Merging Combinatorial Design and Optimization: the Oberwolfach Problem

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Abstract

The Oberwolfach Problem $OP(F)$ — posed by Gerhard Ringel in 1967 —
is a paradigmatic Combinatorial Design problem asking whether the complete graph K_v decomposes into edge-disjoint copies of a 2-regular graph F of order v. In this paper we provide all the necessary equipment to generate solutions to $OP(F)$ for relatively small orders by using so-called difference methods. From the theoretical standpoint, we present new insights on the combinatorial structures involved in the solution of the problem. Computationally, we provide a full recipe whose base ingredients are advanced optimization models and tailored algorithms. This algorithmic arsenal can solve the $OP(F)$ for all possible orders up to 60

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with the modest computing resources of a personal computer. The 20 new orders, from 41 to 60, encompass 241 200 instances of the Oberwolfach Problem, which is 22 times greater than those solved in previous contributions.

1   Introduction

Gerhard Ringel proposed the Oberwolfach Problem (OP) for the first time in 1967 [18], while attending a conference at the Mathematical Research Institute of Oberwolfach, in Germany. In conferences held at the Institute, participants usually dine together in a room with circular tables of different sizes, and each participant has an assigned seat. Ringel asked whether there exists a seating arrangement for an odd number v of people and $(v-1)/2$ meals so that all pairs of participants are seated next to each other exactly once.

Formally, given a spanning 2-regular subgraph (a 2-*factor*) F of K_v (the *complete graph* of v vertices), the Oberwolfach problem $OP(F)$ asks whether K_v with v odd decomposes into $(v-1)/2$ edge-disjoint copies of F. We write $F = [{}^{m_1}\ell_1, \ldots, {}^{m_t}\ell_t]$ whenever F is a 2-regular graph with m_i cycles of length ℓ_i, and refer to m_i as the multiplicity of the cycle length ℓ_i, for $i = 1, \ldots, t$. In this case, we will often write $OP({}^{m_1}\ell_1, \ldots, {}^{m_t}\ell_t)$ in place of $OP(F)$. For the sake of simplicity, we omit any ℓ_i for which $m_i = 0$, and we remove the superscript when $m_i = 1$ for a given i.

In 1979, Huang et al. [21] extended the problem to the case where v is even. Although OP has drawn interest, and much progress has been made over the past few years (see, for instance, [5, 6, 8, 17, 19, 22, 27]), a complete solution has yet to be found. A survey of the most relevant results on this problem, updated to 2006, can be found in [13].

Solutions to OP can often be found by focusing on those having symmetries with a particular action on the vertex set. By knowing the structure of these solutions, the problem of finding edge-disjoint 2-factors turns into finding a few well-structured 2-factors. The so-called *difference methods* – a family of algebraic tools – set the rules to construct such well-structured 2-factors. *Difference methods* were introduced for the first time by Anstice [3] to generalize solutions to Kirkman's 15 schoolgirls problem, one of the paradigmatic problems in Combinatorial Design Theory. Arranging seats around tables is not new for Operations Research as well. García et al. [16], for instance, introduced a table placement problem aiming to maximize a measure of social benefit.

The baseline of this work is the contribution of Deza et al. [15]. There, authors solved OP for $18 \le v \le 40$, modeling difference methods with undisclosed algorithms and tests which were carried out on a high-performance computing cluster [12].

In this paper, we provide all the necessary equipment to generate solutions to an Oberwolfach problem of a relatively small order; not only on the theoretical side but, most importantly and differently from Deza et al. [15], also on the computational side. Indeed we give a full recipe whose base ingredients are advanced optimization

algorithms rather than an exhaustive search. These algorithms allowed us to rapidly obtain the desired solutions for all possible orders $v \leq 60$ with a personal computer which, we point out, is a much lower performance threshold compared to a high-performance computing cluster of Deza et al. [15] in 2008. We also recall that the number of generated solutions increases with the order of the problem, for instance, from 1756 partitions of $v = 40$ to 33552 partitions of $v = 60$.

Our approach blends combinatorial design theory with optimization and computation paradigms. We model difference methods as Constraint Programming (CP) problems, and leverage on state-of-the-art algorithms to find the combinatorial solutions. We were able to generate complete solutions for OP when $v \leq 60$. Our approach solves a generic instance within 5 seconds on a standard computer, compared to the need for a high-performance computing cluster [15]. The extensive computational testing allowed us to derive new theoretical results for OP, in particular, a new necessary condition was detected on the existence of 1-*rotational* solutions. Moreover, an Integer Programming (IP) model verifies the non-existence of solutions for $OP(^23, 5)$.

In a nutshell, the above optimization tools enabled us to solve large OP instances in limited CPU times and at the same time to derive theoretical results for general classes of instances. We believe such an approach could be generalized to a broader class of Combinatorial Design problems.

The paper proceeds as follows. In Section 2, since this work is at the intersection of two distinct domains, Combinatorial Design Theory and Combinatorial Optimization, we introduce a standard tool pertaining to the former (difference methods) by means of an illustrative example. Section 3 presents how to construct well-structured 2-factors for the solution of $OP(F)$. Section 4 shows how to translate into CP models the findings of section 3. Section 5 provides the outcome of the experimental testing. Section 6 concludes the paper with final remarks.

2   Difference methods and $OP(F)$: an illustrative example

Difference methods exploit the symmetries of a 2-factorization and tell us how to construct one well-structured 2-factor which yields, by *translation*, the complete set of 2-factors giving a solution to $OP(F)$. To explain it in the context of the Ringel's informal formulation, we can construct, for instance, the first meal seating arrangement (the desired well-structured 2-factor) and *derive/translate* from it the remaining ones. In the following example, we provide a well-structured 2-factor for $OP(3, 6)$, and show how the remaining meals can be *derived* starting from it.

Figure 1 depicts the seating arrangement of the first meal (see Section 3.1, *Proposition* 3.1) of $OP(F)$, where F contains two cycles (dinner tables) of size 3 and 6, thus $F = [3, 6]$ and $v = 9$.

We label the vertices of F with the elements of $\mathbb{Z}_8 \cup \{\infty\}$ and for each edge incident with two vertices different from ∞, say i and j, the two differences $i - j$ and $j - i$ (both   mod $(v - 1)$) have to be calculated. For instance, if we consider the
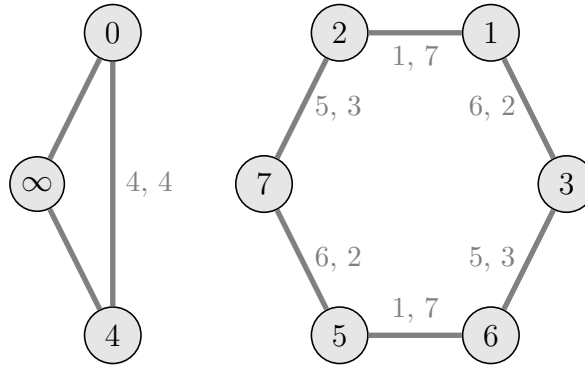
Figure 1: A first meal arrangement for $OP(F = [3, 6])$.

vertices labeled 2 and 1 in Figure 1, we obtain the differences 1 and $-1 \equiv 7 \pmod 8$. The list ΔF of all possible differences between adjacent vertices of F, different from ∞, contains every element in $\{1, 2, \ldots, 7\}$ with multiplicity 2. Furthermore, $F + 4 = F$, where $F + 4$ is the graph obtained from F by adding 4 to every vertex but ∞. In other words, we have found a vertex labeling of F such that ΔF contains every element in $\{1, 2, \ldots, 7\}$ with multiplicity 2, and $F + 4 = F$. These are the two crucial conditions which guarantee that F is the sought-after 2-factor (see *Proposition* 3.1) which will generate a solution to $OP(F)$. Indeed, the set $\mathcal{F} = \{F, F+1, F+2, F+3\}$ of all distinct translates of F (see Figure 2) gives us a set of edge-disjoint copies of F which decompose K_9, that is, a solution to $OP(F)$.

In the following section we provide conditions to find a well-structured 2-factor F which guarantee that all its distinct translates yield a solution to OP(F). In Section 4, these conditions are then reformulated as CP models to be tackled by a solver to generate solutions (i.e. first meal arrangements).

3   Constructions of well-structured 2-factors

A graph has a 2-factorization if and only if it is regular and of even degree, as Petersen [23] shows. However, given a particular 2-factor F, if we ask for a 2-factorization whose factors are all isomorphic to F, then the problem becomes much harder. Our focus is on constructing 2-factorizations of K_v^* which is the complete graph K_v of order v when v is odd, or $K_v - I$, i.e. the complete graph K_v minus the 1-factor I, when v is even. Given a 2-factor F of order v, the Oberwolfach problem on F ($OP(F)$) asks for a 2-factorization of K_v^* into copies of F.

A solution to $OP(F)$ exists whenever the order of F is less than 40 [15], except only when $F \in \{[\,^2 3], [^4 3], [4, 5], [^2 3, 5]\}$. These are the only known cases in which the problem is not solvable. We point out that even though Piotrowski [25] self-cites for a computer-based proof of the non-existence of a solution to $OP(^2 3, 5)$, no published proof exists. $OP(F)$ has also been solved when F is a uniform 2-factor (i.e., F consists of cycles of the same length) [1, 2, 20], when F is bipartite (i.e., F contains only cycles of even lengths) [1, 5], when F has exactly two cycles [27], or for
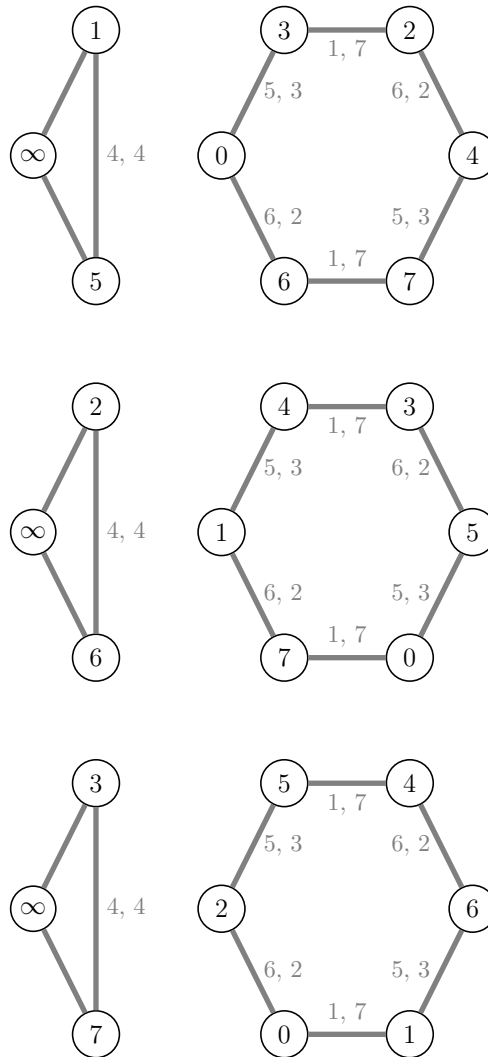
Figure 2: The remaining meals arrangements for $OP(F = [3, 6])$.

an infinite family of prime orders [6]. In addition, [7, 8, 10] studied solutions having symmetries with a prescribed action on the vertices, and existence for sufficiently large v can be found in [17]. Furthermore, a solution to the infinite variant of the Oberwolfach problem can be found in [14]. However, the problem is still open in general, and [13, Section VI.12] provides a detailed survey on this subject, updated to 2006.

3.1   1-*rotational* solutions to $OP(F)$

Buratti and Rinaldi [8] construct 1-*rotational* solutions in the odd case, that is, 2-factorizations of K_v, with v odd, with a well-behaved automorphism group. Let $v = 2n + 1$ be a positive integer, let $\mathbb{Z}_{2n}$ denote the group of integers modulo $2n$, and set $V = \mathbb{Z}_{2n} \cup \{\infty\}$. The list of differences of a subgraph Γ of K_V is the multiset $\Delta\Gamma$ of all possible differences between pairs of adjacent vertices of Γ, excluding the

vertex ∞, namely:

$$\Delta\Gamma = \big\{x - y \mid \lfloor x, y \rfloor \in E(\Gamma \setminus \{\infty\})\big\}. \tag{1}$$

We use the notation $\lfloor x, y \rfloor$ to denote an edge between the nodes x and y. Also, for every $g \in \mathbb{Z}_{2n}$, we denote by $\Gamma + g$ the graph with vertex set V obtained from Γ by replacing each vertex $x \neq \infty$ with $x + g$.

The following result, proven in [8], provides conditions which guarantee the existence of a solution to $OP(F)$.

Proposition 3.1. *Let F be a 2-regular graph satisfying the following properties:*

1. $V(F) = \mathbb{Z}_{2n} \cup \{\infty\}$,

2. $\Delta F \supset \mathbb{Z}_{2n} \setminus \{0\}$,

3. $F + n = F$.

Then $\mathcal{F} = \{F + g \mid g \in \mathbb{Z}_{2n}\}$ is a solution of $OP(F)$.

A factorization $\mathcal{F}$ of K_{2n+1} constructed as in Proposition 3.1 is called 1-*rotational*, since the permutation group $G = \{\tau_g \mid g \in \mathbb{Z}_{2n}\}$ of V, where τ_g fixes ∞ and maps $x \in \mathbb{Z}_{2n}$ to $x + g$, is an automorphism group of $\mathcal{F}$ whose action on $V \setminus \{\infty\}$ creates only one orbit.

In [11, Proposition 2.5] it is shown that if there exists a 2-regular graph $F = [\ell_1, \ell_2 \ldots, \ell_s]$ of order $2n + 1$ satisfying the assumptions of Proposition 3.1, then there exists a solution to $OP([\ell_1 + 1, \ell_2 \ldots, \ell_s])$ where ℓ_1 is the length of the cycle of F through ∞. The following result weakens this condition by showing that ℓ_1 can be the length of any cycle of F provided that it contains a suitable difference.

Proposition 3.2. *Let $F = [\ell_1, \ell_2 \ldots, \ell_s]$ be a 2-regular graph of order $2n + 1$ satisfying the assumptions of Proposition 3.1. If C is an ℓ_1-cycle of F such that ΔC contains an element of $\mathbb{Z}_{2n}$ of order $2 \pmod 4$, then there exists a solution to $OP([\ell_1 + 1, \ell_2 \ldots, \ell_s])$.*

Proof. Let $C = (c_1, c_2, \ldots, c_{\ell_1})$ be the ℓ_1-cycle of F whose list of differences contains, by assumption, an element x of order $u \equiv 2 \pmod 4$. Without loss of generality, we can assume that $x = c_1 - c_2$.

Let $\mathcal{G} = \big\{2x \cdot i + j \mid i \in \{0, 1, \ldots, u/2 - 1\}, j \in \{0, 1, \ldots, 2n/u - 1\}\big\}$ and recall that, by the definition of order, u is the smallest positive integer such that $xu \equiv 0 \pmod{2n}$; hence $xu/2 \equiv n \pmod{2n}$. Therefore, it is not difficult to check that

$$\text{both } \{\mathcal{G}, \mathcal{G} + x\} \text{ and } \{\mathcal{G}, \mathcal{G} + n\} \text{ are partitions of } \mathbb{Z}_{2n}. \tag{2}$$

Considering that F satisfies the assumptions of Proposition 3.1, we have that $\mathcal{F} = \{F + g \mid g \in \mathbb{Z}_{2n}\}$ is a solution of $OP(F)$. By taking into account (2) and recalling that $F + n = F$, it follows that $\mathcal{F} = \{F + g \mid g \in \mathcal{G}\}$.

Now set $C' = (c_1, \infty', c_2, \ldots, c_{\ell_1})$, where $\infty' \notin \mathbb{Z}_{2n} \cup \{\infty\}$, and let F' be the 2-regular graph of order $2n+1$ obtained from F by replacing C with C'. Finally, set $\mathcal{F}' = \{F' + g \mid g \in \mathcal{G}\}$, where $F' + g$ is the graph obtained from F' by replacing each vertex $v \notin \{\infty, \infty'\}$ with $v + g$, and set $V = \mathbb{Z}_{2n} \cup \{\infty, \infty'\}$.

In order to prove that $OP(F')$ has a solution, we will show that $\mathcal{F}'$ is a 2-factorization of $K^*_{2n+2} = K_V - I$, where $I = \{\lfloor \infty, \infty' \rfloor\} \cup \{\lfloor g, x + g \rfloor \mid g \in \mathcal{G}\}$. Note that (2) guarantees that I is a 1-factor of K_V. Also, since F' contains all the edges of F except only for $\lfloor c_1, c_2 \rfloor$, and considering that

$$\{\lfloor c_2 + g, c_1 + g \rfloor \mid g \in \mathcal{G}\} = \{\lfloor g, x + g \rfloor \mid g \in \mathcal{G}\} \subset I,$$

it follows that every edge of $K_V - I$ of the form $\lfloor a, b \rfloor$ with $a \neq \infty' \neq b$ belongs to exactly one cycle of $\mathcal{F}'$. Finally, again by (2) we have that $\{\lfloor \infty', c_1 + g \rfloor, \lfloor \infty', c_2 + g \rfloor \mid g \in \mathcal{G}\} = \{\lfloor \infty', b \rfloor \mid b \in \mathbb{Z}_{2n}\}$, therefore every edge of K^*_{2n+2} of the form $\lfloor \infty', b \rfloor$ with $b \in \mathbb{Z}_{2n}$ belongs to exactly one cycle of $\mathcal{F}'$. Hence, $\mathcal{F}'$ is a 2-factorization of K^*_{2n+2} and this completes the proof. $\qquad \square$

We now provide two necessary conditions for the existence of a 2-regular graph satisfying the assumptions of Proposition 3.1.

Proposition 3.3. *If* $F = [\, ^{a_1}\ell_1, \, ^{a_2}\ell_2, \ldots, \, ^{a_s}\ell_s]$ *is a 2-regular graph of odd order* $2n + 1$ *satisfying the assumptions of Proposition 3.1, then*

$$|\{i \mid a_i \ell_i \text{ is odd}\}| = 1. \tag{3}$$

Proof. In [8, Proposition 3.4], the authors show that

1. the cycle of F passing through ∞ has odd length, and

2. if C is any other cycle of F such that $C + n = C$, then C has even length.

Therefore, if C is an odd length cycle of F not passing through ∞, then $C \neq C + n \in F$. In other words, if a_i denotes the number of cycles of F of odd length ℓ_i, then a_i is even, unless ℓ_i is the length of the cycle through ∞ and the assertion follows. $\quad \square$

Remark 3.1. *Let C be a cycle with vertices in $V = \mathbb{Z}_{2n} \cup \{\infty\}$ such that $C = C + n$. It is not difficult to check that C has one of the following forms:*

(a) $C = (\infty, c_1, \ldots, c_{m-1}, c_m, c_m + n, c_{m-1} + n, \ldots, c_1 + n)$,

(b) $C = (c_1, \ldots, c_{m-1}, c_m, c_m + n, c_{m-1} + n, \ldots, c_1 + n)$, *and* $\infty \notin V(C)$,

(c) $C = (c_1, c_2, \ldots, c_m, c_1 + n, c_2 + n, \ldots, c_m + n)$, *and* $\infty \notin V(C)$.

In the first two cases, the translation by n acts on C as a reflection, while in the latter case such an action is a rotation. In [8, Proposition 3.7], it is shown in particular that a 2-factor of K_V satisfying the assumptions of Proposition 3.1 contains exactly one cycle on which the translation by n acts as a reflection, which then coincides with the cycle through ∞. Therefore, any cycle C of F fixed by n and not passing through ∞ has the same form as in (c).

The following result can be seen as a generalization of [9, Theorem 3.2].

Proposition 3.4. *Let F be a 2-regular graph of order $2n + 1$ and let r denote the number of cycles in F of even length. If F satisfies the assumptions of Proposition 3.1 and its cycle passing through ∞ has length 3, then either $n \equiv 0 \pmod 4$ or $\frac{n-1}{2} + r$ is an even integer.*

Proof. Let F be a 2-regular graph of order $2n + 1$ such that

1. $V(F) = \mathbb{Z}_{2n} \cup \{\infty\}$,

2. $\Delta F \supset \mathbb{Z}_{2n} \setminus \{0\}$,

3. $F + n = F$.

and let C_∞ denote the cycle of F through ∞. By assumption, C_∞ has length 3, and by conditions 1 and 3 we have that $C_\infty + n = C_\infty$; hence $C_\infty = (\infty, g, g + n)$ for some $g \in \mathbb{Z}_{2n}$.

Let $C_1, C_2, \ldots, C_u$ be the list of the cycles in F distinct from C_∞, with $C_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,\ell_i})$ for every $1 \le i \le u$. By condition 3, we can assume without loss of generality that $C_i + n = C_i$ when $1 \le i \le s$, and $C_i + n = C_{i+t}$ when $s+1 \le i \le s+t$, where $u = s + 2t$. Hence, for $1 \le i \le s$ we have that ℓ_i is even, and by Remark 3.1 we obtain that $c_{i,j+\ell_i/2} = c_{i,j} + n$ for every $1 \le j \le \ell_i/2$. Now set $m_i = \ell_i/2$ when $1 \le i \le s$, otherwise set $m_i = \ell_i$. Also, let $d_{i,j} = c_{i,j+1} - c_{i,j}$ (where the subscript j is computed modulo ℓ_i) and set $D_i = \{d_{i,j} \mid 1 \le j \le m_i\}$ for every $1 \le i \le s+t$. Considering that any translation preserves the differences, we have that

$$d \in \Delta C_i, \ \text{if and only if} \ d \in \pm D_i.$$

for every $1 \le i \le s + t$. By recalling condition 2, and considering that $\Delta C_\infty = \{\pm n\}$ and $\sum_{i=1}^{s+t} m_i = n - 1$, it follows that $\mathbb{Z}_{2n} \setminus \{0, n\} = \bigcup_{i=1}^{s+t}(\pm D_i)$. Also, since $c_{i,1} + \sum_{j=1}^{m_i} d_{i,j} = c_{i,m_i+1}$, we have that $\sum_{j=1}^{m_i} d_{i,j} = n$ when $1 \le i \le s$, otherwise $\sum_{j=1}^{m_i} d_{i,j} = 0$; hence $\sum_{i=1}^{s+t} \sum_{j=1}^{m_i} d_{i,j} = sn$. Finally, considering that $\mathbb{Z}_{2n} \setminus \{0, n\}$ contains $2\lfloor \frac{n}{2} \rfloor$ odd integers and $-x \neq x$ for every $x \in \mathbb{Z}_{2n} \setminus \{0, n\}$, it follows that $\bigcup_{i=1}^{s+t} D_i$ contains exactly $\lfloor \frac{n}{2} \rfloor$ odd integers, therefore

$$sn \equiv \left\lfloor \frac{n}{2} \right\rfloor \pmod 2.$$

If n is even, then $n \equiv 0 \pmod 4$. If n is odd, then $\frac{n-1}{2} \equiv s \pmod 2$. Denoting by s' the number of even length cycles in $\{C_{s+1}, C_{s+2}, \ldots, C_{s+t}\}$ and recalling that $C_i \neq C_i + n \in F$ for $s+1 \le i \le s+t$, it follows that the total number of even length cycles in F is $r = s + 2s'$, hence $\frac{n-1}{2} \equiv r \pmod 2$, that is, $\frac{n-1}{2} + r$ is even, and the assertion is proven. □

Proposition 3.1 tells us how to construct 1-*rotational* solutions of order $2n + 1$. These can then be used, following *Proposition* 3.2, to construct solutions of order

$2n+2$. Finally, Propositions 3.3 and 3.4 give us necessary conditions for a 1-*rotational* solution to exist.

We use the above results to construct 1-*rotational* solutions to $OP(F)$ whenever F is a 2-regular graph of order $4t+1$, thus $n = 2t$ and $t \in \mathbb{N}$. Equation (4) defines F as the graph containing u_i cycles of odd length ℓ_i, and w_j cycles of even length m_j, for every $i \in \{1, 2, \ldots, h\}$ and $j \in \{1, 2, \ldots, k\}$. Recalling the necessary condition in Proposition 3.3, we have

$$F = [{}^{u_1}\ell_1, \ldots, {}^{u_h}\ell_h, {}^{w_1}m_1, \ldots, {}^{w_k}m_k] \quad : \quad |\{i \mid u_i \text{ is odd}\}| = 1. \tag{4}$$

The graph F must also satisfy Equation (5), which implements *Proposition* 3.4.

$$\exists! i : (\ell_i = 3 \wedge u_i \text{ is odd}) \Rightarrow$$
$$2t \equiv 0 \pmod{4} \quad \vee \quad \left(\frac{2t-1}{2} + \sum_{i=1}^{k} w_i \right) \equiv 0 \pmod{2}. \tag{5}$$

The symmetries stated in *Remark* 3.1 reduce the labeling problem on F to a simpler one on a new graph F^*, the asymmetric version of F, which can be seen as the union of 2 subsets, namely the set of paths ($\mathcal{P}$) and the set of cycles ($\mathcal{C}$). To better describe the structure of F^*, we assume without loss of generality that u_1 is odd, and the remaining u_i are even. Hence we can write $u_1 = 2a_1 + 1$, $u_i = 2a_i$ for every $i \in \{2, 3, \ldots, h\}$, and $m_j = 2\mu_j$ for every $j \in \{1, 2, \ldots, k\}$. Thus Equation (6) describes the structure of the reduced graph F^*,

$$F^* = \mathcal{C} \cup \mathcal{P}, \tag{6}$$
$$\mathcal{C} = [(\ell_1 + 1)/2, {}^{a_1}\ell_1, {}^{a_2}\ell_2, \ldots, {}^{a_h}\ell_h], \tag{7}$$
$$\mathcal{P} = [[{}^{w_1}\mu_1, \ldots, {}^{w_k}\mu_k]], \tag{8}$$

where $\mathcal{P}$ is the graph containing w_j paths with μ_j edges, for every $j \in \{1, 2, \ldots, k\}$, and they are pairwise vertex-disjoint. $\mathcal{C}$ also contains a cycle with $(\ell_1 - 1)/2$ nodes, namely the one with infinity.

In a more general and intuitive way, the rationale for F^* is as follows. On the one hand, we consider the cycles of odd length in F. If there is an even number $u_i \equiv 0 \pmod{2}$ of cycles of odd length ℓ_i, these contribute in F^* with $u_i/2$ cycles of length ℓ_i in $\mathcal{C}$. The only leftover cycle is of odd length ℓ_1 (since the number of nodes is $4t+1$) and it is the one passing through ∞, which contributes with an open chain of length $(\ell_1 - 1)/2$ in $\mathcal{P}$. On the other hand, an even number $w_j \equiv 0 \pmod{2}$ of cycles of even length m_j are represented in F^* as $w_j/2$ cycles of length m_j in $\mathcal{C}$, similar to an even number of odd cycles. Any cycle of even length m_j with single multiplicity $w_j = 1$ becomes a chain of length $m_j/2$ in $\mathcal{P}$.

Note that the number of edges of F^* is $2t - 1$. We seek to determine a vertex labeling of F^* with the elements of $\mathbb{Z}_{4t}$ such that

1. $V(F^*)$ contains exactly one element in $\{x, x + 2t\}$, for every $x \in \mathbb{Z}_{4t}$,

2. $\Delta F^* = \mathbb{Z}_{4t} \setminus \{0, 2t\}$.

This labeling of the vertices of F^* leads to a labeling of F satisfying *Proposition* 3.1, and hence to a solution for $OP(F)$ (see Figure 3).

3.2   (Almost) 2-*rotational* solutions to $OP(F)$

Here we describe a method to construct solutions to $OP(F)$ in all cases where there is no 1-*rotational* solution and, in particular, when the necessary conditions of Propositions 3.3 and 3.4 do not hold.

Let $v = 2n+1$ be a positive integer, and set $V = \big(\{0,1\} \times \mathbb{Z}_n\big) \cup \{\infty\}$. For every subgraph Γ of K_V and for every $i, j \in \{0,1\}$, let $\Delta_{ij}\Gamma$ be the list of (i,j)-differences of Γ defined below:

$$\Delta\Gamma_{ij} = \big\{x - y \mid \lfloor (i,x), (j,y) \rfloor \in E(\Gamma \setminus \{\infty\})\big\}. \tag{9}$$

For every $g \in \mathbb{Z}_n$ we denote by $\Gamma + g$ the graph with vertex set V obtained from Γ by replacing each vertex (i,x) with $(i, x+g)$.

The following result provides sufficient conditions for the existence of a solution to $OP(F)$.

Proposition 3.5. *Let $F = [\ell_1, \ell_2 \ldots, \ell_s]$ be a 2-regular graph of order $2n + 1$ satisfying the following conditions:*

1. $V(F) = \big(\{0,1\} \times \mathbb{Z}_n\big) \cup \{\infty\}$,

2. the vertices adjacent to ∞ are of the form $(0, x_0)$, $(1, x_1)$ for some $x_0, x_1 \in \mathbb{Z}_n$,

3. if n is odd, then $\Delta_{00}F = \Delta_{11}F = \mathbb{Z}_n \setminus \{0\}$ and $\Delta_{01}F = \mathbb{Z}_n$,

4. if n is even, then:

   (a) F contains the path $P = \lfloor (0,0), (0, n/2), (1, n/2), (1, 0) \rfloor$ in one of its cycles,

   (b) $\Delta_{ij}(F - P) = \mathbb{Z}_n \setminus \{0, n/2\}$ for every $(i,j) \in \{(0,0), (0,1), (1,1)\}$.

Then there exists a solution of $OP([\ell_1, \ell_2 \ldots, \ell_s])$. Furthermore, if C is an ℓ_1-cycle of F such that $\Delta_{01}C$ contains an integer distinct from $n/2$, then there exists a solution to $OP([\ell_1 + 1, \ell_2 \ldots, \ell_s])$.

Proof. Let $\mathcal{F} = \{F + g \mid g \in [1, n]\}$ when n is odd, otherwise let $\mathcal{F} = \{F + g, F^* + (n/2 + g) \mid 1 \leq g \leq n/2\}$, where F^* is the 2-regular graph obtained by replacing the path P (which is contained in F by condition 4a) with $P^* = \lfloor (0,0), (1, n/2), (0, n/2), (1, 0) \rfloor$. It is important to notice that in this case

$$F - P = F^* - P^*. \tag{10}$$

We claim that $\mathcal{F}$ is a solution of $OP(F)$. By condition 1 and considering that the total number of edges (counted with their multiplicity) covered by $\mathcal{F}$ is $n|F| =$

$n(2n + 1) = |E(K_{2n+1})|$, to prove the assertion it is enough to show that every edge of K_V, with $V = (\{0, 1\} \times \mathbb{Z}_n) \cup \{\infty\}$, is contained in some 2-factor of $\mathcal{F}$.

Denoting with $(0, x_0)$ and $(1, x_1)$ the neighbours of ∞ in F (condition 2), we have that $\lfloor \infty, (i, a) \rfloor \in F - x_i + a$ for every $(i, a) \in V \setminus \{\infty\}$. By recalling that (10) holds when n is even, it follows that every edge of K_V incident to ∞ belongs to some 2-factor of $\mathcal{F}$.

Now let (i, a) and (j, b) be two distinct vertices of $V \setminus \{\infty\}$ such that $a - b \neq n/2$ for n even. By conditions 3 and 4b, there exists an edge of F, say $\lfloor (i, a'), (j, b') \rfloor$ such that $a' - b' = a - b$. It follows that $\lfloor (i, a), (j, b) \rfloor = \lfloor (i, a'), (j, b') \rfloor + (b - b') \in F + (b - b')$. By taking into account (10) for n even, we have that $\lfloor (i, a), (j, b) \rfloor$ belongs to some 2-factor of $\mathcal{F}$. It is not difficult to check that every edge of the form $\lfloor (i, a), (j, a + n/2) \rfloor$, with $1 \leq a \leq n/2$, is contained in $P + a$ or $P + (n/2 + a)$. Hence every edge of K_V is contained in some 2-factor of $\mathcal{F}$ which is therefore a solution to $OP(F)$.

Now let $C = (c_0, c_1, \ldots, c_{\ell_1} - 1)$ be the ℓ_1-cycle of F such that $\Delta_{01}C$ contains an element distinct from $n/2$; in other words, C contains an edge of the form $\lfloor (0, y_0), (1, y_1) \rfloor$ with $y_0 - y_1 \neq n/2$; hence, this edge does not belong to P. Without loss of generality, we can assume that $c_0 = (0, y_0)$ and $c_1 = (1, y_1)$.

Set H and H^* be the 2-regular graphs of order $2n + 2$ obtained from F and F^*, respectively, by replacing the edge $\lfloor c_0, c_1 \rfloor$ with the 2-path $\lfloor c_0, \infty', c_1 \rfloor$, where $\infty' \notin V$. Also, $I = \{\lfloor \infty, \infty' \rfloor\} \cup \{\lfloor c_0 + g, c_1 + g \rfloor \mid 1 \leq g \leq n\}$ is clearly a 1-factor of $K_{2n+2} = K_{V \cup \{\infty'\}}$. Finally, let $\mathcal{H} = \{H + g \mid g \in [1, n]\}$ when n is odd, otherwise let $\mathcal{H} = \{H + g, H^* + (n/2 + g) \mid 1 \leq g \leq n/2\}$.

We claim that $\mathcal{H}$ is a solution to $OP([\ell_1 + 1, \ell_2 \ldots, \ell_s])$. Since C is also a cycle of F^* for n even, both H and H^* are 2-regular graphs of K_{2n+2} isomorphic to $[\ell_1 + 1, \ell_2 \ldots, \ell_s]$. Also, considering that $\mathcal{F}$ is a 2-factorization of K_V, every edge of $K_{V \cup \{\infty'\}} - I$ not incident to ∞' is contained in some 2-factor of $\mathcal{H}$. Since $H - P = H^* - P^*$ and $\lfloor \infty', (i, a) \rfloor = \lfloor \infty', c_i \rfloor + (a - y_i)$, it follows that every edge incident to ∞' belongs to some 2-factor of $\mathcal{H}$, therefore $\mathcal{H}$ provides a solution to $OP([\ell_1 + 1, \ell_2 \ldots, \ell_s])$. □

A factorization $\mathcal{F}$ of $K_{2n+1} = K_V$, with $V = (\{0, 1\} \times \mathbb{Z}_n) \cup \{\infty\}$, constructed as in Proposition 3.5, when n is odd, is called 2-*rotational*, since the permutation group $G = \{\tau_g \mid g \in \mathbb{Z}_n\}$ of V, where τ_g fixes ∞ and maps $(i, x) \in V \setminus \{\infty\}$ to $(i, x + g)$, is an automorphism group of $\mathcal{F}$ whose action on $V \setminus \{\infty\}$ creates two orbits of size n.

The idea of constructing 2-factorizations of K_{2n+1} with n even, as described in Proposition 3.5, was first presented in [15].

4 Solving instances of the Oberwolfach Problem

4.1 Computing 1-*rotational* solutions

Recalling the content of Section 3.1, we propose a linear-time algorithm that implements *Proposition* 3.3 and the related *Remark* 3.1, and reduces F to F^*. Afterward,

CP solves the problem over F^*, and therefore the labeling of F is retrieved.

Algorithm 1 starts by reducing F to F^* with lines (5-8), where the only unpaired cycle ($u_i = 1 \mod 2$ as of Equation 4) of odd length l_i reduces to a cycle of length $(l_i - 1)/2$ in F^* (symmetry of case a). This latter cycle contains node ∞. In lines (9-12), a pair of 2 isomorphic ($u_i > 1$) cycles of odd lengths l_a and l_b is reduced to a single cycle of length $l_i = l_a$ in F^* (symmetry of case c). In the same way, in lines (15-17), a pair of 2 isomorphic ($w_i > 1$) cycles of even length $m_a = m_b$ is reduced to a single cycle of length $m_i = m_a$ in F^* (symmetry of case c). In lines (18-21), the remaining unpaired cycles ($w_i = 1$) of even lengths m_i are reduced to open chains of length $m_i/2$ in F^* (symmetry of case b). We may derive F from F^* by an inverse constructive process.

Algorithm 1: Reducing to F^*

1: **Input:** The original graph F
2: **Output:** The transformed version of the graph, F^*
3: infinite=$false$, and $V_{CP} = D_{CP}=0$
4: **for** i in u_i **do**   ▷ Iterate through odd-length cycles
5:   **if** $u_i \equiv 1 \pmod 2$ && infinite=false **then**   ▷ The cycle with infinite
6:     Put a chain of length $(l_i - 1)/2$ in F^* in position 0;
7:     $u_i - -;\quad V_{CP}+ = (l_i - 1)/2;\quad D_{CP} = l_i/2 - 2;$
8:     infinite=$true$;
9:   **else**   ▷ $u_i > 1$. Pair of odd-length cycles. Simplify one.
10:     Put a cycle of length l_i in F^*;
11:     $u_i = u_i - 2;\quad V_{CP}+ = l_i;\quad D_{CP} = l_i;$
12:   **end if**
13: **end for**
14: **for** i in w_i **do**   ▷ Iterate through even-length cycles.
15:   **if** $w_i > 1$ **then**   ▷ Pair of even-length cycles. Simplify one.
16:     Put a cycle of length m_i in F^*;
17:     $w_i = w_i - 2;\quad V_{CP}+ = m_i;\quad D_{CP} = m_i;$
18:   **else** $w_i = 1$ ▷ Treat the cycle as a chain of half length.
19:     Put a chain of length $m_i/2$ in F^*;
20:     $w_i - -;\quad V_{CP}+ = m_i/2;\quad D_{CP} = m_i/2;$
21:   **end if**
22: **end for**
23: **return** F^*

Following the reduction, the F^* labeling problem (F^* LP) aims at finding a labeling for F^* so that there is a solution to the complete $OP(F)$.

Problem 4.1 (F^* Labeling). *Let $F^* = (V, E)$ be a graph of order $|V(F^*)| = \gamma + 1$, and let $v_i \in V$ be an element in $G \cup \{\infty\}$ where $G = \mathbb{Z}_{2\gamma}$. Also, let $F^* = O \cup C$ with O set of open chains and C set of closed cycles. For each node $i \in \bar{V} = V \setminus \{\infty\}$ the F^* Labeling Problem asks to assign a label $n_i \in G$ so that the following conditions hold:*

1. *Each element in the set $\tilde{V}$ receives a unique label from G.*
   $n_\alpha \neq n_\beta$ *for all* $n_\alpha, n_\beta \in \bar{V}$.

2. *The set $\bar{V}$ contains either label n_α or its γ-translated label.*
   $n_\alpha \in \bar{V} \vee n_\beta = n_\alpha + \gamma \pmod{2\gamma} \in \bar{V}$ *with* $n_\alpha \in \mathbb{Z}_{2\gamma}$.

3. ΔF^* *has cardinality* $\lambda = 2\gamma - 2$ *and contains all the elements in* $G\backslash\{0, \gamma\}$ *with multiplicity* 1. $|\Delta F^*| = 2(\gamma - 2) \wedge \Delta F^* = G\backslash\{0, \gamma\}.$

The corresponding CP model (11-17) describes (F^* LP).

We remark that *alldifferent* and *card* are typical CP operators on arrays of elements [4]. The first operator requires all array elements to exhibit different values. The second one, which takes an additional integer argument value c, imposes the cardinality of the integer value c in the given array.

$$V = \{n_i \mid n_i \in G\}, \qquad (11)$$

$$\texttt{alldifferent(V)} \qquad dom(V) = [0, 2\gamma), \qquad (12)$$

$$\texttt{card}(V|n_i) + \texttt{card}(V|(n_i + \gamma \pmod{2\gamma})) = 1 \qquad \text{for all } n_i \in \mathbb{Z}_\gamma, \qquad (13)$$

$$D = dC \cup dO, \qquad (14)$$

$$dC = \{(n_\alpha - n_\beta \pmod{2\gamma})\} \qquad \text{for all } \alpha, \beta \in V \wedge \lfloor \alpha, \beta \rfloor \in C, \qquad (15)$$

$$dO = \{\omega_i - \phi, \phi - \omega_i \pmod{2\gamma}\} \qquad \text{for all } o_i = [\omega_1, \ldots, \omega_i] \in O, \qquad (16)$$

$$\phi = \omega_1 + \gamma \pmod{2\gamma},$$

$$\texttt{alldifferent(D)} \qquad dom(D) = (0, 2\gamma)\backslash\{\gamma\}. \qquad (17)$$

Equations (11)-(12) enforce Condition (1) in (F^* LP), while (13) enforces Condition (2) for γ-translated labels. The difference-set is split into two subsets, as in Equation (14): dC in (15) for the edges over F^*, and dO for open chains in (16). The virtual label ϕ is reported in the latter subset. Finally, Constraint (17) enforces Condition (3) in (F^* LP). Therefore, the problem of labeling F^* collapses to a feasible assignments of set V, represented in (11).

Example 4.1. *(referring to Problem 4.1) Consider an* $OP(F = [5,^2 3,^2 4, 6])$ *of order* $4t + 1 = 25$ *with* $t = 6$. *F reduces to* F^*, *and the simplified instance is* $OP(F^* = [3_\infty, 3, 4, 3_c])$ *where* $O = [3_c]$ *and* 3_∞ *is the cycle with* ∞. *Therefore* $\gamma = 12$ *and nodes in* $V(F^*)$ *acquire their labels from* $\mathbb{Z}_{24} \cup \{\infty\}$. *Figure 3 represents the reduced* F^*, *with the virtual node 12.* $\Delta F^* = G\backslash\{0, 12\}$, *hence the labeling is a valid solution for the* (F^* LP). *Figure 4 represents the corresponding labeling for F.*
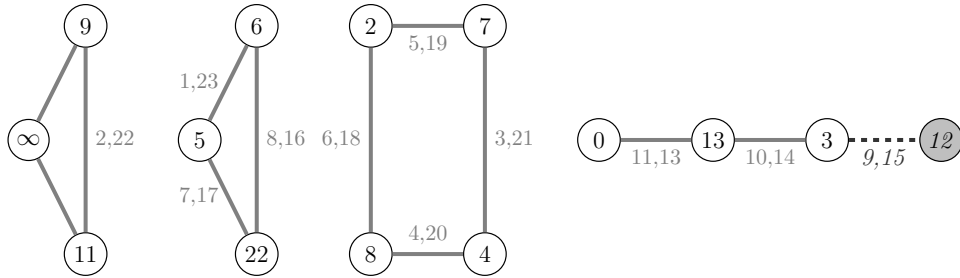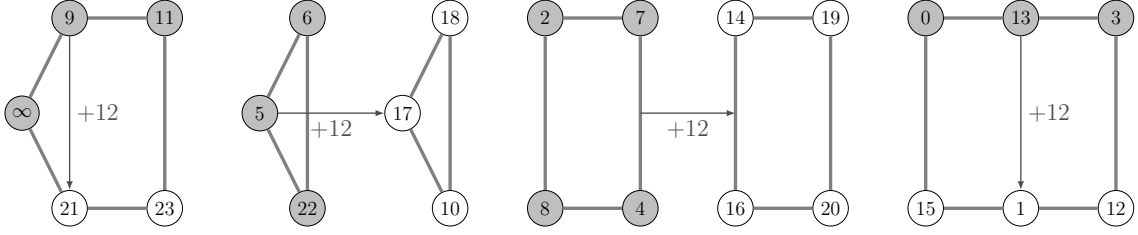

Figure 3: F^* instance for $F = [5,^2 3,^2 4, 6]$.

Figure 4: F instance derived from $F^* = [3_\infty, 3, 4, 3_c]$.

A 2-factor F of order $4t + 1$ which generates a 1-*rotational* solution for $OP(F)$ satisfies *Proposition* 3.1, 3.3, and 3.4. Correspondingly, a solution of order $4t + 2$ can be derived from F, according to *Proposition* 3.2, and its polynomial-time computation works as follows. Given $F = [l_1, l_2, ..., l_s]$ and $F' = [l_1 + 1, l_2, ..., l_s]$, a new node ∞' joins the cycle l_1. The new node replaces a path $P = \lfloor c_1, c_2 \rfloor \in l_1$ with a new path $P^* = \lfloor c_1, \infty', c_2 \rfloor$ in the cycle l_1. Therefore, the difference-set of F' omits values $c_1 - c_2$ and $c_2 - c_1$ (in modulo). For our computational tests, node ∞' is arbitrarily inserted between two nodes c_1, c_2 so that $c_1 - c_2 = c_2 - c_1 \mod 2t$. Solutions of order $4t + 2$ require the same computational effort as 1-*rotational* instances of order $4t + 1$. Moreover, we highlight how multiple instances of order $4t + 2$ originate from the same instance of order $4t + 1$ (see Example 4.2).

Example 4.2. *Consider two instances of order $4t + 2$, such as $OP(19, 4, 3)$ and $OP(18, 5, 3)$ with $t = 6$. Both the instances originate from $OP(18, 4, 3)$, of order $4t + 1$.*

4.2 Computing (almost) 2-*rotational* (n odd)

This class of solutions derives from *Proposition* 3.5 (see also Section 3.2). Since that proposition distinguishes between odd and even values of n, we present the approach for odd values of n, and discard Condition (4) of *Proposition* 3.5. For even values of n, see the Appendix (6.1).

Given the 2-regular graph $F = (V, E)$ of order $2n + 1$, recalling that n is odd, we write $n = 4t + 3$ where $t \in \mathbb{N}$. The set $V = (\{0, 1\} \times G) \cup \{\infty\}$ represents nodes labels, where G is the additive group $\mathbb{Z}_{2t+1}$. Without loss of generality, ∞ lies in the longest cycle of F. There are 3 difference-sets, as Condition 3 of *Proposition* 3.5 states. Each difference $\lfloor \alpha, \beta \rfloor \in E(F)$ goes into a set depending on $\{0, 1\}$ labels of both α and β. We propose a two-step approach that initially determines the first labels and then the second ones. Once the first labels are determined, the problem resembles a 1-*rotational* problem where the decision variables are in a set of integers. On the other side, there are 3 difference-sets, as described by Equation (9). The first-step solution provides information about the type of edge (e.g., $\lfloor \alpha = (i, a), \beta = (i, b) \rfloor \mid a, b \in G$ is in the difference-set ΔF_{ii}), and the second-step exploits such knowledge.

The Binary Labeling Problem (BLP) is the first-step subproblem, and asks to label each node $i \in V(F) \backslash \{\infty\}$ with a coordinate $c_i \in \{0, 1\}$, namely the *binary label*. Once the solution of (BLP) is given, the Group Labeling Problem (GLP) seeks to

assign a second coordinate $n_i \in G$, namely the *group label*, to each node, so that Condition 3 of *Proposition* 3.5 holds. Differences of the type $\Delta_{01}F$ are directed from $c_\alpha = c_x = 0$ to $c_\beta = c_y = 1$. Without loss of generality, the direction is arbitrarily fixed. Besides, $\Delta_{00}F$ and $\Delta_{11}F$ contain both the two undirected differences for each edge (e.g., both $\alpha - \beta$ and $\beta - \alpha$).

According to Condition 1 of *Proposition* 3.5, there are exactly $2t+1$ nodes for each binary label. Condition 2 states that two nodes adjacent to ∞ have different binary labels. Condition 3 defines difference-sets' cardinalities as $|\Delta_{00}F| = |\Delta_{11}F| = 2t$ while $|\Delta_{01}F| = 2t + 1$. (BLP) formalizes such conditions.

Problem 4.2 (Binary Labeling Problem). *Let $F = (V, E)$ be a 2-regular graph of order $|V| = 4t + 3$. (BLP) asks to assign to each node $i \in \bar{V} = V \backslash \{\infty\}$ a binary label $c_i \in \{0, 1\}$ so that the following conditions hold:*

1. *The two nodes $\alpha, \beta \in \bar{V}$ adjacent to ∞ have different binary labels.*
   $\exists \lfloor \alpha, \infty \rfloor \wedge \lfloor \beta, \infty \rfloor : c_\alpha \neq c_\beta.$

2. *There are exactly $2t + 1$ directed differences (edges) between nodes with different binary labels.*
   $|\Delta_{01}F = \{\alpha - \beta \mid \lfloor (0, \alpha), (1, \beta) \rfloor$ *for all* $\alpha, \beta \in G\}| = 2t + 1.$

Equations (18-24) formulate (BLP) in CP.

$$C = \{c_i \mid c_i \in \{0, 1\}\} \qquad \text{for all } i \in \bar{V}, \tag{18}$$

$$dM = \{d_j \mid d_j \in \{0, 1\}\} \qquad \text{for all } j \in 0, 1, \ldots, E(F \backslash \{\infty\}), \tag{19}$$

$$d_j = \begin{cases} 1 & \text{if } c_\alpha = 0, c_\beta = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } \alpha, \beta \in \bar{V} \wedge \lfloor \alpha, \beta \rfloor, \tag{20}$$

$$\mathtt{card}(dM \mid 1) = 2t + 1, \tag{21}$$

$$\mathtt{card}(C \mid 1) = 2t + 1, \tag{22}$$

$$\mathtt{card}(C \mid 0) = 2t + 1, \tag{23}$$

$$c_\alpha = 1 \wedge c_\beta = 0 \qquad \text{if } \lfloor \alpha, \infty \rfloor \wedge \lfloor \beta, \infty \rfloor \wedge \alpha \neq \beta. \tag{24}$$

Example 4.3. *(referring to Problem 4.2) Consider an $OP(5, 6)$ of order $4t + 3 = 11$ with $t = 2$. An example of binary labels assigned according to the (BLP) is in Figure 5).*

Each node $i \in \bar{V}$ acquires a binary label c_i, hence the solution is the set C in Equation 18. Each element $d_j \in dM$ (19) is 1 if and only if the oriented edge $\lfloor \alpha, \beta \rfloor$ connects a node $\alpha : c_\alpha = 0$ with $\beta : c_\beta = 1$, and 0 otherwise. Constraint (21) ensures that Condition (2) of (BLP) holds, while Constraints (22) and (23) bound the cardinality of binary-labeled nodes in V. (24) implements Condition (1) in (BLP) by hard-fixing labels of the two nodes adjacent to ∞.

Computational solutions for 2-*rotational* instances led us to understand the underlying structure of (BLP). Correspondingly, it was possible to devise a general
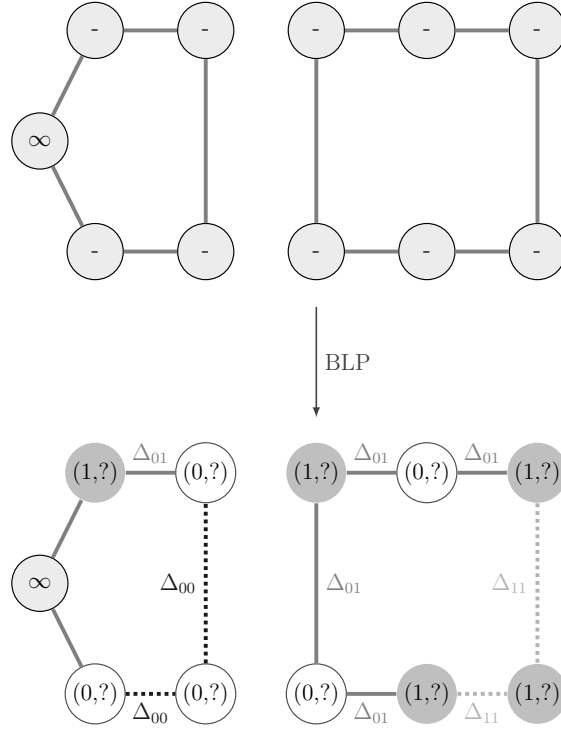
Figure 5: (BLP) over $OP(F = [5, 6])$.

polynomial-time algorithm to solve (BLP). The rationale is to search for known patterns and insert parts of solution (e.g.: label a subset of nodes) until the problem reduces to a basic pattern of the form $F[3]$, $F[5, 6]$, $F[^53]$, and $F[^35]$. Algorithm 2 in the Appendix (see 6.1) presents such a procedure.

Once (BLP) returns the binary labels, solving the $OP(F)$ is matter of a group labeling over the additive group G. Condition (3) of *Proposition* 3.5 holds on the difference-sets. In analogy with the labeling for 1-*rotational* methods, each group-label $n_i \in G \mid i \in V(F)$ translates into two values in (two) difference-set, depending on its binary label. Difference-sets (9) fulfill Equations (25-27).

$$\Delta_{00}F = \{n_\alpha - n_\beta \pmod{2t + 1} : \tag{25}$$
$$\text{for all } \alpha, \beta \in V(F) \wedge \lfloor \alpha, \beta \rfloor \wedge c_\alpha = c_\beta = 0\}$$
$$\Delta_{11}F = \{n_\alpha - n_\beta \pmod{2t + 1} : \tag{26}$$
$$\text{for all } \alpha, \beta \in V(F) \wedge \lfloor \alpha, \beta \rfloor \wedge c_\alpha = c_\beta = 1\}$$
$$\Delta_{01}F = \{n_\alpha - n_\beta \pmod{2t + 1} : \tag{27}$$
$$\text{for all } \alpha, \beta \in V(F) \wedge \lfloor \alpha, \beta \rfloor \wedge c_\alpha = 0, c_\beta = 1\}$$

Condition 2 of *Proposition* 3.5 infers domains on sets so that the desired 2-factor F is a 2-*rotational* solution for $OP(F)$. Therefore, the Group Labeling Problem (GLP) formalizes *Proposition* 3.5.

Problem 4.3 (Group Labeling Problem). *Let $F = (V, E)$ be a 2-regular graph of order $|V| = 4t + 3$. $V(F) = \{\{0, 1\} \times G\} \cup \{\infty\}$ is the set of nodes over F, where*

G is the additive group $\mathbb{Z}_{2t+1}$. For each node $i \in \bar{V} = V\backslash\{\infty\}$ with its binary label $c_i \in \{0,1\}$, the (GLP) asks to assign a group label $n_i \in G$ so that the following conditions hold:

1. *Undirected difference-sets are so that $\Delta_{00}F = \Delta_{11}F = G\backslash\{0\}$,*

2. *The directed difference-set is so that $\Delta_{01}F = G$.*

Equations (28-37) formulate the (GLP) with CP.

$$V = \{A \cup B\}, \tag{28}$$

$$A = \{a_i \mid a_i \in G, c_i = 0\} \qquad B = \{b_i \mid b_i \in G, c_i = 1\}, \tag{29}$$

$$\texttt{alldifferent(A)} \qquad dom(A) = [0, 2t+1), \tag{30}$$

$$\texttt{alldifferent(B)} \qquad dom(B) = [0, 2t+1), \tag{31}$$

$$dA = \{(a_\alpha - a_\beta \mod (2t+1))\} \qquad \text{for all } \alpha, \beta \in A \wedge \lfloor \alpha, \beta \rfloor, \tag{32}$$

$$dB = \{(b_\alpha - b_\beta \mod (2t+1))\} \qquad \text{for all } \alpha, \beta \in B \wedge \lfloor \alpha, \beta \rfloor, \tag{33}$$

$$dAB = \{(a_\alpha - b_\beta \mod (2t+1))\} \qquad \text{for all } \alpha \in A, \beta \in B \wedge \lfloor \alpha, \beta \rfloor, \tag{34}$$

$$\texttt{alldifferent(dA)} \qquad dom(dA) = (0, 2t+1), \tag{35}$$

$$\texttt{alldifferent(dB)} \qquad dom(dB) = (0, 2t+1), \tag{36}$$

$$\texttt{alldifferent(dAB)} \qquad dom(dAB) = [0, 2t+1), \tag{37}$$

$$\texttt{card(dA)} = \texttt{card(dB)} = 2t \qquad \texttt{card(dAB)} = 2t+1. \tag{38}$$

Equation (28) represents the set V as the union of A and B, respectively the subset of nodes with binary label $c_i = 0$, and $c_i = 1$. The solution is a feasible assignment for V. Constraints (30)-(31) force on both A and B domains over G. Difference-sets in (32-34) rewrite sets in Equations (25-26). Finally, constraints and domains in (35-37) enforce Conditions 1 and 2 of (GLP). In particular, Constraint (38) ensures that difference-sets have the required cardinalities. The (GLP) solution generates a complete solution for $OP(F)$, with roto-translation similar to the ones explained for 1-*rotational* methods. *Proposition* 3.5 describes how F generates the other 2-regular copies.

Example 4.4. *(referring to Problem 4.3) Consider an $OP(F = [5,6])$ of order $4t + 3 = 11$ with $t = 2$. Assuming binary labels are assigned, an example of group labels from (GLP) is represented in Figure 6.*

According to *Proposition* 3.5, a solution of order $4t + 3$ generates a solution of order $4(t + 1)$. The process requires polynomial-time, and is as follows. Starting from the $4(t + 1)$ instance, a 2-path $P = [a, i^*, b]$ is selected from the cycle C_∞ (the cycle containing the ∞ node), and replaces the edge $\lfloor a, b \rfloor$. The resulting graph is the 2-regular F^*, of order $|V(F^*)| = 4t + 3$. The 2-*rotational* approach solves the $4t + 3$ instance on F^*. Afterwards, the pruned node i^* goes back to F^*, so that F^* becomes F. Without loss of generality, i^* lies between nodes with different binary labels i^* inside the cycle C_∞, so that $P^* = [\alpha, i^*, \beta] \mid c_\alpha \neq c_\beta$. Node i^* is relabeled as ∞_2 while the original ∞ becomes ∞_1. Therefore, the difference-set $\Delta_{01}F$ on F loses the difference $n_\alpha - n_\beta$ (or $n_\beta - n_\alpha$ if and only if $c_\beta = 0, c_\alpha = 1$).
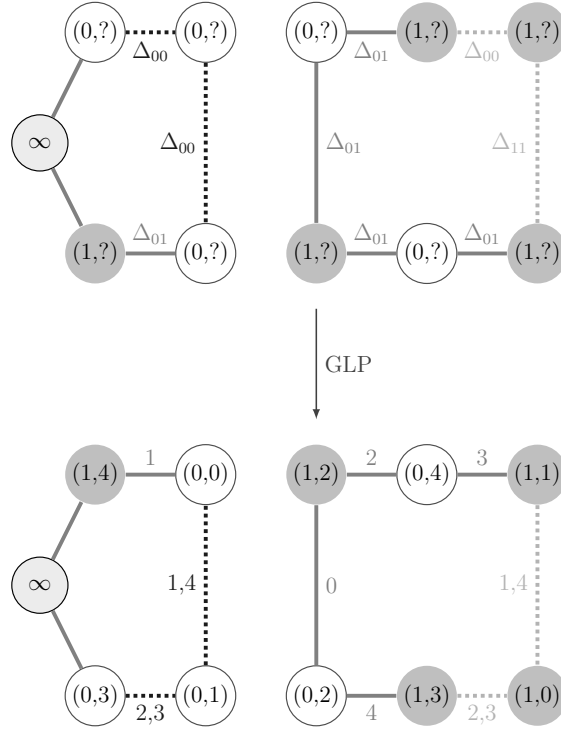
Figure 6: (GLP) over $OP(F = [5, 6])$.

5   Experimental Results

We implemented the proposed algorithms and formulations with Java 1.8 (see Section 6.1 for code), *IBM ILOG CPLEX* and *CP Optimizer* 12.7. Tests ran on a *Intel(R) Core i5-3550 @ 3.30GHz* with $4GB$ of RAM, a computer. Deza et al. [15] solve instances of much smaller size (order ≤ 40), with undisclosed algorithms running on a dedicated cluster machine [12]. The new 20 orders we solve here, from 41 to 60, are 241200, that is 22 times greater than those (10897) solved in [15]. Moreover, they generalize *r-rotational* rules also with $r \notin \{1, 2\}$, while our contribution deals only with $r \in \{1, 2\}$. Table 1 reports computational results for instances with $n \in [40, 60]$, and complete solutions are available online (see Section 6.1). Timelimits for (GLP) and $(F^* \text{ LP})$ are $5 \cdot (1 + |V(F)|/50)$ seconds, and $|V(F^*)|/20$ seconds, respectively, while Algorithm 2 solves (BLP) in negligible time.

The 2-*rotational* approach (see 4.2) with odd n values (see *Proposition* 3.5) solves instances of order $4t + 3$. Solutions of order $4t$ directly derive from the $4t + 3$ ones. The 1-*rotational* approach (see 4.1) solves instances of order $4t + 1$, and hence $4t + 2$. While solving orders $4t + 1$, we discovered that certain instances do not have a 1-*rotational* solution, and, consequently, we derived *Proposition* 3.4.

The formal proof stemmed after this empirical evidence. The 2-*rotational* approach with even n values (see 6.1) solves instances not fulfilling requirements in *Proposition* 3.4. We generated all the order-dependent partitions of integers $i \in [40, 60]$ with at least three or more cycles (tables), since Traetta [27] provides com-

plete solutions to the two-table OP. Each order (1st column of Table 1) is tackled by means of 2-*rotational* and/or 1-*rotational* rules (3rd column). The *time* (4th column) represents the total time required for the class OP, so that all the instances (5th column) have a solution (6th column). Correspondingly, the average time per instance (7th column) is the arithmetic mean. The proposed methodologies solved all the instances. Finally, from our tests, 1-*rotational* methods appear to be faster than 2-*rotational* methods in terms of CPU times, according to Table 1. Also, we can report that single solutions for OP with $n < 120$ could be generated in less than 60 seconds.

#	Type	Method	Time (s)	Partitions	Solved	Avg. Time (s.ms)
40	4t	(Derived from 39)	911	1756	1756	00.519
41	4t+1		807	2056	2056	00.393
		1-*rotational*	90		1433	00.063
		A 2-*rotational*	717		623	01.151
42	4t+2	(Derived from 41)	90	2418	2418	00.037
43	4t+3	A 2-*rotational*	2462	2822	2822	00.872
44	4t	(Derived from 43)	2462	3302	3302	00.746
45	4t+1		3268	3851	3851	00.849
		1-*rotational*	1406		2547	00.552
		A 2-*rotational*	1862		1304	01.428
46	4t+2	(Derived from 45)	1406	4488	4488	00.313
47	4t+3	A 2-*rotational*	6348	5215	5215	01.217
48	4t	(Derived from 47)	6348	6072	6072	01.045
49	4t+1		5587	7033	7033	00.794
		1-*rotational*	460		4417	00.104
		A 2-*rotational*	5127		2616	01.960
50	4t+2	(Derived from 49)	460	8158	8158	00.056
51	4t+3	A 2-*rotational*	16705	9441	9441	01.769
52	4t	(Derived from 51)	16705	10920	10920	01.530
53	4t+1		18998	12600	12600	01.508
		1-*rotational*	4246		7513	00.565
		A 2-*rotational*	14752		5087	02.900
54	4t+2	(Derived from 53)	4246	14552	14552	00.292
55	4t+3	A 2-*rotational*	57043	16753	16753	03.405
56	4t	(Derived from 55)	57043	19296	19296	02.956
57	4t+1		42700	22183	22183	01.925
		1-*rotational*	2519		12557	00.201
		A 2-*rotational*	40181		9626	04.174
58	4t+2	(Derived from 57)	2519	25491	25491	00.099
59	4t+3	A 2-*rotational*	105258	29241	29241	03.600
60	4t	(Derived from 59)	105258	33552	33552	03.137

Table 1: Computational results for the OP with $n \in [40, 60]$, with more than 3 cycles per instance

5.1 *IP* proves the absence of solution for $OP(^2 3, 5)$

The *Handbook of Combinatorial Design* from [13] states that it is well known that $OP(^2 3, 5)$ has no solution, referring to a conference proceeding of Rosa [26]. In a different work, Alspach et al. [2] cite an unpublished paper by Piotrowski [25]. In the latter, the author self-cites an *unpublished* paper [24] from 1979, where he describes a proof made with the *aid of a computer*. We provide a simple proof of non-existence for $OP(^2 3, 5)$ with an *IP* formulation. The $OP(^2 3, 5)$ is the problem of arranging $n = 11$ people in 2 tables of 3 and 1 table of 5 for $M = 5$ meals. Each person has a label in $\mathbb{Z}_{11}$. The *IP* formulation enumerates every feasible combination of labels for tables of 3 (triplets) and tables of 5 (5-sets). Afterwards, *IP* seeks to select for $M = 5$ meals, one 5-set and two triplets so that each node *is seated* next to every other node exactly once during all the meals. There are $\binom{11}{3} = 165$ different triplets, with at least one distinct label, and $\binom{11}{5} \cdot 12 = 5544$ 5-sets with different adjacencies. The *IP* formulation in (39)-(44) models $OP(^2 3, 5)$.

$$min_{F,T} \qquad 0 \qquad\qquad (39)$$
$$\text{subject to}$$
$$\sum_{i \in I} F_{id} = 1 \qquad \text{for all } d \in D, \qquad (40)$$
$$\sum_{j \in J} T_{jd} = 2 \qquad \text{for all } d \in D, \qquad (41)$$
$$\sum_{i \in I} F_{id} \cdot fl_{il} + \sum_{j \in J} T_{jd} \cdot tl_{jl} = 1 \quad \text{for all } d \in D, \text{for all } l \in L, \qquad (42)$$
$$\sum_{d \in D} (\sum_{i \in I} F_{id} \cdot fa_{i\alpha\beta} + \sum_{j \in J} T_{jd} \cdot ta_{i\alpha\beta}) = 1 \quad \text{for all } \alpha, \beta \in L \wedge \alpha \neq \beta, \qquad (43)$$
$$F_{id}, T_{jd} \in \{0, 1\} \quad \text{for all } i \in I, j \in J, d \in D. \qquad (44)$$

The binary variables $F_{id} : i \in I = [1, 5544]$ and $T_{jd} : j \in J = [1, 165]$ with $d \in D = [1, D = 5]$ respectively represent all the different 5-sets and triplets over the 5 meals, respectively. F_{id} and T_{jd} take value of 1 if and only if the corresponding element - the ith 5-set or the jth triplet - is selected for the dth day. Coefficients fl_{il} and tl_{jl} are respectively equal to 1 if the label $l \in L = [1, 11]$ is present in the ith 5-set or i^{th} triplet. Coefficients $fa_{i\alpha\beta}$ and $ta_{j\alpha\beta}$ are equal to 1 if two different labels $\alpha, \beta \in L$ are adjacent in the ith 5-set or i^{th} triplet. The model has no objective function, as of in Equation (39). Equations (40) and (41) enforce the selection of one 5-set and two triplets per each $d \in D$. Constraint (42) enforces that each label appears only once for each $d \in D$. Constraint (43) enforces that two labels $\alpha, \beta \in L$ are adjacent only one time over all the meals. The continuous relaxation of (39)-(44) finds no solution in less than a second: hence $OP(^2 3, 5)$ has no solution. Correspondingly, the following proposition holds.

Proposition 5.1. *Let $OP(^2 3, 5)$ be the Oberwolfach Problem with* 2 *cycles of length* 3 *and a cycle of length* 5*. There is no solution to $OP(^2 3, 5)$.*

6   Concluding remarks

CP – particularly its propagation and inference algorithms – exploits difference methods for the Oberwolfach problem by means of well-posed formulations. 1 and 2-*rotational* methods reduce the complete OP to the problem of labeling a single 2-factor and that problem is efficiently solved in CP. In particular, Constraint propagation exploits the relations of mutual exclusion between labels to smartly guide the search procedure. Computational results prove the effectiveness of the approach, which provided complete solutions for the OP for $n \in [40, 60]$. Moreover, theoretical results such as *Proposition* 3.4 and the proof of absence of a solution for $OP(^2 3, 5)$ constitute a further outcome of this work. The complementarity of Combinatorial Design and Combinatorial Optimization and their positive interaction is, in our view, the main stake pointed out here. The contribution is bidirectional: computational evidence helps to deduce theoretical results, and the latter provides models for the former. We believe that approaches similar to the one presented may help to solve other problems in Combinatorial Design Theory.

6.1   Solutions and code

We complement the paper with the software written to implement the methodologies presented. To make its use accessible, we provide a GUI interface.

The software is available on GitHub at the following repository:
`https://github.com/ALCO-PoliTO/TheOberSolver`
Full solutions are available at:
`https://github.com/ALCO-PoliTO/TheOberSolver/tree/master/OberResults`.
The *IP DinnerFor11* formulation of $OP(^2 3, 5)$ is available at
`https://github.com/ALCO-PoliTO/DinnerFor11`

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Appendix

BLP Algorithm

The graph $F = [^{n_1}l_1, {}^{n_2}l_2, ..., {}^{n_a}l_b]$ is described with n_i the number of cycles of length l_i, $T = \sum_i^a n_i$ the number of cycles, and $t_M = \max_b l$ the longest cycle in F. The input is an unlabeled F and the output is the BLP solution for F, namely F_l. The Algorithm iteratively adds to the incumbent a partial labeling for a known pattern, and terminates when all the nodes have been labeled and transferred from F to F_l. Lines (3-6) iterate through T cycles, and reduce each cycle $l_i \geq 7$ with a cycle of length at most of 6, by labeling patterns of 4 nodes at a time. Cycles with exactly 4 nodes - as of in lines (7-9), are labeled instantly. Lines (12-18) search for more complex patterns (e,g: $F = [3, 5]$). Lines (19-21) label basic patterns in F. The order reported in line (14) is binding, and labeled patterns have different orientations depending on the incumbent labeled nodes. The worst-case time complexity of Algorithm 2 is $\mathcal{O}(T \cdot t_M)$ with an implementation without Lists.

Algorithm 2: (BLP) Algorithm

1: **Input:** The graph $F = [^{n_1}l_1, {}^{n_2}l_2, ..., {}^{n_a}l_b] = [t_1, t_2, ..., t_T]$
2: **Output:** The labeled (colored) graph F_l
3: **for** t_i in F **do**
4:     **while** $t_i \geq 7$ **do**
5:         $F_l \leftarrow$ last labels of t_i are $[1100]$; $t_i \leftarrow (t_i - 4)$ ▷ Color last four elements
6:     **end while**
7:     **if** $t_i = 4$ **then**
8:         $F_l \leftarrow$ labels of t_i are $[1100]$; $t_i \leftarrow (t_i - 4)$ ▷ Color last four elements
9:     **end if**
10: **end for**
11: found=$true$
12: **while** $found$ **do**   ▷ Search for known patterns
13:     $found$=false   ▷ The order of search is strictly as specified
14:     **if** F contains patterns like $F[3, 5]$, $F[^4 5]$, $F[^8 3]$, $F[^2 6]$, $F[^2 3, 6]$ **then**
15:         $found$=true
16:         $F_l \leftarrow$ labels of t are $[Pattern]$; $t_i \leftarrow (t_i - Pattern.length)$
17:     **end if**
18: **end while**
19: **if** the remaining problem t in T_i is of the form of $F[3]$, $F[5, 6]$, $F[^5 3]$, $F[^3 5]$ **then**
20:     $F_l \leftarrow$ labels of t are $[Pattern]$; $t_i \leftarrow (t_i - Pattern.length)$
21: **end if**   ▷ Check for basic solutions. The order of search is as specified
22: **return** F_l

(Almost) 2-*rotational* with n even

The approach to this class of instances is similar to the one presented for odd n. The 2-*rotational* method with even n solves instance not fulfilling *Proposition* 3.3. The methodology is analogous to the one for odd n, but Condition 3 from *Proposition* 3.5 is discarded, while Condition 4 holds. If n is even, F has order of $4t + 1$ with $n = 2t$. The set of vertices is $V = (\{0, 1\} \times G) \cup \{\infty\}$, with G the additive group $\mathbb{Z}_{2t}$. (BLP) and (GLP) slightly vary, according to the Proof of *Proposition* 3.5. In particular,

according to Condition 4 of *Proposition* 3.5, a cycle of F contains a path of the form $P = \lfloor (0,0), (0,n/2), (1,n/2), (1,0) \rfloor$. The modified(BLP) takes into account the path P so that the first two nodes of P take the label 0, and the last two the label 1. We call *critical paths* all the candidates paths in F. Difference-sets, represented Equations (25-27), are considered over the graph $F - P$, with modulo $4t$, and their cardinality is lowered to $2t - 2$. For ease of notation, the introduced new sub-problems are the Even Binary Labeling Problem (eBLP) and the Even Group Labeling Problem (eGLP).

Problem 6.1 (Even Binary Labeling Problem). *Let $F = (V, E)$ be a 2-regular graph of order $|V| = 4t + 1$. The (eBLP) asks to assign to each node $i \in \bar{V} = V \backslash \{\infty\}$ a binary label $c_i \in \{0, 1\}$ so that the following conditions hold:*

1. $\lfloor \alpha, \infty \rfloor \wedge \lfloor \beta, \infty \rfloor \implies c_\alpha \neq c_\beta$,

2. There is at least one critical path P so that
   $P = \lfloor x, y, z, k \rfloor \mid x, y, z, k \in \bar{V} \wedge c_x = c_y = 0, c_z = c_k = 1,$

3. $|\Delta(F - P)_{01} = \{\alpha - \beta \; : \; \lfloor (0, \alpha), (1, \beta) \rfloor \; \forall \, \alpha, \beta \in G\}| = 2t - 2.$

The CP model (18-24) is modified to fit the additional Condition (2) for the (eBLP). Constraints (21-23) are modified to enforce the new cardinality $(2t - 1)$ for both dM and C. Moreover, the following additional Constraints hold:

$$A = \{A_i \mid A_i \in \{0, 1\}\} \qquad \text{for all } i \in PA, \qquad (45)$$

$$A_i = \begin{cases} 1 & \text{if } c_x = c_y = 0 \wedge c_z = c_k = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad PA_i = \lfloor x, y, z, k \rfloor, \qquad (46)$$

$$\texttt{card}(A_i | 1) \geq 1. \qquad (47)$$

The set PA in (45) enumerates all combinations of four consecutive nodes in F. In Constraints (45) and (46), each element $A_i \in A$ is set to 1 if $c_x = c_y = 0 \wedge c_z = c_k = 1$, and hence A_i is a critical path. At least one critical path exists as of (47). Once (eBLP) is solved, the (eGLP) labels are determined considering a single critical path A_i. If no solution is found for the latter sub-problem, a new critical path $A_j \neq A_i$ induces a different (eGLP). In terms of (eGLP), Condition 4a induces four specific group-labels on the critical path A_i.

Problem 6.2 (Even Group Labeling Problem). *Let $F = (V, E)$ be a 2-regular graph of order $|V(F)| = 4t + 1$. $V(F) = \{\{0, 1\} \times G\} \cup \{\infty\}$ is the set of nodes over F, where G is the additive group $\mathbb{Z}_{2t}$. For each node $i \in \bar{V} = V \backslash \{\infty\}$ - given the binary label $c_i \in \{0, 1\}$ of V and a critical path P, the (eGLP) asks to assign a label $n_i \in G$ so that the following conditions hold:*

1. Difference sets are so that $\Delta_{00}(F - P) = \Delta_{11}(F - P) = \Delta_{01}(F - P) = G \backslash \{0, t\}$,

2. F contains the path $P = \lfloor x, y, z, k \rfloor = \lfloor (0, 0), (0, t), (1, t), (1, 0) \rfloor$.

The (eGLP) CP formulation is similar to the one in (28-37), and the critical-path $P = A_i$ constitutes an additional input.

$$V = \{A \cup B\}, \tag{48}$$
$$A = \{a_i \mid a_i \in G, c_i = 0\} \qquad B = \{b_i \mid b_i \in G, c_i = 1\}, \tag{49}$$
$$\texttt{alldifferent(A)} \qquad dom(A) = [0, 2t), \tag{50}$$
$$\texttt{alldifferent(B)} \qquad dom(B) = [0, 2t), \tag{51}$$
$$dA = \{(a_\alpha - a_\beta \mod (2t))\} \qquad \text{for all } \alpha, \beta \in A, \lfloor \alpha, \beta \rfloor \notin P, \tag{52}$$
$$dB = \{(b_\alpha - b_\beta \mod (2t))\} \qquad \text{for all } \alpha, \beta \in B, \lfloor \alpha, \beta \rfloor \notin P, \tag{53}$$
$$dAB = \{(a_\alpha - b_\beta \mod (2t))\} \qquad \text{for all } \alpha \in A, \beta \in B, \lfloor \alpha, \beta \rfloor \notin P, \tag{54}$$
$$\texttt{alldifferent(dA)} \qquad dom(dA) = (0, 2t)\backslash\{t\}, \tag{55}$$
$$\texttt{alldifferent(dB)} \qquad dom(dB) = (0, 2t)\backslash\{t\}, \tag{56}$$
$$\texttt{alldifferent(dAB)} \qquad dom(dAB) = (0, 2t)\backslash\{t\}, \tag{57}$$
$$n_x = n_k = 0, n_y = n_z = t \qquad P = \lfloor x, y, z, k \rfloor, \tag{58}$$
$$\texttt{card(dA)} = \texttt{card(dB)} = \texttt{card(dAB)} \qquad \texttt{card(dAB)} = 2t - 2. \tag{59}$$

Constraints (49-54 and 59) have different cardinalities and modulo arguments in. The value $n/2 = t$ is not in difference-sets of Constraints (52-54). Moreover, edges inside P do not figure in difference-sets. Condition 2 of (eGLP) assigns labels for nodes inside the critical path P, as in Constraint (58).

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