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Steklov spectral problems in a set with a thin toroidal hole

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ABSTRACT

The paper concerns the Steklov spectral problem for the Laplace operator, and some variants in a 3-dimensional bounded domain, with a cavity Γ_ε having the shape of a thin toroidal set, with a constant cross-section of diameter $\varepsilon \ll 1$. We construct the main terms of the asymptotic expansion of the eigenvalues in terms of real-analytic functions of the variable $|\ln\varepsilon|^{-1}$, and we prove that the relative asymptotic error is of much smaller order $O(\varepsilon|\ln\varepsilon|)$ as $\varepsilon \to 0^+$. The asymptotic analysis involves eigenvalues and eigenfunctions of a certain integral operator on the smooth curve Γ, the axis of the cavity Γ_ε.

1. Introduction

1.1. Prelude

The Steklov spectral problem[1] is naturally associated with surface waves over a heavy liquid (see, e.g., Ref. 2) and there exists a vast literature about its spectrum in finite basins and infinite channels; the reader may refer to the review papers,[3–5] and the citations therein. In Section 5.3 of the present paper the analysis performed in the preceding Sections 3 and 4 will be applied to the study of the asymptotic behaviour of the eigenvalues of the water-waves problem in a 3-dimensional basin, where the free water surface corresponds to a thin curved strip Γ_ε, of width $\varepsilon \ll 1$. Fig. 1a suggests to think about a wide deep lake covered with ice, where a narrow path Γ_ε allows kayak rallies.

The spectrum of the Steklov problem under consideration here has a quite peculiar asymptotic feature, namely all suitably normalized eigenvalues λ_p^ε in the, so-called, mid-frequency range $\{\lambda \in \overline{\mathbb{R}}_+ = [0,+\infty) : \lambda \le c\varepsilon^{-1}\}$ have the same limit, i.e.

$$\lim_{\varepsilon\to 0^+} \varepsilon|\ln\varepsilon|\lambda_p^\varepsilon = \Lambda > 0 \tag{1.1}$$

while the correction term of order $|\ln\varepsilon|^{-1}$ in the asymptotic form of λ_p^ε depends on the eigenvalue with index p. More precisely, it is computed by means of the discrete spectrum of an integral pseudo-differential operator defined on the smooth curve

$$\Gamma = \bigcap_{\varepsilon>0} \Gamma_\varepsilon$$

that is, the centreline of the thin sets Γ_ε, and the remainder is estimated by terns of order $O(|\ln\varepsilon|^{-2})$. The construction of the asymptotic expansion and the proof of the error estimate are much more complicated in comparison with traditional regular and singular perturbations of the free surface and/or water domain (see above-mentioned citations and, in particular, the papers[6–10]). Moreover, they are valid under serious restrictions: for example, the cross-section of the strip Γ_ε must be constant, and the curve Γ must be smooth, simple, and closed. Hence, an asymptotic structure of spectrum of the water-wave problem for the iced basin with a crack (see Fig. 1b) is still an open question.

In Section 1.2 we describe in details the geometry of the domain and the main spectral problem under consideration, while the discussion of the state of the art in the existing literature and the plan of the paper follow in Sections 1.3 and 1.4.

1.2. Statement of the problem

Let Γ be a smooth simple closed curve in the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. The Cartesian coordinates in $\mathbb{R}^3$ are denoted by $x = (x_1, x_2, x_3)$ or by $(y, z) = (y_1, y_2, z)$. A neighbourhood $V \subset \mathbb{R}^3$ of Γ is supplied with the local coordinate system (s, n, z), where s is the arc length along Γ and n is the oriented distance from Γ, choosing $n > 0$ outside the plane domain surrounded by Γ in $\mathbb{R}^2 \times \{0\}$. With a slight abuse of notation, we will occasionally write simply $s \in \Gamma$, instead of $(s, 0, 0) \in \Gamma$, to indicate a point of the curve. Without loss of generality, we assume that the length of Γ is equal to 2π, and the Cartesian coordinates and all geometrical parameters are made dimensionless. Let $\omega \subset \mathbb{R}^2$ be a bounded open set and, for any $\varepsilon > 0$, let Γ_ε be the open subset of $\mathbb{R}^3$ defined by

$$\Gamma_\varepsilon = \{x = (s, n, z) \in V : s \in \Gamma, \eta = (\varepsilon^{-1}n, \varepsilon^{-1}z) \in \omega\}. \tag{1.2}$$

We fix a bounded open set $\Omega \subset \mathbb{R}^3$, containing the curve Γ and, therefore, the thin toroidal set (1.2) is contained in Ω for all $\varepsilon \in (0, \varepsilon_0]$,

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Fig. 1. Basins with circular (a) and straight (b) cracks in ice.

$\varepsilon_0 > 0$, and, for simplicity, we assume that the boundaries $\partial\Omega$ and $\partial\omega$ are both smooth, e.g. of class C^2. We introduce the singularly perturbed domain

$$\Omega_\varepsilon = \Omega \setminus \overline{\Gamma_\varepsilon}, \tag{1.3}$$

where we consider the Laplace equation

$$-\Delta_x u^\varepsilon(x) = 0, x \in \Omega_\varepsilon, \tag{1.4}$$

with the spectral Steklov condition

$$\partial_\nu u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), x \in \partial\Gamma_\varepsilon, \tag{1.5}$$

and the Dirichlet condition on the exterior part of the boundary $\partial\Omega_\varepsilon = \Gamma_\varepsilon \cup \partial\Omega$

$$u^\varepsilon(x) = 0, x \in \partial\Omega. \tag{1.6}$$

In (1.5) ∂_ν is the derivative along the outward normal. The variational formulation of problem (1.4)–(1.6) reads

$$(\nabla_x u^\varepsilon, \nabla_x v^\varepsilon)_{\Omega_\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\partial\Gamma_\varepsilon} \quad \forall v^\varepsilon \in H^1_0(\Omega_\varepsilon, \partial\Omega) \tag{1.7}$$

where ∇_x denotes the gradient with respect to the Cartesian coordinates x, $(,)_A$ indicates the natural scalar product in the Lebesgue space $L^2(A)$, and $H^1_0(\Omega_\varepsilon, \partial\Omega)$ is the subspace of functions in the Sobolev space $H^1(\Omega_\varepsilon)$ satisfying the homogeneous Dirichlet condition (1.6) on $\partial\Omega$. Since the embedding $H^1(\Omega_\varepsilon) \subset L^2(\partial\Gamma_\varepsilon)$ is compact, problem (1.3)–(1.6), or (1.7), has discrete spectrum consisting of a positive monotone unbounded sequence

$$\lambda^\varepsilon_1 < \lambda^\varepsilon_2 \le \lambda^\varepsilon_3 \le \ldots \le \lambda^\varepsilon_p \le \ldots \to +\infty, \tag{1.8}$$

where multiplicities are taken into account.

The corresponding eigenfunctions $u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3, \ldots, u^\varepsilon_p \ldots \in H^1_0(\Omega_\varepsilon, \partial\Omega)$ can be chosen satisfying the orthogonality and normalization conditions

$$(u^\varepsilon_p, u^\varepsilon_q)_{\Gamma_\varepsilon} = \delta_{p,q}, \quad p, q \in \mathbb{N}, \tag{1.9}$$

where $\delta_{p,q}$ is the Kronecker symbol. As announced above, the main goal of the paper is to describe the asymptotics of eigenvalues and eigenfunctions in (1.8), (1.9) when $\varepsilon \to 0^+$ and the thin long cavity Γ_ε disappears in the limit.

1.3. State of the art

Asymptotic studies of the Steklov problem having a clear physical interpretation within the linear theory of water-waves (see, e.g., the monographs Ref. 2, 11, and others) have been performed in various formulations and from different points of view. The most investigated case is that of a domain of the type $G(\varepsilon) = G \setminus g_\varepsilon \subseteq \mathbb{R}^d$, $d \ge 3$, where a small cavity or cavern $g_\varepsilon = \{x : \varepsilon^{-1}x \in g\}$ is considered (see Fig. 3a and b).

In[8] it was observed that, in contrast to the majority of other singularly perturbed elliptic problems, the Steklov problem in $G(\varepsilon)$ admits a complete asymptotic analysis of eigenvalues in both the low and mid-frequency range of the spectrum σ_ε. More precisely, the formal expansions

$$\lambda^\varepsilon_k \sim \lambda^0_k + \sum_{j=1}^{\infty} \varepsilon^j \lambda_{kj} \tag{1.10}$$

as well as the estimates

$$|\lambda^\varepsilon_k - \lambda^0_k - \sum_{j=1}^{J} \varepsilon^j \lambda_{kj}| \le c_J \varepsilon^{J+1} \quad \forall j \in \mathbb{N} \tag{1.11}$$

were derived, where $\{\lambda^0_k\}_{k\in\mathbb{N}}$ is nothing but the eigenvalue of the interior Steklov problem in the intact domain G, and λ_{kj}, $j \in \mathbb{N}$, are correction terms constructed by a certain iterative asymptotic procedure. At the same time, there exists another family of eigenvalues with asymptotics

$$\lambda^\varepsilon_{N^\varepsilon(k)} \sim \varepsilon^{-1}\mu_k + \sum_{j=1}^{\infty} \varepsilon^{j-1}\mu_{kj} \tag{1.12}$$

where $\{\mu_k\}_{k\in\mathbb{N}}$ is an eigenvalue sequence of the exterior Steklov problem in $\mathbb{R}^n \setminus \overline{g}$. Proximity estimates to $\lambda^\varepsilon_{N^\varepsilon(k)}$, of the type (1.11), hold true also for partial sums of the infinite series in (1.12). However, the eigenvalue number $N^\varepsilon(k)$ depends on the small parameter $\varepsilon > 0$ because, in view of (1.11) and (1.12), the multiplicity of the spectrum σ_ε in $(0, \varepsilon^{-1}\mu_1)$ grows unboundedly when $\varepsilon \to 0^+$.

In the cavern case, Fig. 3b, the results in[8] are much weaker, and the infinite formal asymptotic series of the type in (1.10) and (1.12) are not constructed yet.

A different approach, based on previous studies[12,13] of spectral Dirichlet and Neumann problems for the Laplace operator, is developed in,[10] for the Steklov problem in the domain $G(\varepsilon)$, Fig. 3a. It is proved that in dimension $d \ge 3$ a simple eigenvalue λ^ε_k is a real analytic function in the small parameter ε while, for $d = 2$, it becomes analytic in two variables ε and $|\ln\varepsilon|^{-1}$. It should be emphasized that asymptotic tools used in[8] do not help to prove the convergence of the infinite series (1.12).

The Steklov problem (1.5) in the domain singularly perturbed by the thin toroidal cavity (1.2), Fig. 2a, is not considered yet in the mathematical literature, hence the core of the present paper (i.e. Sections 3, 4, 5) contains new results. Our investigation of the spectrum (1.8) requires to adapt and to generalize the asymptotic methods developed for the Dirichlet and Neumann problems for the Laplace operator, the stationary ones in[14–18] (see also, Ref. 19 Section 12.2). The Steklov condition (1.5) brings into the asymptotic analysis all complications inherent to the Dirichlet condition on $\partial\Gamma_\varepsilon$, namely, the asymptotic structures are governed by an integral (pseudodifferential) operator J on the curve $\Gamma \subset \mathbb{R}^3$. This integral operator appears in an asymptotic expansion of a singular solution of the Dirichlet problem in Ω, with the Dirac mass δ distributed along Γ with a smooth density γ (see Section 2), and remains the same in many boundary-value problems with various singular perturbations on thin elongated sets, the corresponding asymptotic constructions involve attributes of the operator J. Instead, boundary-value problems in Ω_ε with the Neumann condition on $\partial\Gamma_\varepsilon$ do not require this integral operator, and their asymptotic analysis is much simpler than that of the Dirichlet and Steklov conditions on $\partial\Gamma_\varepsilon$.

In Section 5 we will compare asymptotic results for different variants of boundary conditions on the exterior and interior parts of the boundary of Ω_ε.

The Steklov spectral problem quite often gets peculiar features of asymptotic analyses in other singularly perturbed domains, we refer to the paper[20] and references within it.

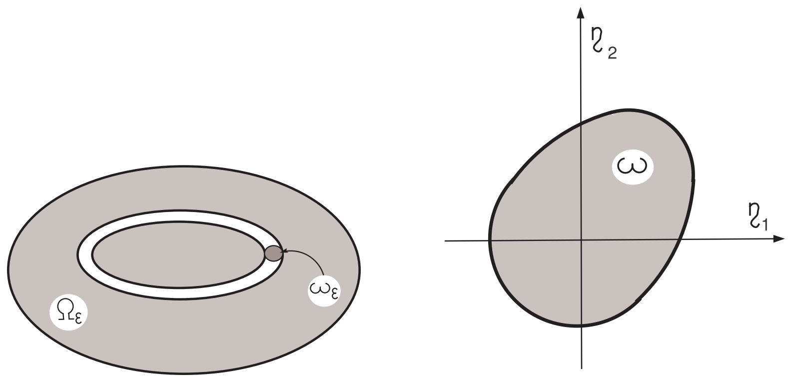

Fig. 2. Domain with thin toroidal cavity and cross-section.

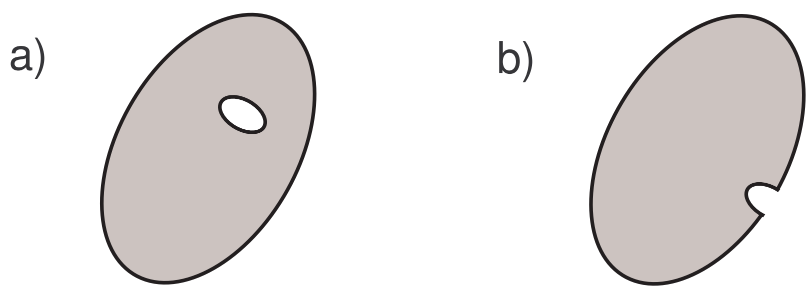

Fig. 3. Domains with singular perturbation: a hole (a) and a cavern (b).

1.4. Structure of the paper

Section 2 is devoted to recall some known results concerning solutions to the Dirichlet problem in Ω and the Neumann problem in $\mathbb{R}^2 \setminus \overline{\omega}$, that are needed to construct the asymptotic expansion of the eigenvalues (1.8). In this section the integral (pseudodifferential) operator J is introduced and its properties are also discussed.

The main asymptotic terms of the eigenpairs of problem (1.4)–(1.6) are formally constructed in Section 3, in terms of real analytic functions in the variable $\zeta = |\ln\varepsilon|^{-1}$.

The statement of the main result, saying that the asymptotic remainders become relatively small is given in Theorem 3.1, whose proof contained in Section 4, that is the most complicated and technical part of the paper.

Finally, in Section 5 we discuss spectral problems for the Laplace equation, with the Steklov, Dirichlet, and Neumann conditions distributed on different parts of the boundary. In particular, there we treat the usual Steklov problem

$$-\Delta_x u^\varepsilon(x) = 0, x \in \Omega_\varepsilon, \quad \partial_\nu u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \partial\Omega_\varepsilon, \tag{1.13}$$

and the water-wave problem mentioned in Section 1.1.

2. Preliminary results: special solutions of the limit problems

In this section we introduce two auxiliary boundary-value problems, usually called *limit problems*, in the framework of the general asymptotic theory of singularly perturbed domains (see, e.g., the monograph[19]). Their solutions will be the essential ingredients of the asymptotic expansions for the eigenpairs of problem (1.4)–(1.6) under consideration.

2.1. Singular solutions to the Dirichlet problem

In this section we introduce solutions of the Dirichlet problem in $\Omega \setminus \Gamma$ with singularities on Γ, in order to describe the behaviour of the eigenfunctions of problem (1.4)–(1.6) far from Γ_ε. For a smooth function $\gamma \in C^\infty(\mathbb{R}^3)$, let us set

$$\mathfrak{V}(\gamma; x) = \int_\Gamma \gamma(s) G(x; s)\, d\sigma(s) \tag{2.1}$$

where $d\sigma$ denotes (here and below) the arc-length measure, $G(x;\xi)$ is the Green function in the domain Ω with singularity at a point $\xi \in \Omega$. $G(x;\xi)$ can be represented by

$$G(x;\xi) = \frac{1}{4\pi}|x-\xi|^{-1} + G_0(x,\xi) \tag{2.2}$$

where the first term is nothing but the fundamental solution of the Laplace operator in $\mathbb{R}^3$ and G_0 is the regular part, i.e., a smooth solution of the following problem

$$-\Delta_x G_0(x;\xi) = 0, x \in \Omega, \quad G_0(x;\xi) = -(4\pi|x-\xi|)^{-1},\ x \in \partial\Omega.$$

In other words, $\mathfrak{V}(\gamma,\cdot) = 0$ on $\partial\Omega$ and $\mathfrak{V}(\gamma,\cdot)$ is the distributional solution to the equation

$$-\Delta\mathfrak{V}(\gamma,\cdot) = \gamma\delta_\Gamma \qquad \text{in } \mathcal{D}'(\Omega)$$

where $\gamma\delta_\Gamma$ denotes the Dirac distribution along the curve Γ with density γ, i.e.,

$$\langle \gamma\delta_\Gamma, \varphi\rangle = \int_\Gamma \varphi(s)\gamma(s)\, d\sigma(s) \qquad \forall\varphi \in C_c^\infty(\Omega).$$

It is known (see, Ref. 19, Section 12.2), that the function in (2.1) admits the decomposition

$$\mathfrak{V}(\gamma; x) = -\frac{1}{2\pi}\gamma(s)\ln r + J(\gamma; s) + O(r(1+|\ln r|)), \qquad r \to 0^+, \tag{2.3}$$

where (s, n, z) are the local coordinates of $x \in \mathcal{V}$, and $r = (n^2+z^2)^{1/2}$ is the distance from x to Γ in $\mathbb{R}^3$ and the integral operator J takes the form

$$J(\gamma; s) = \int_\Gamma (\gamma(\tau) - \gamma(s)) G(\tau, s)\, d\sigma(\tau) + j(s)\gamma(s). \tag{2.4}$$

Here, $G(\tau, s)$ is the trace on $\Gamma \times \Gamma$ of the Green function (2.2) and the factor j in (2.4) is determined as follows:

$$j(s) = \frac{1}{2\pi}\ln 2 - \mathcal{G}^+(s+0, s) - \mathcal{G}^-(s-0, s),$$

while $\mathcal{G}$ is a primitive of the function $s \in \Gamma \mapsto G(s,\tau)$, which, by (2.2), takes the form

$$\mathcal{G}(s,\tau) = \pm\frac{1}{4\pi}\ln|s-\tau| \pm \mathcal{G}^{\pm}(s,\tau). \tag{2.5}$$

with the (bounded) regular part $\mathcal{G}^{\pm}$. Notice that $+$ occurs in (2.5) when the point $s \in \Gamma$ is on the right of $\tau \in \Gamma$ and $-$ occurs if s is on the left of τ.

2.2. The spectrum of the integral operator J

In order to understand the spectrum of the integral operator J, we preliminary consider the first term in (2.4), that we denote by J^0, namely

$$J^0(\gamma;s) = \int_\Gamma (\gamma(\tau)-\gamma(s))G(\tau,s)\,d\tau. \tag{2.6}$$

Due to general properties of the Green function, the kernel G in (2.6) is symmetric and positive. Furthermore, for any smooth function $\kappa \in C^\infty(\mathbb{R}^3)$

$$\begin{aligned} -\int_\Gamma \kappa(s)J^0(\gamma;s)\,d\sigma(s) &= \tfrac{1}{2}\int_\Gamma\int_\Gamma(\gamma(\tau)-\gamma(s))G(\tau,s)\,d\sigma(\tau)\kappa(s)\,d\sigma(s)+\\ &+\tfrac{1}{2}\int_\Gamma\int_\Gamma(\gamma(\tau)-\gamma(s))G(\tau,s)\,d\sigma(s)\kappa(\tau)\,d\tau\\ &= \tfrac{1}{2}\int_\Gamma\int_\Gamma(\gamma(s)-\gamma(\tau))(\kappa(s)-\kappa(\tau))G(s,\tau)\,d\sigma(s)\,d\sigma(\tau). \end{aligned} \tag{2.7}$$

Taking $\kappa = \gamma$ in the preceding integrals, it follows that the expression

$$\int_\Gamma |\gamma(s)|^2\,d\sigma(s) - \int_\Gamma J^0(\gamma;s)\gamma(s)\,d\sigma(s) =$$
$$= \int_\Gamma |\gamma(s)|^2\,d\sigma(s) + \frac{1}{2}\int_\Gamma\int_\Gamma(\gamma(s)-\gamma(\tau))^2G(s,\tau)\,d\sigma(s)\,d\sigma(\tau)$$

is positive for $\gamma \not\equiv 0$. The Hilbert space obtained as the completion of $C^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\gamma; H_{\ln}(\Gamma)\| := \left(\|\gamma; L^2(\Gamma)\|^2 - (J^0\gamma,\gamma)_\Gamma\right)^{1/2}$$

is denoted by $H_{\ln}(\Gamma)$. According to (2.7), J^0 is a negative continuous and symmetric, therefore, self-adjoint operator in $H_{\ln}(\Gamma)$. To make our consideration much more precise, taking into account (2.2), we observe the singularity $O(|s-\tau|^{-1})$ of the kernel in (2.6) and, in view of the results in,[21] we conclude that $H_{ln}(\Gamma)$ is nothing but the Hörmander space[22] generated by the weight function $\mu = (1+\ln|\xi|+|\ln|\xi||)^{1/2}$. In other words, a norm in $H_{ln}(\Gamma)$ may be defined through an appropriate partition of unity on Γ and the following norm in $H_{ln}(\mathbb{R})$:

$$\left(\int_{\mathbb{R}} \mu(\xi)^2\left|(\mathcal{F}_{s\to\xi}\gamma)(\xi)\right|^2 d\xi\right)^{1/2}$$

where $\mathcal{F}_{s\to\xi}$ stands for the Fourier transform. Since μ grows unboundedly when $\xi \to \pm\infty$, the embedding $H_{\ln}(\Gamma) \subset L^2(\Gamma)$ is compact. By a direct calculation, one can also verify that J^0, and therefore J, is a pseudo-differential operator with principal symbol $-(2\pi)^{-1}\,|\ln|\xi\,\|$ (see, Ref. 19, Ch. 12). The scalar product in $H_{\ln}(\Gamma)$

$$(\gamma,\kappa) = \int_\Gamma \gamma(s)\kappa(s)\,d\sigma(s) - \int_\Gamma J^0(\gamma;s)\kappa(s)\,d\sigma(s)$$

involves, first, the natural scalar product in the Lebesgue space $L^2(\Gamma)$ and, second, its extension up to duality between the Hörmander space $H_{\ln}(\Gamma)$ and its adjoint $H_{\ln}(\Gamma)^*$.

Example 2.1 (*See Also Example 2.1 in Ref. 18*). Let Γ be a circle of radius 1. Then, the distance in $\mathbb{R}^3$ between the points s and τ in Γ equals to $2\left|\sin\frac{1}{2}(s-\tau)\right|$. In this special case the operator J^0 in (2.6) takes the form

$$\mathbf{J^0}(\gamma;s) = \frac{1}{8\pi}\int_0^{2\pi}(\gamma(\tau)-\gamma(s))\left|\sin\left(\frac{1}{2}(s-\tau)\right)\right|^{-1} d\sigma(\tau) \tag{2.8}$$

and its eigenvalues and eigenfunctions can be computed explicitly (see, Ref. 19, Ch. 12.2), namely,

$$\begin{aligned} \boldsymbol{\beta}_{2k-1} &= \boldsymbol{\beta}_{2k} = -\frac{1}{16\pi}\sum_{j=0}^{k-2}\frac{1}{1+2j},\\ \boldsymbol{\gamma}_{2k-2} &= \pi^{-1/2}\sin((k-1)s), \quad \boldsymbol{\gamma}_{2k-1} = \pi^{-1/2}\cos((k-1)s), \end{aligned}$$

where $k = 1,2,\ldots$, but $\boldsymbol{\beta}_0 = 0$ and $\boldsymbol{\gamma}_0 = 0$ are skept.

Proposition 2.1. *The operator J defined by (2.4) has discrete spectrum*

$$\beta_1 \ge \beta_2 \ge \cdots \ge \beta_k \ge \ldots \to -\infty \tag{2.9}$$

where eigenvalues are listed according to their multiplicity. The corresponding eigenfunctions $\gamma_0,\ldots,\gamma_k,\ldots$ belong to $C^\infty(\Gamma)$ and can be chosen satisfying the normalization and orthogonality conditions

$$(\gamma_k,\gamma_p)_\Gamma = \delta_{k,p} \quad k,p \in \mathbb{N}_0 = \{0,1,\ldots\}. \tag{2.10}$$

Proof. In view of the above considerations, the quadratic form

$$(J^0\gamma,\gamma)_\Gamma + (j\gamma,\gamma)_\Gamma \tag{2.11}$$

is symmetric, closed, and above-semi-bounded in $H_{\ln}(\Gamma)$. Hence, according to classical results (see, e.g., Ref. 23 Ch.10), the form (2.11) is associated to a semi-bounded self-adjoint unbounded operator $\mathcal{J}$ in $L^2(\Gamma)$ with a domain $D(\mathcal{J}) \subset H_{\ln}(\Gamma)$. It has the discrete spectrum (2.9) because of the compact embedding $H_{ln}(\Gamma) \subset L^2(\Gamma)$ and general results in operator theory (see, Ref. 23, Thm. 10.1.5 and 10.2.2). Since the pseudo-differential operator J, as well as J^0, has unbounded principal symbol $-(2\pi)^{-1}|\ln|\xi||$, it is hypo-elliptic, and therefore its eigenfunctions belong to $C^\infty(\Gamma)$ (see, e.g., Ref. 24). Condition (2.10) is standard. □

Proposition 2.2. *Eigenvalues in (2.9) take the asymptotic form*

$$\beta_k = -2\ln k + O(1), \quad k \to +\infty. \tag{2.12}$$

Proof. The difference $J - \mathbf{J^0}$ of the operators defined by (2.4) and (2.8) takes the form

$$J(\gamma;s) - \mathbf{J^0}(\gamma;s) = (K\gamma)(s) = \int_\Gamma(\gamma(\tau)-\gamma(s))\,\mathcal{K}(s,\tau)\,d\sigma(\tau) + j(s)\gamma(s)$$

where the kernel K is bounded on $\Gamma\times\Gamma$, due to the definitions of J and G, (2.4) and (2.2), and the fact that

$$\frac{1}{8\pi}\left|\sin\left(\frac{1}{2}(s-\tau)\right)\right|^{-1} = \frac{1}{4\pi}|s-\tau|^{-1} + O(1) \quad \text{as } |s-\tau| \to 0.$$

Thus, the mapping $K : L^2(\Gamma) \to L^2(\Gamma)$ is continuous with the norm $\|K\|$. By the max–min principle (see, e.g., Ref. 23, Thm. 10.2.2), applied to the operators $-J$ and $-\mathbf{J^0}$ (with minus) it follows that

$$-\beta_i = \max_{E_i}\inf_{\gamma\in E_i\setminus\{0\}}\frac{-(J\gamma,\gamma)_\Gamma}{(\gamma,\gamma)_\Gamma} \text{ and } -\boldsymbol{\beta}_i = \max_{E_i}\inf_{\gamma\in E_i\setminus\{0\}}\frac{-(\mathbf{J^0}\gamma,\gamma)_\Gamma}{(\gamma,\gamma)_\Gamma} \tag{2.13}$$

for all $i \in \mathbb{N}_0$, where E_i is any subspace of $H_{\ln}(\Gamma)$ with codimension i, i.e., $\dim(H_{\ln}(\Gamma)\ominus E_i) = i$, and $E_0 = H_{\ln}(\Gamma)$. The equalities (2.13) and the above mentioned property of K imply that

$$\boldsymbol{\beta}_i - \|K\| \le \beta_i \le \boldsymbol{\beta}_i + \|K\|.$$

Finally, the asymptotic relation (2.12) follows from the classical formula

$$\sum_{p=0}^{k}\frac{1}{1+2p} = \frac{1}{2}\ln k + O(1), \quad k \to +\infty$$

(see, e.g., Ref. 19, Lemma 12.2.3). □

2.3. The exterior Neumann problem

In this section we introduce a two-dimensional boundary-value problem, and briefly discuss the properties of its solutions, which are needed in Section 3, in the asymptotic expansion (3.1)–(3.2). Let us consider the following exterior Neumann problem

$$-\Delta_\eta W(\eta)=0,\ \ \eta\in\mathbb{R}^2\setminus\overline{\omega},\quad \partial_{\nu(\eta)}W(\eta)=F(\eta),\ \ \eta\in\partial\omega, \tag{2.14}$$

where $\Delta_\eta=\nabla_\eta\cdot\nabla_\eta$, ∇_η denotes the gradient with respect to η, $\partial_{\nu(\eta)}=\nu(\eta)\cdot\nabla_\eta$, and $\nu(\eta)$ is the outward unit normal vector. It is well-known that problem (2.14) has a solution with finite Dirichlet semi-norm $\|\nabla_\eta W;L^2(\mathbb{R}^2\setminus\overline{\omega})\|$ if and only if $F\in L^2(\partial\omega)$ is of mean zero over the boundary $\partial\omega$. This solution is determined up to an additive constant and, therefore, a solution with the decay rate $O(|\eta|^{-1})$ at infinity exists and is unique.

Proposition 2.3. *The exterior Neumann problem (2.14) with the right-hand side*

$$F_0(\eta)=\frac{1}{2\pi}\frac{\partial}{\partial\nu(\eta)}\ln|\eta|+\frac{1}{|\partial\omega|} \tag{2.15}$$

has a unique solution $W_0(\eta)=O(|\eta|^{-1})$ as $|\eta|\to+\infty$, with $\nabla_\eta W_0\in L^2(\mathbb{R}^2\setminus\overline{\omega})$. Here, $|\partial\omega|$ is the length of $\partial\omega$.

Proof. Here we set $\rho=|\eta|$. It suffices to recall that

$$\frac{1}{2\pi}\int_{\partial\omega}\frac{\partial}{\partial\nu(\eta)}\ln\rho\,d\sigma(\eta)=-\frac{1}{2\pi}\int_{\{\eta:\rho=R\}}\frac{\partial}{\partial\nu(\eta)}\ln\rho\,d\sigma(\eta)=-1, \tag{2.16}$$

where $\sigma(\eta)$ denotes the arc-length parametrization of $\partial\omega$. $\square$

In the sequel we will need two integral characteristics of the cross-section ω of the toroidal set Γ_ε defined by (1.2), namely,

$$l(\omega)=\frac{1}{2\pi}\int_{\partial\omega}\ln\rho\,d\sigma(\eta),\quad L(\omega)=\int_{\partial\omega}W_0(\eta)\,d\sigma(\eta) \tag{2.17}$$

where W_0 is the harmonic function introduced in Proposition 2.3.

2.4. The operator formulation of the exterior Neumann problem

Let us denote by $H^{1/2}(\partial\omega)$ is the Sobolev–Slobodetskii space of traces on $\partial\omega$ of functions in $H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\omega)$. The norm can be defined by

$$\|W;H^{1/2}(\partial\omega)\|=\left(\|W;L^2(\partial\omega)\|^2+\int_{\partial\omega}\int_{\partial\omega}\frac{|W(\eta)-W(Y)|^2}{|\eta-Y|^2}\,d\sigma(\eta)\,d\sigma(Y)\right)^{1/2} \tag{2.18}$$

Let us also consider the space

$$L^2_\perp(\partial\omega)=\{F\in L^2(\partial\omega):(F,1)_{\partial\omega}=0\}. \tag{2.19}$$

Thanks to the considerations made in Section 2.3, we can define the mapping

$$R:\ L^2_\perp(\partial\omega)\to H^{1/2}(\partial\omega) \tag{2.20}$$

$$F\quad\mapsto RF=W_{|\partial\omega} \tag{2.21}$$

where W is a decaying solution of the exterior Neumann problem (2.14). This solution is unique and satisfies the estimate

$$\|\nabla_\eta W;L^2(\mathbb{R}^2\setminus\omega)\|+\|W;L^2(\partial\omega)\|\le c\|F;L^2(\partial\omega)\|,$$

and in particular

$$\|W;H^1(B_d\setminus\omega)\|\le c_r\|F;L^2(\partial\omega)\|,$$

where the radius d is chosen such that $\overline{\omega}\subset B_d=\{\eta:|\eta|<d\}$. Hence, the mapping R defined in (2.20), (2.21), is a continuous monomorphism. The map R will be used in Section 3.2 to establish existence and properties of the pair (3.10).

3. Asymptotic behaviour of eigenvalues and eigenfunctions

3.1. The formal asymptotic ansätze

Based on results of previous works[14–17] (see also, Ref. 19, Section 12.2), we guess the following asymptotic ansätze for an eigenpair $(\lambda^\varepsilon_p,u^\varepsilon_p)$ of problem (1.4)–(1.6):

$$u^\varepsilon_p(x)=\mathfrak{V}(\gamma_p;x)+\chi(x)\gamma_p(s)w_p(\varepsilon^{-1}n,\varepsilon^{-1}z;\zeta)+\cdots, \tag{3.1}$$

$$\lambda^\varepsilon_p=\varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_p(\zeta)+\cdots, \tag{3.2}$$

Here γ_p is an eigenfunction of the operator J given by (2.4), $\mathfrak{V}(\gamma_p;x)$ is the singular solution (2.1), $\chi\in C^\infty_c(\Omega)$ is a cut-off function which equals 1 in the 3d-neighbourhood of the curve $\Gamma\subset\mathbb{R}^3$, and the dots stand for higher-order terms, which are neglected here, since they are of no use in our formal asymptotic analysis in this section. Finally, $\mu_p(\zeta)$ and $w_p(\eta;\zeta)$ are a number and a harmonics in $\eta\in\mathbb{R}^2\setminus\overline{\omega}$ (see Section 2.3) which should be determined: as it will be clear from the following computations, both $\mu_p(\zeta)$ and $w_p(\eta;\zeta)$ depend on the small parameter $\zeta=|\ln\varepsilon|^{-1}$. Note that in this section we do not care about normalization of the eigenfunction in (3.1). This will be done in Section 4.4. First, we observe that the Laplace operator Δ_x in the curvilinear coordinated system (s,n,z) reads:

$$\Delta_x=(1+n\varkappa(s))^{-1}\left(\frac{\partial}{\partial n}(1+n\varkappa(s))\frac{\partial}{\partial n}+\frac{\partial}{\partial s}(1+n\varkappa(s))^{-1}\frac{\partial}{\partial s}\right)+\frac{\partial^2}{\partial z^2} \tag{3.3}$$

where $\varkappa(s)$ is the curvature of the arc $\Gamma\subset\mathbb{R}^2$ at a point s. Hence,

$$\Delta_x=\varepsilon^{-2}\Delta_\eta+\cdots$$

where Δ_η is the Laplace operator in the stretched coordinates $\eta=(\eta_1,\eta_2)\in\mathbb{R}^2$ (see (1.2)). In view of the definition (1.2) of Γ_ε, the normal derivative $\partial\Gamma_\varepsilon$ takes the form

$$\partial_\nu=\varepsilon^{-1}\partial_{\nu(\eta)}, \tag{3.4}$$

where $\partial_{\nu(\eta)}$ is the *inward* normal derivative at the boundary $\partial\omega$ of the inflated cross-section ω.

Inserting the asymptotic ansätze (3.1) into the Laplace equation (1.4), thanks to the above relations (3.3) and (3.4), we obtain that

$$\Delta_\eta w(\eta;\zeta)=0,\quad \eta\in\mathbb{R}^2\setminus\overline{\omega}. \tag{3.5}$$

Note that the first term $\mathfrak{V}(\gamma_p;x)$ in (3.1) satisfies the Laplace equation (1.4), and the Dirichlet boundary condition (1.6), while the second term vanishes on the exterior boundary $\partial\Omega$, due to the cut-off function χ. Analysing the Steklov condition (1.5) on the interior boundary $\partial\Gamma_\varepsilon=\partial\Omega_\varepsilon\setminus\partial\Omega$, we recall the asymptotic ansätze (3.2) for λ^ε_p, the representation (2.3) of $\mathfrak{V}(\gamma_p;x)$ near Γ_ε (with $r=\varepsilon|\eta|$) and the equation

$$J(\gamma_p;s)=\beta_p\gamma_p(s),\quad s\in\Gamma,$$

for the eigenpair $\{\beta_p,\gamma_p\}$ of the integral operator J. Neglecting higher-order asymptotic terms and factoring out the eigenfunction $\gamma_p(s)$, we convert the Steklov boundary condition (1.5) into the Neumann condition

$$\partial_{\nu(\eta)}w_p(\eta;\zeta)=f_p(\eta;\zeta) \tag{3.6}$$

where

$$f_p(\eta;\zeta)=\frac{1}{2\pi}\partial_{\nu(\eta)}\ln\rho+\frac{1}{|\ln\varepsilon|}\mu_p(\zeta)\left(w_p(\eta;\zeta)-\frac{1}{2\pi}\ln(\varepsilon\rho)+\beta_p\right) \tag{3.7}$$

with $\eta\in\partial\omega$, $\rho=|\eta|$. The compatibility condition recalled in Section 2.3, namely,

$$\int_{\partial\omega}f_p(\eta;\zeta)\,d\sigma(\eta)=0 \tag{3.8}$$

in the exterior Neumann problem (3.5), (3.6), with right-hand side $f_p(\eta;\zeta)$ fixed for a while, turns out into

$$\mu_p(\zeta)=\frac{2\pi}{|\partial\omega|}-\frac{\mu_p(\zeta)}{|\ln\varepsilon|}\left(\frac{2\pi}{|\partial\omega|}\int_{\partial\omega}w_p(\eta;\zeta)\,d\sigma(\eta)-2\pi\frac{l(\omega)}{|\partial\omega|}+2\pi\beta_p\right). \tag{3.9}$$

Here, we observed that $-\ln\varepsilon=|\ln\varepsilon|$ for $0<\varepsilon\le\varepsilon_0\le1$, and we used formulas (2.16) and (2.15). Hence we will set $\zeta=|\ln\varepsilon|^{-1}$.

3.2. Solving the non-linear system

We regard (3.5), (3.6), (3.7), and (3.9) as a system to determine the pair

$$\{\mu_p(\zeta), w_p(\cdot;\zeta)\} \in \mathcal{B} := \mathbb{R} \times H^{1/2}(\partial\omega) \tag{3.10}$$

for any given ζ. We are now in a position to formulate the main result of this section.

Proposition 3.1. *For any $p \in \mathbb{N}$ there exists a number $\zeta_p > 0$, such that the system (3.5), (3.6), (3.7), and (3.9) has a unique solution $(\mu_p(\zeta), w_p(\cdot;\zeta))$ for any $|\zeta| < \zeta_p$. Both the harmonics $w_p(\cdot;\zeta)$ and the number $\mu_p(\zeta) > 0$ are real analytic functions in the parameter ζ. Furthermore, $w_p(\eta;\zeta)$ decays as $|\eta| \to +\infty$, i.e., for $\zeta \in [0, \zeta_p]$, $\zeta_p > 0$,*

$$|\nabla^j_\eta w_p(\eta;\zeta)| \le c_j(d)|\eta|^{-1-j}, \ j \in \mathbb{N}_0, \ \eta \in \mathbb{R}^2 \setminus B_d, \tag{3.11}$$

where $c_j(d)$ are independent of ζ and $d > 0$ is fixed such that $\overline{\omega} \subset B_d$. Finally, μ_p has the following behaviour as $\varepsilon \to 0^+$

$$\mu_p\left(|\ln\varepsilon|^{-1}\right) = \frac{2\pi}{|\partial\omega|} - \frac{1}{|\ln\varepsilon|}\left(\frac{2\pi}{|\partial\omega|}\right)^2 \left(L(\omega) - l(\omega) + \beta_p|\partial\omega|\right) + O\left(|\ln\varepsilon|^{-2}\right). \tag{3.12}$$

Proof. Let $F(\zeta)$ denote the right-hand side of (3.6) with the factor $\mu_p(\zeta)$ replaced by the right-hand side of (3.9) and let $P_\perp$ be the orthogonal projector of $L^2(\partial\omega)$ onto the subspace $L^2_\perp(\partial\omega)$, introduced in (2.19). In order to rewrite the above mentioned system as a fixed point problem, we introduce the operator

$$T_p(\zeta, \cdot, \cdot) : (\mu, F) \mapsto (\mu_p, F_p) \tag{3.13}$$

where μ_p, F_p are defined by the following formulae

$$\mu_p(\zeta) = \frac{2\pi}{|\partial\omega|} - \zeta\mu\left(\frac{2\pi}{|\partial\omega|}\int_{\partial\omega} RF\, d\sigma(\eta) - 2\pi\frac{l(\omega)}{|\partial\omega|} + 2\pi\beta_p\right),$$

and

$$F_p(\zeta) = \frac{1}{2\pi}P_\perp\partial_\nu \ln\rho + \zeta\mu\left(P_\perp RF - \frac{1}{2\pi}P_\perp \ln\rho\right) \tag{3.14}$$

Then, the problem of solving the system (3.5), (3.6), (3.7), and (3.9) is reduced to solve the fixed point equation

$$\{\mu_p(\zeta), F_p(\zeta)\} = T_p(\zeta, \mu_p(\zeta), F_p(\zeta)) \quad \text{in } \mathcal{B},$$

since we may reconstruct the solution (3.10) of the original system (3.5), (3.6) and (3.9) by solving the exterior Neumann problem (2.14) in the class of decaying harmonics. Hence, to reach the conclusion it is enough to notice that, due to the compact embedding $H^1(B_r \setminus \omega) \subset L^2(\partial\omega)$, the operator T_p in (3.13)–(3.14) is compact. Moreover, it is real analytic in all its arguments. Finally, for any fixed $\rho > 0$, there is a value ζ_p such that for all $|\zeta| < \zeta_p$, the ball

$$\left\{\{\mu, F\} \in \mathcal{B} : |\{\mu, F\} - T_p(0,0,0)| \le \rho\right\} \tag{3.15}$$

is sent to itself. Hence, due to the Banach contraction principle, we conclude that, for all $|\zeta| < \zeta_p$ there is a solution $\{\mu_p(\zeta), F_p(\zeta)\} \in \mathcal{B}$. This solution is analytic in $\zeta \in (-\zeta_p, \zeta_p)$, thanks to basic results on abstract non-linear equations (see, Ref. 25, Ch.5, Ref. 26, Ch.3) and others. Furthermore, the Banach contraction principle ensures the uniqueness of $\{\mu_p(\zeta), F_p(\zeta)\}$ in the ball (3.15), for $\zeta \in (-\zeta_p, \zeta_p)$. We remark that the analytic dependence on the parameter ζ is of course preserved by the solution (3.10).

We finally observe that, according to (3.6) and (2.15), the function $w_p(\cdot;0)$ (note here $\zeta = 0$!) coincides with the harmonics W_0 mentioned in Proposition 2.3 and, moreover, by virtue of (3.9) and (2.17), the main asymptotic term of the eigenvalue (3.2) has the behaviour (3.12). □

Remark 3.1. We note that formula (3.12) is consistent with the relation (1.1); indeed, the main term $\mu_p(0) = 2\pi|\partial\omega|^{-1}$ is independent of $p \in \mathbb{N}$, while the correction term $|\ln\varepsilon|^{-1}\partial_\zeta\mu_p(0)$ involves the eigenvalue β_p of the operator J introduced in (2.4). In this sense, the 'asymptotic splitting' of eigenvalues λ^ε_p mentioned in the introduction, indeed, occurs. Nevertheless, it should be underlined that the presence of the density $\gamma_p(s)$ in the *ansätz* (3.1) yields different asymptotic approximations for the eigenpairs $\{\lambda^\varepsilon_p, u^\varepsilon_p\}$ and $\{\lambda^\varepsilon_q, u^\varepsilon_q\}$ when $p \ne q$, even if $\beta_p = \beta_q$.

3.3. Statements of the main results

At this point, we have all the tools to state the main results concerning the asymptotic behaviour of the eigenvalues λ^ε_p and eigenfunctions u^ε_p of problem (1.4)–(1.6).

Theorem 3.1. *Let β_n be an eigenvalue of the integral operator (2.4) with multiplicity $\kappa \ge 1$, cf. (4.27). Then the entries of the eigenvalue sequence (1.8) of the Steklov–Dirichlet problem (1.4)–(1.6) satisfy the asymptotic formula*

$$|\lambda^\varepsilon_p - \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_n(|\ln\varepsilon|^{-1})| \le C_n \quad \text{for } \varepsilon \in (0, \varepsilon_n] \tag{3.16}$$

where

$$p = n, \dots, n + \kappa - 1, \tag{3.17}$$

C_n and ε_n are some positive numbers, and $\mu_n(\zeta)$ is the first component of the pair (3.10) that solves the non-linear system (3.5), (3.6), (3.7), and (3.9) (see Proposition 3.1).

Theorem 3.2. *For any eigenvalue β_n of the integral operator (2.4) with multiplicity $\kappa \ge 1$, there exist κ unit vectors $a^{\varepsilon n}, \dots, a^{\varepsilon n+\kappa-1} \in \mathbb{R}^\kappa$ and such that*

$$\|u^\varepsilon_p - \sum_{j=p}^{p+\kappa-1} a^{\varepsilon p}_j U^\varepsilon_j; \mathcal{H}^\varepsilon\| \le c_n\varepsilon|\ln\varepsilon| \quad \text{for } \varepsilon \in (0, \varepsilon_n]$$

where $p = n, \dots, p + \kappa - 1$, ε_n is some positive number, $U^\varepsilon_p, \dots, U^\varepsilon_{p+\kappa-1}$ are eigenfunctions of the problem (1.5) verifying the conditions (4.11) and $u^\varepsilon_p, \dots, u^\varepsilon_{p+\kappa-1}$ are determined in (4.15), (4.14), according to the asymptotic procedure in Section 3.

Corollary 3.1. *If the eigenvalue β_n of the integral operator J in (2.4) is simple, i.e. $\kappa = 1$ in Theorem 3.2, then*

$$\|u^\varepsilon_n - U^\varepsilon_n; \mathcal{H}^\varepsilon\| \le c_n\varepsilon|\ln\varepsilon| \quad \textit{for} \quad \varepsilon \in (0, \varepsilon_n).$$

4. Proof of the main results

4.1. Auxiliary inequalities

In this section we will need several weighted estimates presented in two lemmas.

Lemma 4.1. *There exists $\varepsilon_0, c > 0$ such that the inequality*

$$\|r^{-1}(1 + |\ln r|)^{-1}u; L^2\|^2(\Omega_\varepsilon) \le c\left(\|\nabla_x u; L^2(\Omega_\varepsilon)\|^2 + \varepsilon^{-1}(1 + |\ln\varepsilon|)^{-2}\|u; L^2(\partial\Gamma_\varepsilon)\|^2\right) \tag{4.1}$$

with $r = \mathrm{dist}(x, \Gamma)$, is valid for all $u \in H^1_0(\Omega_\varepsilon; \partial\Omega)$.

Proof. Based on the Dirichlet condition (1.6), we write the Friedrichs and trace inequalities

$$\|u; L^2(\partial V_\mathcal{R})\|^2 + \|u; L^2(\Omega \setminus V_\mathcal{R})\|^2 \le c_r\|u; L^2(\Omega \setminus V_\mathcal{R})\|^2, \tag{4.2}$$

where $V_\mathcal{R} \subset \Omega$ is a tubular $\mathcal{R}$-neighbourhood of the curve Γ. We multiply the two-dimensional Poincaré inequality

$$\int_{B_R\setminus\omega} |u|^2\, d\eta \le c\left(\int_{B_R\setminus\omega} |\nabla_\eta u|^2\, d\eta + \int_{\partial\omega} |u|^2\, d\sigma(\eta)\right)$$

by $(1+|\ln\varepsilon|)^{-2}$ and return to the 'slow' variable y. Since

$$\varepsilon^{-2}(1+|\ln\varepsilon|)^{-2}\geq cr^{-2}(1+|\ln r|)^{-2}$$

for $y\in B_{\varepsilon R}\setminus\omega^\varepsilon$, integration along Γ provides the relation

$$\|r^{-1}(1+|\ln r|)^{-1}u;L^2(\mathcal{V}_{\varepsilon R}\setminus\Gamma_\varepsilon)\|^2\leq c\,(1+|\ln\varepsilon|)^{-2}\left(\|\nabla_x u;L^2(\mathcal{V}_{\varepsilon R}\setminus\Gamma_\varepsilon)\|^2 + \varepsilon^{-1}\|u;L^2(\partial\Gamma_\varepsilon)\|^2\right). \tag{4.3}$$

Note that

$$\begin{aligned}&\int_{\mathcal{V}_{\varepsilon R}\setminus\Gamma_\varepsilon} r^{-2}(1+|\ln r|)^{-2}|u(x)|^2\,dx = \\ &=\int_\Gamma(1+n\varkappa(s))\int_{B_{\varepsilon R}\setminus\omega_\varepsilon} r^{-2}(1+|\ln r|)^{-2}|u(y,s)|^2\,dy\,d\sigma(s)= \\ &=(1+O(\varepsilon))\int_\Gamma\int_{B_{\varepsilon R}\setminus\omega_\varepsilon} r^{-2}(1+|\ln r|)^{-2}|u(y,s)|^2\,dy\,d\sigma(s)\end{aligned} \tag{4.4}$$

because $|\varkappa(s)|<c$ and $|\eta|<\varepsilon$ in $B_{\varepsilon R}$. Adding to the sum of (4.2) and (4.3) the relation

$$\|r^{-1}(1+|\ln r|)^{-1}u;L^2(V_{\mathcal{R}}\setminus V_{\varepsilon R})\|^2\leq c\left(\|\nabla_x u;L^2(V_{\mathcal{R}}\setminus\mathcal{V}_{\varepsilon r})\|^2+\|u;L^2(\partial V_{\mathcal{R}})\|^2\right),$$

we arrive at the desired estimate (4.1). In this way, it suffices to use the well-known one-dimensional inequality of Hardy's type

$$\int_{\varepsilon R}^{\mathcal{R}} r^{-1}(1+|\ln r|)^{-2}|U(r)|^2\,dr\leq c\left(\int_{\varepsilon R}^{\mathcal{R}} r\left|\frac{dU}{dr}(r)\right|^2dr+U(\mathcal{R})^2\right). \quad\square$$

Lemma 4.2. *Let $u\in H_0^1(\Omega(\varepsilon);\partial\Omega)$ and*

$$\hat u(s)=\frac{1}{|\partial\omega_\varepsilon|}\int_{\partial\omega_\varepsilon}u(x)\,d\sigma(n,z)\qquad\text{for a.e. } s\in\Gamma. \tag{4.5}$$

Then the difference $u_\perp=u-\hat u$ satisfies the estimate

$$\|u_\perp;L^2(\partial\Gamma_\varepsilon)\|^2\leq c\varepsilon\|\nabla_x u;L^2(\Omega_\varepsilon)\|^2. \tag{4.6}$$

Proof. Recalling the trace inequality

$$\int_{\partial\omega}|u(\eta,s)-\hat u(s)|^2\,d\sigma(\eta)\leq c\int_{B_R\setminus\omega}|\nabla_\eta(u(\eta,s)-\hat u(s))|^2\,d\eta=c\int_{B_R\setminus\omega}|\nabla_\eta u(\eta,s)|^2d\eta,$$

and repeating the argument in (4.4), we get (4.6). $\square$

4.2. Reduction to an abstract equation

In order to prove Theorem 3.1, we express the spectral problem (1.4)–(1.6) in abstract form. To this end, let us denote by $\mathcal{H}^\varepsilon$ the Hilbert space

$$\mathcal{H}^\varepsilon=H_0^1(\Omega_\varepsilon;\partial\Omega)=\{u\in H^1(\Omega_\varepsilon):u=0\text{ on }\partial\Omega\},$$

equipped with the scalar product

$$\langle u,v\rangle=(\nabla u,\nabla v)_{\Omega_\varepsilon}+\varepsilon^{-1}|\ln\varepsilon|^{-1}(u,v)_{\partial\Gamma_\varepsilon}. \tag{4.7}$$

We also introduce the positive, continuous, symmetric, and, therefore, self-adjoint operator $\mathcal{I}^\varepsilon:\mathcal{H}^\varepsilon\to\mathcal{H}^\varepsilon$, defined by

$$\langle\mathcal{I}^\varepsilon u,v\rangle=(u,v)_{\partial\Gamma_\varepsilon}\quad\forall u,v\in\mathcal{H}^\varepsilon. \tag{4.8}$$

According to (4.7) and (4.8), the variational formulation (1.7) of problem (1.4)–(1.6) reduces to the spectral equation

$$\mathcal{I}^\varepsilon u^\varepsilon=\tau^\varepsilon u^\varepsilon\quad\text{ in }\mathcal{H}^\varepsilon \tag{4.9}$$

with the new spectral parameter

$$\tau^\varepsilon=(\lambda^\varepsilon+\varepsilon^{-1}|\ln\varepsilon|^{-1})^{-1}=(\lambda^\varepsilon+\varepsilon^{-1}\zeta)^{-1}. \tag{4.10}$$

The operator $\mathcal{I}^\varepsilon$ is compact and its essential spectrum consists of the only point $\tau=0$ (see, Ref. 23, Thm. 10.1.5), while its discrete spectrum is formed by a positive, monotone, infinitesimal sequence

$$1>\tau_1^\varepsilon>\tau_2^\varepsilon\geq\dots\geq\tau_p^\varepsilon\geq\ldots\to+0,$$

(see (1.8) and (4.10)). It is possible to choose a basis of $\mathcal{H}^\varepsilon$ made of eigenfunctions U_j^ε of the operator $\mathcal{I}^\varepsilon$, satisfying the orthogonality and normalization condition

$$\langle U_j^\varepsilon,U_k^\varepsilon\rangle=\delta_{jk}. \tag{4.11}$$

The main tool for the proof of Theorem 3.1 is the following assertion, known also as the Lemma on "almost eigenvalues and eigenvectors" (see Ref. 27) that follows from the spectral decomposition of the resolvent (see, Ref. 23, Ch.6).

Lemma 4.3. *Let $\boldsymbol{u}^\varepsilon\in\mathcal{H}^\varepsilon$ and $\boldsymbol{t}^\varepsilon>0$ be such that*

$$\|\boldsymbol{u}^\varepsilon;\mathcal{H}^\varepsilon\|=1;\quad\|\mathcal{I}^\varepsilon\boldsymbol{u}^\varepsilon-\boldsymbol{t}^\varepsilon\boldsymbol{u}^\varepsilon;\mathcal{H}^\varepsilon\|=\delta\in[0,\boldsymbol{t}^\varepsilon). \tag{4.12}$$

Then the interval $[\boldsymbol{t}^\varepsilon-\delta,\boldsymbol{t}^\varepsilon+\delta]$ contains at least one eigenvalue τ^ε of the operator $\mathcal{I}^\varepsilon$. Moreover, for any $\delta_+\in(\delta,\boldsymbol{t}^\varepsilon)$, one finds coefficients a_j^ε, $j=N^\varepsilon,\dots,N^\varepsilon+X^\varepsilon-1$, such that

$$\|\boldsymbol{u}^\varepsilon-\sum_{j=N^\varepsilon}^{N^\varepsilon+X^\varepsilon-1}a_j^\varepsilon U_j^\varepsilon;\mathcal{H}^\varepsilon\|\leq 2\frac{\delta}{\delta_+},\qquad\sum_{j=N^\varepsilon}^{N^\varepsilon+X^\varepsilon-1}|a_j^\varepsilon|^2=1, \tag{4.13}$$

where $\tau_{N^\varepsilon}^\varepsilon,\dots,\tau_{N^\varepsilon+X^\varepsilon-1}^\varepsilon$ are all the eigenvalues of $\mathcal{I}^\varepsilon$ in the interval $[\boldsymbol{t}^\varepsilon-\delta_+,\boldsymbol{t}^\varepsilon+\delta_+]$, and $U_{N^\varepsilon}^\varepsilon,\dots,U_{N^\varepsilon+X^\varepsilon-1}^\varepsilon$ are the corresponding eigenvectors subject to the normalization and orthogonality conditions (4.11).

4.3. Calculating discrepancies

In this section we define the pair $\boldsymbol{u}^\varepsilon\in\mathcal{H}^\varepsilon$ and $\boldsymbol{t}^\varepsilon>0$ needed to apply Lemma 4.3, we compute the value δ in (4.12), and we show that if β_n is a multiple eigenvalue of the integral operator J (defined by (2.4)) with multiplicity κ, then at least κ eigenvalues of the operator $\mathcal{I}^\varepsilon$ belong to a small neighbourhood of β_n. This is done below in 4 steps. The complete proof of Theorem 3.1 will be accomplished later on, at the end of Section 4.4. Recalling the asymptotic ansätze given in Section 3.1 we choose the approximate eigenvalue and eigenvector:

$$\boldsymbol{t}_p^\varepsilon=\varepsilon|\ln\varepsilon|\left(1+\mu_p(\zeta)\right)^{-1},\quad\boldsymbol{u}_p^\varepsilon(x)=\|\mathcal{U}^\varepsilon;\mathcal{H}^\varepsilon\|^{-1}\mathcal{U}_p^\varepsilon(x) \tag{4.14}$$

where

$$\mathcal{U}_p^\varepsilon(x)=\mathfrak{V}(\gamma_p;x)+\chi(x)\gamma_p(s)w_p(\varepsilon^{-1}n,\varepsilon^{-1}z;\zeta), \tag{4.15}$$

$\chi\in C_c^\infty(\Omega)$ is the cut-off function appearing in (3.1), $\{\beta_p,\gamma_p\}$ is an eigenpair of the operator J defined by (2.4), found in Proposition 2.1, the pair $(\mu_p(\zeta),w_p(\eta;\zeta))$ is a solution of the non-linear system (3.5), (3.6), (3.7), (3.9) depending on $\zeta=|\ln\varepsilon|^{-1}$, according to Proposition 3.1. We recall, as noted in Remark 3.1, that when $p\neq q$ and $\beta_p=\beta_q$, then $t_p=t_q$, but it may happen that $u_p\neq u_q$ (and linearly independent): below this, in particular, yields that the constant δ_p computed in Step 2 may change with p.

Step 1 We prove that

$$\|\mathcal{U}_p^\varepsilon;\mathcal{H}^\varepsilon\|^2\geq c_p|\ln\varepsilon|,\quad c_p>0. \tag{4.16}$$

We first proceed with the computation of the scalar products

$$\begin{aligned}&\langle\mathcal{U}_p^\varepsilon,\mathcal{U}_q^\varepsilon\rangle=(\nabla\mathcal{U}_p^\varepsilon,\nabla\mathcal{U}_q^\varepsilon)_{\Omega_\varepsilon}+\varepsilon^{-1}|\ln\varepsilon|^{-1}(\mathcal{U}_p^\varepsilon,\mathcal{U}_q^\varepsilon)_{\partial\Gamma_\varepsilon}= \\ &=-(\Delta_x\mathcal{U}_p^\varepsilon,\mathcal{U}_q^\varepsilon)_{\Omega_\varepsilon}+(\partial_\nu\mathcal{U}_p^\varepsilon,\mathcal{U}_q^\varepsilon)_{\partial\Gamma_\varepsilon}+\varepsilon^{-1}|\ln\varepsilon|^{-1}(\mathcal{U}_p^\varepsilon,\mathcal{U}_q^\varepsilon)_{\partial\Gamma_\varepsilon}\end{aligned}$$

of further use in the cases $p=q$ and $p\neq q$. Since $\mathcal{U}_p^\varepsilon$ is defined in Ω_ε and $\mathfrak{V}(\gamma_p;x)$ and $w_p(\eta;\zeta)$ are harmonic in $x\in\Omega\setminus\Gamma$ and in $\mathbb{R}^2\setminus\overline{\omega}$, respectively, we use formulas (2.3) for $\mathfrak{V}(\gamma_p;x)$, (3.11) for $w_p(\eta;\zeta)$, the notation $r=(n^2+z^2)^{1/2}$ for the distance in $\mathbb{R}^3$ of a point x from the set Γ, and formula (3.3) for the Laplacian Δ_x to derive that

$$\begin{aligned}&\Delta_x\mathcal{U}_p^\varepsilon(x)=\left(\frac{\varkappa(s)}{1+n\varkappa(s)}\frac{\partial}{\partial n}+\frac{\partial}{\partial s}\frac{1}{1+n\varkappa(s)}\frac{\partial}{\partial s}+\frac{\partial^2}{\partial z^2}\right)\chi(x)w_p(\eta;\zeta), \\ &|\Delta_x\mathcal{U}_p^\varepsilon(x)|\leq c_p\left(\varepsilon^{-1}(1+\varepsilon^{-1}r)^{-2}+(1+\varepsilon^{-1}r)^{-1}\right),\quad|\mathcal{U}_q^\varepsilon(x)|\leq c_q(1+|\ln r|),\end{aligned} \tag{4.17}$$

Hence,

$$|(\Delta_x \mathcal{U}_p^\varepsilon, \mathcal{U}_q^\varepsilon)_{\Omega_\varepsilon}| \le c_{pq} \int_{B_d\setminus\omega_\varepsilon} \left(\frac{1}{\varepsilon}\left(1+\frac{r}{\varepsilon}\right)^{-2} + \left(1+\frac{r}{\varepsilon}\right)^{-1}\right)(1+|\ln r|)\,dy \le$$
$$\le c_{pq}\varepsilon|\ln\varepsilon|^2.$$

Furthermore, recalling formula (2.10) for γ_k and (2.1) for $\mathfrak{V}(\gamma_k;x)$, and setting $d\Sigma$ for the standard 2-dimensional measure on the smooth surface $\partial\Gamma^\varepsilon$, we have

$$\int_{\partial\Gamma_\varepsilon} \mathcal{U}_p^\varepsilon(x)\mathcal{U}_q^\varepsilon(x)\,d\Sigma(x) = \int_\Gamma (1+n\varkappa(s))\int_{\partial\omega_\varepsilon} \mathcal{U}_p^\varepsilon(x)\mathcal{U}_q^\varepsilon(x)\,d\sigma(n,z)\,d\sigma(s) =$$
$$= (1+O(\varepsilon))\int_\Gamma \gamma_p(s)\gamma_q(s)\,d\sigma(s)\int_{\partial\omega_\varepsilon}\left(-\frac{\ln\varepsilon}{2\pi}+O(1)\right)^2 d\sigma(n,z) =$$
$$= \frac{1}{4\pi^2}|\ln\varepsilon|^2\varepsilon|\partial\omega|\delta_{pq} + O(\varepsilon|\ln\varepsilon|).$$

In a similar way, taking into account the boundary condition (3.6) which in view of (3.7) reads as

$$\partial_{\nu(\eta)} w_p(\eta;\zeta) = \frac{1}{2\pi}\partial_{\nu(\eta)}\ln\rho + \frac{1}{2\pi}\mu_p(\zeta) + O(|\ln\varepsilon|^{-1}), \quad \eta\in\partial\omega,$$

as well as the representation (2.3), we have

$$\int_{\partial\Gamma_\varepsilon} \mathcal{U}_p^\varepsilon(x)\partial_\nu \mathcal{U}_q^\varepsilon(x)\,d\Sigma(x) =$$
$$= \int_\Gamma (1+n\varkappa(s))\int_{\partial\omega_\varepsilon} \mathcal{U}_q^\varepsilon(x)(\partial_\nu\mathfrak{V}(\gamma_p;x)+\gamma_p(s)\partial_\nu w_p(\eta;\zeta))\,d\sigma(n,z)\,d\sigma(s) =$$
$$= (1+O(\varepsilon))\int_\Gamma \gamma_p(s)\gamma_q(s)\,d\sigma(s)\times$$
$$\times\int_{\partial\omega_\varepsilon}\left(-\frac{\ln\varepsilon}{2\pi}+O(1)\right)\left(-\partial_\nu\frac{\ln r}{2\pi}+\frac{1}{2\pi}\partial_\nu\frac{\ln r}{\varepsilon}+\frac{1}{2\pi}\mu_p(\zeta)+O(\frac{1}{|\ln\varepsilon|})\right)$$
$$\times\, d\sigma(n,z) =$$
$$= \frac{1}{4\pi^2}|\partial\omega|\mu_p(\zeta)|\ln\varepsilon|\delta_{pq} + O(1).$$

Thus,

$$\left|\langle \mathcal{U}_p^\varepsilon, \mathcal{U}_q^\varepsilon\rangle - \frac{|\partial\omega|}{4\pi^2}(1+\mu_p(\zeta))|\ln\varepsilon|\delta_{pq}\right| \le c_{pq} \tag{4.18}$$

and, in particular (3.12) in Proposition 3.1 yields the relation (4.16).

Step 2 Let us now evaluate the quantity δ_p obtained from (4.12) and (4.14); namely, we will prove that

$$\delta_p \le c_p\varepsilon^2|\ln\varepsilon|^2. \tag{4.19}$$

We do not indicate the dependence of δ_p on ε explicitly. By one of the definitions of Hilbert norm, we write

$$\delta_p = \sup|\langle \mathcal{I}^\varepsilon \boldsymbol{u}_p^\varepsilon - \boldsymbol{t}_p^\varepsilon \boldsymbol{u}_p^\varepsilon, v\rangle| \tag{4.20}$$

where the supremum is computed over all $v\in\mathcal{H}^\varepsilon$ such that

$$\|v;\mathcal{H}^\varepsilon\| = 1. \tag{4.21}$$

Formulas (4.7), (4.8), and (4.14) provide the relation

$$\delta_p = \boldsymbol{t}_p^\varepsilon\|\mathcal{U}_p^\varepsilon;\mathcal{H}^\varepsilon\|^{-1}\sup|(\Delta_x\mathcal{U}_p^\varepsilon, v)_{\Omega_\varepsilon} - (\partial_\nu\mathcal{U}_p^\varepsilon - \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_p(\zeta)\mathcal{U}_p^\varepsilon, v)_{\partial\Gamma_\varepsilon}|. \tag{4.22}$$

Using (4.17) for $\Delta_x\mathcal{U}_p^\varepsilon$ and (4.21), (4.1) for v, we have

$$|(\Delta_x\mathcal{U}_p^\varepsilon, v)_{\Omega_\varepsilon}| \le c\left(\int_{\Omega_\varepsilon} r^2(1+|\ln r|)^2\left(\frac{1}{\varepsilon^2}\left(1+\frac{r}{\varepsilon}\right)^{-4}+\left(1+\frac{r}{\varepsilon}\right)^{-2}\right)dx\right)^{1/2}\times$$
$$\times\|r^{-1}(1+|\ln r|)^{-1}v;L^2(\Omega_\varepsilon)\| \le c\varepsilon^2|\ln\varepsilon|^2\|v;\mathcal{H}^\varepsilon\| = c\varepsilon^2|\ln\varepsilon|^2,$$

where $r = r(x) = dist(x,\Gamma)$. Moreover, the boundary condition (3.6) and the decomposition (2.3) ensure that, for $x\in\partial\Gamma_\varepsilon$,

$$\partial_\nu\mathcal{U}_p^\varepsilon(x) - \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_p(\zeta)\mathcal{U}_p^\varepsilon(x) = O(|\ln\varepsilon|),$$

and

$$|(\partial_\nu\mathcal{U}_p^\varepsilon - \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_p(\zeta)\mathcal{U}_p^\varepsilon, v)_{\partial\Gamma_\varepsilon}| \le c|\partial\omega_\varepsilon|^{1/2}|\ln\varepsilon|\|v;L^2(\partial\Gamma_\varepsilon)\| \le$$
$$c\varepsilon^{1/2}|\ln\varepsilon|\|v;\mathcal{H}^\varepsilon\|\varepsilon^{1/2}|\ln\varepsilon| = c\varepsilon|\ln\varepsilon|^2.$$

Inserting the derived estimates together with (4.16) and (4.14) into formula (4.22) yields the relation

$$\delta_p \le \varepsilon|\ln\varepsilon|^{-1}(\varepsilon^2|\ln\varepsilon|^2+\varepsilon|\ln\varepsilon|^2) \le c_p\varepsilon^2|\ln\varepsilon|^2, \tag{4.23}$$

which completes the proof of (4.19).

Step 3 We can now deduce that there exists λ_k^ε (note that k may depend on p and ε, i.e., $\lambda_k^\varepsilon = \lambda_{p_\varepsilon}^\varepsilon$, but we use here a simpler notation) such that

$$|\lambda_k^\varepsilon - \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_p(\zeta)| \le C_p \quad \text{for } \varepsilon\in(0,\varepsilon_p]. \tag{4.24}$$

In fact, according to Lemma 4.3, (4.19) guarantees the existence of at least one eigenvalue τ_k^ε of the operator $\mathcal{I}^\varepsilon$ such that

$$\left|\tau_k^\varepsilon - \varepsilon|\ln\varepsilon|(1+\mu_p(\zeta))^{-1}\right| \le c_p\varepsilon^2|\ln\varepsilon|^2 \tag{4.25}$$

and, therefore, (4.24) occurs. The last inference is a direct consequence of the relationship (4.10) between the spectral parameters and the simple calculation

$$\left|\frac{1}{A}-\frac{1}{B}\right| \le \epsilon \Rightarrow A \le \frac{B}{1-\epsilon B} \le 2B \quad \text{and } |A-B| \le 2\epsilon B^2 \quad \text{if } \epsilon B \le \tfrac{1}{2}. \tag{4.26}$$

Moreover, since (3.12) in Proposition 3.1 yields the bound

$$0 \le \mu_p(\zeta) \le \mu_p^0 \quad \text{for } \zeta\in[0,\zeta_p],$$

setting $\epsilon = c_p\varepsilon^2|\ln\varepsilon|^2$ in (4.25), we find that C_p and $\varepsilon_p\in(0,e^{-1/\zeta_p}]$ in (4.24) must verify

$$C_p = 2c_p(1+\mu_p^0)^2, \quad \varepsilon_p|\ln\varepsilon_p| \le (2c_p)^{-1}.$$

Step 4 We prove that if β_n is an eigenvalue of integral operator J defined by (2.4), with multiplicity κ, i.e.

$$\beta_{n-1} < \beta_n = \cdots = \beta_{n+\kappa-1} < \beta_{n+\kappa.} \tag{4.27}$$

and $\delta_+ \ge \delta_p$ for all $p = n,\dots,n+\kappa-1$, where each δ_p is given by (4.20), then there are at least κ eigenvalues τ_j^ε of the operator $\mathcal{I}^\varepsilon$ in the interval $[t_n^\varepsilon-\delta_+, t_n^\varepsilon-\delta_+]$. In fact, for such β_n (3.12) and (4.9) yield the equalities

$$\mu_n(\zeta) = \cdots = \mu_{n+\kappa-1}(\zeta),$$

and

$$\boldsymbol{t}_n^\varepsilon = \cdots = \boldsymbol{t}_{n+\kappa-1}^\varepsilon.$$

In order to prove that formula (4.14) now gives approximations $\{t_p^\varepsilon, u_p^\varepsilon\}$, $p = n,\dots,n+\kappa-1$, to κ *different* eigenpairs of the original problem (1.4)–(1.6), we employ the second part of Lemma 4.3. We introduce a big parameter $\theta > 1$, to be specified later, then set

$$\delta_+ = \theta\max\{\delta_n,\dots,\delta_{n+\kappa-1}\},$$

and consider *all* the eigenvalues $\tau_{N^\varepsilon}^\varepsilon,\dots,\tau_{N^\varepsilon+X^\varepsilon-1}^\varepsilon$ of $\mathcal{I}^\varepsilon$ that fall into the interval $[t_n^\varepsilon-\delta_+, t_n^\varepsilon-\delta_+]$. From (4.19) $\delta_+ \le \tilde{c}_n\theta\varepsilon^2|\ln\varepsilon|^2$ with

$$\tilde{c}_n = \max\{c_n,\dots,c_{n+\kappa-1}\},$$

where each c_p is the constant in (4.25), and hence

$$\tau_{N^\varepsilon}^\varepsilon,\dots,\tau_{N^\varepsilon+X^\varepsilon-1}^\varepsilon \in \Upsilon_n^\varepsilon := [t_n^\varepsilon-\tilde{c}_n\theta\varepsilon^2|\ln\varepsilon|^2, t_n^\varepsilon+\tilde{c}_n\theta\varepsilon^2|\ln\varepsilon|^2] \tag{4.28}$$

Now, by (4.13) in Lemma 4.3, there exist κ unit vectors

$$a^{\varepsilon p} = (a_{N^\varepsilon}^{\varepsilon p},\dots,a_{N^\varepsilon+X^\varepsilon-1}^{\varepsilon p}) \in \mathbb{R}^{X^\varepsilon}, \quad p = n,\dots,n+\kappa-1, \tag{4.29}$$

such that, setting

$$S_p^\varepsilon = \sum_{j=N^\varepsilon}^{N^\varepsilon+X^\varepsilon-1} a_j^{\varepsilon p}U_j^\varepsilon,$$

for u_p^ε, $p = n,\dots,n+\kappa-1$ the following estimate holds true:

$$\|u_p^\varepsilon - S_p^\varepsilon;\mathcal{H}^\varepsilon\| \le 2\frac{\delta_p}{\delta_+} \le \frac{2}{\theta}.$$

Moreover, thanks to (4.11) we have

$$a^{\varepsilon q}\cdot a^{\varepsilon p}=\sum_{j=N^\varepsilon}^{N^\varepsilon+X^\varepsilon-1} a_j^{\varepsilon q}a_j^{\varepsilon p}=\langle S_p^\varepsilon,S_q^\varepsilon\rangle=$$
$$=\langle S_p^\varepsilon-u_p^\varepsilon,S_q^\varepsilon\rangle+\langle u_p^\varepsilon,S_q^\varepsilon-u_q^\varepsilon\rangle+\langle u_p^\varepsilon,u_q^\varepsilon\rangle.$$

In view of (4.29) and (4.18), (4.16), we observe that

$$|a^{\varepsilon q}\cdot a^{\varepsilon p}-\delta_{pq}|\le\frac{2}{\theta}+\frac{2}{\theta}+\frac{c_{pq}}{|\ln\varepsilon|}.$$

Hence, for a small ε and a big θ, the columns (4.29) are "almost orthonormalized" that may happen only in the case $\kappa\le X_n^\varepsilon$. In other words, the interval γ_n^ε in (4.28) contains at least κ eigenvalues of the operator $\mathcal{I}^\varepsilon$, that is,

$$|\tau_j^\varepsilon-\varepsilon|\ln\varepsilon|(1+\mu_n(\zeta))^{-1}|\le\tilde c_n\theta\varepsilon^2|\ln\varepsilon|^2,\quad j=N^\varepsilon,\dots,N^\varepsilon+\kappa-1.\ (4.30)$$

Using again the calculations in (4.26), from (4.30) we can derive proximity estimates for at least κ eigenvalues λ_j^ε of the original problem. However, the exact statement (3.16), (3.17) in Theorem 3.1 is not verified yet.

4.4. Convergence results

Based on the considerations in Section 4.3, we are not able to make the conclusion (3.17) on the eigenvalue indexes in (3.16). In this section we will perform the most technical part of our work, to ensure (3.17) and, therefore, to conclude with Theorem 3.1.

Let $\{\lambda_p^\varepsilon,u_p^\varepsilon\}$ be an eigenpair of the problem (1.4)–(1.6). In Section 4.3 we have verified that, for any entry β_k of the eigenvalue sequence (2.9), there exists its own eigenvalue $\lambda_{N(k)}^\varepsilon$ with the bound

$$\lambda_{N(k)}^\varepsilon\le c_k\varepsilon^{-1}|\ln\varepsilon|^{-1}.$$

This means that

$$\lambda_p^\varepsilon\le\lambda_{N(p)}^\varepsilon\le c_p\varepsilon^{-1}|\ln\varepsilon|^{-1},$$

and, hence, the integral identity (1.7) and the normalization condition (1.9) show that

$$\|\nabla_x u_p^\varepsilon;L^2(\Omega_\varepsilon)\|^2=\lambda_p^\varepsilon\|u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|^2\le c_p\varepsilon^{-1}|\ln\varepsilon|^{-1}.\tag{4.31}$$

The mean-value function $\hat u_p^\varepsilon$ defined by (4.5) belongs to $C^\infty(\Gamma)$, because the eigenfunction u_p^ε is smooth in $\overline{\Omega_\varepsilon}$. Moreover,

$$\|\hat u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|^2\le\|u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|^2=1,\tag{4.32}$$

hence

$$\|\hat u_p^\varepsilon;L^2(\Gamma)\|^2\le c|\partial\omega_\varepsilon|^{-1}\|\hat u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|^2\le c\varepsilon^{-1}.$$

By Lemma 4.2,

$$\|u_p^\varepsilon-\hat u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|^2\le c\varepsilon\|\nabla_x u_p^\varepsilon;L^2(\Omega_\varepsilon)\|^2\le c|\ln\varepsilon|^{-1}\tag{4.33}$$

so that

$$\|u_p^\varepsilon-\hat u_p^\varepsilon;L^2(\partial\Gamma_\varepsilon)\|\to 0.$$

Let us set

$$\hat\gamma_p^\varepsilon=\varepsilon^{1/2}\hat u_p^\varepsilon.\tag{4.34}$$

Thus, from (4.32) we can pass to the limit along an infinitesimal positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and get

$$\hat\gamma_p^\varepsilon=\varepsilon^{1/2}\hat u_p^\varepsilon\rightharpoonup\hat\gamma_p,\quad\text{weakly in }L^2(\Gamma).\tag{4.35}$$

Let us recall an information on λ_p^ε. In view of the asymptotic formulas (4.24) and (3.12) we have

$$\lambda_p^\varepsilon-\varepsilon^{-1}|\ln\varepsilon|^{-1}\frac{2\pi}{|\partial\omega|}\le\lambda_{N(p)}^\varepsilon-\varepsilon^{-1}|\ln\varepsilon|^{-1}\frac{2\pi}{|\partial\omega|}\le c_p\varepsilon^{-1}|\ln\varepsilon|^{-2}$$

and, hence, we can pass to the limit along an infinitesimal sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and get

$$B_p^\varepsilon:=\varepsilon|\ln\varepsilon|^2(\lambda_p^\varepsilon-\varepsilon^{-1}|\ln\varepsilon|^{-1}\frac{2\pi}{|\partial\omega|})\to B_p\in\mathbb{R}.\tag{4.36}$$

Taking this information into account, and the asymptotic formula (3.12) again, we are going to prove that the value

$$\hat\beta_p=:-\frac{1}{|\partial\omega|}\left(\frac{|\partial\omega|^2}{4\pi^2}B_p+L(\omega)-\frac{l(\omega)}{2\pi}\right)\tag{4.37}$$

is an eigenvalue of the operator J. To this end, we fix some density $\kappa\in C^\infty(\Gamma)$ and insert

$$v^\varepsilon(x)=|\ln\varepsilon|\sqrt{\varepsilon}\mathfrak{V}(\kappa;x)+|\ln\varepsilon|\sqrt{\varepsilon}\chi(x)\kappa(s)W_0(\varepsilon^{-1}n,\varepsilon^{-1}z)\tag{4.38}$$

into the integral identity (1.7) as a test function. In (4.38), $\mathfrak{V}(\kappa;x)$ is the singular solution (2.1) with $\gamma=\kappa$ and W_0 is a harmonics in $\mathbb{R}^2\setminus\omega$ introduced in Proposition 2.3. We obtain

$$(u_p^\varepsilon,\Delta_x v^\varepsilon)_{\Omega_\varepsilon}=(u_p^\varepsilon,\partial_\nu v^\varepsilon-\lambda_p^\varepsilon v^\varepsilon)_{\partial\Gamma_\varepsilon}.\tag{4.39}$$

Similarly to the calculation (4.17), we derive that

$$|\Delta_x v^\varepsilon(x)|\le c_\varepsilon|\ln\varepsilon|\sqrt{\varepsilon}\left(\varepsilon^{-1}(1+\varepsilon^{-1}r)^{-2}+(1+\varepsilon^{-1}r)^{-1}\right)$$

and, making use of the relations (4.1) and (4.31), we can estimate the modulus of the left-hand side of (4.39) with

$$c|\ln\varepsilon|\sqrt{\varepsilon}\left(\int_{\Omega_\varepsilon}\left(\frac{1}{\varepsilon}\left(1+\frac{r}{\varepsilon}\right)^{-2}+\left(1+\frac{r}{\varepsilon}\right)^{-1}\right)^2r^2(1+|\ln r|^2)^2\,dx\right)^{1/2}\times$$
$$\times\|r^{-1}(1+|\ln r|)^{-1}u_p^\varepsilon;L^2(\Omega_\varepsilon)\|\le c|\ln\varepsilon|\sqrt{\varepsilon}(\varepsilon^2)^{1/2}\varepsilon^{-1/2}|\ln\varepsilon|^{-1}=C\varepsilon.$$

In this way, we conclude that $(u_p^\varepsilon,\Delta_x v^\varepsilon)_{\Omega_\varepsilon}$ vanishes when $\varepsilon\to 0^+$. Let us compute the limit of the right-hand side of (4.39). According to (2.3) and (2.15), we write on $\partial\Gamma_\varepsilon$

$$\partial_\nu v^\varepsilon(x)-\lambda_p^\varepsilon v^\varepsilon(x)=|\ln\varepsilon|\varepsilon^{-1/2}\left(\frac{1}{2\pi}\frac{\partial}{\partial\nu}\ln\frac{1}{\rho}+\frac{\partial}{\partial\nu}W_0(\eta)\right)\kappa(s)+O(1+|\ln\varepsilon|)+$$
$$-\varepsilon^{-1/2}\left(\frac{2\pi}{|\partial\omega|}+\frac{1}{|\ln\varepsilon|}B_p^\varepsilon\right)\left(\left(\frac{1}{2\pi}|\ln\varepsilon|-\frac{1}{2\pi}\ln\rho+W_0(\eta)\right)\kappa(s)\right.$$
$$\left.+\ J(\kappa;s)+O(1+|\ln\varepsilon|)\right)=$$
$$=-\varepsilon^{-1/2}\left(\frac{1}{2\pi}B_p^\varepsilon-\frac{2\pi}{|\partial\omega|}(\frac{2\pi}{\ln}\rho-W_0(\eta))\right)\kappa(s)+\frac{2\pi}{|\partial\omega|}J(\kappa;s)+O(|\ln\varepsilon|^{-1})$$

Note that $\frac{1}{2\pi}\frac{\partial}{\partial\nu}\ln\frac{1}{\rho}+\frac{\partial}{\partial\nu}W_0(\eta)=\frac{1}{|\partial\omega|}$, see (2.15), and this is cancelled by $-\frac{2\pi}{|\partial\omega}\frac{1}{2\pi}|\ln\varepsilon|$ in the second summand. Neglecting all infinitesimal terms, we rewrite the right-hand side of (4.39) as follows:

$$-\varepsilon^{-1/2}\int_\Gamma(1+n\kappa(s))\int_{\partial\omega_\varepsilon}\left(\frac{B_p^\varepsilon}{2\pi}-\frac{2\pi}{|\partial\omega|}\left(\frac{\ln\rho}{2\pi}-W_0(\eta)\right)\right.$$
$$\left.\times\ \kappa(s)+\frac{2\pi}{|\partial\omega|}J(\kappa;s)+\cdots\right)u_p^\varepsilon(x)\,d\sigma(n,z)\,d\sigma(s).\tag{4.40}$$

The next step in our calculation is to apply formulas (4.5), (4.6), and (4.32), (4.33), in order to replace $u_p^\varepsilon(x)$ with $\hat u_p^\varepsilon$ in (4.40). Indeed, writing $u_p^\varepsilon=\hat u_p^\varepsilon+u_{p\perp}^\varepsilon$ and observing that

$$\left|\varepsilon^{-1/2}\int_\Gamma(1+n\varkappa(s))\int_{\partial\omega_\varepsilon}F_p^\varepsilon(x)u_{p\perp}^\varepsilon(x)\,d\sigma(n,z)\,d\sigma(s)\right|$$
$$\le c\varepsilon^{-1/2}|\partial\omega_\varepsilon|^{1/2}\|u_{p\perp}^\varepsilon;L^2(\partial\omega_\varepsilon)\|\le c|\ln\varepsilon|^{-1/2}$$

where $F_p^\varepsilon(x)$ is a multiplier in the integrand, we pass to the limit in (4.40) by means of the convergence (4.36), (4.35) and formula (2.15). As a result, we obtain

$$0=\int_\Gamma\left(\frac{|\partial\omega|}{2\pi}B_p\kappa(s)-\frac{2\pi}{|\partial\omega|}\left(\frac{1}{2\pi}l(\omega)-L(\omega)\right)\kappa(s)+2\pi J(\kappa;s)\right)\hat\gamma_p(s)\,d\sigma(s)$$

so that, in view of definition (4.37), we derive the integral identity

$$\int_\Gamma(J(\kappa;s)-\hat\beta_p\kappa(s))\hat\gamma_p(s)\,ds=0$$

with $\hat{\gamma}_p \in L^2(\Gamma)$ and any $\kappa \in C^\infty(\Gamma)$. Since J is a hypo-elliptic self-adjoint operator, we conclude that $\hat{\gamma}_p \in C^\infty(\Gamma)$ and

$$J(\hat{\gamma}_p; s) = \hat{\beta}_p \hat{\gamma}_p(s), \quad \forall s \in \Gamma.$$

In this way, if the convergence (4.35) is strong in $L^2(\Gamma)$, formulas (4.35), (4.5) and (1.9) ensure that

$$\|\hat{\gamma}_p; L^2(\Gamma)\| = |\partial\omega|^{-1/2}, \tag{4.41}$$

and, therefore, $\{\hat{\beta}_p, \hat{\gamma}_p\}$ is an eigenpair of the operator J.

Proposition 4.1. *Let $\hat{\gamma}_p^\varepsilon$ be defined by (4.34). Then there exist $\gamma_p^\varepsilon, \tilde{\gamma}_p^\varepsilon$ such that*

$$\hat{\gamma}_p^\varepsilon = \gamma_p^\varepsilon + \tilde{\gamma}_p^\varepsilon \tag{4.42}$$

with

$$\|\gamma_p^\varepsilon; H_{\ln}(\Gamma)\| \le c_p, \quad \|\tilde{\gamma}_p^\varepsilon; L^2(\Gamma)\| \le c_p |\ln\varepsilon|^{-1}. \tag{4.43}$$

Moreover, the convergence (4.35) along an infinitesimal subsequence $\{\hat{\varepsilon}_n\}_{n\in\mathbb{N}}$ is strong in $L^2(\Gamma)$ and the limit $\hat{\gamma}_p$ satisfies the relation (4.41).

Proof. We consider (1.4)–(1.6) as a problem where (1.5) is the Neumann condition with the (fixed) right-hand side $\lambda_p^\varepsilon u_p^\varepsilon(x)$. For $x \in \Omega_\varepsilon$, we write

$$u_p^\varepsilon(x) = \lambda_p^\varepsilon \int_{\partial\Gamma_\varepsilon} \mathfrak{G}^\varepsilon(x, \mathfrak{x}) u_p^\varepsilon(\mathfrak{x})\, d\sigma(\mathfrak{x}), \tag{4.44}$$

while $\mathfrak{G}^\varepsilon(x, \mathfrak{z})$ is the Poisson (resolvent) kernel, namely, the distributional solution of the mixed boundary-value problem

$$-\Delta_x \mathfrak{G}^\varepsilon(x, \mathfrak{x}) = 0, \quad x \in \Omega_\varepsilon, \tag{4.45}$$

$$\mathfrak{G}^\varepsilon(x, \mathfrak{x}) = 0, \quad x \in \partial\Omega, \ \partial_\nu \mathfrak{G}^\varepsilon(x, \mathfrak{x}) = \delta(x - \mathfrak{x}), \ x \in \partial\Gamma_\varepsilon. \tag{4.46}$$

where $\mathfrak{x} \in \partial\Gamma_\varepsilon$. Since the boundary $\partial\Gamma_\varepsilon$ is smooth, the Poisson kernel admits the representation (note that the first summand below is just the fundamental solution multiplied by 2)

$$\mathfrak{G}^\varepsilon(x, \mathfrak{x}) = \frac{1}{2\pi}|x - \mathfrak{x}|^{-1} + \mathfrak{G}_0^\varepsilon(x, \mathfrak{x}),$$

and the regular part $\mathfrak{G}_0^\varepsilon(x, \mathfrak{x})$ is bounded in $\overline{\Omega_\varepsilon}$, however the bound may depend on the small parameter ε because the domain Ω_ε is singularly perturbed. To highlight useful properties of $\mathfrak{G}^\varepsilon$, we construct the asymptotic representation as $\varepsilon \to 0^+$. To this end, we fix a point $\mathfrak{x} \in \partial\Gamma_\varepsilon$ with local coordinates $(\mathfrak{s}, \varepsilon\mathfrak{y}) \in \Gamma \times \partial\omega_\varepsilon$ and make the coordinates dilation

$$x \mapsto \xi = (\eta, \zeta) = (\varepsilon^{-1}n, \varepsilon^{-1}z, \varepsilon^{-1}(s - \mathfrak{s})).$$

In view of formula (3.3) for the Laplacian, we arrive at the limit problem

$$-\Delta_\xi \mathfrak{V}(\xi, \mathfrak{y}) = 0, \xi \in \mathbb{R}^3 \setminus \overline{Q}, \quad \partial_{\nu(\xi)} \mathfrak{V}(\xi, \mathfrak{y}) = \delta(\eta - \mathfrak{y})\delta(\zeta), \ \xi \in \partial Q. \tag{4.47}$$

In the 3-dimensional space with the cylindrical tunnel $Q = \omega \times \mathbb{R}$. Applying general results[1] we deduce the existence of the solution $\mathfrak{V}(\xi, \mathfrak{y})$ obeying the following asymptotic formulas:

$$|\mathfrak{V}(\xi, \mathfrak{y}) - (2\pi)^{-1}(|\eta - \mathfrak{y}|^2 + \zeta^2)^{-1/2}| + |\nabla_x \mathfrak{V}(\xi, \mathfrak{y}) - (2\pi)^{-1}\nabla_\xi(|\eta - \mathfrak{y}|^2 + \zeta^2)^{-1/2}| \le c_R, \quad |\xi| \le R,$$

$$|\mathfrak{V}(\xi, \mathfrak{y}) - (24\pi|\xi|)^{-1}| + |\nabla_x \mathfrak{V}(\xi, \mathfrak{y}) - (4\pi)^{-1}\nabla_\xi|\xi|^{-1}| \le c_R|\xi|^{-2}, \quad |\xi| \ge R. \tag{4.48}$$

[1] The transformation $\xi \mapsto |\xi|^{-2}\xi$ maps $\mathbb{R}^3 \setminus \overline{Q}$ into a bounded domain with two irregular points, the exterior of the three-dimensional cusp (see Fig. 4). Such irregularities of the boundary have been studied in[28,29]; see also Ref. 30, Ch. 9.

Note that the solution of (4.47) gets similar, but distinct, behaviour as $\xi \to (\mathfrak{y}, 0)$ and $\xi \to +\infty$. In particular, at infinity, it is the fundamental solution of the Laplacian in the whole space $\mathbb{R}^3$ perturbed by terms of higher-order decay rate and a boundary layer near the tunnel Q that do not influence the estimate (4.48).

We set

$$\mathcal{G}^\varepsilon(x, \mathrm{x}) = X_d(x, \mathrm{x})\varepsilon^{-1}\mathrm{w}(\varepsilon^{-1}n, \varepsilon^{-1}z, \varepsilon^{-1}(s - \mathrm{s}), \mathrm{y}) + \widetilde{\mathcal{G}}^\varepsilon(x, \mathrm{x}) \tag{4.49}$$

where $X_d(x, \mathrm{x}) = \chi_d(|s - \mathrm{s}|)\chi_d(|y|)$ is a smooth cut-off function, $\chi_d(t) = 1$ for $t < 1/2$ and $\chi_d(t) = 0$ for $t > d$, while $d > 0$ is fixed such that $X_d(x, \mathrm{x}) = 1$ for $x \in \Gamma_\varepsilon$ and $X_d(x, \mathrm{x}) = 0$ for $x \in \partial\Omega$. In view of (4.47) the first term in the right-hand side of (4.49) fulfils the boundary conditions (4.46). Moreover, recalling the representation (2.16) again, we see that the discrepancy left by this term in the differential equation (4.45) is continuously differentiable uniformly in ε. Thus, according to Ref. 19, Ch. 12 and 13, Ref. 18, Sections 5 and 7, we obtain the estimate

$$|\widetilde{\mathcal{G}}^\varepsilon(x, \mathrm{x})| \le c \tag{4.50}$$

for the solution $\widetilde{\mathcal{G}}^\varepsilon$ of our problem in Ω that compensates for that discrepancy we observe that

$$\varepsilon^{-1}\left|\left(|\varepsilon^{-1}y - \mathrm{y}|^2 + \varepsilon^{-2}(s - \mathrm{s}_1)^2\right)^{-1/2} - \left(|\varepsilon^{-1}y - \mathrm{y}|^2 + \varepsilon^{-2}(s - \mathrm{s}_2)^2\right)^{-1/2}\right| =$$
$$= \left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_1)^2\right)^{-1/2}\left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_2)^2\right)^{-1/2} \times$$
$$\times \frac{|(s - \mathrm{s}_1)^2 - (s - \mathrm{s}_2)^2|}{\left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_1)^2\right)^{1/2} + \left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_2)^2\right)^{1/2}} \le$$
$$\le \frac{c|\mathrm{s}_1 - \mathrm{s}_2|}{\left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_1)^2\right)^{1/2}\left(|y - \varepsilon\mathrm{y}|^2 + (s - \mathrm{s}_2)^2\right)^{1/2}}.$$

Hence, for the mean-value function

$$\widetilde{\mathcal{G}}_{\mathrm{irr}}^\varepsilon = \frac{1}{|\partial\omega_\varepsilon|}\int_{\partial\omega_\varepsilon} \mathcal{G}_{\mathrm{irr}}^\varepsilon(x, \tilde{x})\, d\sigma(n, z)$$

we obtain the estimate

$$\left|\widetilde{\mathcal{G}}_{\mathrm{irr}}^\varepsilon(s_1, \mathrm{x}) - \widetilde{\mathcal{G}}_{\mathrm{irr}}^\varepsilon(s_2, \mathrm{x})\right| \le \frac{c|s_1 - s_2|}{(|s_1 - \mathrm{s}|^2 + \varepsilon^2)^{1/2}(|s_2 - \mathrm{s}|^2 + \varepsilon^2)^{1/2}}. \tag{4.51}$$

It should be mentioned that the summands ε^2 in the denominator in (4.51) result from integration in y, taking into account that, under our assumptions, the coordinate origin $\eta = 0$ lays inside the domain ω and, hence, $|y| > c\,\varepsilon$, $c > 0$. Using definitions (4.35) and (4.5), we derive from (4.44) the representation

$$\hat{\gamma}_p^\varepsilon(s) = \varepsilon^{1/2}\frac{\lambda_p^\varepsilon}{|\partial\omega_\varepsilon|}\int_{\partial\omega_\varepsilon}\int_{\Gamma_\varepsilon} \mathcal{G}^\varepsilon(x, \mathrm{x})u_p^\varepsilon(\mathrm{x})\, d\sigma(\mathrm{x})\, d\sigma(n, z).$$

Replacing above $\mathcal{G}^\varepsilon$ with $\widetilde{\mathcal{G}}^\varepsilon$, which was defined in (4.50), we obtain the component $\tilde{\gamma}_p^\varepsilon$ of (4.42) together with the inequality

$$\|\tilde{\gamma}_p^\varepsilon; L^2(\Gamma)\|^2 \le c\varepsilon(\lambda_p^\varepsilon)^2|\partial\omega_\varepsilon|^{-2}\left(\int_{\partial\omega_\varepsilon}\int_{\partial\Gamma_\varepsilon}|u_p^\varepsilon(\mathrm{x})|\, d\sigma(\mathrm{x})\, d\sigma(n, z).\right)^2 \le$$
$$\le c\varepsilon\varepsilon^{-2}|\ln\varepsilon|^{-2}|\partial\omega_\varepsilon|^{-2}|\partial\omega_\varepsilon|^2|\partial\Gamma_\varepsilon|\|u_p^\varepsilon; L^2(\partial\Gamma_\varepsilon)\|^2 \le c|\ln\varepsilon|^{-1}.$$

The second estimate, (4.43) is checked up. To verify the first one, we write

$$\|\gamma_p^\varepsilon; H_{\ln}(\Gamma)\|^2 = \int_\Gamma\int_\Gamma \frac{|\gamma_p^\varepsilon(s_1) - \gamma_p^\varepsilon(s_2)|^2}{|s_1 - s_2|}\, d\sigma(s_1)\, d\sigma(s_2) \le$$
$$\le c\varepsilon|\lambda_p^\varepsilon|^2|\partial\omega_\varepsilon|^{-2}\int_\Gamma\int_\Gamma\left|\int_{\partial\omega_\varepsilon}\int_{\partial\Gamma_\varepsilon}\left(\mathcal{G}^\varepsilon(y, s_1, \mathrm{x}) - \mathcal{G}^\varepsilon(y, s_2, \mathrm{x})\right)u_p^\varepsilon(\mathrm{x})d\sigma_{\mathrm{x}}\, d\sigma(n, z)\right|^2$$
$$\times \frac{d\sigma(s_1)\, d\sigma(s_2)}{|s_1 - s_2|} \le$$
$$\le c\varepsilon\varepsilon^{-2}|\ln\varepsilon|^{-2}\int_\Gamma\int_\Gamma\left|\int_{\partial\Gamma_\varepsilon}\left(\hat{\mathcal{G}}_{\mathrm{irr}}^\varepsilon(s_1, \mathrm{x}) - \hat{\mathcal{G}}_{\mathrm{irr}}^\varepsilon(s_2, \mathrm{x})\right)u_p^\varepsilon(\mathrm{x})d\sigma(\mathrm{x})\right|^2\frac{d\sigma(s_1)\, d\sigma(s_2)}{|s_1 - s_2|} \le$$
$$\le c|\ln\varepsilon|^{-2}\int_\Gamma\int_\Gamma\int_\Gamma\frac{(|s_1 - \mathrm{s}| + |s_2 - \mathrm{s}|)\, d\sigma(s_1)\, d\sigma(s_2)}{((s_1 - \mathrm{s})^2 + \varepsilon^2)((s_2 - \mathrm{s})^2 + \varepsilon^2)} =$$

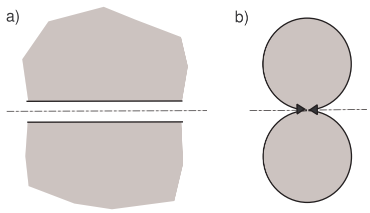

Fig. 4. Inversion of space with a tunnel (a): a bounded domain with two singular points ▲ and ▼ of the cusp exterior type. The rotation axis is dash line.

$$= 2c|\ln\varepsilon|^{-2}\int_\Gamma\int_\Gamma \frac{|s-\mathrm{s}|\,d\sigma(s)}{|s-\mathrm{s}|^2+\varepsilon^2}\int_\Gamma \frac{d\sigma(\mathbf{s})}{|\mathbf{s}-\mathrm{s}|^2+\varepsilon^2} \leq$$
$$c|\ln\varepsilon|^{-2}\int_\Gamma \frac{1+|\ln(\mathrm{s}^2+\varepsilon^2)|}{(|\mathrm{s}|^2+\varepsilon^2)^{1/2}}\,d\sigma(\mathrm{s}) \leq C. \quad \square$$

End of the proof of Theorem 3.1. We are now in the position to finish the proof of Theorem 3.1. Up to now, we have proved that the statement (3.16) holds true for some $p \geq n$. Let us show that it is false for $p > n$. In fact, assuming that $p > n$, we detect an eigenvalue $\lambda^\varepsilon_{p+\kappa}$ whose limit (4.36) defines an eigenvalue $\hat\beta_{p+\kappa}$ of the operator J such that

$$\hat\beta_{p+\kappa} \geq \beta_{p+\kappa-1}.$$

Moreover, the convergence (4.35), strong in $L^2(\Gamma)$ due to Proposition 4.1, defines the eigenfunction $\hat\gamma_{p+\kappa}$ which is normed by (4.41) but is orthogonal in $L^2(\Gamma)$ to $\gamma_1,\dots,\gamma_{p+\kappa-1}$. The latter contradicts our way to compose the eigenvalue sequence (2.9) and to satisfy the orthogonality condition (2.10).

4.5. Asymptotics of eigenfunctions and proof of Theorem 3.2

In order to prove Theorem 3.2 we apply again the second part of Lemma 4.3 but now, thanks to Theorem 3.1, we may take $\delta_* = c_*\varepsilon|\ln\varepsilon|$ and fix $c_* > 0$ such that the interval

$$[t^\varepsilon_n - c\varepsilon|\ln\varepsilon|, t^\varepsilon_n + c\varepsilon|\ln\varepsilon|]$$

contains only the eigenvalues $\tau^\varepsilon_n,\dots,\tau^\varepsilon_{n+\kappa-1}$ of the operator $\mathcal{T}^\varepsilon$. Then, the estimate (4.25) gives the bound $c_n\varepsilon|\ln\varepsilon|$ to the first inequality (4.13) and we complete the proof of Theorem 3.2.

5. Generalizations and variants

5.1. The spectral Steklov condition on the external boundary

Let us consider the Laplace equation (1.4) in Ω_ε with the Steklov condition

$$\partial_\nu u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \partial\Omega, \tag{5.1}$$

and either the Dirichlet or the Neumann condition on the boundary of the thin cavity Γ_ε, namely

$$u^\varepsilon(x) = 0, \quad x \in \partial\Gamma_\varepsilon, \tag{5.2}$$

$$\partial_\nu u^\varepsilon(x) = 0, \quad x \in \partial\Gamma_\varepsilon. \tag{5.3}$$

Asymptotic expansions for eigenpairs of these problems can be derived in the same, or even much simpler, way as in the paper,[18] where the Helmholtz equation

$$\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega_\varepsilon, \tag{5.4}$$

with the boundary conditions (1.6), (5.2) or (1.6), (5.3) was studied. However, the asymptotic procedures and formulas for the Dirichlet problem (5.4), (1.6), (5.2) and the mixed boundary-value problem (5.4), (1.6), (5.3) differ crucially from each other. If $\partial\Gamma_\varepsilon$ is supplied with the Neumann condition, then the asymptotic procedure becomes rather elementary because an eigenfunction u^0_p of the Dirichlet problem in Ω leaves a small discrepancy $O(\varepsilon)$ in (5.3), while a bounded (non necessarily decaying) solution of the exterior Neumann problem in $\mathbb{R}^2\setminus\overline{\omega}$ may play a role of the boundary layer. In this way, an infinite asymptotic series of powers of ε, with coefficients of polynomial type in $|\ln\varepsilon|$, are at hand for the eigenpairs in problem (5.4), (1.6), (5.3).

The Dirichlet condition on $\partial\Gamma_\varepsilon$ tangles the asymptotic procedure seriously and the paper[18] provides for eigenpairs of the problem (5.4), (1.6), (5.2) only series in powers of $\zeta = |\ln\varepsilon|^{-1}$ and it is not known if these series converge or not.

The same asymptotic results can be obtained for the problems (1.6), (5.1), (5.3), and (1.6), (5.1), (5.2), respectively, by repeating *ad litteram* calculations and argumentations in.[18] In other words, passing the spectral parameter λ^ε from the differential equation (5.4) to the Steklov condition (5.1) on $\partial\Omega$ does not trouble the asymptotic procedure.

5.2. The Steklov condition on $\partial\Omega_\varepsilon$

In the same way as in,[8] the spectral problem (1.13) gains two families of eigenvalues with stable asymptotics in the low and mid-frequency range of the spectrum. The first family consists of the eigenvalues

$$\lambda^\varepsilon_p = \lambda^0_p + O(\varepsilon|\ln\varepsilon|), \tag{5.5}$$

where the main term is taken from the eigenvalue sequence $\{\lambda^0_p\}_{p\in\mathbb{N}}$ of the Steklov problem in the entire domain Ω. Moreover, according to the relations $\partial_\nu - \lambda^\varepsilon_p = \varepsilon^{-1}(\partial_{\nu(\eta)} - \varepsilon\lambda^\varepsilon_p)$ and $\lambda^\varepsilon_p \leq c_p$, the Steklov condition (1.5) on $\partial\Gamma_\varepsilon$ must be regarded as a small perturbation of the Neumann condition (5.3) and, therefore, in view of observations made in,[8,18] infinite series of type (1.10), although with coefficients of polynomial type in $|\ln\varepsilon|$, are available for the eigenvalues (5.5), together with slightly modified error estimates (1.12).

The asymptotic expansions

$$\lambda^\varepsilon_{N^\varepsilon(k)} = \varepsilon^{-1}|\ln\varepsilon|^{-1}\mu_k(|\ln\varepsilon|^{-1}) + O(1), \tag{5.6}$$

of eigenvalues in the second family can be constructed and justified in the same way as in Sections 3 and 4. As a matter of fact, the Steklov condition (5.1) on $\partial\Omega$ with the spectral parameter (5.6) transforms into

$$u^\varepsilon_{N^\varepsilon(k)}(x) = \left(\lambda^\varepsilon_{N^\varepsilon(k)}\right)^{-1}\partial_\nu u^\varepsilon_{N^\varepsilon(k)}(x) = O(\varepsilon|\ln\varepsilon|)$$

and, therefore, can be regarded as a small, however irregular, cf.,[27] perturbation of the Dirichlet condition (1.6). In principle, after determining the main asymptotic terms like in Section 3, it is possible to construct infinite asymptotic series for the eigenvalues (5.6) and (3.2) of the problems (5.6) and (1.4)–(1.6), respectively. At the same time, even the main terms are quite complicated and we doubt whether it is worth to add further accessory but laborious computations.

It is still and open question if the spectrum of the problem (5.6) or (1.4)–(1.6) admits other families of eigenvalues with stable asymptotics.

5.3. The water-wave problem

Let Ω^-, Fig. 1a, be a domain in the lower half-space $\mathbb{R}^3_- = \{x = (y,z) : z < 0\}$ bounded by the union $\overline{\Sigma}\cup\partial\Omega^-$ of a smooth surface $\partial\Omega^- \subset \mathbb{R}^3_-$ and the planar one $\Sigma \subset \{x : z = 0\}$. Assuming that the curve Γ belongs to Σ, we introduce the thin set

$$\Gamma_\varepsilon = \{x \in \Sigma\cap V : s \in \Gamma, |n| < \varepsilon\} \tag{5.7}$$

and consider the spectral problem

$$\Delta_x u^\varepsilon_-(x) = 0,\ x \in \Omega^-, \quad \partial_z u^\varepsilon_-(x) = \lambda^\varepsilon u^\varepsilon_-(x),\ x \in \Gamma_\varepsilon, \tag{5.8}$$
$$\partial_\nu u^\varepsilon_-(x) = 0,\ x \in \partial\Omega^-\cup(\Sigma\setminus\overline{\Gamma_\varepsilon}),$$

Its spectrum is discrete and forms the eigenvalue sequence (1.8) where $\lambda_1^\varepsilon = 0$ and the corresponding eigenfunction is constant. Extending u_-^ε as an even function in the variable z, we obtain from (5.8) the spectral Steklov–Neumann problem

$$\Delta_x u_-^\varepsilon(x) = 0,\ x \in \Omega_\varepsilon = \Omega \setminus \overline{\Gamma_\varepsilon} = (\Omega^- \cup \Sigma \cup \Omega^+) \setminus \overline{\Gamma_\varepsilon}, \tag{5.9}$$

$$\pm \partial_z u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Gamma_\varepsilon^\pm,$$

$$\partial_\nu u^\varepsilon(x) = 0,\ x \in \partial\Omega. \tag{5.10}$$

Here, $\Omega^+ = \{x : (y, -z) \in \Omega^-\}$ is the mirror reflection of Ω^- and $\Gamma_\varepsilon^\pm = \{x \in V : z = \pm 0, s \in \Gamma, |n| < \varepsilon\}$ are the upper (+) and lower (-) sides of the two-dimensional surface (5.7), a curved ring of width 2ε.

The Neumann condition (5.10) and the fact that the interior of the Steklov set $\Gamma_\varepsilon = \Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$ is empty require certain modifications in the asymptotic procedure developed for the Steklov–Dirichlet problem (1.4)–(1.6). Let us list them.

The limit Neumann problem in Ω has the Neumann (generalized Green) function $G(x, \xi)$ in the form (2.2) as a distributional solution to the problem

$$-\Delta_x G(x, \xi) = \delta(x - \xi) - |\Omega|^{-1},\ \partial_\nu G(x, \xi) = 0,\ x \in \partial\Omega,$$

and, therefore, the integral (2.1) satisfies the Laplace equation in $\Omega \setminus \Gamma$ if and only if the density γ is orthogonal to 1 in $L^2(\Gamma)$. The Neumann function is defined up to an addendum $C(\xi)$ which can be fixed at $\xi \in \Gamma$ such that $j = 0$ in the representation (2.4) and $J = J^0$ (compare (2.4) and (2.6)). In this way, Eq. (3.8) reduces onto the subspace $L_\perp^2(\Gamma)$, cf. (2.19), i.e. (3.8) becomes $\int_\Gamma f_p(\eta; \zeta) d\sigma(\eta) = 0$.

The Neumann problem (2.14) in the plane with the incision $\overline{\omega}$ replaced by the set $\upsilon = \overline{\omega} = \{\eta \in \mathbb{R}^2 : \eta_2 = 0, |\eta_1| \le 1\}$ is a traditional object in the theory of cracks, see, e.g., Ref. 31–33, and can be solved explicitly by means of conformal mappings. It keeps all the properties mentioned in Section 2.3 and only some specification are needed. For example, in the definition of the Sobolev–Slobodetskii norm (2.18) the curve $\partial\omega$ is the union of two sides $\upsilon^\pm$ of the incision υ with common end-points, while the distance $|\eta - \mathbf{y}|$ is to be measured along the sides. Moreover, the term $(2\pi)^{-1}\partial_\nu(\eta) \ln \rho$ on the right of (3.6) is not null, but is given by the sum of two Dirac masses $\delta(\eta_1)$ located at the centre-points of $\upsilon^\pm$. The simplest way to avoid solving the problem (3.5), (3.6) within the theory of distribution is to consider the problem with transmission conditions

$$\Delta_\eta \mathbf{w}_p(\eta; \zeta) = 0,\ \eta \in (\mathbb{R}^2 \setminus \overline{B}_R) \cup (B_R \setminus \upsilon),$$

$$[\mathbf{w}_p](\varphi; \zeta) = \Psi_{p0}(\varphi; \zeta),\ [\partial_\rho \mathbf{w}_p](\varphi; \zeta) = \Psi_{p1}(\varphi; \zeta), \tag{5.11}$$

$$\partial_\nu(\eta) \mathbf{w}_p(\varphi; \zeta) = \zeta \mu_p(\zeta) \mathbf{w}_p(\varphi; \zeta),\ \eta \in \upsilon^\pm,$$

where $[w](\varphi) = w(R+0, \varphi) - w(R-0, \varphi)$ expresses the jump for a function w written in the polar coordinates (ρ, φ). The right-hand sides in the transmission conditions (5.11) appear as the discrepancies caused by the singular solution $\mathfrak{V}(\gamma_p; x)$ modulo $\gamma_p(s)$, i.e.,

$$\Psi_{p0}(\varphi; \zeta) = (2\pi)^{-1} \ln(\varepsilon R) - \beta_p,$$

$$\Psi_{p1}(\varphi; \zeta) = (2\pi)^{-1} \partial_\rho (\ln(\varepsilon R) - \beta_p)|_{\rho = R} = (2\pi R)^{-1}.$$

All other calculations and argumentation to derive and justify asymptotics of eigenpairs in the Steklov–Neumann problem (5.9)–(5.10) and, therefore, the water-wave problem (5.8), require just evident and minor changes in the material of Sections 3 and 4, as well as in the formulation of Theorems 3.1, 3.2, and Corollary 3.1.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

1. Steklov VA. *Main Problems in Mathematical Physics*. 2nd ed. Moscow: Nauka; 1983:432 [in Russian]. Sur les problèmes fondamentaux de la physique mathématique. *Annales Sci ENS, Sér 3*. 1902;19:191–259, French transl;
2. Kuznetsov NI, Maz'ya VG, Vainberg BR. *Linear Water Waves: A Mathematical Approach*. Cambridge University Press; 2002.
3. Linton CM, McIver P. Embedded trapped modes in water waves and acoustics. *Wave Motion*. 2007;45:16–29. http://dx.doi.org/10.1016/j.wavemoti.2007.04.009.
4. Bonnet-Ben Dhia A-S, Joly P. Mathematical analysis of guided water waves. *SIAM J Appl Math*. 1993;53:1507–1550. http://dx.doi.org/10.1137/0153071.
5. Nazarov SA. Properties of spectra of boundary value problems in cylindrical and quasicylindrical domains. In: Maz'ya V, ed. *Sobolev Spaces in Mathematics. Vol. II*. In: International Mathematical Series, vol. 9, New York: Springer; 2008:261–309.
6. Videman JH, Chiado Piat V, Nazarov SA. Asymptotics of frequency of a surface wave trapped by a slightly inclined barrier in a liquid layer. *Zap Nauchn Sem St-Petersburg Otdel Mat Inst Steklov*. 2011;393:46–79; *J Math Sci*. 2012;185:536–553, English transl.
7. Nazarov SA. Asymptotic behavior of the eigenvalues of the Steklov problem on a junction of domains of different limit dimensions. *Zh Vychisl Mat i Mat Fiz*. 2012;52:2033–2049; *Comput Math Math Phys*. 2012;52:1574–1589, English transl.
8. Nazarov SA. Asymptotic expansions of eigenvalues of the Steklov problem in singularly perturbed domains. *Algebra i analiz*. 2014;26:119–184; *St Petersburg Math J*. 2015;26:273–318, English transl.
9. Chiadò Piat V, Nazarov SA, Taskinen J. Embedded eigenvalues for water-waves in a three dimensional channel with a thin screen. *Quart J Mech Appl Math*. 2018;7:187–220.
10. Lanza de Cristoforis M. Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. A functional analytic approach. *Rev Mat Complut*. 2012;25:369–412.
11. Stoker JJ. *Water Waves the Mathematical Theory with Applications*. New York: John Wiley and Sons, Inc.; 1992 Reprint of the 1957 original.
12. Gryshchuk S, Lanza de Cristoforis M. Simple eigenvalues for the Steklov problem in a domain with a small hole. A functional analytic approach. *Math Methods Appl Sci*. 2013:1–17. http://dx.doi.org/10.1002/mma.2933.
13. Lanza de Cristoforis M. Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach. *Analysis*. 2008;28:63–93.
14. Fedoryuk MV. Asymptotics of the solution of the Dirichlet problem for the Laplace and Helmholtz equations in the exterior of a slender cylinder. *Izv Akad Nauk SSSR Ser Mat*. 1981;45:167–186; *Math USSR Izvestija*. 1982;18:167–186, English transl.
15. Maz'ya WG, Nazarov SA, Plamenevskii BA. On the asymptotics of solutions of the Dirichlet problem in three-dimensional domain with a slender cavity. *Dokl Akad Nauk SSSR*. 1981;256:37–39; *Sov Math Dokl*. 1981;23:32–35, English transl.
16. Maz'ya WG, Nazarov SA, Plamenewskii BA. Asymptotics of solutions of the Dirichlet problem in a domain with a truncated thin tube. *Mat sbornik*. 1981;116:187–217; *Math USSR Sbornik*. 1983;44:167–194, English transl.
17. Nazarov SA. Averaging of boundary value problems in a domain containing a thin cavity with periodically varying cross-section. *Tr Mosk Mat Obs*. 1990;53:98–129; *Trans Mosc Math Soc*. 1991;53:101–134, English transl.
18. Nazarov SA. Asymptotics of the eigenvalues of boundary value problems for the Laplace operator in a three-dimensional domain with a thin closed tube. *Trudy Moskov Mat Obschch*. 2015;76:1–66; *Trans Moscow Math Soc*. 2015;76:1–53, English transl.
19. Maz'ya WG, Nazarov SA, Plamenewskii BA. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. Bd. 1, 2*. Berlin: Akademie-Verlag; 1991 432 S, 319 S. [in German]. *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Vols. 1, 2*. Basel: Birkhäuser Verlag; 2000 English transl.
20. Cardone G, Durante T, Nazarov SA. Water-waves modes trapped in a canal by a body with the rough surface. *Z Angew Math Mech*. 2010;90:983–1004.
21. Nagel J. On equivalent norms in functional spaces H^μ. *Vestnik Leningrad Uni*. 1971;7:41–47.
22. Hörmander L. *Linear Partial Differential Operators*. Berlin: Springer-Verlag; 1963.
23. Birman MS, Solomyak MZ. *Spectral Theory of Self-Adjoint Operators in Hilbert Space*. Dordrecht: Reidel Publishing Company; 1986.
24. Taylor ME. *Pseudodifferntial Operators*. New Jerey: Princeton Uni. Press; 1981.

25. Krasnoselskii MA, Vainikko GM, Zabreiko PP, Rutickii YaB, Stecenko VYa. *Approximate Solution of Operator Equations*. Moscow: Nauka; 1969 [in Russian]. Groningen: Wolters-Noordhoff; 1972. English transl.
26. Hille CE, Phillips R. *Functional Analysis and Semi-Groups*. In: Amer. Math. Soc. Colloq. Publ., rev. ed. vol. 31, Providence, RI: American Mathematical Society; 1957.
27. Vishik MI, Lyusternik LA. Regular degeneration and the boundary layer for linear differential equations with a small parameter. *Usp Mat Nauk*. 1957;12:3–122; *Amer Math Soc Transl*. 1962;20:239–364, English transl.
28. Maz'ya WG, Nazarov SA, Plamenewskii BA. Elliptic boundary-value problems in domains of the exterior-of-a-cusp type. *Probl Mat Anal Leningrad Univ*. 1984;9:105–148; *J Sov Math*. 1986;35:2227–2256, English transl.
29. Nazarov SA. Asymptotics of the solution to the Neumann problem in a domain with singular point of peak exterior type. *Russ J Math Phys*. 1996;4:217–250.
30. Kozlov VA, Maz'ya VG, Rossmann J. *Elliptic Boundary Value Problems in Domains with Point Singularities*. Providence: Amer. Math. Soc.; 1997.
31. Sedov LI. *Mechanics of Continuous Media. Vol. 2*. River Edge: World Scientific; 1997.
32. Broek D. *The Practical Use of Fracture Mechanics*. Kluwer Academic Publishers; 1989.
33. Knott JF. *Fundamentals of Fracture Mechanics*. Butterworths; 1973.