Decay and Smoothness for Eigenfunctions of Localization Operators

DECAY AND SMOOTHNESS FOR EIGENFUNCTIONS OF LOCALIZATION OPERATORS

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Abstract. We study decay and smoothness properties for eigenfunctions of localization operators $A_a^{\varphi_1,\varphi_2}$. Considering symbols a in the wide modulation space $M^{p,\infty}$ (containing the Lebesgue space $L^p(\mathbb{R}^d)$), $p<\infty$, and two general windows φ_1,φ_2 in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, we show that L^2-eigenfuctions with non-zero eigenvalue are indeed highly compressed onto a few Gabor atoms. Similarly, for symbols in the weighted modulation space $M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, $s>0$, the corresponding L^2-eigenfunctions are actually in $\mathcal{S}(\mathbb{R}^d)$.

An important role is played by quasi-Banach Wiener amalgam and modulation spaces. As a tool, new convolution relations for modulation spaces and multiplication relations for Wiener amalgam spaces in the quasi-Banach setting are exhibited.

1. Introduction

The study of localization operators has a longstanding tradition. They have become popular with the papers by I. Daubechies [12, 13] and from then widely investigated by several authors in different fields of mathematics: from signal analysis to pseudodifferential calculus, see, for instance [1, 2, 5, 8, 9, 11, 27, 28, 38, 39, 40, 44]. In quantum mechanics they were already known as Anty-Wick operators, cf. [36] and the references therein.

Localization operators can be introduced via the time-frequency representation known as short-time Fourier transform (STFT). Let us though introduce the STFT. Recall first the modulation M_ω and translation T_x operators of a function f on $\mathbb{R}^d$:

$$M_\omega f(t)=e^{2\pi i t\omega}f(t)\,,\qquad T_x f(t)=f(t-x)\,,\quad \omega,x\in\mathbb{R}^d.$$

For $z=(z_1,z_2)\in\mathbb{R}^{2d}$, we define the time-frequency shift $\pi(z)=M_{z_2}T_{z_1}$. Fix $\varphi\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$. We define the short-time Fourier transform of a tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ as

$$(1)\qquad V_\varphi f(x,\omega)=\langle f,\pi(x,\omega)\varphi\rangle=\mathcal{F}(fT_x\varphi)(\omega)=\int_{\mathbb{R}^d}f(y)\,\overline{\varphi(y-x)}\,e^{-2\pi i y\omega}\,dy.$$

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The localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2 is formally defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega. \tag{2}$$

If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical Anti-Wick operator and the mapping $a \mapsto A_a^{\varphi_1,\varphi_2}$ is a quantization rule in quantum mechanics [6, 14, 36, 44].

If one considers a symbol a in the Lebesgue space $L^q(\mathbb{R}^{2d})$ ($1 \leq q < \infty$) and window functions φ_1, φ_2 in the Feichtinger's algebra $M^1(\mathbb{R}^d)$ (see below for its definition) then the localization operator $A_a^{\varphi_1,\varphi_2}$ is in the Schatten class S_q (see [8]). This implies, in particular, that $A_a^{\varphi_1,\varphi_2}$ is a bounded and compact operator on $L^2(\mathbb{R}^d)$.

Sharp results for localization operators on modulation spaces (Banach case) were obtained in [10], thus concluding the open issues related to this problem.

The focus of this paper is the properties of *eigenfunctions* of localization operators.

The study of eigenvalues and eigenfunctions of a restrict class of localization operators, namely of the type $A_{\chi_\Omega}^{\varphi,\varphi}$, where Ω is a compact domain of the time-frequency plane and the window φ is in $L^2(\mathbb{R}^d)$, was pursued in [1, 3, 4]. The focus of the previous papers, as well as new recent contributions [32, 33], extending the previous results, is the asymptotic behavior of the eigenvalues, depending on the domain Ω.

Our perspective is different: properties of eigenfunctions of a localization operator having a general symbol a, without any requirement on the geometry of the function a. In particular, the symbol a does not need to have a compact support. Besides, the related localization operator $A_a^{\varphi_1,\varphi_2}$ is not necessarily a self-adjoint operator. It is easy to check that the adjoint of a localization operator is given by

$$(A_a^{\varphi_1,\varphi_2})^* = A_{\bar{a}}^{\varphi_2,\varphi_1};$$

hence the self-adjointness property forces the choice $\varphi_1 = \varphi_2$ and the symbol a real valued, as for the case $A_{\chi_\Omega}^{\varphi,\varphi}$ mentioned above. Our framework can allow the use of two different windows φ_1 and φ_2 to analyze and synthesize the signal f, respectively. Moreover, the symbol a can be a complex-valued function.

To chase our goal, the new idea is to use properties of quasi-Banach modulation and Wiener amalgam spaces. Some of these issues are investigated for the first time in this paper.

To give a flavour of these results, we recall the definition of modulation spaces (un-weighted case: for the more general cases see below).

Fix a non-zero window function $g \in \mathcal{S}(\mathbb{R}^d)$ and $p, q \in (0, \infty)$. The modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}} := \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\omega)|^p dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}} < \infty$$

(with natural modifications when $p=\infty$ or $q=\infty$). For $p,q\geq 1$ the function $\|\cdot\|_{M^{p,q}}$ is a norm, and different window functions $g\in\mathcal{S}(\mathbb{R}^d)$ yield equivalent norms, thus the same space. Modulation spaces $M^{p,q}(\mathbb{R}^d)$, $p,q\geq 1$, are Banach spaces, invented by H. Feichtinger in [20], where many of their properties were already investigated. The (quasi-)Banach space $M^{p,q}$, $p,q>0$, were first introduced and studied by Y.V. Galperin and S. Samarah in [23], see the next section for more details. Roughly speaking, the mapping f is in $M^{p,q}(\mathbb{R}^d)$ if locally behaves like a function in $\mathcal{F}L^q(\mathbb{R}^d)$ and "decays" as a function in $L^p(\mathbb{R}^d)$ at infinity.

In this paper we extend convolution relations for Banach modulation spaces exhibited in [8, 42] to the quasi-Banach ones, see Proposition 3.1 below. This result seems remarkable by itself.

Such convolution relations will be crucial for our main result, that can be simplified as follows (cf. Theorem 3.6).

Theorem 1.1. *Consider a symbol $a\in M^{p,\infty}(\mathbb{R}^{2d})$, $0<p<\infty$, and non-zero windows $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d)$. If $f\in L^2(\mathbb{R}^d)$ is an eigenfunction of the localization operator $A_a^{\varphi_1,\varphi_2}$, that is $A_a^{\varphi_1,\varphi_2}f=\lambda f$, with $\lambda\neq 0$, then $f\in\bigcap_{\gamma>0}M^{\gamma}(\mathbb{R}^d)$.*

Roughly speaking, this means that L^2 eigenfunctions of such localization operators reveal to be extremely well-localized. To make this statement more precise, we use Gabor frames. Let $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ be a lattice of the time-frequency plane. The set of time-frequency shifts $\mathcal{G}(g,\Lambda)=\{\pi(\lambda)g:\ \lambda\in\Lambda\}$, for a non-zero $g\in L^2(\mathbb{R}^d)$, is called a Gabor system. The set $\mathcal{G}(g,\Lambda)$ is a Gabor frame if there exist constants $A,B>0$ such that

$$A\|f\|_2^2\leq\sum_{\lambda\in\Lambda}|\langle f,\pi(\lambda)g\rangle|^2\leq B\|f\|_2^2,\qquad\forall f\in L^2(\mathbb{R}^d). \tag{3}$$

If the eigenfunction f above satisfies $f\in\bigcap_{\gamma>0}M^{\gamma}(\mathbb{R}^d)$, then f is highly compressed onto a few Gabor atoms $\pi(\lambda)g$. Indeed, for $N\in\mathbb{N}_+$, its N-term approximation error $\sigma_N(f)$ presents super-polynomial decay. We define

$$\Sigma_N=\{p=\sum_{k,n\in F}c_{k,n}\pi(\alpha k,\beta n)g\ :\ c_{k,n}\in\mathbb{C},\ F\subset\mathbb{Z}^d\times\mathbb{Z}^d,\ \operatorname{card}F\leq N\} \tag{4}$$

(the set of all linear combinations of Gabor atoms consisting of at most N terms). Note that Σ_N is not a linear subspace since $\Sigma_N+\Sigma_N=\Sigma_{2N}$. That is why the approximation of a signal f by elements of Σ_N is often referred to as non-linear approximation. Given a function $f\in L^2(\mathbb{R}^d)$, the N-term approximation error in

$L^2(\mathbb{R}^d)$ is

$$\sigma_N(f) = \inf_{p\in\Sigma_N} \|f-p\|_2. \tag{5}$$

That is, $\sigma_N(f)$ is the error produced when f is approximated optimally by a linear combination of N Gabor atoms.

We shall show in Corollary 3.8 that, if $f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d)$, then, for every $r>0$ there exists $C = C(r,f)$ such that

$$\sigma_N(f) \leq CN^{-r}.$$

Since $\sigma_N(f)$ is the error produced when f is approximated optimally by a linear combination of N Gabor atoms, the decay above shows the high compression of the eigenfunction onto such atoms.

Another main result (Theorem 3.9) states, in the same spirit, that for symbols in the weighted modulation space $M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, $s>0$ (see Definition 2.4 below), the corresponding L^2 eigenfunctions are actually in $\mathcal{S}(\mathbb{R}^d)$. These symbol classes include certain measures and the result applies, in particular, to Gabor multipliers. We leave the precise statement to the interested reader.

In short, the paper is organized as follows. Section 2 is devoted to the function spaces involved in our study. In particular, we show new multiplication relations for Wiener amalgam spaces in the quasi-Banach setting and we prove convolution relations for quasi-Banach modulation spaces. Section 3 represents the core of the paper. We first exhibit continuity results for Weyl operators on modulation spaces (involving the quasi-Banach setting), cf. Theorem 3.2. Then we study the eigenfunctions' properties for Weyl operators (Propositions 3.4 and 3.5 below). Next we show our main result on the eigenfunctions' regularity and smoothness of localization operators: Theorem 3.6 and related consequences in terms of the N-term approximation (Proposition 3.7 and Corollary 3.8). The last Section 4 is devoted to the study of eigenfunctions of localization operators with symbols in the weighted Lebesgue spaces $L^q_m(\mathbb{R}^d)$, $1\leq q<\infty$. This section suggests a wider study of the topic for localization operators on groups [44]. We shall pursue this issue in a subsequent paper.

2. Preliminaries

Notation. We define $t^2 = t\cdot t$, for $t\in\mathbb{R}^d$, and $xy = x\cdot y$ is the scalar product on $\mathbb{R}^d$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f,g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f,g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform of a function

f on $\mathbb{R}^d$ is normalized as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\, dx.$$

The involution g^* is given by $g^*(t) = \overline{g(-t)}$. We use T^* for the adjoint of an operator T. Observe that $\pi(z)^* = \pi(-z)$ and the following commutation relations hold

$$\pi(z)\pi(w) = e^{-2\pi i z_1 w_2}\pi(w)\pi(z). \tag{6}$$

2.1. **Weight functions.** In the sequel v will always be a continuous, positive, submultiplicative weight function on $\mathbb{R}^d$ (or on $\mathbb{Z}^d$), i.e., $v(z_1+z_2) \le v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^d$ (or for all $z_1, z_2 \in \mathbb{Z}^d$). We say that $m \in \mathcal{M}_v(\mathbb{R}^d)$ (or $m \in \mathcal{M}_v(\mathbb{Z}^d)$) if m is a positive, continuous weight function on $\mathbb{R}^d$ (or on $\mathbb{Z}^d$) *v-moderate*: $m(z_1+z_2) \le Cv(z_1)m(z_2)$ for all $z_1, z_2 \in \mathbb{R}^d$ (or for all $z_1, z_2 \in \mathbb{Z}^d$). We will mainly work with polynomial weights of the type

$$v_s(z) = \langle z\rangle^s = (1+|z|^2)^{s/2}, \quad s \in \mathbb{R}, \quad z \in \mathbb{R}^d \text{ (or } \mathbb{Z}^d). \tag{7}$$

Observe that, for $s < 0$, v_s is $v_{|s|}$-moderate.

Given two weight functions m_1, m_2 on $\mathbb{R}^d$ (or $\mathbb{Z}^d$), we write

$$(m_1 \otimes m_2)(x,\omega) = m_1(x)m_2(\omega), \quad x,\omega \in \mathbb{R}^d \quad (\text{or } \mathbb{Z}^d).$$

2.2. **Spaces of sequences.**

Definition 2.1. *For $0 < p, q \le \infty$, $m \in \mathcal{M}_v(\mathbb{Z}^{2d})$, the space $\ell^{p,q}_m(\mathbb{Z}^{2d})$ consists of all sequences $a = (a_{k,n})_{k,n\in\mathbb{Z}^d}$ for which the (quasi-)norm*

$$\|a\|_{\ell^{p,q}_m} = \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|a_{k,n}|^p m(k,n)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

(with obvious modification for $p = \infty$ or $q = \infty$) is finite.

For $p = q$, $\ell^{p,q}_m(\mathbb{Z}^{2d}) = \ell^p_m(\mathbb{Z}^{2d})$, the standard spaces of sequences. Namely, in dimension d, for $0 < p \le \infty$, m a weight function on $\mathbb{Z}^d$, a sequence $a = (a_k)_{k\in\mathbb{Z}^d}$ is in $\ell^p_m(\mathbb{Z}^d)$ if

$$\|a\|_{\ell^p_m} = \left(\sum_{k\in\mathbb{Z}^d}|a_k|^p m(k)^p\right)^{\frac{1}{p}} < \infty.$$

Here there are some properties we need in the sequel [22, 23]:

(i) *Inclusion relations*: If $0 < p_1 \le p_2 \le \infty$, then $\ell^{p_1}_m(\mathbb{Z}^d) \hookrightarrow \ell^{p_2}_m(\mathbb{Z}^d)$, for any positive weight function m on $\mathbb{Z}^d$.

(ii) *Young's convolution inequality*: Consider $m \in \mathcal{M}_v(\mathbb{Z}^d)$, $0 < p, q, r \leq \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \quad \text{for} \quad 1 \leq r \leq \infty \tag{8}$$

and

$$p = q = r, \quad \text{for} \quad 0 < r < 1. \tag{9}$$

Then for all $a \in \ell^p_m(\mathbb{Z}^d)$ and $b \in \ell^q_v(\mathbb{Z}^d)$, we have $a * b \in \ell^r_m(\mathbb{Z}^d)$, with

$$\|a * b\|_{\ell^r_m} \leq C \|a\|_{\ell^p_m} \|b\|_{\ell^q_v}, \tag{10}$$

where C is independent of p, q, r, a and b. If $m \equiv v \equiv 1$, then $C = 1$.

(iii) *Hölder's inequality*: For any positive weight function m on $\mathbb{Z}^d$, $0 < p, q, r \leq \infty$, with $1/p + 1/q = 1/r$,

$$\ell^p_m(\mathbb{Z}^d) \cdot \ell^q_{1/m}(\mathbb{Z}^d) \hookrightarrow \ell^r(\mathbb{Z}^d). \tag{11}$$

2.3. **Wiener Amalgam Spaces** [17, 18, 19, 21, 23, 29, 34, 35].

Definition 2.2. *Consider $p, q \in (0, \infty]$, a weight function $m \in \mathcal{M}_v$ and the compact set $Q = [0,1]^d$. The Wiener amalgam space $W(L^p, L^q_m)(\mathbb{R}^d)$ consists of the functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $f \in L^p_{loc}(\mathbb{R}^d)$ and for which the* control function*:*

$$F^Q_f(k) := \|f \cdot T_k \chi_Q\|_{L^p} \in \ell^q_m\left(\mathbb{Z}^d\right), \qquad k \in \mathbb{Z}^d. \tag{12}$$

The norm on $W(L^p, L^q_m)$ is given by

$$\begin{aligned} \|f\|_{W\left(L^p, L^q_m\right)} &:= \left\| F^Q_f(k) \right\|_{\ell^q_m} \\ &= \left\| \|f \cdot T_k \chi_Q\|_{L^p} \right\|_{\ell^q_m} \\ &= \left(\sum_{k \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} |f(t)|^p \chi_Q(t-k)\, \mathrm{d}t \right)^{\frac{q}{p}} m^q(k) \right)^{\frac{1}{q}}, \end{aligned} \tag{13}$$

with suitable adjustments for the cases $p, q = \infty$.

This special definition allows us to grasp the sense of the amalgam: we first view f "locally" through translations $T_k \chi_Q$ of the sharp cutoff function χ_Q, and measure those local pieces in the L^p-norm, then we measure the global behavior of those local pieces according to the ℓ^q_m-norm. The "window" through which we view f locally need not be a unit d-dimensional cube, cf. [18, 23, 29, 35]. In the sequel we shall use the following properties:

(i) Inclusion relations: For $0 < p_1 \leq p_2 \leq \infty$, $0 < q_2 \leq q_1 \leq \infty$, we have

$$W(L^{p_2}, L^{q_2}_m)(\mathbb{R}^d) \hookrightarrow W(L^{p_1}, L^{q_1}_m)(\mathbb{R}^d). \tag{14}$$

(ii) Convolution relations (for the quasi-Banach case see [23, Lemma 2.9]): Consider $m_i \in \mathcal{M}_v$, $0 < p_i, q_i \leq \infty$, $i \in \{1,2,3\}$, and $p_3 \geq 1$. Assume that $L^{p_1} * L^{p_2} \hookrightarrow L^{p_3}$ and $\ell^{q_1}_{m_1} * \ell^{q_2}_{m_2} \hookrightarrow \ell^{q_3}_{m_3}$, then

$$W(L^{p_1}, L^{q_1}_{m_1}) * W(L^{p_2}, L^{q_2}_{m_2}) \hookrightarrow W(L^{p_3}, L^{q_3}_{m_1}). \tag{15}$$

(iii) For $m \in \mathcal{M}_v$, $0 < p \leq \infty$, we have

$$L^p_m = W(L^p, L^p_m). \tag{16}$$

Proposition 2.3 (Multiplication relations)**.** *Consider $m, w \in \mathcal{M}_v$, $0 < p_i, q_i \leq \infty$, $i = \{1,2,3\}$. Assume $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$, then*

$$W(L^{p_1}, L^{q_1}_m) \cdot W(L^{p_2}, L^{q_2}_{w/m}) \hookrightarrow W(L^{p_3}, L^{q_3}_w). \tag{17}$$

Proof. The result is well known for $1 \leq p_i, q_i \leq \infty$, cf. [17, 29]. Here we show that the same proof works for quasi-Banach spaces. Indeed, since the standard Hölder inequality holds for Lebesgue exponents in $(0, +\infty]$, for $f_1 \in W(L^{p_1}, L^{q_1}_m)$, $f_2 \in W(L^{p_2}, L^{q_2}_{w/m})$ we have

$$\|f_1 f_2 T_k \chi_Q\|_{L^{p_3}} = \|(f_1 T_k \chi_Q)(f_2 T_k \chi_Q)\|_{L^{p_3}} \leq \|f_1 T_k \chi_Q\|_{L^{p_1}} \| \|f_2 T_k \chi_Q\|_{L^{p_2}}.$$

Defining $a_k = \|f_1 T_k \chi_Q\|_{p_1}$ and $b_k = \|f_2 T_k \chi_Q\|_{p_2}$ and using Hölder's inequality for sequences $\ell^{q_1}\ell^{q_2} \hookrightarrow \ell^{q_3}$, for $1/q_1 + 1/q_2 = 1/q_3$ ($0 < q_i \leq \infty$, $i = 1,2,3$), we obtain

$$\|a_k b_k w(k)\|_{\ell^{q_3}} = \|(a_k m(k))(b_k w(k)/m(k))\|_{\ell^{q_3}} \leq \|a_k m(k)\|_{\ell^{q_1}} \|b_k/m(k)\|_{\ell^{q_2}}.$$

This completes the proof. □

2.4. **Modulation Spaces.** We use the extension to quasi-Banach spaces introduced first by Y.V. Galperin and S. Samarah in [23].

Definition 2.4. *Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight $m \in \mathcal{M}_v$ and $0 < p, q \leq \infty$. The modulation space $M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (quasi-)norm*

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} \tag{18}$$

(obvious changes with $p = \infty$ or $q = \infty$) is finite.

The most famous modulation spaces are those $M^{p,q}_m(\mathbb{R}^d)$ with $1 \leq p, q \leq \infty$, invented by H. Feichtinger in [20]. In that paper he proved they are Banach spaces, whose norm does not depend on the window g, in the sense that different window functions in $\mathcal{S}(\mathbb{R}^d)$ yield equivalent norms. Moreover, the window class $\mathcal{S}(\mathbb{R}^d)$ can be extended to the modulation space $M^{1,1}_v(\mathbb{R}^d)$ (so-called Feichtinger algebra).

For shortness, we write $M^p_m(\mathbb{R}^d)$ in place of $M^{p,p}_m(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ if $m \equiv 1$.

The modulation spaces $M^{p,q}_m(\mathbb{R}^d)$, $0<p,q<1$, where introduced almost twenty years later by Y.V. Galperin and S. Samarah in [23] and then studied in [30, 34, 43] (see also references therein). In this framework, it appears that the largest natural class of windows universally admissible for all spaces $M^{p,q}_m(\mathbb{R}^d)$, $0<p,q\leq\infty$ (with m having at most polynomial growth) is the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. There are thousands of papers involving modulation spaces with indices $1\leq p,q\leq\infty$, whereas very few works deal with the quasi-Banach case $0<p,q\leq 1$. Indeed, many properties related to the latter case are still unexplored.

In this paper their contribution is fundamental, since they are the key tool for understanding the properties of eigenfunctions of localization operators having symbols with some decay at infinity, measured in the L^p-mean, $1\leq p<\infty$.

In the sequel we shall use inclusion relations for modulation spaces, we refer to [23, Theorem 3.4] and [24, Theorem 12.2.2].

Theorem 2.5. *Let $m\in\mathcal{M}_v(\mathbb{R}^{2d})$. If $0<p_1\leq p_2\leq\infty$ and $0<q_1\leq q_2\leq\infty$ then $M^{p_1,q_1}_m(\mathbb{R}^d)\hookrightarrow M^{p_2,q_2}_m(\mathbb{R}^d)$.*

The duality properties for modulation spaces with indices $p,q<1$ where studied in [31] and completed in [43, Proposition 6.4, page 163]:

Proposition 2.6. *Let $s\in\mathbb{R}$ and $0<p,q<\infty$. If $p\geq 1$ we denote $1/p+1/p'=1$; if $0<p<1$ we denote $p'=\infty$. Then $(M^{p,q}_{1\otimes v_s}(\mathbb{R}^d))'=M^{p',q'}_{1\otimes v_{-s}}(\mathbb{R}^d)$.*

We will repeatedly use the following result, cf. [23, Theorem 3.3] (see also [24, Theorem 12.2.1] for $p\geq 1$).

Theorem 2.7. *Assume that $m\in\mathcal{M}_v$. For $0<p<1$ let g be a non-zero window in $\mathcal{S}(\mathbb{R}^d)$, though, for $1\leq p\leq\infty$, the function g can be chosen in the larger space $M^1_v(\mathbb{R}^d)$. If $f\in M^p_m(\mathbb{R}^d)$, $0<p\leq\infty$, then $V_gf\in W(L^\infty,L^p_m)$ and there exists $C>0$, independent of f, such that*

$$\|V_gf\|_{W(L^\infty,L^p_m)}\leq C\|V_gf\|_{L^p_m}. \tag{19}$$

We also need to recall the inversion formula for the STFT (see ([24, Proposition 11.3.2]: assume $g\in M^1_v(\mathbb{R}^d)\setminus\{0\}$, $1\leq p,q\leq\infty$, $f\in M^{p,q}_m(\mathbb{R}^d)$, then

$$f=\frac{1}{\|g\|_2^2}\int_{\mathbb{R}^{2d}}V_gf(x,\omega)M_\omega T_xg\,dx\,d\omega, \tag{20}$$

and the equality holds in $M^{p,q}_m(\mathbb{R}^d)$.

2.5. **Gabor Frames.** Let $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ be a lattice of the time-frequency plane. The set of time-frequency shifts $\mathcal{G}(g,\Lambda)=\{\pi(\lambda)g:\ \lambda\in\Lambda\}$, for a non-zero $g\in L^2(\mathbb{R}^d)$, is called a Gabor system. The set $\mathcal{G}(g,\Lambda)$ is a Gabor frame, if there exist constants $A,B>0$ such that (3) holds true.

If $\mathcal{G}(g,\Lambda)$ is a Gabor frame, then it can be shown that the frame operator

$$Sf=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)g,\quad f\in L^2(\mathbb{R}^d)$$

is a topological isomorphism on $L^2(\mathbb{R}^d)$. Moreover, the system $\mathcal{G}(\gamma,\Lambda)$, where the function $\gamma=S^{-1}g\in L^2(\mathbb{R}^d)$ is the canonical dual window of g, is a Gabor frame and we have the reproducing formula

$$f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)\gamma=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)\gamma\rangle\pi(\lambda)g,\quad f\in L^2(\mathbb{R}^d) \tag{21}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. More generally, any window function $\gamma\in L^2(\mathbb{R}^d)$, such that (21) is satisfied, is called alternative dual window for g. In general, given two functions $g,\gamma\in L^2(\mathbb{R}^d)$, it is customary to extend the notion of Gabor frame operator $S_{g,\gamma}$, related to g,γ, as follows

$$S_{g,\gamma}f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)\gamma,\quad f\in L^2(\mathbb{R}^d),$$

whenever the previous operator is well-defined. With this notation the reproducing formula (21) can be rephrased as $S_{g,\gamma}=I$ on $L^2(\mathbb{R}^d)$, with I being the identity operator.

Modulation spaces provide a natural setting for time-frequency analysis, thanks to discrete equivalent norms produced by means of Gabor frames. The key result is the following (see [24, Chapter 12] for $1\le p,q\le\infty$, and [23, Theorem 3.7] for $0<p,q\le 1$).

Theorem 2.8. *Assume $m\in\mathcal{M}_v(\mathbb{Z}^{2d})$, $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$, $g,\gamma\in\mathcal{S}(\mathbb{R}^d)$ such that $S_{g,\gamma}=I$ on $L^2(\mathbb{R}^d)$. Then*

$$f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)\gamma=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)\gamma\rangle\pi(\lambda)g,\quad f\in L^2(\mathbb{R}^d) \tag{22}$$

with unconditional convergence in $M^{p,q}_m(\mathbb{R}^d)$ if $0<p,q<\infty$ and with weak- convergence in $M^\infty_{1/v}(\mathbb{R}^d)$ otherwise. Furthermore, there are constants $0<A\le B$ such that, for all $f\in M^{p,q}_m(\mathbb{R}^d)$,*

$$A\|f\|_{M^{p,q}_m}\le\left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|\langle f,\pi(\alpha k,\beta n)g\rangle|^p m(\alpha k,\beta n)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\le B\|f\|_{M^{p,q}_m}, \tag{23}$$

independently of p,q, and m. Similar inequalities hold with g replaced by γ.

In other words,

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)}\asymp\|(\langle f,\pi(\lambda)g\rangle)_\lambda\|_{\ell^{p,q}_m(\Lambda)}\asymp\|(V_gf(\lambda))_\lambda\|_{\ell^{p,q}_m(\Lambda)}.$$

3. Main Results

We first study convolution relations for modulations spaces. Let us recall that, for the Banach cases, convolution relations were studied in [8] and [41, 42]. Our approach is general, the techniques use Gabor frames via the equivalence (23), plus Hölder's and Young's inequalities for sequences.

Proposition 3.1. *Let $\nu(\omega) > 0$ be an arbitrary weight function on $\mathbb{R}^d$, $0 < p, q, r, t, u, \gamma \leq \infty$, with*

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}, \tag{24}$$

and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \qquad \text{for } 1 \leq r \leq \infty \tag{25}$$

whereas

$$p = q = r, \qquad \text{for } 0 < r < 1. \tag{26}$$

For $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $m_1(x) = m(x, 0)$ and $m_2(\omega) = m(0, \omega)$ are the restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$, and likewise for v. Then

$$M^{p,u}_{m_1 \otimes \nu}(\mathbb{R}^d) * M^{q,t}_{v_1 \otimes v_2 \nu^{-1}}(\mathbb{R}^d) \hookrightarrow M^{r,\gamma}_m(\mathbb{R}^d) \tag{27}$$

with norm inequality

$$\|f * h\|_{M^{r,\gamma}_m} \lesssim \|f\|_{M^{p,u}_{m_1 \otimes \nu}} \|h\|_{M^{q,t}_{v_1 \otimes v_2 \nu^{-1}}}.$$

Proof. We use the key idea in [8, Proposition 2.4] to measure the modulation space norm with respect to the Gaussian windows $g_0(x) = e^{-\pi x^2}$ and $g(x) = 2^{-d/2} e^{-\pi x^2/2} = (g_0 * g_0)(x) \in \mathcal{S}(\mathbb{R}^d)$.

A straightforward computation shows $V_g f(x, \omega) = e^{-2\pi i x \omega}(f * M_\omega g^*)(x)$ (recall that $g^*(x) = \overline{g(-x)}$). Thus, using the identity $M_\omega(g_0^* * g_0^*) = M_\omega g_0^* * M_\omega g_0^*$, we can write the STFT of $f * h$ as follows:

$$V_g(f * h)(x, \omega) = e^{-2\pi i x \omega}\big((f * h) * M_\omega g^*\big)(x) = e^{-2\pi i x \omega}\big((f * M_\omega g_0^*) * (h * M_\omega g_0^*)\big)(x)\,.$$

In the following, we first use the norm equivalence (23), written in terms of the STFT as $\|f\|_{M^{p,q}_m} \asymp \|(V_g f(\lambda))_{\lambda \in \Lambda}\|_{\ell^{p,q}_m(\Lambda)}$, where $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. Then we majorize m by

$$m(\alpha k, \beta n) \leq m(\alpha k, 0) v(0, \beta n) = m_1(\alpha k) v_2(\beta n),$$

and finally use Young's convolution inequality for sequences in the k-variable and Hölder's one in the n-variable. The indices p, q, r, s, t, u fulfil the equalities in the assumptions. In details,

$$
\begin{aligned}
\|f*h\|_{M^{r,\gamma}_m} &\asymp \|(V_g(f*h))(\alpha k,\beta n)m(\alpha k,\beta n))_{k,n}\|_{\ell^{r,\gamma}(\mathbb{Z}^{2d})}\\
&\leq \left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{m\in\mathbb{Z}^d}|(f*M_{\beta n}g_0^*)*(h*M_{\beta n}g_0^*)(\alpha k)|^r m_1(\alpha k)^r\right)^{\gamma/r} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&= \left(\sum_{n\in\mathbb{Z}^d}\|(f*M_{\beta n}g_0^*)*(h*M_{\beta n}g_0^*)\|^\gamma_{\ell^r_{m_1}(\alpha\mathbb{Z}^d)} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{b\in\mathbb{Z}^d}\|f*M_{\beta n}g_0^*\|^\gamma_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\|h*M_{\beta n}g_0^*\|^\gamma_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)} v_2(\beta n)^\gamma\right)^{1/\gamma}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}^d}\|f*M_{\beta n}g_0^*\|^u_{\ell^p_{m_1}(\alpha\mathbb{Z}^d)}\nu(\beta n)^u\right)^{\frac{1}{u}}\left(\sum_{n\in\mathbb{Z}^d}\|h*M_{\beta n}g_0^*\|^t_{\ell^q_{v_1}(\alpha\mathbb{Z}^d)}\frac{v_2(\beta n)^t}{\nu(\beta n)^t}\right)^{\frac{1}{t}}\\
&\asymp \|((V_{g_0}f)(\lambda))_\lambda\|_{\ell^{p,u}_{m_1\otimes\nu}(\Lambda)}\,\|((V_{g_0}h)(\lambda))_\lambda\|_{\ell^{q,t}_{v_1\otimes v_2\nu^{-1}}(\Lambda)}\\
&\asymp \|f\|_{M^{p,u}_{m_1\otimes\nu}}\,\|h\|_{M^{q,t}_{v_1\otimes v_2\nu^{-1}}}.
\end{aligned}
$$

This concludes the proof. □

3.1. **Weyl Operators.** Every continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a pseudodifferential operator in the Weyl form L_σ, with Weyl symbol $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$. The operator is formally defined by

$$L_\sigma f(x)=\int_{\mathbb{R}^d}\sigma\left(\frac{x+y}{2},\omega\right)e^{2\pi i(x-y)\,\omega}f(y)\,dyd\omega. \tag{28}$$

The crucial relation between the action of the Weyl operator L_σ on time-frequency shifts and the short-time Fourier transform of its symbol is contained in [25, Lemma 3.1].

Lemma 3.1. *Consider $g\in\mathcal{S}(\mathbb{R}^d)$, $\Phi=W(g,g)$. Then, for $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$,*

$$|\langle L_\sigma\pi(z)g,\pi(w)g\rangle|=\left|V_\Phi\sigma\left(\frac{z+w}{2},j(w-z)\right)\right|,\quad z,w\in\mathbb{R}^{2d}, \tag{29}$$

where $j(z_1,z_2)=(z_2,-z_1)$.

Theorem 3.2. *(i) Assume $p,q,\gamma\in(0,\infty]$ such that*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{\gamma}. \tag{30}$$

If $\sigma\in M^{p,\min\{1,\gamma\}}(\mathbb{R}^{2d})$, then the pseudodifferential operator L_σ, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, extends uniquely to a bounded operator from $M^q(\mathbb{R}^d)$ to $M^\gamma(\mathbb{R}^d)$.

(ii) If $s,r\geq 0$, $t\geq r+s$, and the symbol $\sigma\in M^{\infty,1}_{v_s\otimes v_t}(\mathbb{R}^{2d})$, then the pseudodifferential operator L_σ extends uniquely to a bounded operator from $M^2_{v_r}(\mathbb{R}^d)$ into $M^2_{v_{r+s}}(\mathbb{R}^d)$.

Proof. (i) Assume first $\gamma\geq 1$, then, by (30), $p\geq\gamma\geq 1$ and $q\geq\gamma\geq 1$ and the claim was proved by Toft in [41, Theorem 4.3].

Consider now the case $\gamma<1$. We set

$$K_{\mu,\lambda}=\langle L_\sigma(\pi(\lambda)g),\pi(\mu)g\rangle,\quad \lambda,\mu\in\Lambda, \tag{31}$$

where $\Lambda=\alpha\mathbb{Z}^{2d}$ is a lattice in $\mathbb{R}^{2d}$ and $g\in\mathcal{S}(\mathbb{R}^d)$. Notice that, by Lemma 3.1,

$$|K_{\mu,\lambda}|=\left|V_\Phi\sigma\left(\frac{\mu+\lambda}{2},j(\mu-\lambda)\right)\right|,\quad \lambda,\mu\in\Lambda. \tag{32}$$

We choose the Gaussian window $g(t)=2^{d/4}e^{-\pi|t|^2}$, $t\in\mathbb{R}^d$. Recall that $\mathcal{G}(g,\Lambda)$ is a frame whenever $\Lambda=\alpha\mathbb{Z}^{2d}$ (with $\alpha<1$) and it admits a dual window $h\in\mathcal{S}(\mathbb{R}^d)$, see [26]. Then, taking $f\in M^q(\mathbb{R}^d)$ we can expand f by means of the Gabor atoms

$$f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)h\rangle\pi(\lambda)g$$

and write

$$L_\sigma f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)h\rangle L_\sigma(\pi(\lambda)g).$$

Expanding now the function $L_\sigma f$ we get

$$L_\sigma f=\sum_{\mu\in\Lambda}\sum_{\lambda\in\Lambda}\langle L_\sigma(\pi(\lambda)g),\pi(\mu)g\rangle\pi(\mu)h.$$

Using the continuity properties of the coefficient operator C_γ from $M^{p,q}$ into $\ell^{p,q}(\Lambda)$ and the synthesis operator from $\ell^{p,q}(\Lambda)$ into $M^{p,q}$, $0<p,q\leq\infty$, cf. [23, Theorems 3.5 and 3.6], we can decompose the Weyl operator as $L_\sigma=C_\gamma\circ K\circ D_\gamma$, with the operator K defined on sequences by the infinite matrix $K_{\mu,\lambda}$ above and the following diagram is commutative:

$$\begin{tikzcd} M^q \arrow[r, "L_\sigma"] \arrow[d, "C_h"'] & M^\gamma \\ \ell^q \arrow[r, "K"] & \ell^\gamma \arrow[u, "D_h"'] \end{tikzcd}$$

where L_σ is viewed as an operator with dense domain $\mathcal{S}(\mathbb{R}^d)$. Whence, it is enough to prove the continuity of the operator K from ℓ^q into ℓ^γ.

For the Gaussian window $g(t)=2^{d/4}e^{-\pi|t|^2}$, $t\in\mathbb{R}^d$, it is a straightforward calculation to show that the related Wigner distribution is the rescaled Gaussian $\Phi=W(g,g)(x,\omega)=2^de^{-2\pi(x^2+\omega^2)}$. Now, the Gabor system $\mathcal{G}(\Phi,\Lambda\times\Lambda)$ is a frame for $L^2(\mathbb{R}^{2d})$, whenever $\Lambda=\alpha\mathbb{Z}^{2d}$, for any $\alpha>0$ satisfying $\alpha^2<1/2$, as shown by M.

de Gosson in [15, Proposition 10] (take $\hbar = 1/(4\pi)$). Hence we choose $\alpha < 1/\sqrt{2}$ from the beginning, so that all the Gabor systems involved are frames.

By assumption, $\sigma \in M^{p,\gamma}(\mathbb{R}^{2d})$, and using the characterization in Theorem 2.8, this is equivalent to saying

$$\|\sigma\|_{M^{p,\gamma}} \asymp \left(\sum_{\lambda\in\Lambda}\left(\sum_{\mu\in\Lambda}|\langle\sigma,\pi(\mu,\lambda)\Phi\rangle|^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}}. \tag{33}$$

This is the main ingredient in the continuity of $K : \ell^p \to \ell^\gamma$. Indeed, for any sequence $c = (c_\lambda)$, we have, using $\|c\|_{\ell^1} \le \|c\|_{\ell^\gamma}$, $0 < \gamma \le 1$, and (32), we can write

$$\begin{aligned}
\|Kc\|_{\ell^\gamma} &\le \left(\sum_{\mu\in\Lambda}\sum_{\lambda\in\Lambda}|K_{\mu,\lambda}|^\gamma|c_\lambda|^\gamma\right)^{\frac{1}{\gamma}}\\
&= \left(\sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda}\left|V_\Phi\sigma\left(\frac{\lambda+\mu}{2}, j(\mu-\lambda)\right)\right|^\gamma|c_\lambda|^\gamma\right)^{\frac{1}{\gamma}}\\
&= \left(\sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda}\left|V_\Phi\sigma\left(\frac{2\mu+j(\lambda)}{2}, \lambda\right)\right|^\gamma|c_{\mu+j(\lambda)}|^\gamma\right)^{\frac{1}{\gamma}}\\
&\le \left(\sum_{\lambda\in\Lambda}\left(\sum_{\mu\in\Lambda}\left|V_\Phi\sigma\left(\frac{2\mu+j(\lambda)}{2}, \lambda\right)\right|^p\right)^{\frac{\gamma}{p}}\left(\sum_{\lambda\in\Lambda}|c_{\mu+j(\lambda)}|^q\right)^{\frac{\gamma}{q}}\right)^{\frac{1}{\gamma}},
\end{aligned}$$

where we applied Hölder's inequality with

$$\frac{1}{\frac{p}{\gamma}}+\frac{1}{\frac{q}{\gamma}} = 1.$$

Hence, performing the change: $\mu + j(\lambda) = \lambda' \in \Lambda$ in the last sum we get $\|c\|_{\ell^q}$, whereas the change of variables $\mu + j(\lambda)/2 = \mu' \in \frac{1}{2}\Lambda$ gives

$$\begin{aligned}
\|Kc\|_{\ell^\gamma} &\le \|c\|_{\ell^q}\left(\sum_{\lambda\in\Lambda}\left(\sum_{\mu\in\Lambda}\left|V_\Phi\sigma\left(\frac{2\mu+j(\lambda)}{2}, \lambda\right)\right|^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}}\\
&\le \|c\|_{\ell^q}\left(\sum_{\lambda\in\Lambda}\left(\sum_{\mu'\in\frac{1}{2}\Lambda}|\langle\sigma,\pi(\mu',\lambda)\Phi\rangle|^p\right)^{\frac{\gamma}{p}}\right)^{\frac{1}{\gamma}} \le \|c\|_{\ell^q}\|\sigma\|_{M^{p,\gamma}},
\end{aligned}$$

where in the last inequality we used (33). This proves the claim.

(ii) Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$. From the inversion formula (20),

$$V_g(L_\sigma f)(w) = \int_{\mathbb{R}^{2d}} \langle L_\sigma \pi(z)g, \pi(w)g\rangle\, V_g f(z)\, dz.$$

The desired result thus follows if we can prove that the map $M(\sigma)$ defined by

$$M(\sigma)G(w) = \int_{\mathbb{R}^{2d}} \langle L_\sigma \pi(z)g, \pi(w)g\rangle\, G(z)\, dz$$

is continuous from $L^2_{v_r}(\mathbb{R}^{2d})$ into $L^2_{v_{r+s}}(\mathbb{R}^{2d})$. Using (29), we see that it is sufficient to prove that the integral operator with integral kernel

$$\left| V_\Phi \sigma\left(\frac{z+w}{2}, j(w-z)\right)\right| \langle z\rangle^{-r}\langle w\rangle^{r+s}$$

is bounded on $L^2(\mathbb{R}^{2d})$. This follows from Shur's test. Indeed, by assumption $\sigma \in M^{\infty,1}_{v_s\otimes v_t}$, so that

$$\sup_{w\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| V_\Phi \sigma\left(\frac{z+w}{2}, j(w-z)\right)\right| \langle z+w\rangle^s \langle w-z\rangle^t dz < \infty$$

and similarly

$$\sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| V_\Phi \sigma\left(\frac{z+w}{2}, j(w-z)\right)\right| \langle z+w\rangle^s \langle w-z\rangle^t dw < \infty.$$

Hence it is sufficient to prove that for some positive constant $C > 0$ we have

$$\langle z+w\rangle^{-s}\langle w-z\rangle^{-t}\langle z\rangle^{-r}\langle w\rangle^{r+s} \leq C, \quad \forall w, z \in \mathbb{R}^{2d}. \tag{34}$$

Let us prove the estimate (34). Setting $x = z + w$, $y = w - z$, the inequality (34) can be rephrased as

$$\langle x\rangle^{-s}\langle y\rangle^{-t}\langle x-y\rangle^{-r}\langle x+y\rangle^{r+s} \leq C, \quad \forall x, y \in \mathbb{R}^{2d}. \tag{35}$$

For $|x| < 2|y|$, observe that $|x+y| < 3|y|$ and since $t \geq r+s$ we get the estimate (35). For $|x| \geq 2|y|$, we use $\langle x+y\rangle \asymp \langle x-y\rangle \asymp \langle x\rangle$ and (35) immediately follows. $\square$

We remark that

$$M^2_{v_s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d),$$

the Shubin-Sobolev spaces, cf. [7, 36]. In particular, for $s = 0$, $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Thus,

Corollary 3.3. *If $s, r \geq 0$, $t \geq r + s$, and the symbol $\sigma \in M^{\infty,1}_{v_s\otimes v_t}(\mathbb{R}^{2d})$, then the pseudodifferential operator L_σ extends uniquely to a bounded operator from $Q_r(\mathbb{R}^d)$ into $Q_{r+s}(\mathbb{R}^d)$.*

An application of the previous theorem concerns the study of eigenfunctions' properties for Weyl operators.

Proposition 3.4. *Consider a Weyl symbol $\sigma \in M^{p,\gamma}$ for some $1 \le p < \infty$ and every $\gamma > 0$. Then, any eigenfunction $f \in L^2(\mathbb{R}^d)$ such that $L_\sigma f = \lambda f$, with $\lambda \neq 0$, is in $\cap_{\gamma>0} M^\gamma(\mathbb{R}^d)$.*

Proof. By Theorem 3.2, if the symbol σ is in $M^{p,\gamma}(\mathbb{R}^{2d})$, for every $\gamma > 0$, then the Weyl operator acts continuously from $M^2(\mathbb{R}^d)$ into $M^{\gamma_1}(\mathbb{R}^d)$, with $1/p + 1/2 = 1/\gamma_1$ and, since $p < \infty$, we have $\gamma_1 < 2$. Thus, for $f \in M^2(\mathbb{R}^d)$ eigenfunction with eigenvalue $\lambda \neq 0$, we have $f = \frac{1}{\lambda} L_\sigma f \in M^{\gamma_1}(\mathbb{R}^d)$. Starting with $f \in M^{\gamma_1}(\mathbb{R}^d)$, we repeat the same argument, obtaining that the eigenfunction f is in the smaller modulation space $M^{\gamma_2}(\mathbb{R}^d)$, with

$$\frac{1}{p} + \frac{1}{\gamma_1} = \frac{1}{\gamma_2}$$

(observe $\gamma_2 < \gamma_1$ since $p < \infty$). Continuing this way we construct a decreasing sequence of indices $\gamma_n > 0$ and such that $\lim_{n\to\infty} \gamma_n = 0$. This proves the claim. □

Proposition 3.5. *Consider a Weyl symbol $\sigma \in M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$ for some $s > 0$ and every $t > 0$. Then, any eigenfunction $f \in L^2(\mathbb{R}^d)$ such that $L_\sigma f = \lambda f$, with $\lambda \neq 0$, is in $\mathcal{S}(\mathbb{R}^d)$.*

Proof. By Theorem 3.2, if the symbol σ is in $M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$, for some $s > 0$ and every $t > 0$, then the Weyl operator acts continuously from $L^2(\mathbb{R}^d)$ into $M^2_{v_s}(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$. Starting now with the eigenfunction f in $Q_s(\mathbb{R}^d)$ and repeating the same argument with $t \ge s$ we obtain that the eigenfunction is in $Q_{2s}(\mathbb{R}^d)$. Proceeding this way we infer that $f \in \cap_{n\in\mathbb{N}_+} Q_{ns}(\mathbb{R}^d)$. The inclusion relations for Shubin-Sobolev spaces and the property

$$\cap_{r>0} Q_r(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d),$$

prove the claim. □

3.2. **Localization Operators.** The study of eigenfunctions for a localization operator $A_a^{\varphi_1,\varphi_2}$ uses its representation as a Weyl one:

$$A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}$$

(cf. [8] and references therein). Therefore, the Weyl symbol of $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1). \tag{36}$$

We will deduce properties for $A_a^{\varphi_1,\varphi_2}$ using its Weyl form $L_{a*W(\varphi_2,\varphi_1)}$, as detailed below. Precisely, we shall focus on properties of eigenfunctions of localization operators.

Theorem 3.6. *Consider a symbol $a \in M^{p,\infty}(\mathbb{R}^{2d})$, for some $1 \le p < \infty$, and non-zero windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of the localization operator $A_a^{\varphi_1,\varphi_2}$, that is $A_a^{\varphi_1,\varphi_2} f = \lambda f$, with $\lambda \neq 0$, then $f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d)$.*

Proof. Since the windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, the cross-Wigner distribution is in $\mathcal{S}(\mathbb{R}^{2d}) \subset M^{1,\gamma}(\mathbb{R}^{2d})$, for every $0 < \gamma \leq \infty$. We next apply the convolution relations for modulation spaces (27) with $q = 1$, so that $r = p$, and we obtain that $\sigma \in M^{p,\gamma}(\mathbb{R}^{2d})$, for every $\gamma > 0$. Hence the claim immediately follows by Proposition 3.4. □

As a consequence, the eigenfunctions are extremely concentrated on the time-frequency space, having very few Gabor coefficients large whereas all the others are negligible.

Consider a Gabor frame $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbb{R}^d)$, with $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and $g \in \mathcal{S}(\mathbb{R}^d)$. For $N \in \mathbb{N}_+$, we defined in (4) the space Σ_N to be the set of all linear combinations of Gabor atoms consisting of at most N terms.

Given a function $f \in L^2(\mathbb{R}^d)$, the N-term approximation error in $L^2(\mathbb{R}^d)$ is recalled in (5). Namely, $\sigma_N(f)$ is the error produced when f is approximated optimally by a linear combination of N Gabor atoms.

Assume $f \in M^p(\mathbb{R}^d)$ for some $0 < p < 2$ (thus, in particular, $f \in L^2(\mathbb{R}^d)$). The series of Gabor coefficients in (3) are absolutely convergent, hence also unconditionally convergent. Thus we can rearrange the Gabor coefficients $|\langle f, \pi(\lambda)g\rangle|$ in a decreasing order. Precisely, set $c_{k,n} = \langle f, \pi(\alpha k, \beta n)g\rangle$, $k, n \in \mathbb{Z}^d \times \mathbb{Z}^d$, and let $\iota : \mathbb{N}_+ \to \mathbb{Z}^d \times \mathbb{Z}^d$ be any bijection satisfying

$$|c_{\iota(1)}| \geq |c_{\iota(2)}| \geq \cdots \geq |c_{\iota(m)}| \geq |c_{\iota(m+1)}| \geq \cdots$$

The sequence $(\tilde{c}_m)_{m\in\mathbb{N}_+} = (|c_{\iota(m)}|)_{m\in\mathbb{N}_+}$ is called the non-increasing rearrangement of $(c_{k,n})_{k,n}$ above. With this notations, the best approximation of f in Σ_N is

$$p_{opt} = \sum_{m=1}^{N} c_{\iota(m)} \pi(\iota(m))g$$

and the the N-term approximation error becomes

$$\sigma_N(f) = \inf_{p\in\Sigma_N} \|f - p\|_2 = \|f - p_{opt}\|_2 = \left(\sum_{m=N+1}^{\infty} |c_{\iota(m)}|^2 \right)^{\frac{1}{2}}.$$

By abuse of notation, given $a = (a_m)_m$, $a_m \geq 0$ for every m, a non-increasing sequence ($a_1 \geq a_2 \geq \cdots \geq a_m \geq a_{m+1} \geq \cdots$) we write

$$\sigma_N(a) = \left(\sum_{m=N+1}^{\infty} a_m^2 \right)^{\frac{1}{2}}.$$

The key tool is now the following lemma, see [37] and [16] (we also refer to [24, Lemma 12.4.1]):

Lemma 3.2. *Let $a = (a_m)_m$, $a_m \geq 0$ for every m, be a non-increasing sequence and consider $0 < p < 2$. Set*

$$\gamma = \frac{1}{p} - \frac{1}{2} > 0. \tag{37}$$

Then there exists a constant $C = C(p) > 0$, such that

$$\frac{1}{C}\|a\|_{\ell^p} \leq \left(\sum_{N=1}^{\infty} (N^\gamma \sigma_{N-1}(a))^p \frac{1}{N} \right)^{\frac{1}{p}} \leq C\|a\|_{\ell^p}. \tag{38}$$

Proposition 3.7. *Assume $f \in M^p(\mathbb{R}^d)$ for some $0 < p < 2$. Then, there exists $C = C(p) > 0$ such that the N-term approximation error satisfies*

$$\sigma_N(f) \leq C\|f\|_{M^p(\mathbb{R}^d)} N^{-\gamma}, \tag{39}$$

where $\gamma > 0$ is defined in (37).

Proof. If $\mathcal{G}(g, \Lambda)$ is a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$, and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta > 0$, then the sequence of Gabor coefficients of f, given by $(\langle f, \pi(\alpha k, \beta n)g\rangle)_{k,n \in \mathbb{Z}^d \times \mathbb{Z}^d}$, are in $\ell^p_m(\Lambda)$ by Theorem 2.8, with

$$\|f\|_{M^p} \asymp \|(\langle f, \pi(\alpha k, \beta n)g\rangle)_{k,n}\|_{\ell^p(\Lambda)}$$

and the sequence $(|\langle f, \pi(\alpha k, \beta n)g\rangle|)_{k,n}$ can be rearranged in a non-increasing one (a_m), as explained above. Applying Lemma 3.2 to such a sequence, from the right-hand side inequality in (38) we infer (39). □

Corollary 3.8. *Consider a Gabor frame $\mathcal{G}(g, \Lambda)$, with $g \in \mathcal{S}(\mathbb{R}^d)$, and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta > 0$. Under the assumptions of Theorem 3.6, any f eigenfunction of $A_a^{\varphi_1,\varphi_2}$ (with eigenvalue $\lambda \neq 0$) is highly compressed onto a few Gabor atoms $\pi(\lambda)g$, in the sense that its N-term approximation error satisfies the following property: for every $r > 0$ there exists $C = C(r) > 0$ with*

$$\sigma_N(f) \leq C N^{-r}. \tag{40}$$

Proof. By Theorem 3.6, the eigenfunction fulfils $f \in M^p(\mathbb{R}^d)$, for every $p > 0$. Hence the assumptions of Proposition 3.7 are satisfied for every $0 < p < 2$. This immediately yields the claim. □

We next consider the case of localization operators with symbols $a \in M^\infty_{v_s \otimes 1}(\mathbb{R}^{2d})$, $s > 0$. In this case L^2 eigenfunctions reveal to be Schwartz functions, as shown below.

Theorem 3.9. *Consider a symbol $a \in M^\infty_{v_s \otimes 1}(\mathbb{R}^{2d})$, for some $s > 0$, and non-zero windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of the localization operator $A_a^{\varphi_1,\varphi_2}$, that is $A_a^{\varphi_1,\varphi_2} f = \lambda f$, with $\lambda \neq 0$, then $f \in \mathcal{S}(\mathbb{R}^d)$.*

Proof. Tha assumption $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ implies $W(\varphi_2, \varphi_1) \in \mathcal{S}(\mathbb{R}^{2d}) \subset M^{1,1}_{v_r \otimes v_t}(\mathbb{R}^{2d})$, for every $r, t > 0$. We next apply the convolution relations for modulation spaces (27), obtaining that $A_a^{\varphi_1,\varphi_2} = L_\sigma$ with $\sigma \in M^{\infty,1}_{v_s \otimes v_t}(\mathbb{R}^{2d})$, for some $s > 0$ and every $t > 0$. Hence the claim immediately follows by Proposition 3.5. $\square$

4. Symbols in $L^p(\mathbb{R}^{2d})$ spaces

We now consider localization operators with symbols in weighted Lebesgue spaces.

Theorem 4.1. *Let $m \in \mathcal{M}_v$, $m(z) \geq 1$ for every $z \in \mathbb{R}^{2d}$, $a \in L^q_m(\mathbb{R}^{2d})$, $1 \leq q < \infty$, and non-zero windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ is an eigenfunction of the localization operator $A_a^{\varphi_1,\varphi_2}$, that is $A_a^{\varphi_1,\varphi_2} f = \lambda f$, with $\lambda \neq 0$, then $f \in \bigcap_{p>0} M^p_m(\mathbb{R}^d)$.*

Proof. By assumption and using (16), we start with a symbol a in $L^q_m(\mathbb{R}^{2d}) = W(L^q, L^q_m)(\mathbb{R}^{2d})$. Consider the eigenvector $f \in L^2(\mathbb{R}^d)$ and the window $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$. Then by Theorem 2.7 the STFT $V_{\varphi_1} f$ is in the Wiener amalgam space $W(L^\infty, L^2)(\mathbb{R}^{2d})$. Proposition 2.3 yields that $aV_{\varphi_1} f \in W(L^q, L^{p_1}_m)(\mathbb{R}^{2d})$, with

$$\frac{1}{q} + \frac{1}{2} = \frac{1}{p_1},$$

so that the index p_1 satisfies $p_1 < \min\{q, 2\}$. Consider now a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$. Using (6),

$$\begin{aligned} V_g(A_a^{\varphi_1,\varphi_2} f)(w) = \langle A_a^{\varphi_1,\varphi_2} f, \pi(w)g\rangle &= \int_{\mathbb{R}^{2d}} \left(aV_{\varphi_1} f\right)(z) \langle \pi(z)\varphi_2, \pi(w)g\rangle \, dz \\ &= \int_{\mathbb{R}^{2d}} \left(aV_{\varphi_1} f\right)(z) \langle \varphi_2, \pi(-z)\pi(w)g\rangle \, dz \\ &= \int_{\mathbb{R}^{2d}} \left(aV_{\varphi_1} f\right)(z) e^{-2\pi i z_1 w_2} \langle \varphi_2, \pi(w-z)g\rangle \, dz \end{aligned}$$

so that,

$$|V_g(A_a^{\varphi_1,\varphi_2} f)(w)| \leq \int_{\mathbb{R}^{2d}} |(aV_{\varphi_1} f)(z)| \, |V_g\varphi_2(w-z)| dz = |aV_{\varphi_1} f| * |V_g \varphi_2|(w). \tag{41}$$

Taking the modulus of the above expression, we estimate

$$|V_g(A_a^{\varphi_1,\varphi_2} f)(w)| \leq |aV_{\varphi_1} f| * |V_g\varphi_2|(w) \in W(L^q, L^{p_1}_m)(\mathbb{R}^{2d}) * W(L^\infty, L^1_v)(\mathbb{R}^{2d}).$$

Observing that $W(L^\infty, L^1_v)(\mathbb{R}^{2d}) \hookrightarrow W(L^{q'}, L^1_v)(\mathbb{R}^{2d})$ and applying the convolution relations (15) we infer $|V_g(A_a^{\varphi_1,\varphi_2} f)| \in W(L^\infty, L^{p_1}_m) \hookrightarrow L^{p_1}_m$. This proves that $A_a^{\varphi_1,\varphi_2} f \in M^{p_1}_m(\mathbb{R}^d)$.

Recalling the assumption $A_a^{\varphi_1,\varphi_2} f = \lambda f$, $\lambda \neq 0$, we infer $f \in M^{p_1}_m(\mathbb{R}^d)$.

We now repeat the previous argument starting with $f \in M^{p_1}_m(\mathbb{R}^d)$. By Theorem 2.7 the STFT $V_{\varphi_1} f \in W(L^\infty, L^{p_1}_m)(\mathbb{R}^{2d})$ and $aV_{\varphi_1} f \in W(L^q, L^{p_2}_{m^2}) \hookrightarrow W(L^q, L^{p_2}_m)$, (since $m^2 \geq m$), with

$$\frac{1}{q} + \frac{1}{p_1} = \frac{1}{p_2},$$

so that $p_2 < p_1$. Arguing as above we infer $|V_g(A^{\varphi_1,\varphi_2}_a f)(w)| \in W(L^\infty, L^{p_2}_m) \hookrightarrow L^{p_2}_m$. Thus, the eigenfunction f belongs to the smaller space $M^{p_2}_m$.

Continuing this way we construct a strictly decreasing sequence of indices $p_n > 0$ and such that

$$\lim_{n\to\infty} p_n = 0.$$

By induction and using the same argument as above one immediately obtains that if $f \in M^{p_n}_m(\mathbb{R}^d)$ then $f \in M^{p_{n+1}}_m(\mathbb{R}^d)$. This concludes the proof. □

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