Architectural Heritage between Mathematics and Representation: studying the geometry of a barrel vault with lunettes at a first year Bachelor's in Architecture

Third conference of the
International Network for
Didactic Research in University Mathematics

INDRUM 2020

September 12-19

Cyberspace-Virtually from Bizerte

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Architectural Heritage between Mathematics and Representation: studying the geometry of a barrel vault with lunettes at a first year Bachelor's in Architecture

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We present an interdisciplinary activity, directed to students of the first year Bachelor's in Architecture, about the visualization of mathematical objects through physical and digital models, with specific regard to surfaces generated by intersections of cylinders. We focus on the geometrical structure of barrel vaults with lunettes and propose the use of various kind of models to improve the accessibility of interdisciplinary elements and to translate specialistic knowledge, building a common language for students with different backgrounds. The work is a result of a broader research project, dedicated to enhancing the relationship between Mathematics and Architecture through Geometry and Representation.

Keywords: Geometry for Architects, Surfaces, Models, Representation, Visualization.

INTRODUCTION

In the field of Italian university Mathematics education, the need to set up a dialogue within other disciplines in which Mathematics is taught (Architecture, Engineering, Biology, etc.) is increasingly felt. Teaching Mathematics to non-Mathematics learners means teaching it as a service subject, hence as a tool to model systems of the domain and to solve the associated problems (Howson, Kahane, Lauginie & Tuckheim 1988).

We are interested in the role played by Mathematics in courses of the bachelor's degree program in Architecture. In this context, basic mathematical tools are conveyed by the first year Calculus course: here students learn topics which are preparatory and in support of the parallel course of Architectural Drawing and Survey Laboratory (and of the subsequent courses of building physics, real estate evaluation, as well as of the structural matters).

TEACHING PROBLEMS

One of the main teaching problems is that students, in their first year of academic study, come from diversified educational backgrounds and have different skills (or lack of them) about mathematical and graphic language; as for Mathematics, they may not understand where and when the proposed abstract mathematical topics will be concretely applied. It should be necessary to create a perception of their use value in the wider sense, in order to avoid the traditional insularity of Maths in the technical faculties (see e.g. Harris, Black, Hernandez-Martinez, Pepin, Williams, & TransMaths Team 2015; or Rasmussen, Marrongelle, & Borba 2014) and to promote a synergistic relationship that fills the gaps between the various languages. For example, the increased relevance of digital and parametric modelling in Architecture (see e.g. Calvano 2019) has created a need for developing the education of future architects through a greater integration of mathematics and disciplines which are specific of Architecture. Another problem is students' previous mathematical experience in facing university studies, depending on whether they were more or less oriented to the formation of spatial visualization capacity. Important difficulties appear in the transition from 2D to 3D analytical representations of geometrical objects (Cumino, Pavignano, Spreafico, & Zich 2019): for example, from the 2D analytical representation of a straight line or, more generally, of a plane curve to the representation of a plane or cylinder having that curve as a directrix. Students have the concept of cartesian equation based on their experience from high school Mathematics, but some of them may not accept that the same equation represents different set of points passing from 2D to 3D. Such phenomena can be related to those of «tacit models» mentioned by Fischbein (1989), who refers to representations of certain mathematical, abstract notions developed at an initial stage of the learning process which continue to influence, tacitly, reasoning and interpretations of the learner, hence the need to help students to control the (possibly negative) impact of these models.

So, a myriad of problems of cognitive nature (type of background, crystallized information, gaps between disciplinary languages) or of didactic nature (teaching strategies, disciplinary content and courses organization) are to be taken in consideration, before being able to concretize any proposal for teaching interventions.

Investigations in this direction, related to this particular context, seem to have been somewhat limited, mainly in connection with other disciplines, while correlations between spatial imagery information processing, spatial visualization and geometrical figure apprehension have been the subject of a number of studies, see e.g. the comprehensive review by Jones and Tzekaki (2016) and Kovačević, N. (2017), about recent research in Geometry education. On the other hand, it is fundamental to overcome the aforementioned problems, not only from the point of view of basic mathematical education: when approaching the study of the built form, the architecture student should acquire interdisciplinary knowledge -thus developing basic skills- for its analysis and it is important to underline how the knowledge that 'flows' into an architectural design is not only related to its specific disciplines but includes a variety of methodologies and interpretations derived from other subjects. Therefore, in students' educational path, it seems worthwhile to set up study activities having an interdisciplinary approach.

In this contribution, we present an attempt in this direction, focusing on the Geometry education of architecture students and its role in connection to geometrical comprehension of architectural shapes and spatial visualization ability and we highlight the importance of mathematical thinking in the formalization of architectural structures, in particular of roofing systems constituted by vaults.

We also refer to Duval's analysis of visualization process and its interactions with geometrical reasoning, adapting it to the particular educational context; in this sense we follow the idea of construction as a process dependent only on the connections between mathematical properties and technical constraints of the used tools.

VAULTS AND GEOMETRY

Most of the Italian Architectural Heritage was built with masonry structures and vaults are roofing elements mainly used for covering rooms of a building. Original vaults were as simple as the surfaces of rotation and/or of translation that were used to design them. Then they became more complex systems, reaching different forms (subtended by as many geometries).

In Architecture, the use of Geometry and its elements, such as points, lines, planes, then surfaces and solids [1], as the result of a learning path has its roots in the ancient past; i.e. Leon Battista Alberti, Francesco di Giorgio Martini, Andrea Palladio, Vincenzo Scamozzi and Guarino Guarini wrote about these issues in their treatises (Spallone-Vitali 2017, pp. 88-90), see Figure 1. The variety of the vaults compositions is briefly exemplified in Figure 2a, b.

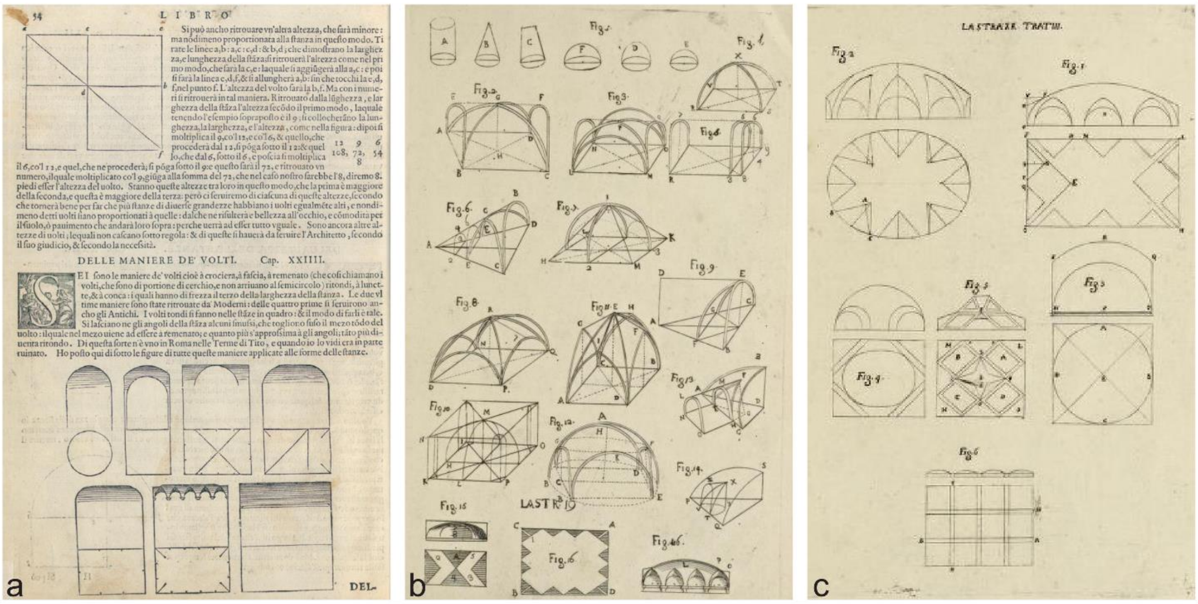

Figure 1: Historical examples of geometric and graphic description of some vaults. Palladio 1570, p. 54; b) Guarini 1737, tav. XXVII, c) Guarini 1737, tav. XXVIII.

The study of these constructive elements plays an important role: we should provide students with some critical tools useful for the conceptual investigation of these structures, even for starting from their graphic formalization.

During the 17th century Guarino Guarini, architect and mathematician, in his treatise, *Architettura Civile*, states that: «vaults […] are hardest to be invented, to put in drawing and to build» (Guarini 1737, p. 183). Again, Guarini states that: «in each of its operations Architecture uses measures, then it relays on Geometry and deserves to know at least its fundamentals» (Guarini 1737, p. 3). His ideas clearly highlight the close relationship between geometry and the architectural artefact.

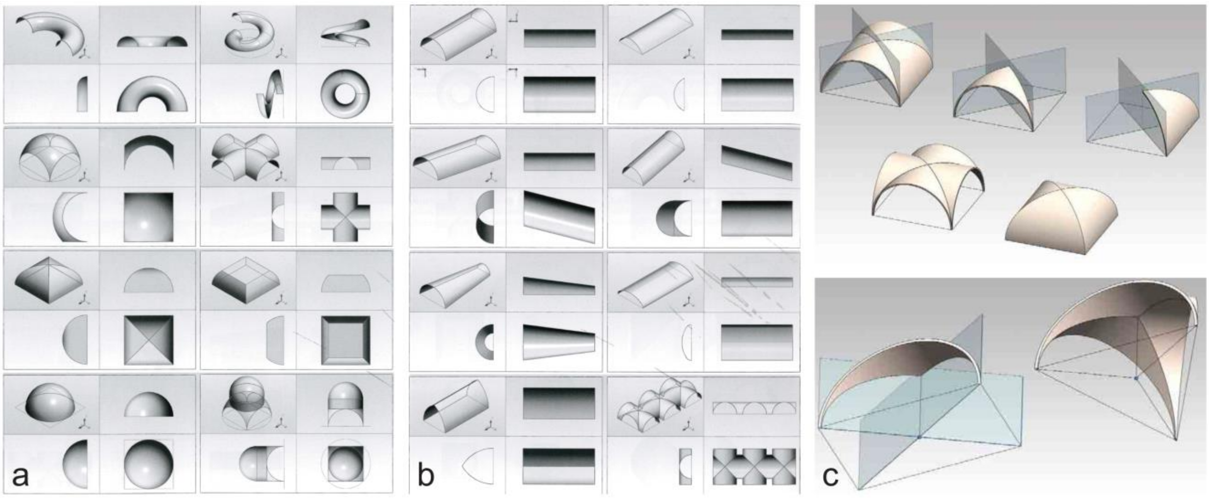

Figure 2: simple and compound vaulted systems: a), b) Bertocci, Bini 2012, pp. 265, 266; c) Fallavollita 2009, pp. 453, 455.

For example, Guarini sometimes used to describe the composition of a vaulted system by junction of parts of different surfaces cut by the same plane (Spallone-Vitali 2017). This idea is still used today. Figure 2c shows this kind of construction applied to square based groin and cloister vaults and to triangle-based groin vault, without caring about the problem of intersections. Recent bibliographical references for the geometric-conceptual study of vaults offer an almost always 3D or pseudo-3D set of views, with textual descriptions. Many authors represent these elements with axonometric views (see e.g. Docci, Gaiani, Maestri 2011; Fallavollita 2009) which immediately convey the idea of the three-dimensionality of these elements, but often neglect its geometrical genesis.

METHODOLOGY TO STUDY A BARREL VAULT WITH LUNETTES

In light of previous considerations, the complexity of the form needs a discretization of its characteristic and descriptive elements through the graphic language that, in being an expression of synthesis, risks to become an excessive simplification of its features. Consequently, at the base of the training path of the architectural student, there is the study conducted between the representation of a theoretical model and its analytical description.

We propose four teaching tools to enhance students' critical shape-reading skills referring to architectural heritage: a tangible model to introduce the problem; a graphic representation to investigate the relationship between drawing and shapes; a virtual model obtained by a dynamic geometric software (DGS), constructed through an analytical description; a physical model as outcome of the preceding three steps. These tools are in order to promote students' ability in switching between different registers of representation (see Duval 1999).

We identified a family of composite vaults -which can be traced back to intersections of barrel vaults with coplanar and orthogonal axes, see Figure 3- as having an explicit formative value in this context.

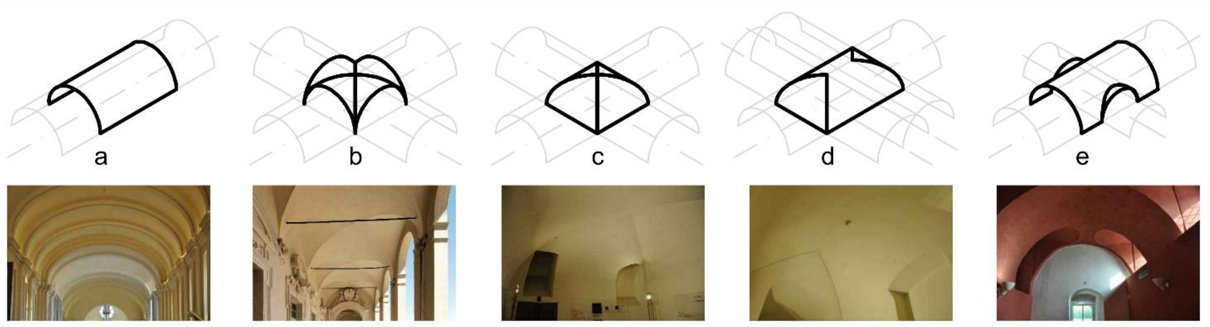

Figure 3: Cylinders intersections. a) barrel vault; b) groin vault; c) cloister vault; d) barrel vault with cloister heads; e) barrel vault with lunettes. Pictures of Royal Residence of Venaria Reale (Torino).

Graphic Analysis

Figure 4 shows the graphic study of intersection between two circular right cylinders of different radius with coplanar and orthogonal axes. The random nature of the choice of which cutting planes could be used for this study leads to subjective results, affected also by different graphic tools (for example manual drawing or Computer Aided Architectural Design (CAAD) bring different typologies of error) [2]. The auxiliary planes, here, were chosen to respect the uniformly distribution of the information about the development of the surface starting from the angular subdivision of the circular section.

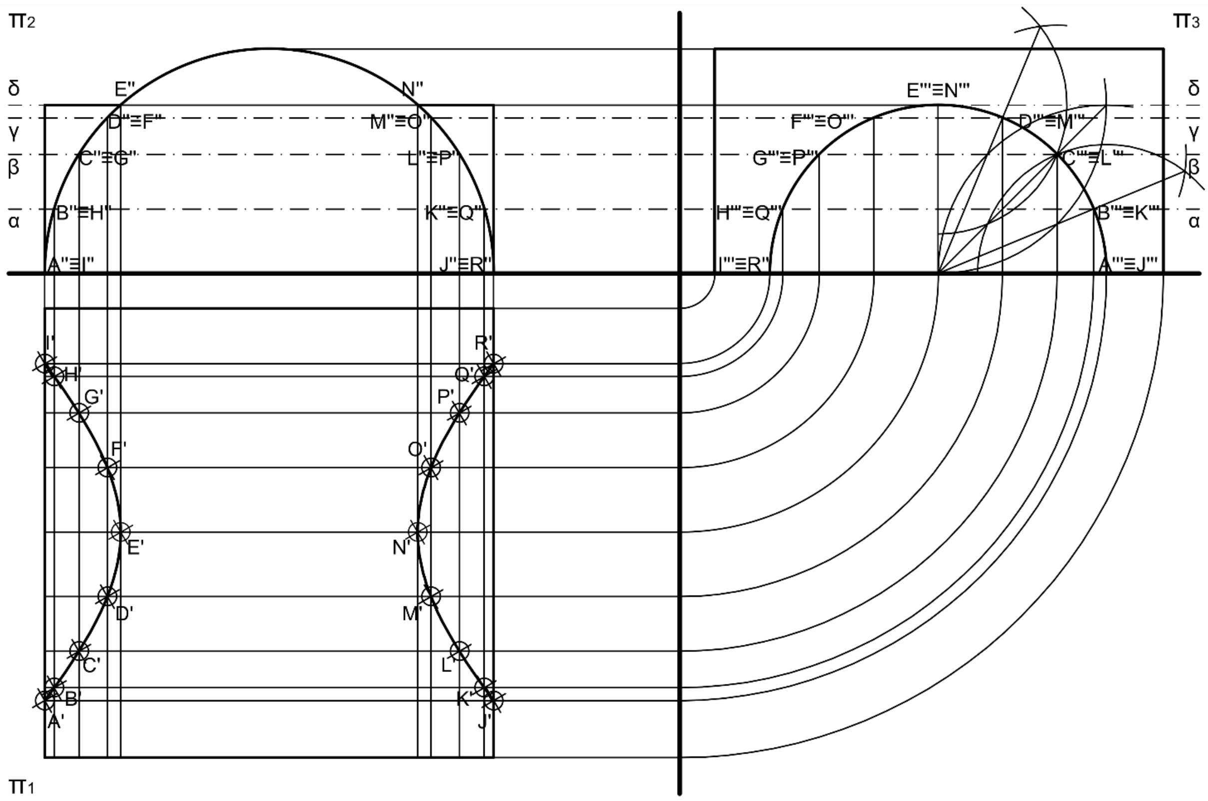

Figure 4: CAAD orthographic projection of the intersection between two barrel vaults.

DGS Analysis

Over the last few decades, the appearance of DGS has renewed Mathematics education providing computational and visual tools available in software environments like *Geogebra*, (Arzarello, F., Bartolini Bussi., M. G., Leung, A., Mariotti, M. A., & Stevenson, I. 2012).

In our context visualization is a crucial matter, in particular in 3D Geometry, but analytic methods may lead to heavy and unilluminating computations and students' mathematical background does not allow them to be autonomous in the visualization of mathematical objects. In the specific case, a surface generated by an intersection of cylinders can be studied in an interdisciplinary activity, making a joint usage of CAAD together with a DGS, which gives the possibility of approaching problems from different perspectives, connecting algebraic and geometrical views, facilitating constructions of mathematical objects and also the direct manipulation of them.

Due to the learners' small knowledge about 3D Analytic Geometry, a ready-made GeoGebra model is proposed, which is realized using an analytical description based on parametric equations of the involved geometric objects.

Let C_1 and C_2 be two semicircular cylinders with coplanar orthogonal axes and different radiuses $R_1 > R_2$. Let $\gamma = C_1 \cap C_2$ be the intersection curve of the two cylinders (see Figure 5), where C_1 is the cylinder generated by translation along the x-axis of the semi-circumference with center in the origin and radius R_1, in the plane (yz) and C_2 the cylinder generated by translation along the y-axis of the semi-circumference with center in the origin and radius R_2, in the plane (xz); taking parametric representations $C_1: (x, y, z) = (v, R_1 \sin(u), R_1 \cos(u))$ and $C_2: (x, y, z) = (R_2 \sin(u), v, R_2 \cos(u)), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, the quartic skew intersection curve $\gamma = C_1 \cap C_2$ may be represented by:

$$\gamma: (x, y, z) = \left(\pm\sqrt{R_2^2 - R_1^2 \cos^2(u)}, R_1 \sin(u), R_1 \cos(u)\right)$$

In order to visually understand the surface, GeoGebra dynamics are exploited, see Figure 5. As the software enables to rotate a still picture using the mouse, the drawing in Figure 6 shows how to visualize the orthographic projections of the intersection curve $\gamma = C_1 \cap C_2$.

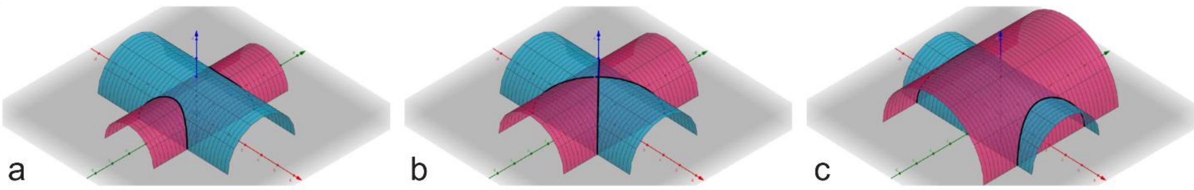

Figure 5: Frames from the dynamic DGS model using a slider to change the red cylinder radius.

We may compare the graphic outcome above with the GeoGebra one: it is clear that the DGS representation is univocal in contrast with the graphic one that is affected by approximations due to drawing choices.

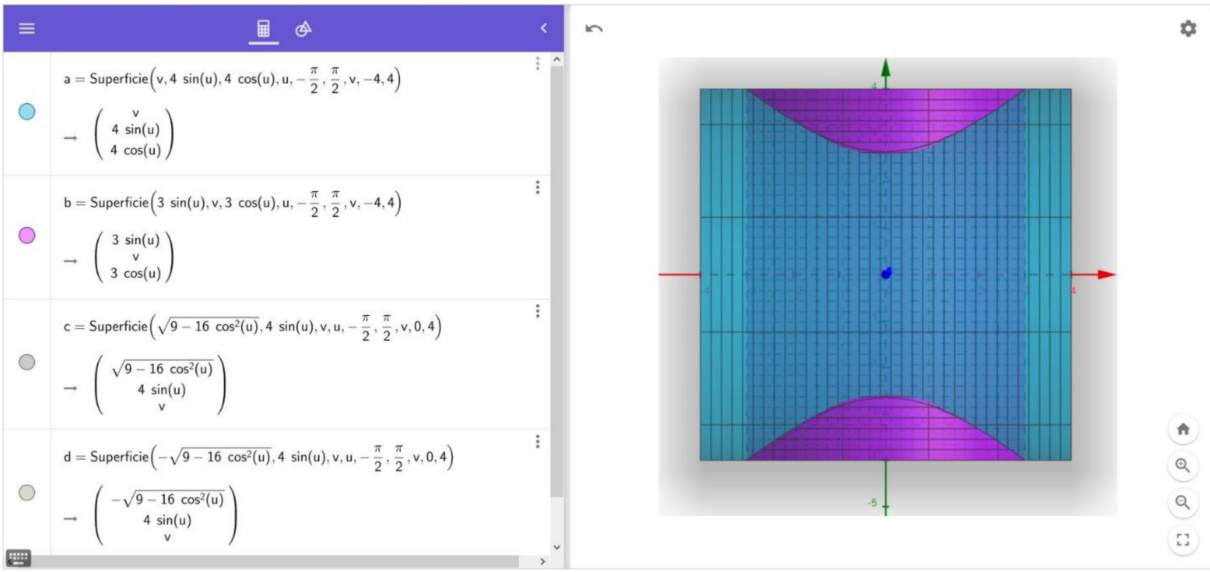

Figure 6: Top view of the barrel vault with lunettes obtained by GeoGebra.

Another way to enhance students understanding is to introduce a real parameter *t*, replacing the given cylinder C_2 with a 1-parameter family of cylinders: in this way, a slider bar enables to manipulate directly the surface thus generated (see Figure 5, 6) and to follow in particular the deformations of the curve γ, as the parameter varies, while, by suitably rotating this dynamic picture, as it was done in Figure 6, the orthographic projection of γ on the plane *(xy)* appears easily as subset of a 1-parameter family of hyperbolas.

Physical Models

Starting from the same analytic description as above, a physical model of a barrel vault with lunettes has been realized, using acetate or paper (Figure 7a, c).

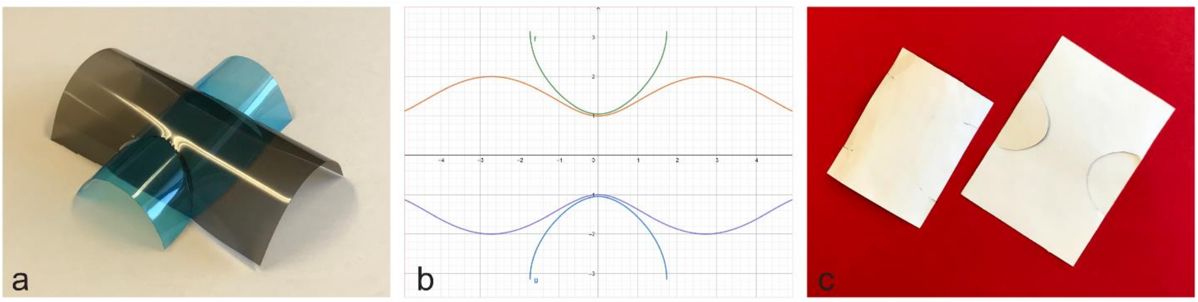

Figure 7: Physical model of a barrel vault with lunettes. a) acetate model; b) developments of the same intersection curve thought as belonging to C_1 or C_2; c) disassembled paper model (similar to the acetate one).

If the semicircular cylinders have the same radius, their intersection is a plane curve (generating a cloister or a groin vault, see Figure 3, b and c) and in this case students can realize that its development is a sinusoidal curve using basic elements of trigonometry (see Cumino et alii 2018); if the radiuses are of different lengths, the intersection curve of the cylinders is in general a skew curve, therefore, to obtain its development in the plane requires more sophisticated mathematical tools (e.g. the concept of development of a spatial curve on the plane and the arc-length calculation, see e.g. M. P. Do Carmo (1976), which do not belong to a standard Calculus course program. Nevertheless, the model may be used as a tangible object to communicate the particular shape of the surface under consideration, because it facilitates the geometric perception of the shape as a whole and also the understanding of the geometric properties of the intersection curve; in fact, disassembling the model one observes that this skew curve develops on the plane in two distinct ways (see Figure 8b), depending on whether it is considered belonging to the cylinder C_1 or to cylinder C_2. Then introducing the arc-length formula, the development of γ on the plane *(xy)*, as a curve on C_1, may be represented by

$$\gamma': (x, y, z) = \left(\pm\sqrt{R_2^2 - R_1^2\cos^2(u)}, R_1\mathrm{u}, 0\right)$$

In a similar way the development of γ as a curve on C_2, may be represented by

$$\gamma": (x, y, z) = \left(R_2\mathrm{u}, \pm\sqrt{R_1^2 - R_2^2\cos^2(u)}, 0\right)$$

With respective cartesian equations

$$\gamma': y = \pm\, R_1 \cos^{-1}\frac{\sqrt{R_2^2 - x^2}}{R_1} \quad \text{and} \quad \gamma": y = \pm\sqrt{R_1^2 - R_2^2\cos^2\left(\frac{x}{R_2}\right)}$$

CONCLUSIONS AND OUTLOOK

In the present paper, we considered how mathematical thinking may contribute in the formalization of architectural structures, using the specific case of barrel vault with lunettes. Our activity was born in an Architectural Drawing and Survey Laboratory, whose main purpose is to make students understand the use of drawing as a tool of analysis and synthesis and as a means to communicate and visualize geometrical objects. The dialogue between Mathematics and Architecture has brought to light a way of interpreting and using Geometry different from that usually practiced by Mathematics teachers.

We took into account Duval's analysis of visualization process and its interactions with geometrical reasoning, adapting it to the particular educational context in regard to 2D/3D graphical representation of a real object and exploration of geometrical situations via physical or digital model; we exploited the experience of teaching

Mathematics in order to make students aware of the variability of representations of an object (depending of the variety of physical or semiotic systems producing them) versus the invariance of the object itself. We present the geometrical reasoning that underpin strategic choices in the graphical description process. To do this, the teacher has to choose a specific set of points and lines that best describe the considered shape; to disclose how to make these choices, physical or digital models are employed, constructed according to a mathematical recipe: the same model which is used by the Mathematics teacher as a tool to teach Geometry. Therefore, the model appears to be, in a sense, a translator between the two disciplinary approaches. Further research in this direction may provide new means to promote a stronger interdisciplinary educational system and to enhance students' spatial and visualization abilities.

NOTES

1. There might be a connection between the idea of tacit model and the architectural design process: if we think of architectural artefacts as results of the creative application of Geometry and its basic elements. In this sense, such elements could act as tacit models for the first architectural composition/shapes recognition exercises.

2. «It is not possible to consider all the points of a surface when you need to represent it. The same is true for all the lines that belong to it. In order to proceed, it is necessary to transform the surface into a discrete set of lines: those that best lend themselves to describe its geometry», Migliari 2001, p. 160.

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