Diversity-Multiplexing Tradeoff of Multi-Layer Scattering MIMO Channels

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Giorgio Taricco

Abstract

Multi-layer (or multi-cluster) scattering Multiple-Input Multiple-Output (MIMO) channels are considered in the framework of the *diversity-multiplexing* tradeoff (DMT). This MIMO channel model finds application for indoor networks, typical of 5G architectures, in which the signal propagates from the transmitter to the receiver through the walls and floors of a building (represented by scattering layers). These results extend the seminal work by Zheng and Tse from the independent identically distributed (iid) Rayleigh fading MIMO channel to a channel matrix which is the product of iid Rayleigh fading matrix components. It is worth noting that the resulting product channel matrix elements are not independent. It is shown that the presence of multiple scattering layers eventually degrades the DMT performance of a MIMO system by an amount depending only on the *three minimum dimensions* of the matrices characterizing the product channel matrix.

I. Introduction

Multi-layer (or multi-cluster) scattering Multiple-Input Multiple-Output (MIMO) channels with a channel matrix that is the product of several independent identically distributed (iid) Rayleigh matrix components received considerable attention in recent years. This model has been applied to 5G architectures where the signal propagates through a sequence of scattering layers representing the different walls or floors in a building (see, *e.g.*, [1]–[6]). Many results have been published in this framework. In particular, the joint singular value distribution of the matrix product has been derived in [7,8] when the matrix components are rectangular iid Rayleigh matrices of compatible sizes. The joint probability density function (pdf) is expressed by a determinant of Meijer G-functions. This expression has been applied to derive the outage capacity of orthogonal space–time block codes in [3]. Other applications have been presented in [4]–[6] to determine the ergodic and outage capacities with one-sided spatial correlation under constraints on the component matrix sizes.

On the other hand, there is a wide literature concerning the diversity-multiplexing tradeoff (DMT). The concept was brought to the general attention in the case of iid Rayleigh fading MIMO channels by [9]. Subsequently, the results were extended to other types of iid fading MIMO channels [10]. The DMT has been investigated in many areas of wireless communications, such as cooperative communications, relaying, and ARQ MIMO channels [11]–[14]. A non-asymptotic framework was developed in [15], encompassing one-sided spatial correlation and Rayleigh fading. Recent results dealt with the DMT of wireless energy harvesting channels [16].

In this work we investigate the DMT of the multi-layer scattering MIMO channel in the asymptotic Signal-to-Noise power Ratio (SNR) regime. Our results extend the ones in the literature from the basic iid Rayleigh case to the multi-layer scattering scenario with iid Rayleigh components determining the channel matrix.

II. System Model

We consider a MIMO channel characterized by the linear equation

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z} \tag{1}$$

Here, $\boldsymbol{x} \in \mathbb{C}^{n_{\mathsf{T}}\times 1}$ is the transmitted signal vector, $\boldsymbol{y} \in \mathbb{C}^{n_{\mathsf{R}}\times 1}$ is the received signal vector, $\boldsymbol{z} \in \mathbb{C}^{n_{\mathsf{R}}\times 1}$ is the received noise vector, and $\boldsymbol{H} \in \mathbb{C}^{n_{\mathsf{R}}\times n_{\mathsf{T}}}$ is the channel matrix, which is characterized as a product form:

$$\boldsymbol{H} = \boldsymbol{G}_1 \cdots \boldsymbol{G}_M. \tag{2}$$

Here, the matrices $\boldsymbol{G}_m \in \mathbb{C}^{n_{m-1}\times n_m}$, for $m = 1, \dots, M$, with $n_0 = n_{\mathsf{R}}, n_M = n_{\mathsf{T}}$, are independent with iid entries distributed as $\mathcal{CN}(0,1)$, and are referred to as *Ginibre matrices*[1]. If $\boldsymbol{x} \sim \mathcal{CN}(\boldsymbol{0}, \gamma \boldsymbol{I}_{n_T})$ and $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_{n_R})$, then the mutual information, conditionally on the channel matrix $\boldsymbol{H}$, is given by

$$I_{\boldsymbol{H}}(\boldsymbol{x};\boldsymbol{y}) \triangleq \log_2\left(\boldsymbol{I}_{n_R} + \frac{\rho}{n_1\cdots n_M}\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}}\right), \tag{3}$$

where ρ is the channel SNR:

$$\rho = \frac{\gamma\,\mathbb{E}[\|\boldsymbol{H}\|^2]}{n_{\mathsf{R}}} = \gamma \prod_{m=1}^{M} n_m. \tag{4}$$

[1]It is interesting to notice that the entries of the matrix $\boldsymbol{H}$ are identically distributed but not independent.

A. *Joint pdf of the eigenvalues of* $\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}}$

The joint pdf of the *increasingly ordered* eigenvalues of $\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}}$ is given by [7,8]:

$$p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \frac{\Delta(\boldsymbol{\lambda}) \det \left[G_{0,M}^{M,0}\left(\lambda_j \middle| \begin{matrix} - \\ \boldsymbol{\nu}_i \end{matrix} \right) \right]_{i,j=1}^{n_{\min}}}{\prod_{n=0}^{n_{\min}-1} \prod_{i=0}^{M} (n+\nu_i)!} \tag{5}$$

over the domain

$$\mathcal{R}_+ \triangleq \{\boldsymbol{\lambda} : 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_{\min}}\}, \tag{6}$$

where we defined the Vandermonde determinant function

$$\Delta(\boldsymbol{\lambda}) \triangleq \prod_{i>j} (\lambda_i - \lambda_j) \tag{7}$$

and the ordered matrix dimension offsets $\nu_i \triangleq n_{\pi(i)} - n_{\min}$ for $i = 0, \ldots, M$, with the permutation π satisfying the inequalities $n_{\pi(i-1)} \ge n_{\pi(i)}$ for $i = 1, \ldots, M$,

$$\boldsymbol{\nu}_i \triangleq (\nu_0, \ldots, \nu_{M-1} + i - 1), \tag{8}$$

and the Meijer's G-Function is defined in [17,18].

The following series expansions of the Meijer G-function will be used in the sequel. According to [17, p. 212], we have

$$G_{0,M}^{M,0}\left(z \middle| \begin{matrix} - \\ \boldsymbol{b}_M \end{matrix} \right) = O(|z|^{\min(\boldsymbol{b}_M)}) \tag{9}$$

for $z \to 0$. According to [18, 5.9.1,Th.5], we have

$$\begin{aligned} G_{0,M}^{M,0}\left(z \middle| \begin{matrix} - \\ \boldsymbol{b}_M \end{matrix} \right) \sim \frac{(2\pi)^{(M-1)/2}}{\sqrt{M}} \exp(-M\, z^{1/M}) \\ \times\, z^{(2\sigma(\boldsymbol{b}_M)-M+1)/(2M)} \sum_{k=0}^{\infty} \beta_k z^{-k/M} \end{aligned} \tag{10}$$

for $|z| \to \infty$, where $\sigma(\boldsymbol{b}_M)$ is the sum of the elements of the vector $\boldsymbol{b}_M$, $\beta_0 = 1$ and the other β_k are suitable finite coefficients.

III. Diversity-Multiplexing tradeoff

Following the approach of [9], we define a *scheme* $\{\mathcal{C}(\rho)\}$ as a family of codes with length l and rate $R(\rho)$. The scheme achieves spatial multiplexing gain

$$r = \lim_{\rho\to\infty} \frac{R(\rho)}{\log_2 \rho} \tag{11}$$

and, if $P_e(\rho)$ is the average error probability, diversity gain

$$d(r) = \lim_{\rho\to\infty} \frac{-\log_2 P_e(\rho)}{\log_2 \rho}. \tag{12}$$

Following [9], we shall use the exponential equality

$$f(\rho) \doteq g(\rho) \;\Leftrightarrow\; \lim_{\rho\to\infty} \frac{\ln|f(\rho)|}{\ln|g(\rho)|} = 1. \tag{13}$$

Now, we consider the asymptotic outage probability

$$\begin{aligned} P_{\mathsf{out}}(\rho) &= P\{I_{\boldsymbol{H}}(\boldsymbol{x};\boldsymbol{y}) < r\log_2\rho\} \\ &\doteq P\{\log_2\det(\boldsymbol{I}+\rho\boldsymbol{H}\boldsymbol{H}^{\mathsf{H}}) < r\log_2\rho\} \\ &= P\bigg\{\prod_{i=1}^{n_{\min}}(1+\rho^{1-\alpha_i}) < \rho^r\bigg\} \\ &\doteq P\bigg\{\sum_{i=1}^{n_{\min}}(1-\alpha_i)_+ < r\bigg\} \end{aligned} \tag{14}$$

where $(x)_+ \triangleq \max(0,x)$ and $\lambda_i \triangleq \rho^{-\alpha_i}$. The last probability can be calculated by integrating the joint pdf of the vector $\boldsymbol{\alpha} \triangleq (\alpha_1,\dots,\alpha_{n_{\min}})$, which is given by $p_{\boldsymbol{\lambda}}(\rho^{-\boldsymbol{\alpha}})(\ln\rho)^{n_{\min}}\rho^{-\sigma(\boldsymbol{\alpha})}$, over the domain

$$\mathcal{A}(r) \triangleq \bigg\{\boldsymbol{\alpha} : \alpha_1 \geq \cdots \geq \alpha_{n_{\min}}, \sum_{i=1}^{n_{\min}}(1-\alpha_i)_+ < r\bigg\}. \tag{15}$$

Above, we set $\rho^{-\boldsymbol{\alpha}} \triangleq (\rho^{-\alpha_1},\dots,\rho^{-\alpha_{n_{\min}}})$. The integration domain can be restricted to the positive orthant $\mathbb{R}_+^{n_{\min}}$:

$$\mathcal{A}_+(r) \triangleq \mathcal{A}(r) \cap \mathbb{R}_+^{n_{\min}} \tag{16}$$

when $\rho\to\infty$. In fact, from the asymptotic expansion (10), we can see that the joint pdf in (5) satisfies

$$p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) \sim \mathsf{poly}(\boldsymbol{\lambda}) \prod_{i=1}^{n_{\min}} \exp(-M\lambda_i^{1/M}) \tag{17}$$

for some multivariate polynomial function $\mathsf{poly}(\boldsymbol{\lambda})$. Therefore, the contribution of the part of the integration domain corresponding to $\mathcal{A}(r)\setminus\mathcal{A}_+(r)$ vanishes, as $\rho\to\infty$, because of the presence of the vanishing negative exponential function. The outage probability can be calculated as follows:

$$\begin{aligned} P_{\mathsf{out}}(\rho) \doteq \int_{\mathcal{A}_+(r)} &\det\Big[G_{0,M}^{M,0}\Big(\rho^{-\alpha_j}\Big|\begin{matrix}-\\ \boldsymbol{\nu}_i\end{matrix}\Big)\Big]_{i,j=1}^{n_{\min}} \\ &\times \Delta(\rho^{-\boldsymbol{\alpha}})\rho^{-\sigma(\boldsymbol{\alpha})}d\boldsymbol{\alpha}, \end{aligned} \tag{18}$$

where we neglected the factor $(\ln\rho)^{n_{\min}} \doteq 1$. Then, since $\alpha_i \geq \alpha_{i+1}, i = 1, \ldots, n_{\min} - 1$,

$$\begin{aligned}\Delta(\rho^{-\boldsymbol{\alpha}})\rho^{-\sigma(\boldsymbol{\alpha})} &= \prod_{i=1}^{n_{\min}} \rho^{-\alpha_i} \prod_{j=1}^{i-1} (\rho^{-\alpha_i} - \rho^{-\alpha_j}) \\ &\doteq \prod_{i=1}^{n_{\min}} \rho^{-i\alpha_i}. \end{aligned} \tag{19}$$

As far as concerns the determinant factor, we have from (9):

$$\begin{aligned} &\det\left[G_{0,M}^{M,0}\left(\rho^{-\alpha_j}\middle|\begin{matrix}-\\ \boldsymbol{\nu}_i\end{matrix}\right)\right]_{i,j=1}^{n_{\min}} \\ &= \sum_{\pi} \operatorname{sign}(\pi) \prod_{i=1}^{n_{\min}} G_{0,M}^{M,0}\left(\rho^{-\alpha_i}\middle|\begin{matrix}-\\ \boldsymbol{\nu}_{\pi(i)}\end{matrix}\right) \\ &\doteq \sum_{\pi} \operatorname{sign}(\pi) \prod_{i=1}^{n_{\min}} \rho^{-\alpha_i \min(\boldsymbol{\nu}_{\pi(i)})} \\ &\doteq \exp\left\{-\left[\min_{\pi} \sum_{i=1}^{n_{\min}} \alpha_i \min(\boldsymbol{\nu}_{\pi(i)})\right] \ln\rho\right\} \\ &= \exp\left\{-\left[\sum_{i=1}^{n_{\min}} \alpha_i \min(\boldsymbol{\nu}_i)\right] \ln\rho\right\}, \end{aligned} \tag{20}$$

where we applied the *Sequence Product Lemma* (Lemma 1 from Appendix A) in the last step. Thus,

$$P_{\mathsf{out}}(\rho) \doteq \int_{\mathcal{A}_+(r)} \prod_{i=1}^{n_{\min}} \rho^{-\mu_i\alpha_i} d\boldsymbol{\alpha}, \tag{21}$$

where

$$\mu_i \triangleq i + \min(\boldsymbol{\nu}_i), \qquad i = 1, \ldots, n_{\min}. \tag{22}$$

The asymptotic evaluation of this integral can be carried out as in [9] and we get

$$P_{\mathsf{out}}(\rho) \doteq \rho^{-d_{\mathsf{out}}(r)}, \tag{23}$$

where

$$d_{\mathsf{out}}(r) \triangleq \inf_{\boldsymbol{\alpha}\in\mathcal{A}_+(r)} \sum_{i=1}^{n_{\min}} \mu_i\alpha_i. \tag{24}$$

Remark 1 This outage formulation is equivalent to the definition in (12), as implied by the arguments of [9, III-B] applied to the present context. Likewise, the use of arbitrary input covariance is equivalent to the iid power assumption. Therefore, we have $d(r) = d_{\mathsf{out}}(r)$. $\square$

Remark 2 Since ν_i is a nonincreasing sequence, we have, for $i = 0, \dots, M$,

$$\min(\boldsymbol{\nu}) = \min(\nu_{M-2}, \nu_{M-1} + i - 1) \tag{25}$$

since $\nu_i \geq \nu_{M-2}$ for every $i \leq M - 2$. Therefore,

$$\mu_i = i + \min(\nu_{M-2}, \nu_{M-1} + i - 1). \tag{26}$$

It is worth noting that the least three dimensions in the matrix product characterize the multi-layer scattering MIMO channel DMT completely. □

The derivation of (24) requires the solution of an optimization problem depending on the channel parameters through the integer coefficients μ_i and on the multiplexing gain r. The solution is summarized in the following

Theorem 1 *The outage diversity $d_{\mathsf{out}}(r)$ specified in* (24) *is given as follows. If $r \geq n_{\min}$, then $d_{\mathsf{out}}(r) = 0$. Otherwise,*

$$d_{\mathsf{out}}(r) = \sum_{i=1}^{n_{\min}-1-\lfloor r \rfloor} \mu_i + \mu_{n_{\min}-\lfloor r \rfloor}(1 - \{r\}) \tag{27}$$

where $\{r\} \triangleq r - \lfloor r \rfloor$ is the ***fractional part*** *of r.*

Proof: See Appendix B. ■

A. Single layer: $M = 1$

In the particular case, we have $\nu_0 = |n_R - n_T|$ and $\mu_i = |n_R - n_T| + 2i - 1$ for $i = 1, \dots, n_{\min}$. Then, applying Theorem 1, we can see that, if $r < n_{\min}$,

$$\begin{aligned} d_{\mathsf{out}}(r) = & \sum_{i=1}^{n_{\min}-1-\lfloor r \rfloor} (|n_R - n_T| + 2i - 1) \\ & + [|n_R - n_T| + 2(n_{\min} - \lfloor r \rfloor) - 1](\lfloor r \rfloor + 1 - r) \\ & = (n_{\max} - r)(n_{\min} - r) + \{r\}(1 - \{r\}) \end{aligned} \tag{28}$$

where $n_{\max} = \max(n_R, n_T)$. This result agrees with [9] when r is an integer $\leq n_{\min}$ and is plainly linear in r when $r \in [r_0, r_0 + 1)$ with integer r_0.

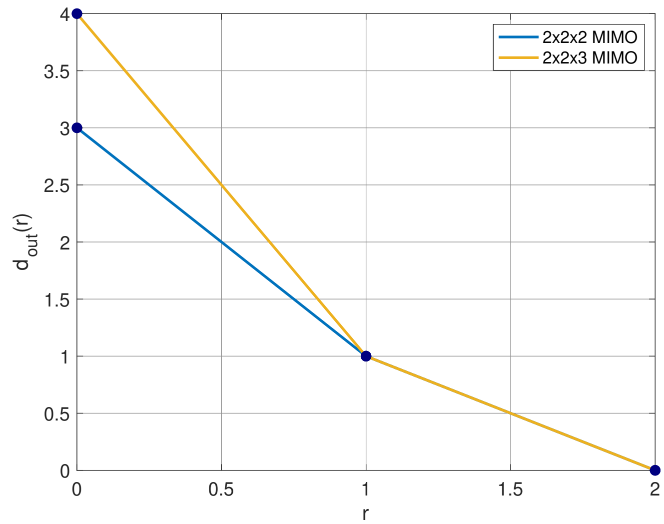

Fig. 1. DMT of a multi-layer scattering MIMO channel with ordered dimensions $2 \times 2 \times m \times \cdots$ with $m = 2, 3, \ldots$ (the DMT doesn't change for $m \geq 3$).

IV. Numerical results

The DMT of the multi-layer scattering MIMO channel derived from Theorem 1 is illustrated in Figs. 1 and 2 for the $2 \times 2 \times m \times \cdots, m \geq 2$ and $4 \times 4 \times m \times \cdots, m \geq 4$ cases, respectively. We can see that the optimum tradeoff is achieved only if the third lowest number of channel dimensions, m, is sufficiently large (more precisely, only if $\nu_{M-2} \geq \nu_{M-1} + n_{\min} - 1$). In that case, the tradeoff coincides with that of the iid Rayleigh fading MIMO channel with a number of antennas equal to the two lowest dimensions of the multi-layer scattering MIMO channel. Otherwise, there is a penalty depending on $\nu_{M-2} - \nu_{M-1}$.

In the worst case, corresponding to $\nu_{M-2} = \nu_{M-1}$, we have $\mu_i = i + \nu_{M-1}$ and then

$$d_{\mathsf{out}}(r) \leq d_{\mathsf{out}}(0) = \frac{n_{\min}(2\nu_{M-1} + n_{\min} + 1)}{2}. \tag{29}$$

V. Conclusions

In this letter we obtained the DMT of the multi-layer scattering MIMO channel with iid Rayleigh matrix components, which extends the seminal result developed in [9] for the basic iid Rayleigh fading MIMO channel.

We showed that the DMT depends on the least three dimensions in the channel matrix product and characterizes the degradation with respect to the basic iid Rayleigh MIMO channel with the two least matrix dimensions from the matrix product.

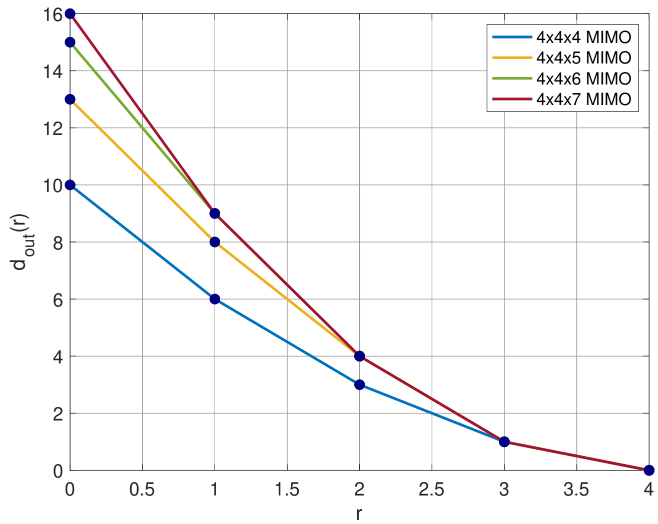

Fig. 2. DMT of a multi-layer scattering MIMO channel with ordered dimensions $4 \times 4 \times m \times \cdots$ with $m = 4, 5, 6, 7, \ldots$ (the DMT doesn't change for $m \geq 7$).

The degradation vanishes when the third least matrix dimension is greater than the second least dimension plus the minimum dimension minus one, i.e.,

$$\nu_{M-2} \geq \nu_{M-1} + n_{\min} - 1. \tag{30}$$

Appendix A
Sequence Product Lemma

Lemma 1 *Given any two real nonnegative sequences $\alpha_i, \beta_i, i = 1, \ldots, n$ such that $\alpha_i \geq \alpha_{i+1}$ and $\beta_i \leq \beta_{i+1}$, for $i = 1, \ldots, n-1$, we have, for every permutation π, the following inequality:*

$$\sum_{i=1}^{n} \alpha_i \beta_i \leq \sum_{i=1}^{n} \alpha_i \beta_{\pi(i)}. \tag{31}$$

Proof: Since every permutation $\pi \in S_n$ can be expressed as a product of disjoint *cycles* [19, Sec. III.70], we have to prove the result only when π is a cycle and then apply it to any $\pi \in S_n$ after proper relabeling of the indexes. Let us assume, w.l.o.g., that $\pi = (1, \ldots, n)$, i.e., the permutation $1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto n \mapsto 1$. Then, we have to show that

$$\alpha_1(\beta_1 - \beta_2) + \alpha_2(\beta_2 - \beta_3) + \cdots + \alpha_n(\beta_n - \beta_1) \leq 0. \tag{32}$$

The inequality stems from the fact that

$$\begin{aligned}
&\alpha_1(\beta_1-\beta_2)+\alpha_2(\beta_2-\beta_3)+\cdots+\alpha_n(\beta_n-\beta_1)\\
&=(\alpha_1-\alpha_n)(\beta_1-\beta_2)+\cdots+(\alpha_{n-1}-\alpha_n)(\beta_{n-1}-\beta_n)\\
&\leq 0, \qquad (33)
\end{aligned}$$

since $\alpha_i-\alpha_n\geq 0$ and $\beta_i-\beta_{i+1}\leq 0$ for every $i=1,\ldots,n-1$. ■

APPENDIX B
PROOF OF THEOREM 1

Proof: First, we notice that the coefficients μ_i defined in (22) form a strictly increasing sequence because the minimum entry of each vector $\boldsymbol{\nu}_i$ defined in (8) is nondecreasing with i for $i=1,\ldots,M$. On the other hand, the elements of the vector $\boldsymbol{\alpha}\in\mathcal{A}_+(r)$ are nondecreasing by assumption, i.e., $\alpha_i\geq\alpha_{i+1}$. Moreover, they satisfy the inequalities $\alpha_i\geq 0$ for $i=1,\ldots,n_{\min}$, and

$$\sum_{i=1}^{n_{\min}}(1-\alpha_i)_+<r, \qquad (34)$$

the multiplexing gain constraint. Since we are looking for the infimum of $\sum_{i=1}^{n_{\min}}\mu_i\alpha_i$, increasing any α_i leads away from the solution so that $\alpha_i>1$ is impossible at the solution since we have the coefficients $(1-\alpha_i)_+$ which are identically equal to 0 for every $\alpha_i\geq 1$. Then, we can restrict the solution space to encompass only the vectors $\boldsymbol{\alpha}\in[0,1]^{n_{\min}}$. If $r\geq n_{\min}$, the point $\boldsymbol{\alpha}=(0,\ldots,0)$ satisfies the multiplexing gain constraint so that $d_{\mathsf{out}}(r)=0$. If $r<n_{\min}$, we can adopt a *greedy* approach and increase the coordinate of $\boldsymbol{\alpha}$ corresponding to the smallest multiplier μ_i in order to limit the corresponding increase of the outage diversity. Then, we start by increasing α_1. If $n_{\min}-1\leq r<n_{\min}$, we can set $\alpha_1=n_{\min}-r\leq 1, \alpha_2=\cdots=\alpha_{n_{\min}}=0$ and obtain

$$d_{\mathsf{out}}(r)=\mu_1(n_{\min}-r). \qquad (35)$$

If $n_{\min}-2\leq r<n_{\min}-1$, we can set $\alpha_1=1,\alpha_2=n_{\min}-1-r\leq 1,\alpha_3=\cdots=\alpha_{n_{\min}}=0$ and obtain

$$d_{\mathsf{out}}(r)=\mu_1+\mu_2(n_{\min}-1-r). \qquad (36)$$

By generalizing this procedure, we can see that, if the integer floor of r satisfies $\lfloor r\rfloor<n_{\min}$, we obtain the solution to the minimization problem reported in eq. (24). ■

REFERENCES

[1] R. Müller, "On the asymptotic eigenvalue distribution of concatenated vector-valued fading channels," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 2086–2091, Jul. 2002.

[2] J. B. Andersen and I. Z. Kovacs, "Power distributions revisited," *COST 273 TD (02) 004*, Jan. 2002.

[3] L. Wei, Z. Zheng, J. Corander, and G. Taricco, "On the outage capacity of orthogonal space–time block codes over multi–cluster scattering MIMO channels," *IEEE Trans. on Commun.*, vol. 63, no. 5, pp. 1700–1711, May 2015.

[4] L. Wei, "Ergodic Capacity of Spatially Correlated Multi-cluster Scattering MIMO Channels," *Proc. IEEE ISIT 2015*, Hong Kong, 14-19 June 2015.

[5] G. Taricco and G. Alfano, "Outage information rate of spatially correlated multi-cluster scattering MIMO channels," *IEEE ISIT 2017*, Aachen, Germany, 25–30 June, 2017.

[6] G. Taricco and G. Alfano, "Outage information rate of doubly correlated multi-cluster scattering MIMO channels," *IEEE Wireless Communications Letters*, vol. 7, no. 6, pp. 1042–1045, December 2018.

[7] G. Akemann, J.R. Ipsen, and M. Kieburg, "Products of rectangular random matrices: Singular values and progressive scattering," *PhysṘev. E, Stat. Nonlin. Soft Matter Phys.*, vol. 88, no. 5, Nov. 2013.

[8] J.R. Ipsen and M. Kieburg, "Weak commutation relations and eigenvalue statistics for products of rectangular random matrices," *Phys. Rev. E, Stat. Nonlin. Soft Matter Phys.*, vol. 89, no. 3, Mar. 2014.

[9] L. Zheng and D.N.C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[10] L. Zhao, W. Mo, Y. Ma, and Z. Wang, "Diversity and multiplexing tradeoff in general fading channels," *IEEE Trans. Inf. Theory*, vol. 53, no. 4, pp. 1549–1557, April 2007.

[11] K. Azarian, H. El Gamal and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4152-4172, Dec. 2005.

[12] H. El Gamal, G. Caire and M. O. Damen, "The MIMO ARQ Channel: Diversity–Multiplexing–Delay Tradeoff," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3601-3621, Aug. 2006

[13] M. Yuksel and E. Erkip, "Multiple-Antenna Cooperative Wireless Systems: A Diversity–Multiplexing Tradeoff Perspective," *IEEE Transactions on Information Theory*, vol. 53, no. 10, pp. 3371-3393, Oct. 2007.

[14] Y. Fan, C. Wang, J. Thompson and H. V. Poor, "Recovering Multiplexing Loss through Successive Relaying Using Repetition Coding," in IEEE Transactions on Wireless Communications, vol. 6, no. 12, pp. 4484-4493, December 2007.

[15] R. Narasimhan, "Finite-SNR Diversity–Multiplexing Tradeoff for Correlated Rayleigh and Rician MIMO Channels," *IEEE Transactions on Information Theory*, vol. 52, no. 9, pp. 3965-3979, Sept. 2006

[16] Y. S. Rao, A. S. Madhukumar and S. R. Prasad, "Wireless Energy Harvesting-Based Relaying: A Finite-SNR Diversity-Multiplexing Tradeoff Perspective," *IEEE Transactions on Green Communications and Networking*, vol. 4, no. 1, pp. 277-288, March 2020.

[17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions (The Bateman Manuscript Project) vol. I.* New York: McGraw-Hill, 1953.

[18] Y.D. Luke, *Mathematical Functions and their Approximations*. Academic Press, 1975.

[19] T. Gowers, *The Princeton Companion to Mathematics*. Princeton University Press, 2008.