operators $X \to Y$ with the topology of bounded convergence is denoted by $\mathcal{L}(X,Y)$, whereas we write $\mathcal{L}_s(X,Y)$ for the same set endowed with the strong operator topology. The space of trace-class operators on H is denoted by $\mathcal{L}^1(H)$.

The space of smooth E-valued functions with bounded derivatives of any order larger than $k \in \mathbb{N}$ is

$$C^\infty_{\geq k}(\mathbb{R}^d, E) \coloneqq \left\{ f \in C^\infty(\mathbb{R}^d, E) \,:\, |\partial^\alpha f| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^d, |\alpha| \geq k \right\}.$$

Notice that $C^\infty_{\geq 0}(\mathbb{R}^d)$ coincides with the well-known Hörmander class $S^0_{0,0}(\mathbb{R}^d)$ [31, 33]. We will occasionally make use of the Dirac notation for projection operators: given $\phi, \psi \in H$, we define

$$|\psi\rangle\langle\phi| : H \to H, \quad |\psi\rangle\langle\phi|(w) = (w,\phi)_H \psi.$$

Given a triple E_1, E_2 and E_3 of complex Banach spaces, we say that the map

$$\bullet : E_1 \times E_2 \to E_3, \quad (x_1, x_2) \mapsto x_3 = x_1 \bullet x_2$$

is a *multiplication* [2] if it is a continuous bilinear operator such that $\|\bullet\|_{\mathcal{L}(E_1\times E_2, E_3)} \leq 1$. The following are common examples of multiplications that will be used below:

1. multiplication with scalars: $\mathbb{C} \times E \to E$, $(\lambda, x) \mapsto \lambda x$;
2. the duality pairing: $E' \times E \to \mathbb{C}$, $(u, x) \mapsto u(x)$;
3. the evaluation map: $\mathcal{L}(E_1, E_2) \times E_1 \to E_2$, $(T, x) \mapsto Tx$;

(4) multiplication in a Banach algebra.

Although neither the concrete expressions of the Dirac matrices nor deep aspects related to the Clifford algebra representation theory are relevant for our purposes, we point out that the conditions (3) force n to be even and we may assume without loss of generality that

$$\alpha_0 = \begin{pmatrix} I_{n/2} & 0 \\ 0 & -I_{n/2} \end{pmatrix}.$$

We refer the interested reader to [37, 46] for further details.

2.2. **Vector-valued function spaces and operators.** The notation and the basic results of analysis on infinite-dimensional spaces are rather standard [2, 28, 35] and we will not linger over the subtleties arising from the infinite-dimensional context. For the convenience of the reader we briefly collect the main facts of harmonic analysis in the vector-valued context. In what follows we consider functions $f : \mathbb{R}^d \to E$, where $\mathbb{R}^d$ is provided with the Lebesgue measure μ_L.

The family of *Lebesgue-Bochner spaces* is the natural analogue of Lebesgue spaces of scalar-valued functions. When there is no risk of confusion, we will write $L^p_s(E)$ for $L^p_s(\mathbb{R}^d, E)$ and $L^p(E)$ when $s = 0$. Notice that $f = (f_1, \ldots, f_n) \in L^p_s(\mathbb{R}^d, \mathbb{C}^n)$ if and only if $f_j \in L^p_s(\mathbb{R}^d)$ for any $j = 1, \ldots, n$. Most of the usual properties from the scalar-valued case extend in a natural way (with the remarkable exception of duality [35]).

Proposition 2.1 ([2, 35]). *(i) For any $1 \le p \le \infty$, $L^p(\mathbb{R}^d, E)$ is a Banach space with the norm $\|f\|_{L^p(\mathbb{R}^d,E)} = \||f(\cdot)|_E\|_{L^p}$.*
(ii) $L^2(\mathbb{R}^d, H)$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_{L^2(H)} = \int_{\mathbb{R}^d} (f(t), g(t))_H dt.$$

(iii) (Hölder inequality) Given a multiplication $\bullet : E_1 \times E_2 \to E_3$, $s_1, s_2 \in \mathbb{R}$ and $1 \le p_1, p_2, p \le \infty$ such that $1/p_1 + 1/p_2 = 1/p$, if $f \in L^{p_1}_{s_1}(\mathbb{R}^d, E_1)$ and $g \in L^{p_2}_{s_2}(\mathbb{R}^d, E_2)$ then $f \bullet g \in L^p_{s_1+s_2}(\mathbb{R}^d, E_3)$ and $\|f \bullet g\|_{L^p_{s_1+s_2}(E_3)} \le \|f\|_{L^{p_1}_{s_1}(E_1)} \|g\|_{L^{p_2}_{s_2}(E_2)}$.

Distributions and Fourier transform ([2, 35]**).** Recall that the Schwartz class of E-valued rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d, E)$ is a Fréchet space with the topology induced by the family of seminorms $\{p_{m,E}\}_{m\in\mathbb{N}}$, where

$$p_{m,E}(f) := \sup_{\substack{t\in\mathbb{R}^d \\ |\alpha|+|\beta|<m}} \left|t^\alpha \partial^\beta f(t)\right|_E < \infty,$$

and is a dense subset of $L^p(\mathbb{R}^d, E)$ for any $1 \le p < \infty$.

The space of E-valued temperate distributions $\mathcal{S}'(\mathbb{R}^d, E)$ consists of bounded (conjugate-)linear maps from $\mathcal{S}(\mathbb{R}^d)$ to E, that is $\mathcal{S}'(\mathbb{R}^d, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^d), E)$.

For $1 \leq p \leq \infty$ any p-integrable E-valued function f can be identified with a E-valued temperate distribution as usual:

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt, \qquad g \in \mathcal{S}(\mathbb{R}^d).$$

Notice that this is a further meaning for the brackets $\langle \cdot, \cdot \rangle$.

The Fourier transform can be initially defined as a Bochner integral for $f \in L^1(\mathbb{R}^d, E)$ and its restriction to $\mathcal{S}(\mathbb{R}^d, E)$ yields a continuous automorphism that enjoys the usual properties (e.g., the Riemann-Lebesgue lemma, the inversion theorem, the relations with translation, modulation and differentiation). There is a notable exception: while $\mathcal{F} : L^1(\mathbb{R}^d, E) \to L^\infty(\mathbb{R}^d, E)$, the Hausdorff-Young inequality does not hold in general [35]. In particular, it is a deep result by Kwapień [42] that the Parseval-Plancherel theorem yields the extension of $\mathcal{F}$ to a unitary operator on $L^2(\mathbb{R}^d, E)$ if and only if E is isomorphic to a Hilbert space.

Nevertheless, the Fourier transform extends to an isomorphism on $\mathcal{S}'(\mathbb{R}^d, E)$ as follows:

$$\langle \hat{f}, \hat{g}\rangle \equiv \langle f, g\rangle, \qquad f \in \mathcal{S}'(\mathbb{R}^d, E),\ g \in \mathcal{S}(\mathbb{R}^d).$$

For future convenience we define the (Bochner-)Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d, E)$ consisting of distributions $f \in \mathcal{S}'(\mathbb{R}^d, E)$ such that

$$\|f\|_{\mathcal{F}L^q_s(E)} \coloneqq \left\|\mathcal{F}^{-1} f\right\|_{L^q_s(E)} < \infty.$$

The following Bernstein-type lemma can be proved just as in the scalar-valued case; cf. [59, Prop. 1.11].

Lemma 2.2. *Let $N > d/2$ be an integer and $\partial_j^k f \in L^2(\mathbb{R}^d, H)$ for any $j = 1, \ldots, d$ and $0 \leq k \leq N$. Then*

$$\|f\|_{\mathcal{F}L^1(H)} \lesssim \|f\|_{L^2(H)}^{1-d/2N} \left(\sum_{j=1}^d \left\|\partial_j^N f\right\|_{L^2(H)}\right)^{d/2N}. \tag{14}$$

Convolution and Fourier multipliers. The convolution of vector-valued functions can be meaningfully defined as soon as the target spaces are provided with a multiplication structure [2, 35]. The convolution of $f \in \mathcal{S}'(\mathbb{R}^d, E)$ with a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ is the distribution $f * g \in \mathcal{S}'(\mathbb{R}^d, E)$ such that

$$\langle f * g, \phi\rangle \equiv \langle f, g^* * \phi\rangle, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

In fact, $f * g \in C^\infty(\mathbb{R}^d, E)$ is a function of polynomial growth together with all its derivatives. Moreover, for $f \in L^p(\mathbb{R}^d, E)$ and $g \in L^1(\mathbb{R}^d)$ we recover the ordinary convolution

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy,$$

which is a well-defined Bochner integral for a.e. $x \in \mathbb{R}^d$. In particular, $f * g \in L^p(\mathbb{R}^d, E)$ with $\|f * g\|_{L^p(E)} \le \|f\|_{L^p(E)} \|g\|_{L^1}$. The $\bullet$-convolution $f_1 *_\bullet f_2$ of $f_1 \in \mathcal{S}(\mathbb{R}^d, E_1)$ and $f_2 \in \mathcal{S}'(\mathbb{R}^d, E_2)$ can be similarly defined as a smooth E_3-valued function for any multiplication $\bullet : E_1 \times E_2 \to E_3$ [2, Thm. 1.9.1]. We state some results that will be needed below. The proofs of more general versions of these facts can be found in [2, Sec. 1.9]. See also [41].

Proposition 2.3. *(i) (Young inequality) Let $1 \le p, q, r \le \infty$ satisfy $1/p + 1/q = 1 + 1/r$ and $s_1, s_2, s_3 \in \mathbb{R}$ satisfy*

$$s_1 + s_3 \ge 0, \quad s_2 + s_3 \ge 0, \quad s_1 + s_2 \ge 0.$$

If $f \in L^p_{s_1}(\mathbb{R}^d, E_1)$ and $g \in L^q_{s_2}(\mathbb{R}^d, E_2)$, then $f *_\bullet g \in L^r_{-s_3}(\mathbb{R}^d, E_3)$, with

$$\|f *_\bullet g\|_{L^r_{-s_3}(E_3)} \lesssim \|f\|_{L^p_{s_1}(E_1)} \|g\|_{L^q_{s_2}(E_2)}.$$

(ii) For any $f \in \mathcal{S}'(\mathbb{R}^d, E_1)$ and $g \in \mathcal{S}(\mathbb{R}^d, E_2)$:

$$\mathcal{F}(f *_\bullet g) = \hat{f} \bullet \hat{g}.$$

We then introduce the *Fourier multiplier* with symbol $\mu \in \mathcal{S}'(\mathbb{R}^d, E_1)$ as the linear map

$$\mu(D)f \coloneqq \mathcal{F}^{-1}(\mu \bullet \hat{f}) = \mathcal{F}^{-1}\mu *_\bullet f \in \mathcal{S}'(\mathbb{R}^d, E_3),$$

the domain consisting of all $f \in \mathcal{S}'(\mathbb{R}^d, E_2)$ such that the latter convolution is well defined [2].

2.3. **Vector-valued time-frequency analysis.** The short-time Fourier transform of a vector-valued distribution $f \in \mathcal{S}'(\mathbb{R}^d, E)$ with respect to a non-zero window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined [57] as the distribution

$$V_g f(x,\xi) \coloneqq \langle f, M_\xi T_x g\rangle. \tag{15}$$

Equivalent representations of $V_g f$ are the following ones, whenever meaningful (assume for instance $f \in L^2(\mathbb{R}^d, E)$):

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\xi} f(y)\, \overline{g(y-x)}\, dy \tag{16}$$

$$= \mathcal{F}(f \cdot \overline{T_x g})(\xi) \tag{17}$$

$$= e^{-2\pi i x\cdot\xi}(f * M_\xi g^*)(x) \tag{18}$$

$$= \langle \hat{f}, T_\xi M_{-x}\hat{g}\rangle \tag{19}$$

$$= e^{2\pi i x\cdot\xi} V_{\hat{g}}\hat{f}(\xi, -x). \tag{20}$$

It can be proved [57, Lem. 2.1] that $V_g f \in C^\infty(\mathbb{R}^{2d}, E)$ and

$$|V_g f(x,\xi)|_E \le C(1+|x|+|\xi|)^N,$$

for some $C>0$, $N\in\mathbb{N}$ and any $x,\xi\in\mathbb{R}^d$.

Definition 2.4. *Let $1\le p,q\le\infty$ and $r,s\in\mathbb{R}$. The E-valued modulation space $M^{p,q}_{r,s}(\mathbb{R}^d,E)$ consists of distributions $f\in\mathcal{S}'(\mathbb{R}^d,E)$ such that*

$$\|f\|_{M^{p,q}_{r,s}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\xi)|_E^p \langle x\rangle^{rp}dx\right)^{q/p}\langle\xi\rangle^{sq}d\xi\right)^{1/q} < \infty, \tag{21}$$

for some $g\in\mathcal{S}(\mathbb{R}^d)$, with suitable modification for $p=\infty$ or $q=\infty$.

If $r=s=0$ we omit the indices and write $M^{p,q}$. Furthermore, we write M^p for $M^{p,p}$ and $M^{p,q}(E)$ for $M^{p,q}(\mathbb{R}^d,E)$ when there is no risk of confusion. We remark that more general weights may be taken into account [57].

Most of the ordinary theory extends to the vector-valued context by simply substituting $|\cdot|$ with $|\cdot|_E$ in the proofs. For our purposes, it is enough to mention the following properties.

Proposition 2.5. *Let $1\le p,q\le\infty$ and $r,s\in\mathbb{R}$.*

(i) $M^{p,q}_{r,s}(\mathbb{R}^d,E)$ is a Banach space with the norm (21), which is independent of the window function g (i.e., different windows yield equivalent norms).
(ii) If $p,q<\infty$ the Schwartz class $\mathcal{S}(\mathbb{R}^d,E)$ is dense in $M^{p,q}_{r,s}(\mathbb{R}^d,E)$.
(iii) If $p_1\le p_2$, $q_1\le q_2$ and $r_2\le r_1$, $s_2\le s_1$, then $M^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d,E)\hookrightarrow M^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d,E)$.
(iv) If $E=\mathbb{C}^{a\times b}$, then $f\in M^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C}^{a\times b})$ if and only if $f_{ij}\in M^{p,q}_{r,s}(\mathbb{R}^d,\mathbb{C})$ for any $i=1,\dots,a$, $j=1,\dots,b$.

Remark 2.6. *In contrast to the aforementioned properties, duality is a quite subtle question (cf. [57]). In order to avoid related issues, which usually occur when $p,q\in\{1,\infty\}$, it is convenient to introduce the space $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,E)$, namely the closure of $\mathcal{S}(\mathbb{R}^d,E)$ with respect to the $M^{p,q}_{r,s}$ norm. In particular we have $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,E) = M^{p,q}_{r,s}(\mathbb{R}^d,E)$ for $1\le p,q<\infty$.*

By reversing the order of integrals in the definition of modulation spaces one obtains a new family of spaces.

Definition 2.7. *Let $1\le p,q\le\infty$ and $r,s\in\mathbb{R}$. The E-valued modulation space $W^{p,q}_{r,s}(\mathbb{R}^d,E)$ consists of distributions $f\in\mathcal{S}'(\mathbb{R}^d,E)$ such that*

$$\|f\|_{W^{p,q}_{r,s}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\xi)|_E^p \langle\xi\rangle^{rp}d\xi\right)^{q/p}\langle x\rangle^{sq}dx\right)^{1/q} < \infty,$$

for some $g\in\mathcal{S}(\mathbb{R}^d)$, with suitable modification for $p=\infty$ or $q=\infty$.

From (20) we immediately get $\left\|\hat{f}\right\|_{M^{p,q}_{r,s}} = \|f\|_{W^{p,q}_{r,s}}$, that is $\mathcal{F}M^{p,q}_{r,s}(\mathbb{R}^d, E) = W^{p,q}_{r,s}(\mathbb{R}^d, E)$. This should not come as a surprise, since Feichtinger originally designed modulation spaces as Wiener amalgam spaces on the Fourier side [21, 24]. Furthermore, the results stated in Proposition 2.5 have an identical counterpart for Wiener amalgam spaces, it is enough to replace $M^{p,q}_{r,s}$ with $W^{p,q}_{r,s}$ in the claim.

As already noted by Wahlberg [57], the spaces $W^{p,q}_{r,s}(\mathbb{R}^d, E)$ are in fact Wiener amalgam spaces in the broadest sense, namely

$$W^{p,q}_{r,s}(\mathbb{R}^d, E) = W(\mathcal{F}L^p_r(\mathbb{R}^d, E), L^q_s(\mathbb{R}^d)),$$

hence they inherit certain properties from their local and global components [22]. In order to exploit this connection we introduce a useful equivalent discrete norm for amalgam spaces. Recall that a bounded uniform partition of function (BUPU) $(\{\psi_i\}_{i\in I}, (x_i)_{i\in I}, U)$ consists of a family of non-negative functions in $\mathcal{F}L^1_{|r|}(\mathbb{R}^d)$ $\{\psi_i\}_{i\in I}$ such that the following conditions are satisfied:

(1) $\sum_{i\in I} \psi_i(x) = 1$, for any $x \in \mathbb{R}^d$;
(2) $\sup_{i\in I} \|\psi_i\|_{\mathcal{F}L^1_{|r|}} < \infty$;
(3) there exist a discrete family $(x_i)_{i\in I}$ in $\mathbb{R}^d$ and a relatively compact set $U \subset \mathbb{R}^d$ such that $\mathrm{supp}(\psi_i) \subset x_i + U$ for any $i \in I$, and
(4) $\sup_{i\in I} \#\{j \,:\, x_i + U \cap x_j + U \neq \emptyset\} < \infty$.

A general result in the theory of amalgam spaces is the following norm equivalence in the spirit of decomposition spaces [22, 25, 23]:

$$\|f\|_{W^{p,q}_{r,s}(\mathbb{R}^d,E)} \asymp \left(\sum_{i\in I} \|f\,\psi_i\|^q_{\mathcal{F}L^p_r(\mathbb{R}^d,E)} \langle x_i\rangle^{sq} \right)^{1/q}. \tag{22}$$

A similar characterization holds for modulation spaces [21, 61], providing a norm comparable to that of Besov spaces:

$$\|f\|_{M^{p,q}_{r,s}(\mathbb{R}^d,E)} \asymp \left(\sum_{i\in I} \|\Box_i f\|^q_{L^p_r(\mathbb{R}^d,E)} \langle x_i\rangle^{sq} \right)^{1/q} \tag{23}$$

where we introduced the frequency-uniform decomposition operators

$$\Box_i \coloneqq \mathcal{F}^{-1}\psi_i\mathcal{F}, \qquad i \in I.$$

Many properties satisfied by modulation spaces carry over to Wiener amalgam spaces in view of the isomorphism established by the Fourier transform. In particular, a Young type result can be obtained after a suitable modification of the proof of [22, Thm. 3].

Theorem 2.8. *Let $\bullet : E_1 \times E_2 \to E_3$ be a multiplication for the triple of Banach spaces (E_1, E_2, E_3). For any $1 \le p_1, p_2, p_3, q_1, q_2, q_3 \le \infty$ and $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{R}$ such that*

$$\mathcal{F}L^{p_1}_{r_1}(\mathbb{R}^d, E_1) *_\bullet \mathcal{F}L^{p_2}_{r_2}(\mathbb{R}^d, E_2) \hookrightarrow \mathcal{F}L^{p_3}_{r_3}(\mathbb{R}^d, E_3),$$
$$L^{q_1}_{s_1}(\mathbb{R}^d) * L^{q_2}_{s_2}(\mathbb{R}^d) \hookrightarrow L^{q_3}_{s_3}(\mathbb{R}^d),$$

the following inclusion holds:

$$W^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d, E_1) *_\bullet W^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d, E_2) \hookrightarrow W^{p_3,q_3}_{r_3,s_3}(\mathbb{R}^d, E_3). \tag{24}$$

Proof. For the benefit of the reader we sketch here a short proof in the spirit of [32, Thm. 11.8.3]. We consider as BUPU for $W^{p,q}_{r,s}(\mathbb{R}^d, E)$ the family $\{\psi_k\}_{k\in\mathbb{Z}^d} \subset \mathcal{F}L^1_{|r|}(\mathbb{R}^d)$ defined by

$$\psi_k(t) = \frac{\phi(t-k)}{\sum_{k\in\mathbb{Z}^d} \phi(t-k)}, \quad t \in \mathbb{R}^d,$$

for a fixed $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(t) = 1$ for $t \in [0,1]^d$ and $\phi(t) = 0$ for $t \in \mathbb{R}^d \setminus [-1,2]^d$. After introducing the control functions

$$\Psi_{f,p,r,E}(k) := \|f\,\psi_k\|_{\mathcal{F}L^p_r(\mathbb{R}^d,E)}, \quad k \in \mathbb{Z}^d,$$

the equivalent norm (22) becomes

$$\|f\|_{W^{p,q}_{r,s}(\mathbb{R}^d,E)} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|f\,\psi_k\|^q_{\mathcal{F}L^p_r(\mathbb{R}^d,E)} \langle k\rangle^{qs}\right)^{1/q} \asymp \|\Psi_{f,p,r,E}\|_{\ell^q_s(\mathbb{Z}^d)}.$$

For $f \in W^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d, E_1)$ and $g \in W^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d, E_2)$ set $f_m = f\psi_m$, $g_n = g\psi_n$ for $m, n \in \mathbb{Z}^d$. In view of the support property [2, Rem. 1.9.6(f)] and the properties of BUPUs, we have

$$\operatorname{supp}(f_m *_\bullet g_n) \subset \operatorname{supp}(f_m) + \operatorname{supp}(g_n) = m + n + 2\operatorname{supp}\psi.$$

It is then clear that the cardinality of the set $J_k := \{(m,n) \in \mathbb{Z}^{2d} : \operatorname{supp}((f_m *_\bullet g_n)\psi_k) \neq \emptyset\}$ is finite for any $k \in \mathbb{Z}^d$ and is uniformly bounded with respect to m, n, k. In fact, notice that

$$J_k = \{(m,n) \in \mathbb{Z}^{2d} : m = k - n + \alpha,\ |\alpha| \le N(d)\},$$

for a fixed constant $N(d) \in \mathbb{N}$ depending only on the dimension d. Therefore, an easy computation yields

$$\Psi_{f *_\bullet g, p_3, r_3, E_3}(k) = \sum_{|\alpha|\le N(d)} \Psi_{f,p_1,r_1,E_1} * \Psi_{g,p_2,r_2,E_2}(k+\alpha),$$

and hence

$$\|f *_\bullet g\|_{W^{p_3,q_3}_{r_3,s_3}(\mathbb{R}^d,E_3)} \lesssim \|f\|_{W^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d,E_1)} \|g\|_{W^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d,E_2)},$$

that is the claim. $\square$

Remark 2.9. *In view of the relation with modulation spaces and Young inequality for convolution, under the same assumptions of the previous theorem we also have*

$$M^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d,E_1) \bullet M^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d,E_2) \hookrightarrow M^{p_3,q_3}_{r_3,s_3}(\mathbb{R}^d,E_3). \tag{25}$$

An interesting relation between modulation and Wiener amalgam spaces is given by the following generalized Hausdorff-Young inequality, which is a direct consequence of Minkowski's integral inequality:

$$M^{p,q}_{r,s}(\mathbb{R}^d,E) \hookrightarrow W^{q,p}_{s,r}(\mathbb{R}^d,E), \quad 1 \le q \le p \le \infty,\ r,s \in \mathbb{R}. \tag{26}$$

2.4. **Fourier multipliers.** We now provide sufficient conditions on the symbol of a Fourier multiplier in order for it to be bounded on modulation and Wiener amgalgam spaces.

Proposition 2.10. *Let* $\bullet : E_0 \times E_1 \to E_2$ *be a multiplication and* $\mu \in W^{1,\infty}_{|r|,\delta}(\mathbb{R}^d,E_0)$ *for some* $r,\delta \in \mathbb{R}$*. The Fourier multiplier* $\mu(D)$ *is bounded from* $M^{p,q}_{r,s}(\mathbb{R}^d,E_1)$ *to* $M^{p,q}_{r,s+\delta}(\mathbb{R}^d,E_2)$ *for any* $1 \le p,q \le \infty$ *and* $s \in \mathbb{R}$*. In particular,*

$$\|\mu(D)f\|_{M^{p,q}_{r,s+\delta}(E_2)} \lesssim \|\mu\|_{W^{1,\infty}_{|r|,\delta}(E_0)} \|f\|_{M^{p,q}_{r,s}(E_1)}, \quad f \in M^{p,q}_{r,s}(E_1).$$

Proof. The proof is a straightforward generalization of the argument used in the scalar-valued case; see for instance [6, Lem. 8]. We remark that Theorem 2.8 and the associativity of $\bullet$-convolutions [2, Rem. 1.9.6(c)] are required. $\square$

A similar result holds for Fourier multipliers on Wiener amalgam spaces.

Proposition 2.11. *Let* $\bullet : E_0 \times E_1 \to E_2$ *be a multiplication and* $\mu \in M^{\infty,1}_{\delta,|s|}(\mathbb{R}^d,E_0)$ *for some* $s,\delta \in \mathbb{R}$*. The Fourier multiplier with symbol* μ *is bounded from* $W^{p,q}_{r,s}(\mathbb{R}^d,E_1)$ *to* $W^{p,q}_{r+\delta,s}(\mathbb{R}^d,E_2)$ *for any* $1 \le p,q \le \infty$ *and* $r \in \mathbb{R}$*. In particular,*

$$\|\mu(D)f\|_{W^{p,q}_{r+\delta,s}(E_2)} \lesssim \|\mu\|_{M^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)}, \quad f \in W^{p,q}_{r,s}(E_1).$$

Proof. Recall that $W^{p,q}_{r,s}(\mathbb{R}^d,E) = W(\mathcal{F}L^p_r(\mathbb{R}^d,E), L^q_s(\mathbb{R}^d))$. Theorem 2.8 and the relation $\mathcal{F}M^{p,q}_{r,s} = W^{p,q}_{r,s}$ thus yield

$$\begin{aligned} \|\mu(D)f\|_{W^{p,q}_{r+\delta,s}(E_2)} &= \left\|\mathcal{F}^{-1}\mu *_\bullet f\right\|_{W^{p,q}_{r+\delta,s}(E_2)} \\ &\lesssim \left\|\mathcal{F}^{-1}\mu\right\|_{W^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)} \\ &\lesssim \|\mu\|_{M^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)}. \end{aligned}$$

$\square$

2.5. **The Wigner distribution and the Weyl transform.** Given $f, g \in L^2(\mathbb{R}^d, H)$, the Wigner distribution $W(f,g)(x,\xi) \in \mathcal{L}(H)$, $x, \xi \in \mathbb{R}^d$, is defined as follows:

$$W(f,g)(x,\xi) = [\mathcal{F}\mathfrak{T}_s P(f,g)(x,\cdot)](\xi), \tag{27}$$

where we introduced the projector-valued function

$$P(f,g) : \mathbb{R}^{2d} \to \mathcal{L}^1(H), \qquad P(f,g)(x,y) := |f(x)\rangle\langle g(y)|,$$

and $\mathfrak{T}_s$ is the linear transformation acting on $F : \mathbb{R}^{2d} \to H$ as

$$\mathfrak{T}_s F(x,y) = F\left(x + \frac{y}{2}, x - \frac{y}{2}\right).$$

It is therefore clear that $W(f,g) : \mathbb{R}^{2d} \to \mathcal{L}^1(H)$ and in particular [27, 57]

$$(W(f,g)(x,\xi)u, v)_H = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} \left(f(x+y/2), v\right)_H \overline{(g(x-y/2), u)_H} dy,$$

for any $u, v \in H$. More concisely, we have

$$(W(f,g)(x,\xi)u, v)_H = W(\widetilde{f_v}, \widetilde{g_u})(x,\xi),$$

where on the right-hand side we have the ordinary Wigner distribution of the functions

$$\widetilde{f_v}(t) = (f(t), v)_H, \quad \widetilde{g_u}(t) = (g(t), u)_H.$$

The following properties of the Wigner distributions are well known in the standard setting [30] and can be easily derived in the vector-valued context.

Proposition 2.12. *For any $f, g \in \mathcal{S}(\mathbb{R}^d, H)$ and $x, \xi \in \mathbb{R}^d$:*

(i) $W(f,g) \in \mathcal{S}(\mathbb{R}^{2d}, \mathcal{L}^1(H))$.
(ii) $W(f,g)(x,\xi) = W(\hat{f}, \hat{g})(\xi, -x)$.
(iii) $\int_{\mathbb{R}^d} W(f,g)(x,\xi)dx = |\hat{f}(\xi)\rangle\langle\hat{g}(\xi)|$.
(iv) $\int_{\mathbb{R}^d} W(f,g)(x,\xi)d\xi = |f(x)\rangle\langle g(x)|$.

The Wigner transform can be extended to $f, g \in \mathcal{S}'(\mathbb{R}^d, H)$ as follows [57]. Let $\Phi = W(\phi_1, \phi_2)$ for $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$; then $W(f,g) \in \mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}^1(H))$ is such that

$$(\langle W(f,g), \Phi\rangle u, v)_H = (\langle f, \phi_1\rangle, v)_H \overline{(\langle g, \phi_2\rangle, u)_H}, \quad u, v \in H.$$

Assume now $\sigma \in \mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}(H))$. The Weyl transform $\sigma^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d, H) \to \mathcal{S}'(\mathbb{R}^d, H)$ is defined by duality as

$$\langle \sigma^{\mathrm{w}} f, g\rangle = \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[\sigma(x,\xi) W(g,f)(x,\xi)\right] dx d\xi, \quad f, g \in \mathcal{S}(\mathbb{R}^d, H). \tag{28}$$

For further details see [27, pp. 135–137] and [57].

A classical, remarkable result in the scalar-valued case is the boundedness of Weyl transforms with symbols in the Sjöstrand class on any modulation and Wiener amalgam space [30, Thm. 14.5.2]. This property still holds in the vector valued case.

Theorem 2.13. *Let $1 \le p,q \le \infty$, $\gamma \ge 0$ and $r,s \in \mathbb{R}$ be such that $|r|+|s| \le \gamma$; denote by X any of the spaces $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,H)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d,H)$. If $\sigma \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d},\mathcal{L}(H))$ then the Weyl operator σ^{w} is bounded on X.*

Proof. The case $X = \mathcal{M}^{p,q}_{r,s}(H)$ is covered by [57, Cor. 4.8], and it is stated here with small modifications in the spirit of [30, Thm. 14.5.6] in order to take the weights into account. For the case $X = \mathcal{W}^{p,q}_{r,s}(H)$ we need an extension of the well-known *symplectic covariance* property of the Weyl calculus [16, 30], namely

$$\mathcal{F}\sigma^{\mathrm{w}} = \sigma^{\mathrm{w}}_{J^{-1}}\mathcal{F}, \qquad \sigma \in \mathcal{S}'(\mathbb{R}^{2d},\mathcal{L}(H)),$$

where $\sigma_{J^{-1}} = \sigma \circ J^{-1}$; the proof is a straightforward application of Proposition 2.12 above. In view of this property, consider the following diagram:

$$\begin{tikzcd}
\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,H) \arrow[r, "\sigma^{\mathrm{w}}_J"] & \mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,H) \arrow[d, "\mathcal{F}"] \\
\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d,H) \arrow[u, "\mathcal{F}^{-1}"] \arrow[r, "\sigma^{\mathrm{w}}"] & \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d,H)
\end{tikzcd}$$

It is easy to prove that if $\sigma \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d},\mathcal{L}(H))$ then $\sigma_J \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d},\mathcal{L}(H))$ too (cf. for instance the proof of [14, Lem. 5.2]), hence the preceding case implies that σ^{w}_J is bounded on $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,H)$ for any $1 \le p,q \le \infty$ and $r,s \in \mathbb{R}$ such that $|r|+|s| \le \gamma$. $\square$

The relevance of the Sjöstrand class is also enforced by the following characterization - the proof goes exactly as that of [30, Thm. 14.5.3] and [31, Lem. 6.1] with $|\cdot|$ replaced by $|\cdot|_E$.

Proposition 2.14. *The following characterization holds:*

$$S^0_{0,0}(\mathbb{R}^d,E) = \bigcap_{s\ge 0} M^{\infty}_{0,s}(\mathbb{R}^d,E) = \bigcap_{s \ge 0} M^{\infty,1}_{0,s}(\mathbb{R}^d,E).$$

Corollary 2.15. *Let $\sigma \in S^0_{0,0}(\mathbb{R}^{2d},\mathcal{L}(H))$. The Weyl operator σ^{w} is bounded on $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d,H)$ for any $1 \le p,q \le \infty$ and $r,s \in \mathbb{R}$.*

2.6. **Narrow convergence.** Convergence in $M^{\infty,1}$ norm is a very strong requirement. For instance it is well known that C^∞_c is not dense $M^{\infty,1}$ with the norm topology [49]; this fact inhibits the standard approximation arguments and leads to restrict to subspaces such as $\mathcal{M}^{\infty,1}$. Another way to cope with this problem consists in weakening the notion of convergence as follows [13, 55].

Definition 2.16. *Let Ω be a subset of some Euclidean space and $s \in \mathbb{R}$. The map $\Omega \ni \nu \mapsto \sigma_\nu \in M^{\infty,1}_{0,s}(\mathbb{R}^d,E)$ is said to be continuous for the narrow convergence if:*

(1) it is a continuous map in $\mathcal{S}'(\mathbb{R}^d,E)$ (weakly), and

(2) there exists a function $h \in L^1_s(\mathbb{R}^d)$ such that for some (hence any) nonzero window $g \in \mathcal{S}(\mathbb{R}^d)$ one has $\sup_{z\in\mathbb{R}^d} |V_g\sigma_\nu(x,\xi)|_E \le h(\xi)$ for any $\nu \in \Omega$ and a.e. $\xi \in \mathbb{R}^d$.

The benefits of narrow continuity in the scalar-valued case carry over to the Hilbert-valued case. The following property will be used below.

Theorem 2.17. *For any $1 \le p, q \le \infty$ and $\gamma \ge 0$, $r, s \in \mathbb{R}$ such that $|r| + |s| \le \gamma$, let X denote either $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$. If $\Omega \ni \nu \mapsto \sigma_\nu \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^d, \mathcal{L}(H))$ is continuous for the narrow convergence then the corresponding map of operators $\nu \mapsto \sigma^{\mathrm{w}}_\nu$ is strongly continuous on X.*

Proof. The proof for $X = \mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ is a suitable adaption of the one given in [13, Prop. 3]. For the strong continuity on $X = \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$ we reduce to the latter case by the same arguments in the proof of Proposition 2.13, which imply that $\sigma^{\mathrm{w}}_\nu u = \mathcal{F}(\sigma_\nu)^{\mathrm{w}}_J \mathcal{F}^{-1} u$ for $u \in \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$. The claimed result easily follows from the continuity of the map $\nu \mapsto (\sigma_\nu)^{\mathrm{w}}_J \mathcal{F}^{-1} u$ on $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$. ☐

3. Estimates for the Dirac propagator

3.1. **The free case.** Consider the Cauchy problem for the free Dirac equation, namely (1) with $V = 0$:

$$\begin{cases} i\partial_t \psi(t,x) = \mathcal{D}_m \psi(t,x), \\ \psi(0,x) = \psi_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{29}$$

The solution can be recast in terms of the free Dirac propagator:

$$\psi(t,x) = \psi_0(x), \qquad U_0(t) = e^{-it\mathcal{D}_m}. \tag{30}$$

We can take advantage from the framework developed insofar by noticing that $U_0(t)$ is an operator-valued Fourier multiplier on the Hilbert space $H = \mathbb{C}^n$, $\mathcal{L}(\mathbb{C}^n) \simeq \mathbb{C}^{n\times n}$, with symbol

$$\mu_t(\xi) = \exp\left[-2\pi i t \left(m\alpha_0 + \sum_{j=1}^d \xi_j \alpha_j \right)\right].$$

An explicit expression for this matrix can be derived. After setting $C_j = -2\pi t \xi_j$, $j = 1, \dots, d$, and $C_0 = -2\pi t m$ we have $\mu_t(\xi) = \sum_{n\ge 0} \frac{i^n}{n!} (\sum_{j=0}^d C_j \alpha_j)^n$. The identities (3) satisfied by the Dirac matrices imply that

$$\begin{cases} (\sum_{j=0}^d C_j\alpha_j)^n = (-1)^k (\sum_{j=0}^d C_j^2)^k I_n & (n = 2k), \\ (\sum_{j=0}^d C_j\alpha_j)^n = i(-1)^k (\sum_{j=0}^d C_j^2)^k (\sum_{j=0}^d C_j\alpha_j) & (n = 2k+1). \end{cases}$$

A straightforward computation finally yields

$$\mu_t(\xi) = \cos(2\pi t\langle\xi\rangle_m)I_n - 2\pi i \frac{\sin(2\pi t\langle\xi\rangle_m)}{2\pi\langle\xi\rangle_m}\left(m\alpha_0 + \sum_{j=1}^{d} \xi_j\alpha_j\right), \tag{31}$$

from which it is clear that $\mu_t \in S^0_{0,0}(\mathbb{R}^d, \mathbb{C}^{n\times n})$ for any fixed $t \in \mathbb{R}$.

Proof of Theorem 1.1. The proof is a direct application of Proposition 2.10 ($X = M^{p,q}_{r,s}(\mathbb{C}^n)$) or Proposition 2.11 ($X = W^{p,q}_{r,s}(\mathbb{C}^n)$), after noticing that

$$\mu_t \in S^0_{0,0}(\mathbb{R}^d, \mathbb{C}^{n\times n}) \hookrightarrow M^{\infty,1}_{0,|r|}(\mathbb{R}^d, \mathbb{C}^{n\times n}) \hookrightarrow W^{1,\infty}_{|r|,0}(\mathbb{R}^d, \mathbb{C}^{n\times n}), \quad \forall r \in \mathbb{R},$$

the latter embedding being given by the Hausdorff-Young inequality (26). □

Proof of estimate (8). In order to determine the time dependence of the constant $C_X(t)$, $X = M^{p,q}_{0,s}(\mathbb{C}^n)$, we provide a different proof by making use of the discrete norm (23) for modulation spaces. Consider the BUPU in the proof of Theorem 2.8. In view of (23) we need to provide an estimate for $\left\| \|\square_k U(t)f\|_{L^p(\mathbb{C}^n)} \right\|_{\ell^q_s}$. We have

$$\|\square_k U(t)f\|_{L^p(\mathbb{C}^n)} = \sum_{|\ell|_\infty\le 1} \left\|\sigma_{k+\ell}\mu_t\sigma_k\hat{f}\right\|_{\mathcal{F}L^p(\mathbb{C}^n)} \le \sum_{|\ell|_\infty\le 1} \|\sigma_{k+\ell}\mu_t\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})} \|\square_k f\|_{L^p(\mathbb{C}^n)},$$

where we used the approximate orthogonality of the frequency-uniform decomposition operators:

$$\square_k = \sum_{|\ell|_\infty\le 1} \square_k\square_{k+\ell}, \quad k \in \mathbb{Z}^d.$$

The multiplier estimate (14) implies

$$\|\sigma_{k+\ell}\mu_t\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})} = \left\|\sigma_0 T_{-(k+\ell)}\mu_t\right\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})} \lesssim (1+|t|)^{d/2},$$

and complex interpolation with the conservation law $\|\square_k U(t)f\|_{L^2(\mathbb{C}^n)} = \|\square_k f\|_{L^2(\mathbb{C}^n)}$ yields

$$\|\square_k U(t)f\|_{L^p(\mathbb{C}^n)} \lesssim (1+|t|)^{d|1/2-1/p|} \|\square_k f\|_{L^p(\mathbb{C}^n)}.$$

□

This behaviour is not surprising, given that any component of a solution of the free Dirac equation is also a solution of the free Klein-Gordon equation, for which similar estimates hold [59, Prop. 6.8]. This connection can be exploited in many ways, as already mentioned in the Introduction; as an example one can easily prove a smoothing estimate for the free Dirac propagator.

Theorem 3.1. *Let $\psi(t,x)$ be the solution of* (29). *For any $t > 1$, $1 \le p,q \le \infty$ and $s \in \mathbb{R}$,*

$$\|\psi(t,\cdot)\|_{M^{p,q}_{0,s}(\mathbb{C}^n)} \lesssim \|\psi_0\|_{M^{p,q}_{0,s}(\mathbb{C}^n)} + |t|^\gamma \|\psi_0\|_{M^{p,q}_{0,s-\gamma}(\mathbb{C}^n)}, \qquad \gamma = d|1/2-1/p|. \tag{32}$$

Proof. Following the same strategy of [40, Thm. 1.1], namely projection onto the so-called positive and negative energy subspaces of the Dirac operator (cf. [53]), it turns out that the free Dirac equation (29) is unitarily equivalent to a pair of $(n/2)$-dimensional square-root Klein-Gordon equations, namely

$$\begin{cases} i\partial_t\psi_\pm(t,x) = \pm\langle D\rangle_m\psi_\pm(t,x), \\ \psi_\pm(0,x) = (\psi_0)_\pm(x), \end{cases} \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d.$$

It is then enough to replace the estimate (3.2) in that paper for the Klein-Gordon semigroup $e^{it\langle D\rangle_m}$ with the smoothing one proved in [17, Thm. 1.4]. The proof then proceeds in the same way. □

3.2. **The case where V is a rough bounded potential.** For any $1\le p,q\le\infty$, $\gamma\ge 0$ and $r,s\in\mathbb{R}$ such that $|r|+|s|\le\gamma$, let X denote either $\mathcal{M}^{p,q}_{r,s}(\mathbb{C}^n)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{C}^n)$. Let $T>0$ be fixed and consider now the Cauchy problem for the Dirac equation with potential

$$\begin{cases} i\partial_t\psi(t,x) = \left(\mathcal{D}_m + V(t)\right)\psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases} \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \tag{33}$$

where $V(t)=\sigma(t,\cdot)^{\mathrm{w}}$, $t\in[0,T]$, and the map $t\mapsto\sigma(t,\cdot)$ is continuous in $M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d},\mathbb{C}^{n\times n})$ for the narrow convergence. Standard arguments from the theory of operators semigroups (cf. [18, Cor. 1.5]) and Theorem 2.13 imply that for any fixed $t\in\mathbb{R}$ the propagator $U(t)$ is bounded on X.

Proof of Theorem 1.2. The argument is standard, we sketch the strategy for the sake of clarity. Set $\Xi_T = C\left([0,T];\mathcal{L}_s(X)\right)$; the assumptions on σ and Theorem 2.17 imply that $V\in\Xi_T$. A straightforward computation shows that the propagator $U(t)$ corresponding to (33) satisfies the following Volterra integral equation:

$$U(t)\psi_0 = U_0\left(t\right)\psi_0 - i\int_0^t U_0(t-s)V(s)U(s)\psi_0 ds. \tag{34}$$

A solution is given by an iterative scheme: let $\{U_n\}_{n\in\mathbb{N}}$ the sequence of operators

$$U_0(t)\equiv e^{-it\mathcal{D}_m}, \qquad U_n(t)\psi_0 \coloneqq \int_0^t U_0(t-s)V(s)U_{n-1}(s)\psi_0\, ds.$$

We have that $\{U_n\}\subset\Xi_T$, since $U_n = U_0 * VU_{n-1}$ and both convolution and composition are bounded operators on Ξ_T; cf. [18, Ex. 1.17.1 and Lem. B.15]. Furthermore, the following estimates hold:

$$\|U_n(t)\|_{\mathcal{L}(X)} \le K(t)^{(n+1)}\frac{t^n}{n!}, \qquad K(t) = \sup_{s\in[0,t]}\|U_0(s)\|\,\|V(s)\|\,.$$

It then follows that the Dyson-Phillips series $\sum_n U_n(t)$ converges with respect to the operator norm on $\mathcal{L}(X)$ and also uniformly on $[0,T]$. Therefore $U(t)=\sum_n U_n(t)\in \Xi_T$ and $U(t)$ is a propagator for (33). Uniqueness follows by Gronwall's lemma after noticing that a different solution $P(t)$ of (34) would satisfy

$$\|(U(t)-P(t))\psi_0\|_X \le K(t)\int_0^t \|(U(\tau)-P(\tau))\psi_0\|_X \, d\tau.$$

$\square$

3.3. **The case where V is a rough quadratic potential.** Theorem 1.3 involves a rough potential V with at most quadratic growth as in (9). A key ingredient for the proof of Theorem 1.3 is the following lemma, which is a qualitative generalization of [45, Lem. 3.3].

Proposition 3.2. *Let $f:\mathbb{R}^d\to E$ be such that $\partial^\alpha f\in M^{\infty,1}(\mathbb{R}^d,E)$ for any $\alpha\in\mathbb{N}^d$, $|\alpha|=k$ for some $k\in\mathbb{N}$. Then there exist $f_1\in C^\infty_{\ge k}(\mathbb{R}^d,E)$ and $f_2\in M^{\infty,1}(\mathbb{R}^d,E)$ such that $f=f_1+f_2$.*

Proof. Fix a smooth cut-off function $\chi\in C_c^\infty\left(\mathbb{R}^d\right)$ supported in a neighbourhood of the origin and such that $\chi=1$ near zero, then consider the Fourier multiplier $\chi(D)$ with symbol χ. Set $f_1=\chi(D)f$ and $f_2=(I-\chi(D))f$. Clearly $f=f_1+f_2$ and we argue that f_1 and f_2 satisfy the claimed properties.

Indeed, $f_1\in C^\infty(\mathbb{R}^d,E)$ and for any $\alpha\in\mathbb{N}^d$, $|\alpha|=k$, we have

$$\partial^\alpha f_1=\partial^\alpha(\chi(D)f)=\chi(D)(\partial^\alpha f)\in M^{\infty,1}(\mathbb{R}^d,E),$$

since $\partial^\alpha\chi(D)$ is a Fourier multiplier with symbol $(2\pi i\xi)^\alpha\chi(\xi)\in C_c^\infty\left(\mathbb{R}^d\right)$, hence $\partial^\alpha\chi(D)=\chi(D)\partial^\alpha$ and $\chi(D)$ is continuous on $M^{\infty,1}(E)$ by Proposition 2.10. Furthermore, similar arguments imply that for any $\alpha\in\mathbb{N}^d$, $|\alpha|\ge k$,

$$\partial^\alpha f_1=\partial^{\alpha-\beta}\partial^\beta(\chi(D)f)=(\partial^{\alpha-\beta}\chi(D))(\partial^\beta f)\in M^{\infty,1}(\mathbb{R}^d,E).$$

where $\beta\in\mathbb{N}^d$ satisfies $|\beta|=k$.

In order to prove the claim for f_2 consider the finite smooth partition of unity $\{\varphi_j\}_{j=1}^N$ of the unit sphere $S^{d-1}\subset\mathbb{R}^d$ subordinated to the open cover $\{U_j\}_{j=1}^d$, where

$$U_j=\{x\in S^{d-1}\,:\,x_j\neq 0\}.$$

Then we extend each function φ_j on $\mathbb{R}^d\setminus\{0\}$ by zero-degree homogeneity, namely

$$\sum_{j=1}^d \varphi_j\left(x\right)=1,\qquad \varphi_j\left(\alpha x\right)=\varphi_j\left(x\right),\qquad \forall x\in S^{d-1},\,\alpha>0.$$

This procedure gives a finite partition of unity $\{\varphi_j\}_{k=1}^d$ on $\mathbb{R}^d\setminus\{0\}$. Then

$$\begin{aligned} f_2(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}(1-\chi(\xi))\hat{f}(\xi)d\xi \\ &= \sum_{j=1}^d \left[\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\left(\frac{1-\chi(\xi)}{(2\pi i\xi_j)^k}\varphi_j(\xi)\right)\widehat{\partial_j^k f}(\xi)d\xi\right] \\ &= \sum_{j=1}^d \widetilde{\chi_j}(D)(\partial_j^k f)(x) \end{aligned}$$

and thus $f_2 \in M^{\infty,1}(\mathbb{R}^d, E)$ since each $\widetilde{\chi_j}(D)$ is a Fourier multiplier with symbol $(1-\chi(\xi))\,\varphi_j(\xi)/(2\pi i\xi_j)^k \in S^0_{0,0}\left(\mathbb{R}^d\right)$, hence bounded on $M^{\infty,1}(\mathbb{R}^d, E)$. □

Proof of Theorem 1.3. We apply Proposition 3.2 twice, namely to L and Q. We get

- $L = L_1 + L_2$, where $L_1 \in C^\infty_{\geq 1}(\mathbb{R}^d, \mathbb{C}^{n\times n})$ and $L_2 \in M^{\infty,1}(\mathbb{R}^d, \mathbb{C}^{n\times n})$, and
- $Q = Q_1 + Q_2$, where $Q_1 \in C^\infty_{\geq 2}(\mathbb{R}^d)$ and $Q_2 \in M^{\infty,1}(\mathbb{R}^d)$.

The RHS of (1) then becomes

$$\mathcal{H} = (\mathcal{D}_m + L_1 + Q_1) + (L_2 + Q_2 + \sigma^{\mathrm{w}}) =: \mathcal{H}_0 + V'.$$

We see that $e^{-it\mathcal{H}_0}$ is a semigroup of bounded operators on $\mathcal{M}^p(\mathbb{R}^d, \mathbb{C}^n)$ as a consequence of [40, Thm. 1.2]. It is understood that we identify the multiplication by a function $f \in M^{\infty,1}(\mathbb{R}^d, \mathbb{C})$ on $M^{p,q}(\mathbb{C}^n)$ with the operator $fI_n \in \mathbb{C}^{n\times n}$, hence by Remark 2.9 we have

$$\|fu\|_{M^{p,q}(\mathbb{C}^n)} \leq \|fI_n\|_{M^{\infty,1}(\mathbb{C}^{n\times n})} \|u\|_{M^{p,q}(\mathbb{C}^n)} \asymp \|f\|_{M^{\infty,1}} \|u\|_{M^{p,q}(\mathbb{C}^n)}.$$

The boundedness of $e^{-it\mathcal{H}}$ on $\mathcal{M}^p(\mathbb{R}^d, \mathbb{C}^n)$ then follows from the fact that V' is a bounded perturbation of $\mathcal{H}_0$ [18, Cor. 1.5] by Proposition 2.10 and Theorem 2.13. The case where $Q = 0$ follows by the same arguments. □

4. The nonlinear equation

A standard tool in the study of local well-posedness is the following abstract result.

Theorem 4.1 ([52, Prop. 1.38])**.** *Let X and Y be two Banach spaces and $D : X \to Y$ be a bounded linear operator such that*

$$\|Du\|_Y \leq C_0 \|u\|_X, \tag{35}$$

for all $u \in X$ and some $C_0 > 0$. Consider then a nonlinear operator $F : Y \to X$, $F(0) = 0$, such that

$$\|F(u) - F(v)\|_X \leq \frac{1}{2C_0}\|u - v\|_Y, \tag{36}$$

for all u, v in the ball $B_\epsilon(0) = \{u \in Y : \|u\|_Y \le \epsilon\}$ for some $\epsilon > 0$. Then for any $u_0 \in B_{\epsilon/2}$ there exists a unique solution $u \in B_\epsilon$ to the equation

$$u = u_0 + DF(u),$$

and the map $u_0 \mapsto u$ is Lipschitz with constant at most 2, that is $\|u\|_Y \le 2\|u_0\|_Y$.

With that in mind, for the sake of clarity we anticipate some estimates for the nonlinearity (11).

Lemma 4.2. *Let $r, s \ge 0$, $1 \le p \le \infty$ and $\epsilon > 0$, and consider a nonlinear function F as in (11). Denote by X any of the spaces $M^{p,1}_{0,s}(\mathbb{C}^n)$ or $W^{1,p}_{r,s}(\mathbb{C}^n)$. If $\psi_0 \in X$ then $F(\psi) \in X$ and, for any $\psi, \phi \in B_\epsilon(0) \subset X$ there exists a constant $C_\epsilon > 0$ such that*

$$\|F(\psi) - F(\phi)\|_X \le C_\epsilon \|\psi - \phi\|_X .$$

Proof. In view of Proposition (2.5) (iv) and its counterpart for amalgam spaces the first claim is an easy consequence of the algebra property of X under pointwise multiplication [11, Lem. 2.1-2.2] and the series expansion of each component. The estimate in the second part follows from a straightforward computation (cf. the proof of [11, Thm. 4.1]), that is

$$\begin{aligned} F_j(\psi) - F_j(\phi) &= \int_0^1 \frac{d}{dt} F_j(t\psi + (1-t)\phi) dt \\ &= \sum_{k=1}^n \Bigg[(\psi_k - \phi_k) \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} c^{j,k}_{\alpha,\beta,\gamma,\delta} \psi^\alpha \bar{\psi}^\beta \phi^\delta \bar{\phi}^\gamma \\ &\quad + (\bar{\psi}_k - \bar{\phi}_k) \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} \tilde{c}^{j,k}_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} \psi^\alpha \bar{\psi}^\beta \phi^\delta \bar{\phi}^\gamma \Bigg]. \end{aligned}$$

Again by Proposition 2.5 (iv) we have

$$\|F(\psi) - F(\phi)\|_X \lesssim \|\psi - \phi\|_X \sum_{j,k=1}^n \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} C^{j,k}_{\alpha,\beta,\gamma,\delta} \|\psi\|_X^{|\alpha+\beta|} \|\phi\|_X^{|\gamma+\delta|},$$

with $C^{j,k}_{\alpha,\beta,\gamma,\delta} = |c^{j,k}_{\alpha,\beta,\gamma,\delta}| + |\tilde{c}^{j,k}_{\alpha,\beta,\gamma,\delta}|$, and the latter expression is $\le C_\epsilon \|\psi - \phi\|_X$ whenever $\psi, \phi \in B_\epsilon(0)$. ☐

Proof of Theorem 1.4. The proof is an application of the iteration scheme given in Theorem 4.1. In particular we choose either $X = M^{p,1}_{0,s}(\mathbb{C}^n)$ or $X = W^{1,p}_{r,s}(\mathbb{C}^n)$, then $Y = C^0([0,T], X)$, and convert (10) in integral form:

$$\psi(t) = U_0(t)\psi_0 - i \int_0^t U_0(t-s) F(\psi(s)) ds,$$

where $U_0 = e^{-it\mathcal{D}_m}$ is the free propagator. It is then enough to prove (35) and (36) in this setting, where D is the Duhamel operator $D = \int_0^t U_0(t-s)\cdot ds$. First, notice that from Theorem 1.1 we have that

$$\|U_0(t)\psi_0\|_X \leq C_T \|\psi_0\|_X, \quad \forall t \in [0,T].$$

Therefore,

$$\left\| \int_0^t U_0(t-s)u(s)ds \right\|_X \leq \int_0^t \|U_0(t-s)u(s)\|_X \, ds \leq TC_T \sup_{t\in[0,T]} \|u(t)\|_X .$$

Lemma 4.2 then provides (35) with a constant $C_0 = O(T)$ and also (36). The claim follows after choosing $T = T(\|\psi_0\|_X)$ sufficiently small. $\square$

Remark 4.3. *A more general version of Theorem 1.4, namely a nonlinear variant of Theorem 1.2, can be stated. For any $1 \leq p \leq \infty$ and $\gamma \geq 0$ let X denote either $\mathcal{M}^{p,1}_{0,s}(\mathbb{C}^n)$ with $0 \leq s \leq \gamma$ or $\mathcal{W}^{1,p}_{r,s}(\mathbb{C}^n)$ with $r,s \geq 0$ such that $r+s \leq \gamma$. The differential operator $L = i\partial_t - \mathcal{D}_m$ in* (10)*, namely $L\psi = F(\psi)$, is now extended to $L = i\partial_t - \mathcal{D}_m - \sigma_t^{\mathrm{w}}$, where the symbol map $[0,T] \ni t \mapsto \sigma(t,\cdot) \in M^{\infty,1}_{0,2\gamma}(\mathbb{C}^{n\times n})$ is continuous for the narrow convergence and the nonlinear term is* (11)*. We recast the problem in integral form as*

$$\psi(t) = U(t,0)\psi_0 - i\int_0^t U(t,\tau)F(\psi(\tau))d\tau,$$

where $U(t,\tau)$, $0 \leq \tau \leq t \leq T$ is the linear propagator constructed in the proof of Theorem 1.2 corresponding to initial data at time τ. In order for the iteration scheme in Theorem 4.1 to work it is enough to prove that $U(t,\tau)$ is strongly continuous on X jointly in (t,τ), $0 \leq \tau \leq t \leq T$; the latter condition would imply a uniform bound for the operator norm with respect to t,τ as a consequence of the uniform boundedness principle. Theorem 1.2 yields strong continuity of $U(t,\tau)$ in t for fixed τ. The time-reversibility enjoyed by the equation implies that the same holds after switching τ and t. Furthermore, for $\tau' \leq \tau \leq t$ we have

$$\|U(t,\tau)\psi_0 - U(t,\tau')\psi_0\|_X \leq C \|\psi_0 - U(\tau,\tau')\psi_0\|_X ,$$

hence the map $\tau \mapsto U(t,\tau)\psi_0$ is continuous in X, uniformly with respect to t and this gives the desired result.

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References

[1] Amann, Herbert. *Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory.* Monographs in Mathematics, 89. Birkhäuser Boston, 1995.

[2] Amann, Herbert. *Linear and quasilinear parabolic problems. Vol. II. Function spaces.* Monographs in Mathematics, 106. Birkhäuser Basel, 2019.

[3] Arendt, Wolfgang; Batty, Charles J. K.; Hieber, Matthias; Neubrander, Frank. *Vector-valued Laplace transforms and Cauchy problems.* Second edition. Monographs in Mathematics, 96. Birkhäuser/Springer Basel AG, Basel, 2011.

[4] Bejenaru, Ioan; Herr, Sebastian. The cubic Dirac equation: small initial data in $H^{\frac{1}{2}}(\mathbb{R}^2)$. *Comm. Math. Phys.* **343** (2016), no. 2, 515–562.

[5] Bényi, Árpád; Okoudjou, Kasso A. Local well-posedness of nonlinear dispersive equations on modulation spaces. *Bull. Lond. Math. Soc.* **41** (2009), no. 3, 549–558.

[6] Bényi, Árpád; Gröchenig, Karlheinz; Okoudjou, Kasso A.; Rogers, Luke G. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.* **246** (2007), no. 2, 366–384.

[7] Cacciafesta, Federico; D'Ancona, Piero. Endpoint estimates and global existence for the nonlinear Dirac equation with potential. *J. Differential Equations* **254** (2013), no. 5, 2233–2260.

[8] Cacciafesta, Federico; Fanelli, Luca. Dispersive estimates for the Dirac equation in an Aharonov-Bohm field. *J. Differential Equations* **263** (2017), no. 7, 4382–4399.

[9] Chen, Jiecheng; Fan, Dashan. Estimates for wave and Klein-Gordon equations on modulation spaces. *Sci. China Math.* **55** (2012), no. 10, 2109–2123.

[10] Cordero, Elena; Nicola, Fabio. Some new Strichartz estimates for the Schrödinger equation. *J. Differential Equations* **245** (2008), no. 7, 1945–1974.

[11] Cordero, Elena; Nicola, Fabio. Remarks on Fourier multipliers and applications to the wave equation. *J. Math. Anal. Appl.* **353** (2009), no. 2, 583–591.

[12] Cordero, Elena; Nicola, Fabio. On the Schrödinger equation with potential in modulation spaces. *J. Pseudo-Differ. Oper. Appl.* **5** (2014), no. 3, 319–341.

[13] Cordero, Elena; Nicola, Fabio; Rodino, Luigi. Schrödinger equations with rough Hamiltonians. *Discrete Contin. Dyn. Syst.* **35** (2015), no. 10, 4805–4821.

[14] Cordero, Elena; Nicola, Fabio; Trapasso, S. Ivan. Almost Diagonalization of τ-Pseudodifferential Operators with Symbols in Wiener Amalgam and Modulation Spaces. *J. Fourier Anal. Appl.* **25** (2019), no. 4, 1927–1957.

[15] D'Ancona, Piero; Fanelli, Luca. Decay estimates for the wave and Dirac equations with a magnetic potential. *Comm. Pure Appl. Math.* **60** (2007), no. 3, 357–392.

[16] de Gosson, Maurice A. *Symplectic methods in harmonic analysis and in mathematical physics.* Pseudo-Differential Operators. Theory and Applications, 7. Birkhäuser/Springer Basel AG, Basel, 2011.

[17] Deng, Qingquan; Ding, Yong; Sun, Lijing. Estimate for generalized unimodular multipliers on modulation spaces. *Nonlinear Anal.* **85** (2013), 78–92.

[18] Engel, Klaus-Jochen; Nagel, Rainer. *A short course on operator semigroups.* Universitext. Springer, New York, 2006.

[19] Erdoğan, M. Burak; Goldberg, Michael; Green, William R. Limiting absorption principle and Strichartz estimates for Dirac operators in two and higher dimensions. *Comm. Math. Phys.* 367 (2019), no. 1, 241–263.

[20] Feichtinger, Hans G. On a new Segal algebra. *Monatsh. Math.* **92**(4) (1981), 269–289.

[21] Feichtinger, Hans G. Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, 1983.

[22] Feichtinger, Hans G. Banach convolution algebras of Wiener type. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, 509–524, Colloq. Math. Soc. János Bolyai, 35, North-Holland, Amsterdam, 1983.

[23] Feichtinger, Hans G. Banach spaces of distributions defined by decomposition methods. II. *Math. Nachr.* **132** (1987), 207–237.

[24] Feichtinger, Hans G. Modulation spaces: looking back and ahead. *Sampl. Theory Signal Image Process.* **5** (2006), no. 2, 109–140.

[25] Feichtinger, Hans G.; Gröbner, Peter. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.* **123** (1985), 97–120.

[26] Feichtinger, Hans G.; Hörmann, Wolfgang. A distributional approach to generalized stochastic processes on locally compact Abelian groups. In *New perspectives on approximation and sampling theory*, 423–446, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2014.

[27] Folland, Gerald B. *Harmonic analysis in phase space.* Annals of Mathematics Studies, 122. Princeton University Press, Princeton, NJ, 1989.

[28] Girardi, Maria; Weis, Lutz. Vector-valued extensions of some classical theorems in harmonic analysis. In *Analysis and applications—ISAAC 2001 (Berlin)*, 171–185, Int. Soc. Anal. Appl. Comput., 10, Kluwer Acad. Publ., Dordrecht, 2003.

[29] Gröchenig, Karlheinz. Time-frequency analysis of Sjöstrand's class. *Rev. Mat. Iberoam.* **22** (2006), no. 2, 703–724.

[30] Gröchenig, Karlheinz. *Foundations of time-frequency analysis.* Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2001.

[31] Gröchenig, Karlheinz; Rzeszotnik, Ziemowit. Banach algebras of pseudodifferential operators and their almost diagonalization. *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 7, 2279–2314.

[32] Heil, Christopher. An introduction to weighted Wiener amalgams. In *Wavelets and their Applications (Chennai, January 2002)*, M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Dehli (2003), pp. 183–216.

[33] Hörmander, Lars. *The analysis of linear partial differential operators. III. Pseudo-differential operators.* Reprint of the 1994 edition. Classics in Mathematics. Springer, Berlin, 2007.

[34] Huh, Hyungjin. Global strong solution to the Thirring model in critical space. *J. Math. Anal. Appl.* **381** (2011), no. 2, 513–520.

[35] Hytönen, Tuomas; van Neerven, Jan; Veraar, Mark; Weis, Lutz. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory.* A Series of Modern Surveys in Mathematics, 63. Springer, Cham, 2016.

[36] Hytönen, Tuomas; Portal, Pierre. Vector-valued multiparameter singular integrals and pseudodifferential operators. *Adv. Math.* **217** (2008), no. 2, 519–536.

[37] Kalf, Hubert; Yamada, Osanobu. Essential self-adjointness of n-dimensional Dirac operators with a variable mass term. *J. Math. Phys.* **42** (2001), no. 6, 2667–2676.

[38] Kato, Keiichi; Kobayashi, Masaharu; Ito, Shingo. Representation of Schrödinger operator of a free particle via short-time Fourier transform and its applications. *Tohoku Math. J. (2)* **64** (2012), no. 2, 223–231.

[39] Kato, Keiichi; Kobayashi, Masaharu; Ito, Shingo. Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials. *J. Funct. Anal.* **266** (2014), no. 2, 733–753.

[40] Kato, Keiichi; Naumkin, Ivan. Estimates on the modulation spaces for the Dirac equation with potential. *Rev. Mat. Complut.* **32** (2019), no. 2, 305–325.

[41] Kerman, Ronald A. Convolution theorems with weights. *Trans. Amer. Math. Soc.* **280** (1983), no. 1, 207–219.

[42] Kwapień, Stanisław. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Math.* **44** (1972), 583–595.

[43] Machihara, Shuji; Nakanishi, Kenji; Ozawa, Tohru. Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation. *Rev. Mat. Iberoamericana* **19** (2003), no. 1, 179–194.

[44] Naumkin, I. P. Initial-boundary value problem for the one dimensional Thirring model. *J. Differential Equations* **261** (2016), no. 8, 4486–4523.

[45] Nicola, Fabio; Trapasso, S. Ivan. On the pointwise convergence of the integral kernels in the Feynman-Trotter formula. To appear in *Comm. Math. Phys.*, 2019.

[46] Ozawa, Tohru; Yamauchi, Kazuyuki. Structure of Dirac matrices and invariants for nonlinear Dirac equations. *Differential Integral Equations* **17** (2004), no. 9-10, 971–982.

[47] Pelinovsky, Dmitry. Survey on global existence in the nonlinear Dirac equations in one spatial dimension. *Harmonic analysis and nonlinear partial differential equations*, 37–50, RIMS Kôkyûroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.

[48] Reich, Maximilian; Sickel, Winfried. Multiplication and composition in weighted modulation spaces. In *Mathematical analysis, probability and applications—plenary lectures*, 103–149, Springer Proc. Math. Stat., 177, Springer, Cham, 2016.

[49] Sjöstrand, Johannes. An algebra of pseudodifferential operators. *Math. Res. Lett.* **1** (1994), no. 2, 185–192.

[50] Soler, Mario. Classical, stable, nonlinear spinor field with positive rest energy. *Phys. Rev. D* **1** (1970), 2766-–2769.

[51] Sugimoto, Mitsuru; Tomita, Naohito; Wang, Baoxiang. Remarks on nonlinear operations on modulation spaces. *Integral Transforms Spec. Funct.* **22** (2011), no. 4-5, 351–358.

[52] Tao, Terence. *Nonlinear dispersive equations. Local and global analysis.* CBMS Reg. Conf. Ser. Math., Amer. Math. Soc., 2006

[53] Thaller, Bernd. *The Dirac equation.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[54] Thirring, Walter E. A soluble relativistic field theory. *Ann. Physics* **3** (1958), 91–112.

[55] Toft, Joachim. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.* **207** (2004), no. 2, 399–429.

[56] Toft, Joachim. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. *Ann. Global Anal. Geom.* **26** (2004), no. 1, 73–106.

[57] Wahlberg, Patrik. Vector-valued modulation spaces and localization operators with operator-valued symbols. *Integral Equations Operator Theory* **59** (2007), no. 1, 99–128.

[58] Wang, Baoxiang; Hudzik, Henryk. The global Cauchy problem for the NLS and NLKG with small rough data. *J. Differential Equations* **232** (2007), no. 1, 36–73.

[59] Wang, Baoxiang; Huo, Zhaohui; Hao, Chengchun; Guo, Zihua. *Harmonic analysis method for nonlinear evolution equations. I.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[60] Weis, Lutz. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.* **319** (2001), no. 4, 735–758.

[61] Zhao, Guoping; Chen, Jiecheng; Guo, Weichao. Klein-Gordon equations on modulation spaces. *Abstr. Appl. Anal.* 2014, Art. ID 947642, 15 pp.

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