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Chaos

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ABSTRACT

We extend the recently developed generalized Floquet theory [Phys. Rev. Lett. **110**, 170602 (2013)] to systems with infinite memory, i.e., a dependence on the whole previous history. In particular, we show that a lower asymptotic bound exists for the Floquet exponents associated to such cases. As examples, we analyze the cases of an ideal 1D system, a Brownian particle, and a circuit resonator with an ideal transmission line. All these examples show the usefulness of this new approach to the study of dynamical systems with memory, which are ubiquitous in science and technology.

Although Floquet theory is a powerful tool in the solution of linear differential equations with periodic coefficients, its generalization to dynamical systems with finite memory (generalized Floquet theory) has only recently been obtained. However, also of interest are those systems that support memory over the whole time evolution (infinite memory). Here, the generalized Floquet theory is extended to such systems, and analytical properties of the corresponding Floquet exponents are derived. Several examples, chosen for both their fundamental and applicative relevances, are then analyzed to illustrate the theoretical results. Owing to the prevalence of dynamical systems with memory, we expect this work to greatly expand the reach of Floquet theory.

I. INTRODUCTION

Floquet theory is a fundamental tool for expressing the solution of linear differential equations that have periodic coefficients.[1,2] These types of equations are very common in several areas of science and technology, such as quantum[3–10] and classical[11,12] physics, chemistry,[13–15] electronics,[16–19] noise analysis,[20–23] applied mathematics,[24,25] and in general dynamical systems.[26] In particular, it provides a versatile tool for the stability analysis of physical systems characterized by a periodic steady state. However, until very recently, the Floquet theorem was limited to systems whose features depend instantaneously on time, i.e., described by memoryless equations.

On the other hand, systems with memory are by far more common than memoryless ones,[27] with an extremely wide range of applications.[28,29] In general, a system with memory is characterized by a dependence of the dynamical equations on a portion or even on the whole previous state history. Even systems with delay are a special case of memory systems, where the dependence on the past history is limited to a countable set of time instants. The ubiquity of memory systems led some of us (F.L.T., M.D.V., and F.B.) to prove a generalization of the Floquet theory, extended to a wide class of systems described by *linear* memory operators.[30] In that paper, we have only provided the fundamental theorem of this generalized Floquet theory, and as a corollary, we have proved Bloch's theorem for non-local (in space) potentials.

In the present contribution, we apply the general theorem proved in Ref. 30 to the case of systems with linear memory, providing a formal extension of the theorem of Ref. 30 to systems whose memory is infinite and showing that a lower asymptotic

bound exists for the Floquet exponents associated to such cases. Furthermore, we also provide a general numerical tool based on the harmonic balance method[31,32] aimed at the numerical assessment of stability. As practical examples, we discuss the analysis of an ideal one-dimensional dynamical system with memory, of a more realistic Brownian particle with memory, and of a circuit resonator with an ideal transmission line.

We consider here a nonlinear dynamical system expressed as

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{f}(\mathbf{z}(t), t) + \int_{-\infty}^{t} \mathbf{K}(t,\tau)\mathbf{z}(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\mathbf{z}, \mathbf{f} \in \mathbb{R}^n$, and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the kernels representing the memory effects. Notice that (1) is very general, as it may include both autonomous and forced systems such as the periodically driven spin-boson system studied in Ref. 8. We assume that (1) admits a solution $\mathbf{z}_S(t)$ periodic of period T (limit cycle), and that $\mathbf{K}$ is such that

$$\mathbf{K}(t,\tau) = \mathbf{K}(t+T, \tau+T), \tag{2a}$$

$$\int_{-\infty}^{t} \|\mathbf{K}(t,\tau)\|\,\mathrm{d}\tau < \infty \quad \forall t, \tag{2b}$$

where $\|\cdot\|$ is a properly defined norm. Notice that if the memory part is time-invariant, i.e., if $\mathbf{K}(t,\tau) = \mathbf{K}(t-\tau)$, condition (2a) is always satisfied.

The stability of the limit cycle $\mathbf{z}_s(t)$ is defined by the variational problem

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{y}(t) + \int_{-\infty}^{t} \mathbf{K}(t,\tau)\mathbf{y}(\tau)\,\mathrm{d}\tau, \tag{3}$$

where $\mathbf{y}(t) = \mathbf{z}(t) - \mathbf{z}_S(t)$ is cycle perturbation, and $\mathbf{A}(t)$ is the T-periodic Jacobian matrix of $\mathbf{f}(\mathbf{z}(t), t)$, with respect to $\mathbf{z}$, calculated in the limit cycle.

II. GENERALIZED FLOQUET THEOREM

The generalization of the Floquet theorem proved in Ref. 30 shows that the state transition matrix of (3) can be expressed as

$$\Phi(t;t_0) = \mathbf{M}(t;t_0)\,\mathrm{e}^{\mathbf{F}(t-t_0)}, \tag{4}$$

where $\mathbf{M}(t;t_0)$ is an $x \times p$ matrix that is T-periodic with respect to both time variables, and $\mathbf{F}$ is a constant $p \times p$ matrix whose eigenvalues constitute the cycle Floquet exponents. With respect to memoryless systems, however, the size p of $\mathbf{F}$ may be larger than n and even infinite.

Thus, the general solution of (3) can be expressed as a linear combination of p exponential functions, characterized by the Floquet exponent λ, times a T periodic function $\mathbf{r}(t) = \mathbf{r}(t+T)$, the (direct) Floquet eigenvector. This means that we seek solutions of (3) in the form

$$\mathbf{y}(t) = \mathbf{r}(t)\,\mathrm{e}^{\lambda t} \quad \mathbf{r}(t) = \mathbf{r}(t+T). \tag{5}$$

Due to the T-periodicity of the Floquet eigenvector, we have actually a set of p different *classes* of values for λ. In fact if λ_0 is one of the eigenvalues of $\mathbf{F}$, each $\lambda_0 + ki2\pi/T$ ($k \in \mathbb{Z}$) spans the same eigenspace: we call this phenomenon the *splitting* of the eigenvalues. To reduce to a minimum the number of significant quantities, it is customary to define the Floquet multipliers $\mu = \exp(\lambda T)$, since the exponential function eliminates the splitting phenomenon. Of course, the stability of the solution of (1) depends on the sign of the real part of λ or equivalently on the magnitude of μ.

The defining equation for the Floquet eigenvalues (and direct eigenvectors) can be found substituting (5) into (3), obtaining

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \lambda\mathbf{r}(t) = \mathbf{A}(t)\mathbf{r}(t) + \mathbf{q}(\mathbf{r}, t, \lambda), \tag{6}$$

where

$$\mathbf{q}(\mathbf{r}, t, \lambda) = \int_{-\infty}^{t} \mathbf{K}(t,\tau)\mathbf{r}(\tau)\,\mathrm{e}^{\lambda(\tau-t)}\,\mathrm{d}\tau \tag{7}$$

is a T-periodic function of t because all the other terms of (6) are.

The convergence of the integral defining $\mathbf{q}(\mathbf{r}, t, \lambda)$ is not trivially derived from (2b). In order to clarify the matter, we start by proving the following Lemma:

Lemma 1. *Let us consider (3) where the memory kernel satisfies (2). We consider a real $\bar{s}$ and any $s > \bar{s}$. Then, the solutions $\mathbf{y}(t) = \mathbf{r}(t)\exp(\lambda t)$ and $\bar{\mathbf{y}}(t) = \bar{\mathbf{r}}(t)\exp(\bar{\lambda} t)$ to the variational problems (with finite memory)*

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \lambda\mathbf{r}(t) = \mathbf{A}(t)\mathbf{r}(t) + \int_{t-s}^{t} \mathbf{K}(t,\tau)\mathbf{r}(\tau)\,\mathrm{e}^{\lambda(\tau-t)}\,\mathrm{d}\tau, \tag{8a}$$

$$\frac{\mathrm{d}\bar{\mathbf{r}}}{\mathrm{d}t} + \bar{\lambda}\bar{\mathbf{r}}(t) = \mathbf{A}(t)\bar{\mathbf{r}}(t) + \int_{t-\bar{s}}^{t} \mathbf{K}(t,\tau)\bar{\mathbf{r}}(\tau)\,\mathrm{e}^{\bar{\lambda}(\tau-t)}\,\mathrm{d}\tau \tag{8b}$$

satisfy

$$\left|\lambda - \bar{\lambda}\right| \le M \int_{t-s}^{t-\bar{s}} \max_{t\in[0,T]} \|\mathbf{K}(t,\tau)\|\,\mathrm{d}\tau \tag{9a}$$

and for $\bar{s} \gg 0$

$$\left|\lambda - \bar{\lambda}\right| \le M_- \int_{t-s}^{t-\bar{s}} \frac{\max_{t\in[0,T]} \|\mathbf{K}(t,\tau)\|}{t-\tau}\,\mathrm{d}\tau \quad \text{if } \mathrm{Re}\left\{\bar{\lambda}\right\} < 0, \tag{9b}$$

$$\left|\lambda - \bar{\lambda}\right| \le M_+ \int_{t-s}^{t-\bar{s}} \max_{t\in[0,T]} \|\mathbf{K}(t,\tau)\|\,\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}(t-\tau)}\,\mathrm{d}\tau \quad \text{if } \mathrm{Re}\left\{\bar{\lambda}\right\} > 0, \tag{9c}$$

where $0 < M_+, M_- \le M < +\infty$.

Proof. See Appendix A. □

Lemma 1, starting from (2b), shows that for $\bar{s}$ large enough, for every s, $\left|\lambda - \bar{\lambda}\right| \to 0$ faster than $\int_{t-s}^{t-\bar{s}} \|\mathbf{K}(t,\tau)\|\,\mathrm{d}\tau$. This means that the eigenvalue λ becomes independent of $\bar{s}$, i.e., of any finite approximation of the system memory length. The first consequence is numerical: for large enough $\bar{s}$, the eigenvalues of an infinite memory system can be calculated with a prescribed accuracy. Second, from a theoretical standpoint, we have

Theorem 1. *The Floquet exponents λ of (3) satisfy*

$$\mathrm{Re}\,\{\lambda\} > -\min_{t\in[0,T]} k_c(t), \tag{10}$$

where the critical exponent $k_c(t)$ is defined as

$$k_c(t) = \lim_{\tau\to-\infty} \frac{\ln \|\mathbf{K}(t,\tau)\|}{\tau}. \tag{11}$$

Proof. Exploiting a procedure akin to that in the proof of Lemma 1, it is easy to show that both $\delta\mathbf{v}(t)$ and $\delta\mathbf{r}(t)$ tend to zero for large $\bar{s}$. In particular, for $\bar{s} \gg 0$ and $s > \bar{s}$,

$$\|\mathbf{r}(t) - \bar{\mathbf{r}}(t)\| \le H_- \int_{t-s}^{t-\bar{s}} \max_{t\in[0,T]} \|\mathbf{K}(t,\tau)\|\,\mathrm{d}\tau \quad \text{if } \mathrm{Re}\left\{\bar{\lambda}\right\} < 0, \tag{12a}$$

$$\|\mathbf{r}(t) - \bar{\mathbf{r}}(t)\| \le H_+ \int_{t-s}^{t-\bar{s}} \mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}(t-\tau)} \times \max_{t\in[0,T]} \|\mathbf{K}(t,\tau)\|\,\mathrm{d}\tau \quad \text{if } \mathrm{Re}\left\{\bar{\lambda}\right\} > 0, \tag{12b}$$

where H_- and H_+ are positive constants. This result, together with Lemma 1, implies that for $\bar{s} \gg 0$ and for any $s > \bar{s}$, the solution of

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \lambda\mathbf{r}(t) = \mathbf{A}(t)\mathbf{r}(t) + \int_{t-s}^{t} \mathbf{K}(t,\tau)\mathbf{r}(\tau)\,\mathrm{e}^{\lambda(\tau-t)}\,\mathrm{d}\tau \tag{13}$$

becomes independent of s. Since this may happen only if the integral is independent of s, the latter should converge even if $\mathrm{Re}\,\{\lambda\} < 0$. Defining the critical exponent as in (11), the integral may converge independently of s if (10) is met. □

III. FLOQUET EXPONENTS COMPUTATION

Clearly, (6) represents a generalized eigenvalue problem whose solution provides the required Floquet quantities for the limit cycle. The explicit expression for such a generalized eigenvalue problem depends on the features of the memory kernel $\mathbf{K}(t,\tau)$. Since all the terms of (6) are T-periodic, a viable solution strategy is the use of frequency-domain approaches such as the harmonic balance (HB) technique,[31–34] here summarized in Appendix B. Frequency transformation of (6) yields

$$\Omega\tilde{\mathbf{r}} + \lambda\tilde{\mathbf{r}} = \tilde{\mathbf{A}}\tilde{\mathbf{r}} + \tilde{\mathbf{q}}(\tilde{\mathbf{r}},\omega,\lambda), \tag{14}$$

where ω is the set of all the frequencies multiple of the fundamental one $\omega_0 = 2\pi/T$. Equation (14) is a generalized, transcendental eigenvalue problem in λ and $\tilde{\mathbf{r}}$, the collection of the harmonic amplitudes of the Floquet eigenvector $\mathbf{r}(t)$.

The explicit form of (14) depends on the type of $\mathbf{K}(t,\tau)$. However, in general, it can be transformed, at least approximately, into a polynomial eigenvalue problem by formally developing $\tilde{\mathbf{q}}(\tilde{\mathbf{r}};\lambda)$ into Taylor series as a function of λ,

$$\Omega\tilde{\mathbf{r}} + \lambda\tilde{\mathbf{r}} = \tilde{\mathbf{A}}\tilde{\mathbf{r}} + \sum_{k=0}^{+\infty} \left.\frac{\partial\tilde{\mathbf{q}}(\tilde{\mathbf{r}},\omega,\lambda)}{\partial\lambda}\right|_{\lambda=0} \frac{\lambda^k}{k!}. \tag{15}$$

Several techniques are available to tackle the polynomial eigenvalue problems, such as for instance those discussed in Refs. 35–37.

Notice that (14) is an exact representation of the generalized, time-domain eigenvalue problem (6) only if infinite Fourier series are considered. Clearly, for practical calculations, the series is truncated to a finite number of harmonics N_H (see Appendix B) and, equivalently, the time domain problem is time-sampled. The truncation affects the accuracy of the Floquet quantities, especially on the exponents, as discussed, e.g., in Refs. 23 and 32. However, according to intuition, accurate results can be obtained by properly choosing N_H.

IV. EXAMPLES

A. A simple 1D dynamical system with memory

The first example is an extremely simple dynamical system with memory, characterized by the 1D variational equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay(t) + \int_{t-s}^{t} \mathrm{e}^{\tau-t} y(\tau)\,\mathrm{d}\tau, \tag{16}$$

where a is a real parameter and $s > 0$ defines the "length" of the memory part of the system. Using the harmonic balance approach discussed in Sec. III, we find the explicit form of (14):

$$(\lambda + \mathrm{i}\omega_j)r_j = ar_j + \frac{1 - \mathrm{e}^{-s(1+\lambda+\mathrm{i}\omega_j)}}{1+\lambda+\mathrm{i}\omega_j} r_j. \tag{17}$$

Since (16) is scalar, we can simplify r_j from (17), and consider the case $\omega_j = 0$ since the roots of the general eigenvalue equation are simply those for $\omega_j = 0$ plus a shift equal to $-\mathrm{i}\omega_j$, where i is the imaginary unit. In other words, we have to study the transcendental eigenvalue equation

$$\lambda = a + \frac{1 - \mathrm{e}^{-s(1+\lambda)}}{1+\lambda}. \tag{18}$$

Studying (18) is not an easy task, as in general for finite memory (i.e., finite s) it admits of infinite solutions in the complex plane. However, results are more compact in the limit case of infinite memory (i.e., for $s \to +\infty$). Let us start by expressing (18) in the form

$$(a-\lambda)(\lambda+1) + 1 - \mathrm{e}^{-s(1+\lambda)} = 0. \tag{19}$$

Notice that (19) is not fully equivalent to (18), as $\lambda = -1$ solves (19), but this value cannot satisfy (18) since

$$\lim_{\lambda\to-1} \frac{1 - \mathrm{e}^{-s(1+\lambda)}}{1+\lambda} = s. \tag{20}$$

We now put in evidence the real and imaginary components of $\lambda = \lambda_\mathrm{r} + \mathrm{i}\lambda_\mathrm{i}$. By separating the real and imaginary parts of (19), we find

$$\lambda_\mathrm{i}^2 - \lambda_\mathrm{r}^2 + (a-1)\lambda_\mathrm{r} + a + 1 - \mathrm{e}^{-s(1+\lambda_\mathrm{r})}\cos(s\lambda_\mathrm{i}) = 0, \tag{21a}$$

$$(a-1)\lambda_\mathrm{i} - 2\lambda_\mathrm{i}\lambda_\mathrm{r} + \mathrm{e}^{-s(1+\lambda_\mathrm{r})}\sin(s\lambda_\mathrm{i}) = 0. \tag{21b}$$

Deriving λ_i from (21b) and substituting into (21a), we find a second-order algebraic equation in $\cos(s\lambda_i)$,

$$\frac{1}{R^2}\cos^2(s\lambda_\mathrm{i}) + \mathrm{e}^{-s(1+\lambda_\mathrm{r})}\cos(s\lambda_\mathrm{i}) - N = 0, \tag{22}$$

where $R = (a - 1 - 2\lambda_\mathrm{r})\exp[s(1+\lambda_\mathrm{r})]$ and $N = R^{-2} - \lambda_\mathrm{r}^2 + (a - 1)\lambda_\mathrm{r} + a + 1$. Notice also that the exponential function tends to

different limit values for infinite memory,

$$\lim_{s\to+\infty} \mathrm{e}^{-s(1+\lambda_\mathrm{r})} = \begin{cases} +\infty & \text{if } \lambda_\mathrm{r} < -1 \\ 1 & \text{if } \lambda_\mathrm{r} = -1 \\ 0 & \text{if } \lambda_\mathrm{r} > -1. \end{cases} \tag{23}$$

In order to discuss the solutions of (18) in the infinite memory limit $s \to +\infty$, we separate the analysis as a function of λ_r:

1. $\lambda_\mathrm{r} > -1$. Due to (23), $1/R \to 0$ and (22) reduces to $N = 0$. Equation (21b) becomes

$$(a - 1 - 2\lambda_\mathrm{r})\lambda_\mathrm{i} = 0, \tag{24}$$

implying $\lambda_\mathrm{i} = 0$ or $\lambda_\mathrm{r} = (a-1)/2$. The second solution must be discarded, since substituted into (21a) would require an imaginary value for λ_i. On the other hand, using the first solution $\lambda_\mathrm{i} = 0$ into (21a) leads to the only viable root

$$\lambda_\mathrm{r} = \lambda_\infty = \frac{1}{2}\left[a - 1 + \sqrt{(a+1)^2 + 4}\right] \tag{25}$$

since the other solution of the obtained second-order algebraic equation is lower than -1.

2. $\lambda_\mathrm{r} \le -1$. We consider the formal solution of (22),

$$\lambda_\mathrm{i} = \frac{1}{s}\arccos(A/2), \tag{26}$$

where the argument of the inverse cosine function

$$A = -R^2\,\mathrm{e}^{-s(1+\lambda_\mathrm{r})} \pm \sqrt{R^4\,\mathrm{e}^{-2s(1+\lambda_\mathrm{r})} + 4R^2N} \tag{27}$$

is a limited quantity for each s value. This implies that in the long memory limit $\lambda_\mathrm{i} \to 0$ and, thus, (21a) becomes

$$-\lambda_\mathrm{r}^2 + (a-1)\lambda_\mathrm{r} + a + 1 - \mathrm{e}^{-s(1+\lambda_\mathrm{r})} = 0. \tag{28}$$

The solution $(\lambda_\mathrm{r} = -1, \lambda_\mathrm{i} = 0)$ must be excluded as discussed above; therefore, we have to study the case $\lambda_\mathrm{r} < -1$. A Taylor series development of the terms in (28) around $\lambda_{\mathrm{r}0} = -1$ leads to

$$-a - 1 + (\lambda_\mathrm{r} + 1) - s \times \left[1 - \frac{s}{2}(\lambda_\mathrm{r} + 1) + \sum_{j=3}^{+\infty} \frac{s^{j-1}}{j!}|\lambda_\mathrm{r} + 1|^j\right] = 0, \tag{29}$$

where, due to the initial constraint of this analysis, $\lambda_\mathrm{r} + 1 < 0$. In these conditions, the bracketed term is positive $\forall s > 0$, and therefore, (29) cannot have any solution $\lambda_\mathrm{r} < -1$ for $s \to +\infty$.

In order to discuss the behavior of the 1D system for infinite memory, we, therefore, limit our analysis to the real solution of (18) only, whose dependence on the s and a parameters is sketched in Fig. 1. As expected from Lemma 1, for each value of the a parameter and for "long" enough memory, the solution λ reaches the asymptotic value λ_∞ defined in (25). On the other hand, since the memory kernel is exponential, the critical exponent is easily calculated as $k_\mathrm{c} = 1$, thus setting the lower bound for the real part of λ (see Fig. 1).

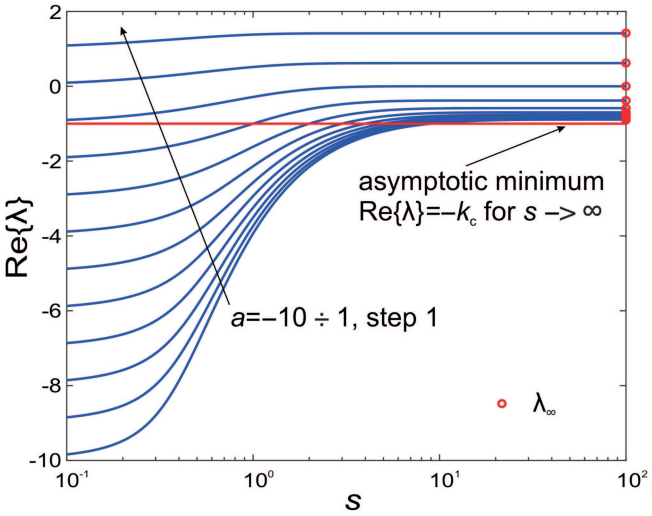

FIG. 1. 1D system with memory: dependence on s of the real solution of (18) as a function of the model parameters. The horizontal line represents the asymptotic possible minimum of Re $\{\lambda\}$ according to Theorem 1. Circles show the asymptotic value λ_∞ according to (25).

B. Brownian particle with memory

The second example of application that we consider is the 2D Brownian particle with memory originally introduced in Ref. 38, extending the results presented in Ref. 39. From the physics standpoint, we study a stochastic system representing the motion of a particle of mass m subject to a Langevin force modeled as an Ornstein–Uhlenbeck process. The finite time correlation of the Langevin source implies a memory effect in the particle dynamics that according to Refs. 40 and 41, is described by the friction retardation function $\Gamma(t,\tau)$, shown here in the deterministic system corresponding to the ensemble average of the stochastic description

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(t), \tag{30a}$$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -m\int_{-\infty}^{t} \Gamma(t,\tau)\mathbf{v}(\tau)\,\mathrm{d}\tau - \nabla U(\mathbf{x}), \tag{30b}$$

where the external potential U represents a central force as in Ref. 38,

$$\nabla U(\mathbf{x}) = m\bar{\boldsymbol{\omega}}^2\mathbf{x}, \tag{31}$$

being $\bar{\boldsymbol{\omega}} = \mathrm{diag}\,\{\bar{\omega}_1, \bar{\omega}_2\}$ and $\bar{\omega}_1, \bar{\omega}_2 > 0$ two real parameters.

According to the discussion in Ref. 39, the friction retardation function reads[41]

$$\Gamma(t,\tau) = \gamma(v)k\,\mathrm{e}^{-k|t-\tau|}, \tag{32}$$

where γ is instantaneously dependent on the particle velocity[40] v (the magnitude of the particle velocity $\mathbf{v}$)

$$\gamma[v(\tau)] = -\alpha + \beta v(\tau)^2 + \frac{g}{k} \tag{33}$$

and $k = 1/\tau_n$, τ_n being the noise correlation time of the Ornstein–Uhlenbeck process. We remark that γ dependence on the velocity magnitude makes the model fully 2D.

Coefficients α, β, g, and k are model parameters. As discussed in Ref. 39, the case of the memoryless particle is obtained by letting $k \to +\infty$.

System (30) admits a periodic limit cycle in the phase space $(\mathbf{x}_S(t), \mathbf{v}_S(t))$ for several values of the parameters. The corresponding stability is assessed following the procedure outlined in Sec. III. The 4D Floquet eigenvector $\mathbf{r}(t)$ is decomposed in the 2D position $\mathbf{r}_x(t)$ and velocity $\mathbf{r}_v(t)$ components. Substituting into (6), we evaluate the integral in (7) taking into account the Fourier expansion of $\mathbf{r}(t)$ and Theorem 1, which guarantees $\mathrm{Re}\{\lambda\} > -k$. Thus, we get a closed, albeit in infinite series form, expression

$$\frac{\mathrm{d}\mathbf{r}_x}{\mathrm{d}t} + \lambda \mathbf{r}_x = \mathbf{r}_v, \tag{34a}$$

$$\frac{\mathrm{d}\mathbf{r}_v}{\mathrm{d}t} + \lambda \mathbf{r}_v = -k \sum_{j,h} \frac{\tilde{\gamma}_{e,j-h} \tilde{\mathbf{r}}_{v,h}\, \mathrm{e}^{\mathrm{i}\omega_j t}}{k + \lambda + \mathrm{i}\omega_j} - \bar{\boldsymbol{\omega}}^2 \mathbf{r}_x, \tag{34b}$$

where $\tilde{\mathbf{r}}_{\alpha,j}$ is the jth harmonic amplitude (in exponential form) for function $\mathbf{r}_\alpha(t)$ $(\alpha = x, v)$, $j, h \in \mathbb{Z}$, and $\omega_j = j2\pi/T$.

Equation (34) is further transformed into (14) by exploiting the Fourier series expansion of all the T-periodic terms. By balancing the harmonic components, we find the ideally infinite set (as a function of the harmonic index j) of order 2 polynomial eigenvalue problems (PEPs),

$$(\lambda + \mathrm{i}\omega_j)\tilde{\mathbf{r}}_{x,j} = \tilde{\mathbf{r}}_{v,j}, \tag{35a}$$

$$(\lambda + \mathrm{i}\omega_j)(k + \lambda + \mathrm{i}\omega_j)\tilde{\mathbf{r}}_{v,j} = -(k + \lambda + \mathrm{i}\omega_j)\bar{\boldsymbol{\omega}}^2 \tilde{\mathbf{r}}_{x,j} - k \sum_h \tilde{\gamma}_{e,j-h} \tilde{\mathbf{r}}_{v,h}. \tag{35b}$$

As discussed in Refs. 36 and 37, an rth order PEP for an equation of size m has $r \times m$ eigenvalues; thus, (35) provides a set of $n/2 + 2(n/2) = 3n/2$ eigenvalues (for each harmonic j), where for the 4D phase space of the 2D particle $n = 4$, i.e., a total of 6 Floquet multipliers.

We implemented a numerical solution of the second-order polynomial eigenvalue problem using $N_H = 30$ harmonics, thus truncating (35) into a system for $j = -N_H, \ldots, N_H$. Furthermore, the limit cycle solution is calculated in the frequency domain exploiting the harmonic balance[32] numerical technique, again making use of 30 harmonics.

With respect to the memoryless case (i.e., $k \to +\infty$), a stable equilibrium (originally unstable) is found in the origin of the phase space. The bifurcation diagram in the parameter space $(\bar{\omega}_1/\bar{\omega}_2, \alpha)$ is plotted in Fig. 2 for $k = 1$ and several values of g. As expected, Arnold tongues are found. The area of the parameter space below the almost horizontal curve (not present in Ref. 38) corresponds to the stable equilibrium previously discussed. Above this line, we have either two symmetric, stable limit cycles (the inner part of the Arnold tongue) or a strange attractor (the outer part): both of these have a shape close to that of the memoryless case.[38] The boundaries of the Arnold tongue correspond to fold bifurcations, thus implying

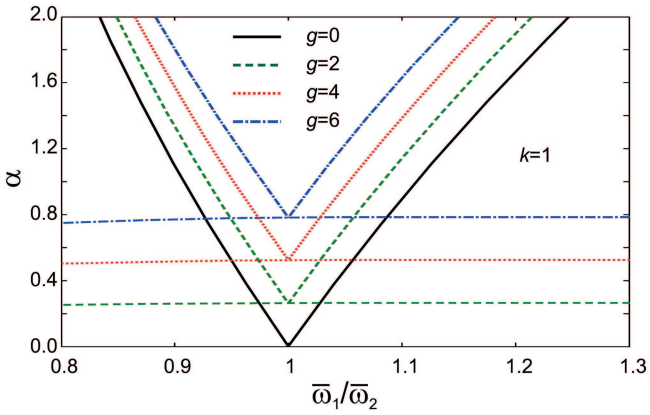

FIG. 2. Brownian particle with memory: Arnold tongue in the $(\bar{\omega}_1/\bar{\omega}_2, \alpha)$ plane as a function of g. Parameters: $\beta = 1$, $\bar{\omega}_1 = 2$, and $k = 1$.

that (as for the memoryless case) the strange attractor is generated by the collapse of the two limit cycles. The crossing point between the Arnold tongue and the boundary of the stable equilibrium defines a *triple* point[42] for the particle dynamics.

Figure 3 represents the bifurcation curve in the (k, α) plane for the triple point as a function of g. Consistently, the bifurcation curves tend to zero for disappearing memory $(k \to +\infty)$.

C. Circuit resonator with ideal transmission line

The last example we provide considers the presence of a lossless transmission line (TL) in an electronic circuit. TLs are ubiquitous in circuits for high frequency applications, e.g., in the RF and microwave frequency range.[43] The presence of the lossless TL implies a memory effect that, as we shall demonstrate in the following, corresponds to a delay system. Despite being well studied, we include it here to highlight the general applicability of out methodology even beyond the exponential kernels discussed in the

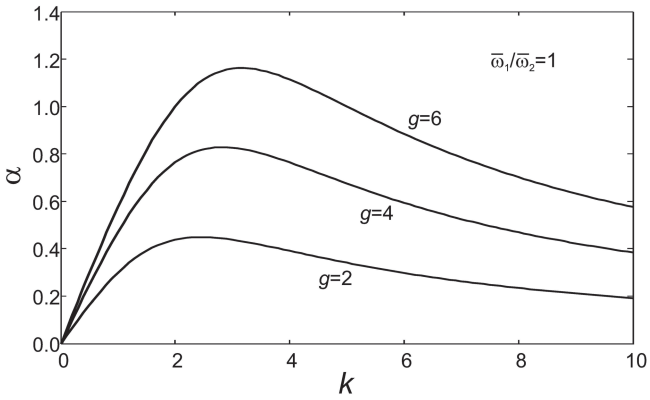

FIG. 3. Brownian particle with memory: bifurcation curve in the (k, α) plane as a function of g for the triple point. Parameters: $\beta = 1$, $\bar{\omega}_1/\bar{\omega}_2 = 1$, and $\bar{\omega}_1 = 2$.

previous examples and to obtain, via a non-standard approach, the well-known oscillation condition.

The lossless TL is a distributed circuit element characterized by the following linear partial differential system of equations

$$\frac{\partial v}{\partial x} = -l\frac{\partial i}{\partial t}, \tag{36a}$$

$$\frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t}, \tag{36b}$$

where $v(x,t)$ and $i(x,t)$ are, respectively, the voltage and current at time t and position x along the TL (see Fig. 4), l is the TL inductance per unit length, and c is the TL capacitance per unit length. The general solution of (36) is expressed as the sum of a progressive and a regressive wave,[43]

$$v(x,t) = v_-(t - x/v_\mathrm{f}) + v_+(t + x/v_\mathrm{f}), \tag{37a}$$

$$i(x,t) = Y_0 v_-(t - x/v_\mathrm{f}) - Y_0 v_+(t + x/v_\mathrm{f}), \tag{37b}$$

where $v_\mathrm{f} = 1/\sqrt{lc}$ is the TL phase velocity and $Y_0 = 1/Z_0 = \sqrt{c/l}$ is the TL characteristic admittance. The specific shape of v_- and v_+ depends on the line boundary and initial conditions, i.e., on the circuit in which the TL is embedded. If we consider the TL embedded into a nonlinear circuit characterized by a set of state variables collectively denoted as $\mathbf{y}(t)$, the circuit equations to be solved take the form

$$\mathcal{L}_0\left\{v(0,t), i(0,t), \mathbf{y}, \dot{\mathbf{y}}\right\} = 0, \tag{38a}$$

$$\mathcal{L}_L\left\{v(L,t), i(L,t), \mathbf{y}, \dot{\mathbf{y}}\right\} = 0, \tag{38b}$$

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{q}(\mathbf{y}, t), \tag{38c}$$

where $\dot{\mathbf{y}} = \mathrm{d}\mathbf{y}/\mathrm{d}t$, $\mathcal{L}_0$ and $\mathcal{L}_L$ are linear operators, and $\mathbf{q}$ is a vector function describing the embedding circuit state equations. System (38) can be easily transformed into a nonlinear system with memory of type (1). In fact, from (37),

$$v(0,t) = v_-(t) + v_+(t), \tag{39a}$$

$$i(0,t) = Y_0 v_-(t) - Y_0 v_+(t), \tag{39b}$$

$$v(L,t) = \int_{-\infty}^{t} v_-(\tau)\delta(t - \tau_\mathrm{f} - \tau)\,\mathrm{d}\tau + \int_{t}^{\infty} v_+(\tau)\delta(t + \tau_\mathrm{f} - \tau)\,\mathrm{d}\tau, \tag{39c}$$

$$i(L,t) = Y_0\int_{-\infty}^{t} v_-(\tau)\delta(t - \tau_\mathrm{f} - \tau)\,\mathrm{d}\tau - Y_0\int_{t}^{\infty} v_+(\tau)\delta(t + \tau_\mathrm{f} - \tau)\,\mathrm{d}\tau, \tag{39d}$$

where $\tau_\mathrm{f} = L/v_\mathrm{f}$ represents the delay associated to the TL. Notice that the time anticipation included in the second integral of (39c) and of (39d) is only apparent, see Ref. 44, Chap. 8. In fact, reversing (37), we find

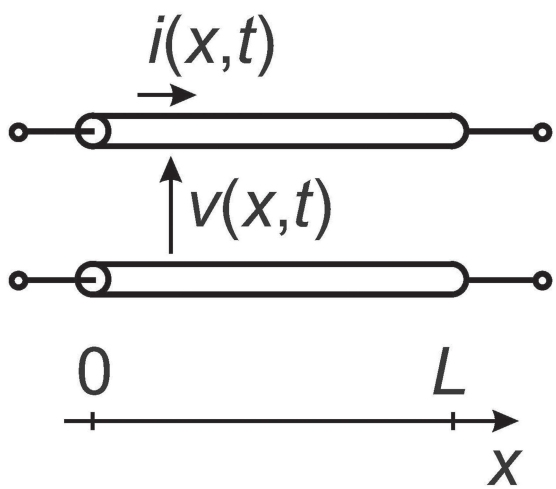

FIG. 4. Circuit representation of a transmission line (TL).

$$v_-(t - x/v_\mathrm{f}) = \frac{1}{2}v(x,t) + \frac{1}{2Y_0}i(x,t), \tag{40a}$$

$$v_+(t + x/v_\mathrm{f}) = \frac{1}{2}v(x,t) - \frac{1}{2Y_0}i(x,t). \tag{40b}$$

The variational problem defining the stability of the solution of (38) can thus be derived by linearizing the full system and looking for solutions of type (5), i.e., $\delta v(x,t) = r_v(x,t)\exp(\lambda t)$ and $\delta i(x,t) = r_i(x,t)\exp(\lambda t)$. Correspondingly, from (37), we find $r_{v_-}(t - x/v_\mathrm{f}) = r_{v_-}(t)\exp[\lambda(t - x/v_\mathrm{f})]$ and $r_{v_+}(t + x/v_\mathrm{f}) = r_{v_+}(t)\exp[\lambda(t + x/v_\mathrm{f})]$. Finally, we get the generalized eigenvalue problem

$$\mathcal{L}_0'\left\{r_{v_-}, r_{v_+}, \mathbf{r}_\mathrm{y}, \dot{\mathbf{r}}_\mathrm{y} + \lambda\mathbf{r}_\mathrm{y}\right\} = 0, \tag{41a}$$

$$\mathcal{L}_L'\left\{r_{v_-}\,\mathrm{e}^{-\lambda\tau_\mathrm{f}}, r_{v_+}\,\mathrm{e}^{\lambda\tau_\mathrm{f}}, \mathbf{r}_\mathrm{y}, \dot{\mathbf{r}}_\mathrm{y} + \lambda\mathbf{r}_\mathrm{y}\right\} = 0, \tag{41b}$$

$$\frac{\mathrm{d}\mathbf{r}_\mathrm{y}}{\mathrm{d}t} + \lambda\mathbf{r}_\mathrm{y} = \mathbf{A}(t)\mathbf{r}_\mathrm{y}, \tag{41c}$$

where $\mathbf{A}(t)$ is the Jacobian of $\mathbf{q}$ with respect to $\mathbf{y}$ and the linear operators $\mathcal{L}_0'$ and $\mathcal{L}_L'$ are derived from $\mathcal{L}_0$ and $\mathcal{L}_L$ exploiting the linear transformation (37).

As an example of application, we consider the distributed resonating circuit in Fig. 5: the active device is used to provide the necessary energy to overcome the dissipation included in resistance R. For the sake of simplicity, we assume here that the electrical equivalent of the active element is simply a constant resistance R_a, whose value however may become negative.

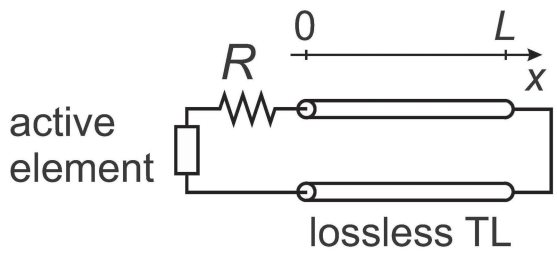

FIG. 5. Example of distributed resonating circuit containing a lossless TL. The active device is here represented as a constant resistance R_a that may assume negative values.

In this case, (38c) disappears, and the two linear operators $\mathcal{L}_0$ and $\mathcal{L}_L$ simply read

$$v(0,t) + (R + R_a)i(0,t) = 0, \tag{42a}$$

$$v(L,t) = 0. \tag{42b}$$

Using v_- and v_+, (42) provides

$$v_+(t) = \Gamma_0 v_-(t), \tag{43a}$$

$$v_-(t) = -v_+(t - 2\tau_f), \tag{43b}$$

where $\Gamma_0 = [(R + R_a)Y_0 - 1]/[(R + R_a)Y_0 + 1]$ is the reflection coefficient[43] measured at the left of position $x = 0$ and calculated using $1/Y_0$ as reference impedance. Clearly, the previous conditions lead to

$$v_+(t) = -\Gamma_0 v_+(t - 2\tau_f). \tag{44}$$

The eigenvalue problem (41), finally, is

$$e^{2\lambda\tau_f} = -\Gamma_0; \tag{45}$$

therefore, the bifurcation ($\mathrm{Re}\{\lambda\} = 0$) takes place for $|\Gamma_0| = 1$. Notice that for finite R, the only possible realization is $\Gamma_0 = -1$, i.e., as expected, $R_a = -R$. In this case, the admitted oscillation frequencies are defined by the complex roots of

$$e^{2\lambda\tau_f} = 1, \tag{46}$$

i.e., $\lambda = i2k\pi f$, where $f = 1/(2\tau_f)$ and $k \in \mathbb{Z}$.

V. CONCLUSION

In this paper, we have proved an important extension of the recently developed generalized Floquet theory[30] to systems supporting infinite memory. In particular, we have proved that a lower asymptotic bound exists for the Floquet exponents of such cases.

We have then analyzed three cases of systems with memory: an ideal 1D system, a Brownian particle, and a circuit resonator with an ideal transmission line. While in the first two cases the memory kernel $\mathbf{K}(t,\tau)$ is characterized by an exponential dependence on $t - \tau$, for the delay system in the third example, the memory kernel cannot be represented in a similar way. This makes such a system difficult to be treated with system enlargement approaches such as those proposed in the recent literature.[6,8] In this sense, our approach is characterized by a wider generality; however, the enlargement approach appears advantageous for the study of the allowed systems, as the enlargement procedure results in a (wider) memoryless description to which, of course, well-known techniques apply.

Owing to the fact that dynamical systems with memory are ubiquitous in science and technology, we expect that our generalized Floquet theory will find numerous applications in diverse fields.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF LEMMA 1

Let us consider a time increment $\delta s \ll \bar{s}$. The solution for $\bar{s} + \delta s$ is expressed as $\mathbf{r}(t) = \bar{\mathbf{r}}(t) + \delta\mathbf{r}(t)$ and $\lambda = \bar{\lambda} + \delta\lambda$. Since δs is small, by linearizing (8a), we find the (first order) relationship between the Floquet quantity variations

$$\mathbf{p}(t)\delta\lambda + \delta\mathbf{v}(t) = \int_{t-\bar{s}-\delta s}^{t-\bar{s}} \mathbf{K}(t,\tau)\mathbf{r}(\tau)\, e^{\bar{\lambda}(\tau-t)}\, d\tau, \tag{A1}$$

where we have defined the T periodic functions

$$\mathbf{p}(t) = \mathbf{r}(t) - \int_{t-\bar{s}}^{t} \mathbf{K}(t,\tau)\bar{\mathbf{r}}(\tau)(\tau - t)\, e^{\bar{\lambda}(\tau-t)}\, d\tau, \tag{A2}$$

$$\delta\mathbf{v}(t) = \frac{d\delta\mathbf{r}}{dt} + \bar{\lambda}\delta\mathbf{r}(t) - \mathbf{A}(t)\delta\mathbf{r}(t) - \int_{t-\bar{s}}^{t} \mathbf{K}(t,\tau)\delta\mathbf{r}(\tau)\, e^{\bar{\lambda}(\tau-t)}\, d\tau. \tag{A3}$$

As δs is small, we can approximate the integral in (A1),

$$\int_{t-\bar{s}-\delta s}^{t-\bar{s}} \mathbf{K}(t,\tau)\mathbf{r}(\tau)\, e^{\bar{\lambda}(\tau-t)}\, d\tau \approx \mathbf{K}(t, t-\bar{s}-\delta s/2)\mathbf{r}(t-\bar{s}-\delta s/2)\, e^{-\bar{\lambda}(\bar{s}+\delta s/2)}. \tag{A4}$$

We now define the scalar product between T-periodic functions

$$(\mathbf{a}(t), \mathbf{b}(t)) = \frac{1}{T}\int_0^T \mathbf{a}^\dagger(t)\mathbf{b}(t)\, dt, \tag{A5}$$

where † denotes Hermitian conjugation, and we consider a versor $\mathbf{e}(t)$ orthogonal to $\delta\mathbf{v}(t)$. Equations (A1) and (A4) yield

$$(\mathbf{e}(t), \mathbf{p}(t))\delta\lambda \approx \left(\mathbf{e}(t), \mathbf{K}(t, t-\bar{s}-\delta s/2)\mathbf{r}(t-\bar{s}-\delta s/2) \times e^{-\bar{\lambda}(\bar{s}+\delta s/2)}\right)\delta s. \tag{A6}$$

Defining $|\boldsymbol{\alpha}|$ as the vector made of the collection of the absolute values of the components of $\boldsymbol{\alpha}(t)$, (A6) implies

$$\begin{aligned}\left|(\mathbf{e}(t),\mathbf{p}(t))\delta\lambda\right| &\le \Big|(|\mathbf{e}(t)|, \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\| \\ &\quad\times |\mathbf{r}(t-\bar{s}-\delta s/2)|)\,\mathrm{e}^{-\bar{\lambda}(\bar{s}+\delta s/2)}\Big| \\ &\le (|\mathbf{e}(t)|, |\mathbf{r}(t-\bar{s}-\delta s/2)|)\,\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}(\bar{s}+\delta s/2)} \\ &\quad\times \max_{t\in[0,T]} \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\|, \end{aligned} \tag{A7}$$

therefore

$$\begin{aligned}|\delta\lambda| &\le \frac{(|\mathbf{e}(t)|, |\mathbf{r}(t-\bar{s}-\delta s/2)|)}{\left|(\mathbf{e}(t),\mathbf{p}(t))\right|}\,\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}(\bar{s}+\delta s/2)} \\ &\quad\times \delta s \max_{t\in[0,T]} \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\|. \end{aligned} \tag{A8}$$

Since $\alpha\exp(\mathrm{Re}\{\bar{\lambda}\}\alpha)$ is a monotonic function of α, $\bar{\tau} \in [t-\bar{s}, t]$ exists such that

$$\begin{aligned}&\int_{t-\bar{s}}^{t} \mathbf{K}(t,\tau)\bar{\mathbf{r}}(\tau)(\tau-t)\,\mathrm{e}^{\bar{\lambda}(\tau-t)}\,\mathrm{d}\tau \\ &= \mathbf{K}(t,\bar{\tau})\bar{\mathbf{r}}(\bar{\tau})\,\mathrm{e}^{\mathrm{iIm}\{\bar{\lambda}\}(\bar{\tau}-t)} \\ &\quad\times \int_{t-\bar{s}}^{t} (\tau-t)\,\mathrm{e}^{\mathrm{Re}\{\bar{\lambda}\}(\tau-t)}\,\mathrm{d}\tau \\ &= \mathbf{K}(t,\bar{\tau})\bar{\mathbf{r}}(\bar{\tau})\,\mathrm{e}^{\mathrm{iIm}\{\bar{\lambda}\}(\bar{\tau}-t)}\,\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}\bar{s}} \\ &\quad\times \left[\frac{\bar{s}}{\mathrm{Re}\{\bar{\lambda}\}} + \frac{1}{\mathrm{Re}\{\bar{\lambda}\}^2} - \frac{\mathrm{e}^{\mathrm{Re}\{\bar{\lambda}\}\bar{s}}}{\mathrm{Re}\{\bar{\lambda}\}^2}\right]. \end{aligned} \tag{A9}$$

Defining $\mathbf{p}'(t) = \mathbf{p}(t)\exp\left(\mathrm{Re}\{\bar{\lambda}\}\bar{s}\right)$, from (A2) and (A9), we find that for $\bar{s} \gg 0$,

$$\mathbf{p}'(t) \approx -\frac{\mathbf{K}(t,\bar{\tau})\bar{\mathbf{r}}(\bar{\tau})\,\mathrm{e}^{\mathrm{iIm}\{\bar{\lambda}\}(\bar{\tau}-t)}}{\mathrm{Re}\{\bar{\lambda}\}}\bar{s} \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} < 0, \tag{A10a}$$

$$\mathbf{p}'(t) \approx \left[\bar{\mathbf{r}}(t) + \frac{\mathbf{K}(t,\bar{\tau})\bar{\mathbf{r}}(\bar{\tau})\,\mathrm{e}^{\mathrm{iIm}\{\bar{\lambda}\}(\bar{\tau}-t)}}{\mathrm{Re}\{\bar{\lambda}\}^2}\right]\mathrm{e}^{\mathrm{Re}\{\bar{\lambda}\}\bar{s}} \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} > 0. \tag{A10b}$$

Since $\mathbf{K}(t,\tau)$ satisfies (2a), $0 < M < +\infty$ exists such that

$$\begin{aligned}\frac{|(|\mathbf{e}|, |\bar{\mathbf{r}}(t-\bar{s}-\delta s/2)|)|}{\left|(\mathbf{e},\mathbf{p})\right|} &= \frac{|(|\mathbf{e}|, |\bar{\mathbf{r}}(t-\bar{s}-\delta s/2)|)|}{\left|(\mathbf{e},\mathbf{p}')\right|} \\ &\quad\times \mathrm{e}^{\mathrm{Re}\{\bar{\lambda}\}\bar{s}} \le M\,\mathrm{e}^{\mathrm{Re}\{\bar{\lambda}\}\bar{s}}. \end{aligned} \tag{A11}$$

Therefore, because of (A10), for $\bar{s} \gg 0$, $0 < M_+, M_- \le M$ exist such that

$$\frac{|(|\mathbf{e}|, |\bar{\mathbf{r}}(t-\bar{s}-\delta s/2)|)|}{\left|(\mathbf{e},\mathbf{p}')\right|} \le \frac{M_-}{\bar{s}} \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} < 0, \tag{A12a}$$

$$\frac{|(|\mathbf{e}|, |\bar{\mathbf{r}}(t-\bar{s}-\delta s/2)|)|}{\left|(\mathbf{e},\mathbf{p}')\right|} \le M_+\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}\bar{s}} \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} > 0. \tag{A12b}$$

Accordingly, to the first order in δs, (A8) becomes

$$|\delta\lambda| \le M\delta s \max_{t\in[0,T]} \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\|, \tag{A13a}$$

and for $\bar{s} \gg 0$,

$$|\delta\lambda| \le \frac{M_-}{\bar{s}}\delta s \max_{t\in[0,T]} \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\| \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} < 0, \tag{A13b}$$

$$|\delta\lambda| \le M_+\mathrm{e}^{-\mathrm{Re}\{\bar{\lambda}\}\bar{s}}\delta s \max_{t\in[0,T]} \|\mathbf{K}(t,t-\bar{s}-\delta s/2)\| \quad \text{if } \mathrm{Re}\{\bar{\lambda}\} > 0. \tag{A13c}$$

Let us now consider any $s > \bar{s}$. We divide the interval $[\bar{s}, s]$ into N sub-intervals of size $\delta s = (s-\bar{s})/N$ and denote as $\delta\lambda_j$ the Floquet exponent variation due to the jth interval (with respect to the value attained at the beginning of the interval itself), such that

$$\left|\lambda - \lambda_j\right| = \left|\sum_{j=1}^{N} \delta\lambda_j\right| \le \sum_{j=1}^{N} \left|\delta\lambda_j\right|. \tag{A14}$$

Taking the limit for $N \to +\infty$ and using (A13), we find (9).

APPENDIX B: THE HARMONIC BALANCE APPROACH

Harmonic balance (HB) is a powerful numerical technique used to transform differential equations into algebraic systems that can be applied when the terms and the solution in the differential equation are time-periodic. As such, it is widely used in the circuit analysis and design tools, see, e.g., Ref. 31. In other words, HB seeks directly for the time-periodic solution without any explicit time-domain integration. Thus, the transient part of the solution is avoided altogether.

Consider first a scalar, real function $\alpha(t)$, whose frequency domain representation is built by means of the (truncated) exponential Fourier series

$$\alpha(t) = \sum_{h=-N_\mathrm{H}}^{N_\mathrm{H}} \tilde{\alpha}_h\,\mathrm{e}^{\mathrm{i}h\omega_0 t}, \tag{B1}$$

where $\tilde{\alpha}_h$ is the hth harmonic amplitude associated to the (angular) frequency $h\omega_0 = h2\pi/T$ (hth harmonic). Since $\alpha(t)$ is real, $\tilde{\alpha}_{-h} = \tilde{\alpha}_h^\star$ (* denotes complex conjugation); therefore, only $2N_\mathrm{H}+1$ real coefficients fully define the Fourier series. For numerical implementation, (B1) is replaced by the more effective trigonometric series representation.[31] However, we will stick here to the exponential form for what concerns theoretical developments.

The $]0,T]$ fundamental period is discretized in a set of $2N_\mathrm{H}+1$ time samples t_k ($k = 1,\ldots,2N_\mathrm{H}+1$), and the collection of the time sampled variable $\breve{\boldsymbol{\alpha}} = [\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_{2N_\mathrm{H}+1})]^\mathrm{T}$ is put in relation with the collection of harmonic amplitudes $\tilde{\boldsymbol{\alpha}} = [\tilde{\alpha}_{-N_\mathrm{H}}, \tilde{\alpha}_{-N_\mathrm{H}+1}, \ldots, \tilde{\alpha}_0, \ldots, \tilde{\alpha}_{N_\mathrm{H}}]^\mathrm{T}$ by means of the discrete Fourier transform (DFT) invertible linear operator Γ^{-1},

$$\breve{\boldsymbol{\alpha}} = \Gamma^{-1}\tilde{\boldsymbol{\alpha}} \Longleftrightarrow \tilde{\boldsymbol{\alpha}} = \Gamma\breve{\boldsymbol{\alpha}}. \tag{B2}$$

Clearly, for $N_{\rm H} \to \infty$, Γ^{-1} is the matrix representation of the operator defining the Fourier series representation of a T-periodic function.

In the frequency domain, the time derivative is represented by a diagonal matrix $\Omega \in \mathbb{C}^{(2N_{\rm H}+1)\times(2N_{\rm H}+1)}$ proportional to ω_0,[31]

$$\tilde{\dot{\boldsymbol{\alpha}}} = \Gamma \breve{\boldsymbol{\alpha}} = \Omega \tilde{\boldsymbol{\alpha}}, \tag{B3}$$

where $\dot{\alpha}(t) = {\rm d}\alpha/{\rm d}t$.

For a vector variable $\boldsymbol{\alpha}(t) \in \mathbb{R}^n$, (B2) and (B3) are easily generalized by expanding each time sample $\alpha(t_i)$ into a vector $\boldsymbol{\alpha}(t_i) \in \mathbb{R}^n$ and therefore defining the collection $\breve{\boldsymbol{\alpha}} = [\boldsymbol{\alpha}^{\rm T}(t_1), \boldsymbol{\alpha}^{\rm T}(t_2), \ldots, \boldsymbol{\alpha}^{\rm T}(t_{2N_{\rm H}+1})]^{\rm T} \in \mathbb{R}^{n(2N_{\rm H}+1)}$. Correspondingly, the HB representation is $\tilde{\boldsymbol{\alpha}} = [\tilde{\boldsymbol{\alpha}}_{-N_{\rm H}}^{\rm T}, \ldots, \tilde{\boldsymbol{\alpha}}_0^{\rm T}, \ldots, \tilde{\boldsymbol{\alpha}}_{N_{\rm H}}^{\rm T}]^{\rm T} \in \mathbb{C}^{n(2N_{\rm H}+1)}$. This allows to formally maintain (B2) and (B3) by defining two block diagonal matrices Γ_n^{-1} and Ω_n built replicating n times the fundamental operators Γ^{-1} and Ω,

$$\breve{\boldsymbol{\alpha}} = \Gamma_n^{-1}\tilde{\boldsymbol{\alpha}} \quad \tilde{\dot{\boldsymbol{\alpha}}} = \Omega_n \tilde{\boldsymbol{\alpha}}. \tag{B4}$$

The HB representation of $\boldsymbol{\beta}(t) = \mathbf{T}(t)\boldsymbol{\alpha}(t)$ (i.e., the convolution in frequency domain) and of its time derivative, where $\mathbf{T}(t)$ is a T-periodic matrix and $\boldsymbol{\alpha}(t)$ is a T-periodic vector, is derived as follows. Denoting as $\breve{\mathbf{T}}$ the $n \times n$ block diagonal matrix built expanding each element $t_{h,k}(t)$ of $\mathbf{T}(t)$ as a $(2N_{\rm H}+1) \times (2N_{\rm H}+1)$ diagonal matrix formed by the time samples $\breve{\mathbf{t}}_{h,k}$, we have

$$\tilde{\boldsymbol{\beta}} = \tilde{\mathbf{T}}\tilde{\boldsymbol{\alpha}} \quad \tilde{\dot{\boldsymbol{\beta}}} = \Omega_n \tilde{\boldsymbol{\beta}} = \Omega_n \tilde{\mathbf{T}}\tilde{\boldsymbol{\alpha}}, \tag{B5}$$

where $\tilde{\mathbf{T}} = \Gamma_n \breve{\mathbf{T}} \Gamma_n^{-1}$ is a Toepliz matrix built assembling the Fourier coefficients of the elements of $\mathbf{T}(t)$.[31]

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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