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A NOTE ON PRODUCT SETS OF RANDOM SETS

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Abstract. Given two sets of positive integers A and B, let $AB := \{ab : a \in A,\, b \in B\}$ be their *product set* and put $A^k := A \cdots A$ (k times A) for any positive integer k. Moreover, for every positive integer n and every $\alpha \in [0, 1]$, let $\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $A \subseteq \{1, \ldots, n\}$ is constructed by choosing independently every element of $\{1, \ldots, n\}$ with probability α. We prove that if $A_1, \ldots, A_s$ are random sets in $\mathcal{B}(n_1, \alpha_1), \ldots, \mathcal{B}(n_s, \alpha_s)$, respectively, $k_1, \ldots, k_s$ are fixed positive integers, $\alpha_i n_i \to +\infty$, and $1/\alpha_i$ does not grow too fast in terms of a product of $\log n_j$; then $|A_1^{k_1} \cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!} \cdots \frac{|A_s|^{k_s}}{k_s!}$ with probability $1 - o(1)$. This is a generalization of a result of Cilleruelo, Ramana, and Ramaré, who considered the case $s = 1$ and $k_1 = 2$.

1. Introduction

Given two sets of positive integers A and B, let $AB := \{ab : a \in A,\, b \in B\}$ be their *product set* and put $A^k := A \cdots A$ (k times A) for any positive integer k.

Problems involving the cardinalities of product sets have been considered by many researchers. For example, the study of $M_n := |\{1, \ldots, n\}^2|$ as $n \to +\infty$ is known as the "multiplicative table problem" and was started by Erdős [2, 3]. The exact order of magnitude of M_n was determined by Ford [4] following earlier work of Tenenbaum [8]. Furthermore, Koukoulopoulos [7] provided uniform bounds for $|\{1, \ldots, n_1\} \cdots \{1, \ldots, n_s\}|$ holding for a wide range of $n_1, \ldots, n_s$. Cilleruelo, Ramana, and Ramaré [1] proved asymptotics or bounds for $|(A \cap \{1, \ldots, n\})^2|$ when A is the set of shifted prime numbers, the set of sums of two squares, or the set of shifted sums of two squares.

For every positive integer n and every $\alpha \in [0, 1]$, let $\mathcal{B}(n, \alpha)$ denote the probabilistic model in which a random set $A \subseteq \{1, \ldots, n\}$ is constructed by choosing independently every element of $\{1, \ldots, n\}$ with probability α. Cilleruelo, Ramana, and Ramaré [1] proved the following:

Theorem 1.1. *Let A be a random set in $\mathcal{B}(n, \alpha)$. If $\alpha n \to +\infty$ and $\alpha = o((\log n)^{-1/2})$, then $|A^2| \sim \frac{|A|^2}{2}$ with probability $1 - o(1)$.*

The contribution of this paper is the following generalization of Theorem 1.1.

Theorem 1.2. *Let $A_1, \ldots, A_s$ be random sets in $\mathcal{B}(n_1, \alpha_1), \ldots, \mathcal{B}(n_s, \alpha_s)$, respectively; and let $k_1, \ldots, k_s$ be fixed positive integers. If $\alpha_i n_i \to +\infty$ and*

$$\alpha_i = o\left(\left((\log n_1)^{k_1 - 1} \prod_{i=2}^{s} (\log n_i)^{k_i}\right)^{-(k_1 + \cdots + k_s - 1)/2}\right),$$

for $i = 1, \ldots, s$, then $|A_1^{k_1} \cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!} \cdots \frac{|A_s|^{k_s}}{k_s!}$ with probability $1 - o(1)$.

2. Notation

We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbol $\ll$, with their usual meanings. Any dependence of implied constants is explicitly stated or indicated with subscripts. For real random variables X and

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Y, we say that "$X = o(Y)$ with probability $1 - o(1)$" if $\mathbb{P}(|X| \geq \varepsilon|Y|) = o_\varepsilon(1)$ for every $\varepsilon > 0$, and that "$X \sim Y$ with probability $1 - o(1)$" if $X = Y + o(Y)$ with probability $1 - o(1)$.

3. Preliminaries

In this section we collect some preliminary results not directly related with product sets.

Lemma 3.1. *Let m be a positive integer. We have*

$$\sum_{a_1 \cdots a_m \leq x} \frac{1}{a_1 \cdots a_m} \ll_m (\log x)^m,$$

for all $x \geq 2$.

Proof. This is a standard application of Rankin's method: For $t := m/\log x$, we have

$$\begin{aligned}\sum_{a_1 \cdots a_m \leq x} \frac{1}{a_1 \cdots a_m} &\leq x^t \sum_{a_1 \cdots a_m \leq x} \frac{1}{(a_1 \cdots a_m)^{1+t}} \leq x^t \left(\sum_{a=1}^{\infty} \frac{1}{a^{1+t}}\right)^m \\ &\leq x^t \left(1 + \frac{1}{t}\right)^m \ll_m (\log x)^m,\end{aligned}$$

as claimed. ☐

The next lemma is an upper bound on the number of matrices of positive integers with bounded products of rows and columns.

Lemma 3.2. *Let m and n be positive integers. Then, for all $x_1, \dots, x_n, y_1, \dots, y_m \geq 2$, the number of $m \times n$ matrices $(c_{i,j})$ of positive integers satisfying $\prod_{i=1}^m c_{i,h} \leq x_h$ and $\prod_{j=1}^n c_{k,j} \leq y_k$, for $h = 1, \dots, n$ and $k = 1, \dots, m$, is at most*

$$O_{m,n}\left(\left(\prod_{i=1}^{n} x_i \prod_{j=1}^{m} y_j\right)^{1/2} \left(\prod_{i=1}^{n-1} \log x_i\right)^{m-1}\right) \tag{1}$$

Proof. We follow the same arguments of [5, p. 380], where the case $m = n$ and $x_1 = \cdots = x_n = y_1 = \cdots = y_m$ is proved.

The number of choices for $c_{m,n}$ is at most

$$\min\left(\frac{x_n}{\prod_{i=1}^{m-1} c_{i,n}}, \frac{y_m}{\prod_{j=1}^{n-1} c_{m,j}}\right) \leq \left(\frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}}\right)^{1/2}.$$

We shall sum this latter quantity over all the choices of $c_{i,n}$ and $c_{m,j}$, with $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$. Since $c_{i,n} \leq y_i / \prod_{k=1}^{n-1} c_{i,k}$ and $c_{m,j} \leq x_j / \prod_{h=1}^{m-1} c_{h,j}$, we have

$$\sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}} \ll \left(\frac{y_i}{\prod_{k=1}^{n-1} c_{i,k}}\right)^{1/2} \quad \text{and} \quad \sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}} \ll \left(\frac{x_j}{\prod_{h=1}^{m-1} c_{h,j}}\right)^{1/2},$$

for $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$. Consequently,

$$\begin{aligned}\sum_{\substack{c_{1,n}, \dots, c_{m-1,n} \\ c_{m,1}, \dots, c_{m,n-1}}} \left(\frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}}\right)^{1/2} &\leq (x_n y_m)^{1/2} \prod_{i=1}^{m-1} \left(\sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}}\right) \prod_{j=1}^{n-1} \left(\sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}}\right) \\ &\ll_{m,n} \left(\prod_{j=1}^{n} x_j \prod_{i=1}^{m} y_i\right)^{1/2} \left(\prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k}\right)^{-1}.\end{aligned}$$

It remains only to sum over all the possibilities for $c_{h,k}$, with $h = 1, \dots, m-1$ and $k = 1, \dots, n-1$. Thanks to Lemma 3.1, we have

$$\sum_{c_{h,k}} \left(\prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k} \right)^{-1} \leq \prod_{k=1}^{n-1} \sum_{c_{1,k} \cdots c_{m-1,k} \leq x_k} \frac{1}{c_{1,k} \cdots c_{m-1,k}} \ll_{m,n} \left(\prod_{k=1}^{n-1} \log x_k \right)^{m-1},$$

and the desired result follows. □

The next lemma is an upper bound for the number of solutions of a certain multiplicative equation with bounded factors.

Lemma 3.3. *Let m and n be positive integers. Then, for all $x_1, \dots, x_n, y_1, \dots, y_m \geq 2$, the number of solutions of the equation $a_1 \cdots a_n = b_1 \cdots b_m$, where $a_1, \dots, a_n, b_1, \dots, b_m$ are positive integers satisfying $a_i \leq x_i$ and $b_j \leq y_j$, for $i = 1, \dots, n$ and $j = 1, \dots, m$, is at most (1).*

Proof. If $a_1 \cdots a_n = b_1 \cdots b_m$ then there exists a $m \times n$ matrix of positive integers $(c_{i,j})$ such that $a_h = \prod_{i=1}^m c_{i,h}$ and $b_k = \prod_{j=1}^n c_{k,j}$, for $h = 1, \dots, n$ and $k = 1, \dots, m$. Indeed, $a_1 \mid \prod_{i=1}^m b_i$ implies the existence of positive integers $c_{1,1}, \dots, c_{m,1}$ such that $a_1 = \prod_{i=1}^m c_{i,1}$ and $c_{i,1} \mid b_i$, for $i = 1, \dots, m$. Then $a_2 \mid \prod_{i=1}^m b_i / c_{i,1}$, which similary implies the existence of positive integers $c_{1,2}, \dots, c_{m,2}$ such that $a_2 = \prod_{i=1}^m c_{i,2}$ and $c_{i,1} c_{i,2} \mid b_i$, for $i = 1, \dots, m$. Then $a_3 \mid \prod_{i=1}^m b_i / (c_{i,1} c_{i,2})$, and so on, until $a_n = \prod_{i=1}^m b_i / (\prod_{j=1}^{n-1} c_{i,j})$, when we set $c_{i,n} := b_i / \prod_{j=1}^{n-1} c_{i,j}$ for $i = 1, \dots, m$. Applying Lemma 3.2 we get the desired result. □

4. Proof of Theorem 1.2

First, we need an asymptotic for the kth power of the size of a random set A in $\mathcal{B}(n, \alpha)$.

Lemma 4.1. *Let A be a random set in $\mathcal{B}(n, \alpha)$, and fix an integer $k \geq 1$. If $\alpha n \to +\infty$, then:*

(i) $\mathbb{E}(|A|^k) \sim (\alpha n)^k$*; and*

(ii) $|A|^k \sim (\alpha n)^k$ *with probability* $1 - o_k(1)$.

Proof. Clearly, $|A|$ follows a binomial distribution with n trials and probability of success α. Consequently, (i) is known (see, e.g., [6, Eq. (4.1)]). In turn, (i) implies that

$$\mathbb{V}(|A|^k) = \mathbb{E}(|A|^{2k}) - \mathbb{E}(|A|^k)^2 = o_k(\mathbb{E}(|A|^k)^2).$$

Hence, by Chebyshev's inequality, for every $\varepsilon > 0$ we have

$$\mathbb{P}\Big(\big| |A|^k - \mathbb{E}(|A|^k) \big| \geq \varepsilon \, \mathbb{E}(|A|^k) \Big) \leq \frac{\mathbb{V}(|A|^k)}{(\varepsilon \, \mathbb{E}(|A|^k))^2} = o_{k,\varepsilon}(1),$$

so that $|A|^k \sim \mathbb{E}(|A|^k) \sim (\alpha n)^k$ with probability $1 - o_k(1)$. □

The next lemma is an easy bound on the size of a product set.

Lemma 4.2. *Let $A_1, \dots, A_s$ be finite sets of positive integers, and let $k_1, \dots, k_s \geq 1$ be integers. Then*

$$\left| \prod_{i=1} A_i^{k_i} \right| \leq \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i}.$$

Proof. The claim follows easily considering that $\binom{|A|+k-1}{k}$ is the number of unordered k-tuples of elements from a set A. □

For the rest of this section, let $A_1, \dots, A_s$ be random sets in $\mathcal{B}(n_1, \alpha_1), \dots, \mathcal{B}(n_s, \alpha_s)$, respectively; and let $k_1, \dots, k_s$ be fixed positive integers. Also, assume $\alpha_i n_i \to +\infty$ and

$$\alpha_i = o\left(\left((\log n_1)^{k_1 - 1} \prod_{i=2}^{s} (\log n_i)^{k_i} \right)^{-(k_1 + \cdots + k_s - 1)/2} \right), \tag{2}$$

for $i = 1, \dots, s$. For brevity, we will omit the dependence of implied constants from $k_1, \dots, k_s$.

Lemma 4.3. *We have* $\mathbb{E}(|A_1^{k_1}\cdots A_s^{k_s}|) \sim \frac{(\alpha_1 n_1)^{k_1}}{k_1!}\cdots\frac{(\alpha_s n_s)^{k_s}}{k_s!}$.

Proof. Hereafter, in operator subscripts, let $\boldsymbol{a} := (\boldsymbol{a}_1, \dots, \boldsymbol{a}_s)$, where each $\boldsymbol{a}_i := \{a_{i,1}, \dots, a_{i,k_i}\}$ runs over the unordered k_i-tuples of elements of $\{1, \dots, n_i\}$. Also, put $\|\boldsymbol{a}\| := \prod_{i=1}^s \prod_{j=1}^{k_i} a_{i,j}$. With this notation, for each positive integer x, we have

$$\mathbb{P}(x \in A_1^{k_1}\cdots A_s^{k_s}) = \mathbb{P}\left(\bigvee_{\|\boldsymbol{a}\|=x} E_{\boldsymbol{a}}\right),$$

where

$$E_{\boldsymbol{a}} := \bigwedge_{i=1}^s (\boldsymbol{a}_i \subseteq A_i).$$

Consequently, by Bonferroni inequalities, we have

$$\begin{aligned}\mathbb{P}(x \in A_1^{k_1}\cdots A_s^{k_s}) &= \mathbb{P}\left(\sideset{}{^*}\bigvee_{\|\boldsymbol{a}\|=x} E_{\boldsymbol{a}}\right) + O\left(\sideset{}{^{**}}\sum_{\|\boldsymbol{a}\|=x} \mathbb{P}(E_{\boldsymbol{a}})\right) \\ &= \sideset{}{^*}\sum_{\|\boldsymbol{a}\|=x} \mathbb{P}(E_{\boldsymbol{a}}) + O\left(\sideset{}{^*}\sum_{\substack{\boldsymbol{a}\neq\boldsymbol{a}' \\ \|\boldsymbol{a}\|=\|\boldsymbol{a}'\|=x}} \mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'})\right) + O\left(\sideset{}{^{**}}\sum_{\|\boldsymbol{a}\|=x} \mathbb{P}(E_{\boldsymbol{a}})\right),\end{aligned}$$

where the superscript * denotes the constraint $|\boldsymbol{a}_i| = k_i$ for every $i \in \{1, \dots, s\}$, the superscript ** denotes the complementary constrain $|\boldsymbol{a}_i| < k_i$ for at least one $i \in \{1, \dots, k\}$, and $\boldsymbol{a}' := (\boldsymbol{a}_1', \dots, \boldsymbol{a}_s')$ follows the same conventions of $\boldsymbol{a}$. Therefore,

$$\begin{aligned}\mathbb{E}(|A_1^{k_1}\cdots A_s^{k_s}|) &= \sum_{x \le n_1^{k_1}\cdots n_s^{k_s}} \mathbb{P}(x \in A_1^{k_1}\cdots A_s^{k_s}) \\ &= \sideset{}{^*}\sum_{\boldsymbol{a}} \mathbb{P}(E_{\boldsymbol{a}}) + O\left(\sideset{}{^*}\sum_{\substack{\boldsymbol{a}\neq\boldsymbol{a}' \\ \|\boldsymbol{a}\|=\|\boldsymbol{a}'\|}} \mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'})\right) + O\left(\sideset{}{^{**}}\sum_{\boldsymbol{a}} \mathbb{P}(E_{\boldsymbol{a}})\right).\end{aligned} \tag{3}$$

Since $A_1, \dots, A_s$ are independent and each A_i belongs to $\mathcal{B}(n_i, \alpha_i)$, we have

$$\mathbb{P}(E_{\boldsymbol{a}}) = \bigwedge_{i=1}^s \mathbb{P}(\boldsymbol{a}_i \subseteq A_i) = \prod_{i=1}^s \alpha_i^{|\boldsymbol{a}_i|}.$$

Hence, for every positive integers $m_1, \dots, m_s$, with $m_i \le k_i$, we have

$$\sum_{\boldsymbol{a}\,:\,|\boldsymbol{a}_i|=m_i} \mathbb{P}(E_{\boldsymbol{a}}) = \sum_{\boldsymbol{a}\,:\,|\boldsymbol{a}_i|=m_i} \prod_{i=1}^s \alpha_i^{m_i} = \prod_{i=1}^s \alpha_i^{m_i} \sum_{|\boldsymbol{a}_i|=m_i} 1 = \prod_{i=1}^s \alpha_i^{m_i}\binom{n_i}{m_i}\binom{k_i-1}{m_i-1},$$

where we used the fact that the number of unordered k-tuples of elements of $\{1, \dots, n\}$ having cardinality equal to m is $\binom{n}{m}\binom{k-1}{m-1}$. Therefore,

$$\sideset{}{^*}\sum_{\boldsymbol{a}} \mathbb{P}(E_{\boldsymbol{a}}) \sim \prod_{i=1}^s \frac{(\alpha_i n_i)^{k_i}}{k_i!} \quad \text{and} \quad \sideset{}{^{**}}\sum_{\boldsymbol{a}} \mathbb{P}(E_{\boldsymbol{a}}) = o\left(\prod_{i=1}^s (\alpha_i n_i)^{k_i}\right), \tag{4}$$

as $\alpha_i n_i \to +\infty$, for $i = 1, \dots, s$. We have

$$\mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'}) = \prod_{i=1}^s \mathbb{P}(\boldsymbol{a}_i \cup \boldsymbol{a}_i' \subseteq A_i) = \prod_{i=1}^s \alpha_i^{|\boldsymbol{a}_i \cup \boldsymbol{a}_i'|}. \tag{5}$$

Suppose that $\boldsymbol{a}$ and $\boldsymbol{a}'$, with $\boldsymbol{a} \neq \boldsymbol{a}'$ and $\|\boldsymbol{a}\| = \|\boldsymbol{a}'\|$, satisfy the condition of *, that is, $|\boldsymbol{a}_i| = |\boldsymbol{a}_i'| = k_i$ for $i = 1, \dots, s$. We shall find an upper bound for (5). Clearly, $|\boldsymbol{a}_i \cup \boldsymbol{a}_i'| \ge |\boldsymbol{a}_i| \ge k_i$ for $i = 1, \dots, s$. Moreover, since $\boldsymbol{a} \neq \boldsymbol{a}'$, there exists $i_1 \in \{1, \dots, s\}$ such that $\boldsymbol{a}_{i_1} \neq \boldsymbol{a}_{i_1}'$.

Since $|\boldsymbol{a}_{i_1}| = |\boldsymbol{a}'_{i_1}| = k_i$, it follows that $|\boldsymbol{a}_{i_1} \cup \boldsymbol{a}'_{i_1}| \geq k_{i_1} + 1$. On the one hand, if there exists $i_2 \in \{1, \dots, s\} \setminus \{i_1\}$ such that $\boldsymbol{a}_{i_2} \neq \boldsymbol{a}'_{i_2}$, then, similarly, we have $|\boldsymbol{a}_{i_2} \cup \boldsymbol{a}'_{i_2}| \geq k_{i_2} + 1$. Hence,

$$\mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'}) \leq \alpha_{i_1} \alpha_{i_2} \prod_{i=1}^{s} \alpha_i^{k_i}.$$

On the other hand, if $\boldsymbol{a}_i = \boldsymbol{a}'_i$ for every $i \in \{1, \dots, s\} \setminus \{i_1\}$, then from $\|\boldsymbol{a}\| = \|\boldsymbol{a}'\|$ it follows that $\prod_{j=1}^{k_{i_1}} a_{i_1,j} = \prod_{j=1}^{k_{i_1}} a'_{i_1,j}$. In turn, this implies that $|\boldsymbol{a}_{i_1} \cup \boldsymbol{a}'_{i_1}| \geq k_{i_1} + 2$. Hence,

$$\mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'}) \leq \alpha_{i_1}^2 \prod_{i=1}^{s} \alpha_i^{k_i}.$$

Therefore, using Lemma 3.3 and recalling (2), we obtain

$$\begin{aligned}
(6)\qquad \sideset{}{^*}\sum_{\substack{\boldsymbol{a} \neq \boldsymbol{a}' \\ \|\boldsymbol{a}\| = \|\boldsymbol{a}'\|}} \mathbb{P}(E_{\boldsymbol{a}} \wedge E_{\boldsymbol{a}'}) &\leq \left(\max_{1 \leq i,j \leq s} \alpha_i \alpha_j\right) \prod_{i=1}^{s} \alpha_i^{k_i} \sum_{\|\boldsymbol{a}\| = \|\boldsymbol{a}'\|} 1 \\
&\ll \left(\max_{1 \leq i,j \leq s} \alpha_i \alpha_j\right) \left((\log n_1)^{k_1 - 1} \prod_{i=2}^{s} (\log n_i)^{k_i}\right)^{k_1 + \cdots + k_s - 1} \prod_{i=1}^{s} (\alpha_i n_i)^{k_i} \\
&= o\left(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\right).
\end{aligned}$$

Finally, putting together (3), (4), and (6), we obtain the desired claim. □

Proof of Theorem 1.2. Define the random variable

$$X := \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i} - \left|\prod_{i=1}^{s} A_i^{k_i}\right|.$$

Thanks to Lemma 4.2, we know that X is nonnegative. Moreover, from Lemma 4.1(i) and Lemma 4.3, it follows that

$$\mathbb{E}(X) = o\left(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\right).$$

Hence, for every $\varepsilon > 0$, by Markov's inequality, we get

$$\mathbb{P}\left(X \geq \varepsilon \prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\right) \leq \frac{\mathbb{E}(X)}{\varepsilon \prod_{i=1}^{s} (\alpha_i n_i)^{k_i}} = o_\varepsilon(1),$$

which in turn implies $X = o\big(\prod_{i=1}^{s} (\alpha_i n_i)^{k_i}\big)$ with probability $1 - o(1)$. Therefore, by Lemma 4.1(ii),

$$\left|\prod_{i=1}^{s} A_i^{k_i}\right| = \prod_{i=1}^{s} \binom{|A_i| + k_i - 1}{k_i} - X = \prod_{i=1}^{s} \frac{|A_i|^{k_i}}{k_i!} + o\left(\prod_{i=1}^{s} |A_i|^{k_i}\right),$$

with probability $1 - o(1)$, as claimed. □

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