GENERALIZED BORN-JORDAN DISTRIBUTIONS AND APPLICATIONS

ELENA CORDERO, MAURICE DE GOSSON, MONIKA DÖRFLER, AND FABIO NICOLA

Abstract. One of the most popular time-frequency representations is certainly the Wigner distribution. Its quadratic nature is, however, at the origin of unwanted interferences or artefacts. The desire to suppress these artefacts is the reason why engineers, mathematicians and physicists have been looking for related time-frequency distributions, many of them being members of the Cohen class. Among these, the Born-Jordan distribution has recently attracted the attention of many authors, since the so-called "ghost frequencies" are grandly damped, and the noise is, in general, reduced; it also seems to play a key role in quantum mechanics The central insight relies on the kernel of such a distribution, which contains the *sinus cardinalis* sinc, the Fourier transform of the first B-Spline B_1. The idea is to replace the function B_1 with the spline or order n, denoted by B_n, yielding the function $(\mathrm{sinc})^n$ when Fourier transformed, whose speed of decay at infinity increases with n. The related Cohen kernel is given by $\Theta^n(z_1, z_2) = \mathrm{sinc}^n(z_1 \cdot z_2)$, $n \in \mathbb{N}$, and the corresponding time-frequency distribution is called *generalized Born-Jordan distribution of order* n. We show that this new representation has a great potential to damp unwanted interference effects and this damping effect increases with n. Our proofs of these properties require an interdisciplinary approach, using tools from both microlocal and time-frequency analysis. As a by-product, a new quantization rule and a related pseudodifferential calculus are investigated.

1. Introduction

The time-frequency analysis of real-world signals is an intrinsically interdisciplinary topic, involving engineering, physics, and mathematics. It is an essential topic in various applications (see for instance the papers [6, 7, 25, 35]). In the present paper we introduce a new family of time-frequency representations defined by exponentiating the *sinus cardinalis* kernel; we call the members of this family *generalized Born-Jordan distributions*. These new distributions form a subclass of the Cohen class containing several important and well-known distributions (Wigner and Born–Jordan). The interest of this new class of time-frequency distributions

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comes from the fact that its members efficiently damp the artefacts stemming from the interaction between distinct time-frequency components in a given signal, which are due to the bilinear nature of Cohen class distributions. These damping properties will be made explicit by a precise study of the smoothing effects induced by our generalized Born–Jordan distributions.

Now, one of the most popular time-frequency representations of a signal f is the Wigner distribution

$$Wf(x,\omega) = \int_{\mathbb{R}^d} f\big(x+\frac{y}{2}\big)\overline{f\big(x-\frac{y}{2}\big)}e^{-2\pi i y\omega}\,dy, \qquad x,\omega\in\mathbb{R}^d, \tag{1.1}$$

where the signal f can be thought of as a function in $L^2(\mathbb{R}^d)$ (or more generally as a tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$). It is however well-known that the quadratic nature of the Wigner distribution generates undesired (usually oscillatory) interferences between signal components separated in time-frequency. To overcome this issue, the so-called Cohen class of time-frequency distributions was introduced in [6] and widely studied by many authors (see [2, 7, 35] and references therein). The Cohen class members Qf are generated by convolving the Wigner distribution of a signal f with a smoothing distribution $\theta\in\mathcal{S}'(\mathbb{R}^{2n})$ (Cohen kernel) in order to try to suppress the oscillatory artefacts:

$$Qf = Wf*\theta. \tag{1.2}$$

Choosing $\theta=\mathcal{F}_\sigma\Theta^1$, where $\mathcal{F}_\sigma\Theta^1$ is the symplectic Fourier transform of

$$\Theta^1(x,\omega)=\operatorname{sinc}(x\omega)=\begin{cases}\dfrac{\sin(\pi x\omega)}{\pi x\omega} & \text{for } x\omega\neq 0\\ 1 & \text{for } x\omega=0\end{cases} \tag{1.3}$$

leads to the Born-Jordan distribution:

$$Q^1 f = Wf*\mathcal{F}_\sigma(\Theta^1),\quad f\in L^2(\mathbb{R}^d), \tag{1.4}$$

see [2, 6, 7, 8, 11, 27, 30, 35] and references therein.

In the present paper we introduce new Cohen kernels and related distributions using the B-spline functions B_n. Recall that the sequence of B-splines $\{B_n\}_{n\in\mathbb{N}_+}$, is defined inductively as follows: the first B-Spline is

$$B_1(t)=\chi_{[-\frac12,\frac12]}(t).$$

Assuming that we have defined B_n, for some $n\in\mathbb{N}_+$, the spline B_{n+1} is then defined by

$$B_{n+1}(t)=(B_n*B_1)(t)=\int_{\mathbb{R}} B_n(t-y)B_1(y)dy=\int_{-\frac12}^{\frac12} B_n(t-y)dy. \tag{1.5}$$

B_n is a piecewise polynomial of degree at most $n-1$, $n\in\mathbb{N}_+$, and satisfying $B_n\in\mathcal{C}^{n-2}(\mathbb{R})$, $n\geq 2$. For the main properties of B_n we refer, *e.g.*, to [4].

Observe that $\mathrm{sinc}(\xi) = \mathcal{F}B_1(\xi)$ hence by induction on n

$$\mathrm{sinc}^n(\xi) = \mathcal{F}B_n(\xi), \quad n \in \mathbb{N}_+. \tag{1.6}$$

Definition 1.1. *For $n \in \mathbb{N}$, the n^{th} Born-Jordan kernel is the function Θ^n on $\mathbb{R}^{2d}$ defined by*

$$\Theta^n(x,\omega) = \mathrm{sinc}^n(x\omega), \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{1.7}$$

The Born-Jordan distribution of order n (BJDn) is given by

$$Q^n f = Wf * \mathcal{F}_\sigma(\Theta^n), \quad f \in L^2(\mathbb{R}^d). \tag{1.8}$$

The cross-BJDn is given by

$$Q^n(f,g) = W(f,g) * \mathcal{F}_\sigma(\Theta^n), \quad f,g \in L^2(\mathbb{R}^d). \tag{1.9}$$

We write $Q^n(f,f) = Q^n f$ for every $f \in L^2(\mathbb{R}^d)$.

Remark 1.2. *Note that $\Theta^0 \equiv 1$, hence $\mathcal{F}_\sigma(\Theta^0) = \delta$ and $Q^0 f = Wf$, the Wigner distribution of f.*

In the sequel we study central properties of the newly introduced distributions and thereby address the following issues.

(i) *Regularity and Smoothness Properties of Q^n*;
(ii) *Damping of interferences in comparison with the Wigner distribution*;
(iii) *Visual comparison in dimension $d = 1$ between Q^n and the Wigner Distribution*;
(iv) *Born–Jordan quantization of order n and related pseudodifferential calculus.*

The most suitable framework to handle these aspects is provided by *modulation spaces* (see [19] and also the textbook [32]), recalled in Subsection 2.3. Their definition is based on the *the short-time Fourier transform (STFT) $V_g f$*, defined, for a fixed Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, by

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{g(y-x)}\, e^{-2\pi i y\omega}\, dy, \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{1.10}$$

For $1 \le p,q \le \infty$, the (unweighted) modulation space $M^{p,q}(\mathbb{R}^d)$ is then the subspace of tempered distributions f such that

$$\|f\|_{M^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\, dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

(with standard modifications for $p = \infty$ or $q = \infty$).

Regularity of Q^n. While it seems intuitively clear that a signal's Born-Jordan distribution of order n cannot be rougher than the corresponding Wigner distribution, we will prove several related precise statements. In Proposition 4.2 we will show that the n-th Born-Jordan kernel belongs to the Wiener amalgam space

$W(\mathcal{F}L^1, L^\infty)$, defined in subsection 2.3 below, for every $n \in \mathbb{N}_+$. This observation is the key tool for proving the following result:

Theorem 1.3. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \le p, q \le \infty$. Then $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$, for every $n \in \mathbb{N}_+$.*

The previous statement holds in more generality and can be rephrased for members in the Cohen class as follows.

Theorem 1.4. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \le p, q \le \infty$ and the Cohen kernel θ defined in* (1.2) *belonging to the modulation space $M^{1,\infty}(\mathbb{R}^{2d})$. Then the corresponding Cohen member Qf belongs to $M^{p,q}(\mathbb{R}^{2d})$.*

Our central concern is the discussion of the new distributions' capacity for the **damping of interferences in comparison with the Wigner distribution,** a topic connected with the smoothness of Q^n and measured using the Fourier-Lebesgue wave-front set.

The notion of wave-front set of a distribution is nowadays a standard technique in the study of singularities for solutions to partial differential (or pseudodifferential) equations. The basic idea is to detect the location and orientation of the singularities of a distribution f by looking at which directions the Fourier transform of φf fails to decay rapidly, where φ is a cut-off function supported in a neighbourhood of any given point x_0. This test is performed in the framework of edge detection, where often the Fourier transform is replaced by other transforms, see *e.g.* [39] and the references therein.

We shall use the Fourier–Lebesgue wave-front set, introduced in [42, 43, 44], and related to the Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \le q \le \infty$. Recall that the norm in the space $\mathcal{F}L^q_s(\mathbb{R}^d)$, $1 \le q \le \infty$, is given by

$$\|f\|_{\mathcal{F}L^q_s(\mathbb{R}^d)} = \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^q(\mathbb{R}^d)}, \tag{1.11}$$

with $\langle\omega\rangle = (1+|\omega|^2)^{1/2}$. Inspired by this definition, given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ its wave-front set $WF_{\mathcal{F}L^q_s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, is the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \neq 0$, where the following condition *is not satisfied*: for some cut-off function φ (i.e., φ is smooth and compactly supported on $\mathbb{R}^d$), with $\varphi(x_0) \neq 0$, and some open conic neighbourhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0 it holds

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^q(\Gamma)} < \infty. \tag{1.12}$$

Observe that $WF_{\mathcal{F}L^2_s}(f) = WF_{H^s}(f)$ is the standard H^s wave-front set (see [36, Chapter XIII] and Section 2 below). Roughly speaking, $(x_0, \omega_0) \notin WF_{\mathcal{F}L^q_s}(f)$ means that f has regularity $\mathcal{F}L^q_s$ at x_0 and in the direction ω_0. We are interested in the $\mathcal{F}L^q_s$ wave-front set of the Born-Jordan distribution of order n of a given signal $f \in L^2(\mathbb{R}^d)$.

Here is the mathematical explanation of the Q^n's smoothing effects:

Theorem 1.5. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{\infty,q}(\mathbb{R}^{2d})$ for some $1 \leq q \leq \infty$. Let $(z,\zeta) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, with $\zeta = (\zeta_1, \zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. Then*

$$(z,\zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^n f).$$

This means that if the Wigner distribution Wf has $\mathcal{F}L^q$ local regularity and is somewhat controlled at infinity, then $Q^n f$ is smoother, having $s = 2n$ **additional derivatives**, at least in the directions $\zeta = (\zeta_1, \zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. In dimension $d = 1$ this condition reduces to $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$. Hence this result explains the smoothing property of such distributions, which involves all the possible directions except those of the coordinates axes. That is why the interferences of two components which do not share the same time or frequency localization come out substantially reduced. Observe that for $n = 1$ we recapture the damping phenomenon of the classical Born–Jordan distribution (cf. [13, Theorem 1.2]).

For signals in $L^2(\mathbb{R}^d)$, the previous result can be rephrased in terms of the Hörmander's wave-front set as follows:

Corollary 1.6. *Let $f \in L^2(\mathbb{R}^d)$, so that $Wf \in L^2(\mathbb{R}^{2d})$. Let (z,ζ) be as in the statement of Theorem 1.5. Then $(z,\zeta) \notin WF_{H^{2n}}(Q^n f)$, i.e., $Q^n f$ has regularity H^{2n} at z and in the direction ζ.*

The pictorial examples below suggest that the smoothing effects of the BJDn do not occur in the directions $\zeta_1 \cdot \zeta_2 = 0$. From a mathematical point of view, this is explained by the following theorem.

Theorem 1.7. *Suppose that for some $1 \leq p, q_1, q_2 \leq \infty$, $n \in \mathbb{N}_+$ and $C > 0$, we have*

$$\|Q^n f\|_{M^{p,q_1}} \leq C \|Wf\|_{M^{p,q_2}}, \tag{1.13}$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. Then $q_1 \geq q_2$.

In other words, for a general signal, the BJDn is not everywhere smoother than the Wigner distribution. As expected, the problems arise in the directions $\zeta = (\zeta_1, \zeta_2)$ such that $\zeta_1 \cdot \zeta_2 = 0$.

Visual Comparison in dimension $d = 1$ between Q^n and the Wigner Distribution. We now illustrate the effect of using higher order cross-term suppression by means of the generalized BJDn. We display the time-frequency distributions of both synthetic and real signals. More precisely, Figure 1 shows a comparison of the Wigner transform, the Born-Jordan transform and generalized Born-Jordan transform of the sum of four rotated Gaussian windows. It is clearly visible that the amount of cross-term suppression increases for higher-order smoothing.
The second example, shown in Figure 2, depicts the Wigner transform, the Born–Jordan transform and two versions of generalized Born-Jordan transform ($n = 10$

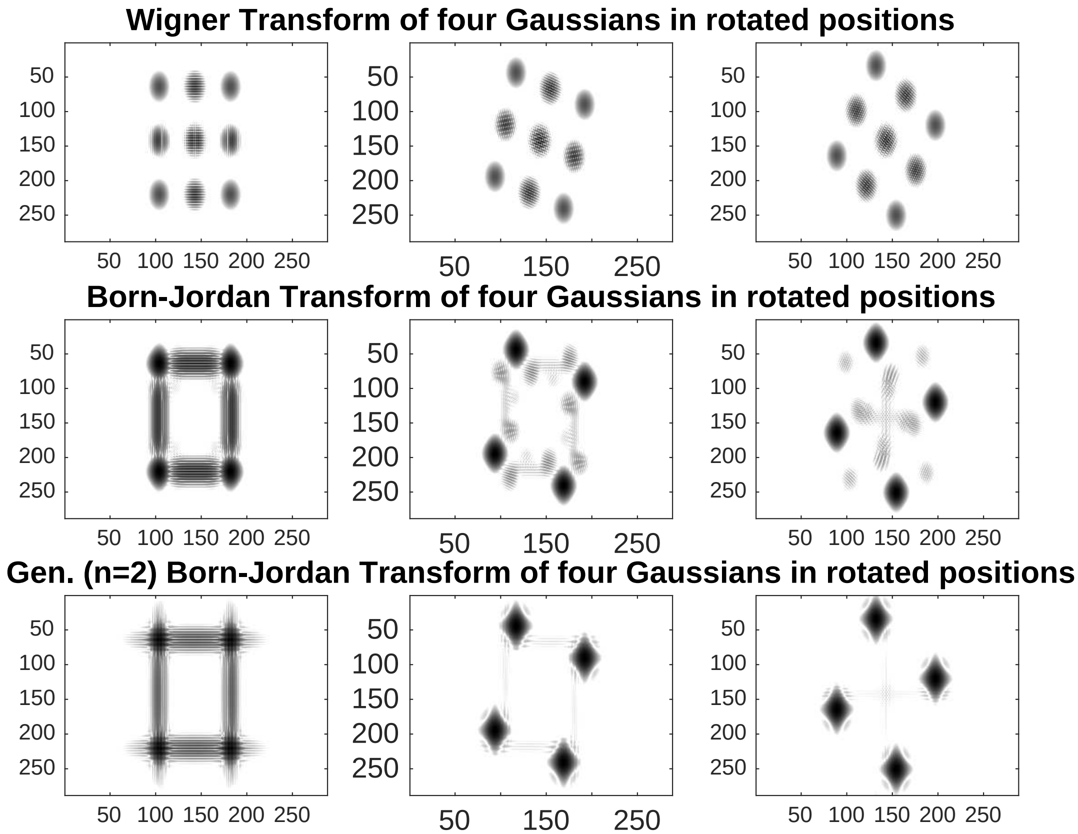

Figure 1. Four Gaussian Windows in rotated positions: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

and $n = 100$) of another synthetic signal consisting of two linear chirps. Note that the geometry of this example is different from the previous one in the sense of that it lacks symmetry around zero.

As a final example, shown in Figure 3, we applied the Wigner transform, the Born–Jordan transform and two versions of generalized Born–Jordan transform to a classical real signal, namely a bat call. As in the first example, the cross-term suppression increases for exponent $n = 2$, while, when applying even higher order smoothing, we observe a loss of concentration in time-frequency. As in the case of the two chirps, the geometry of this example lacks central symmetry.

The Born-Jordan quantization of order n. This procedure arises as the natural extension of the $n = 1$ case (that is, the usual Born–Jordan quantization).

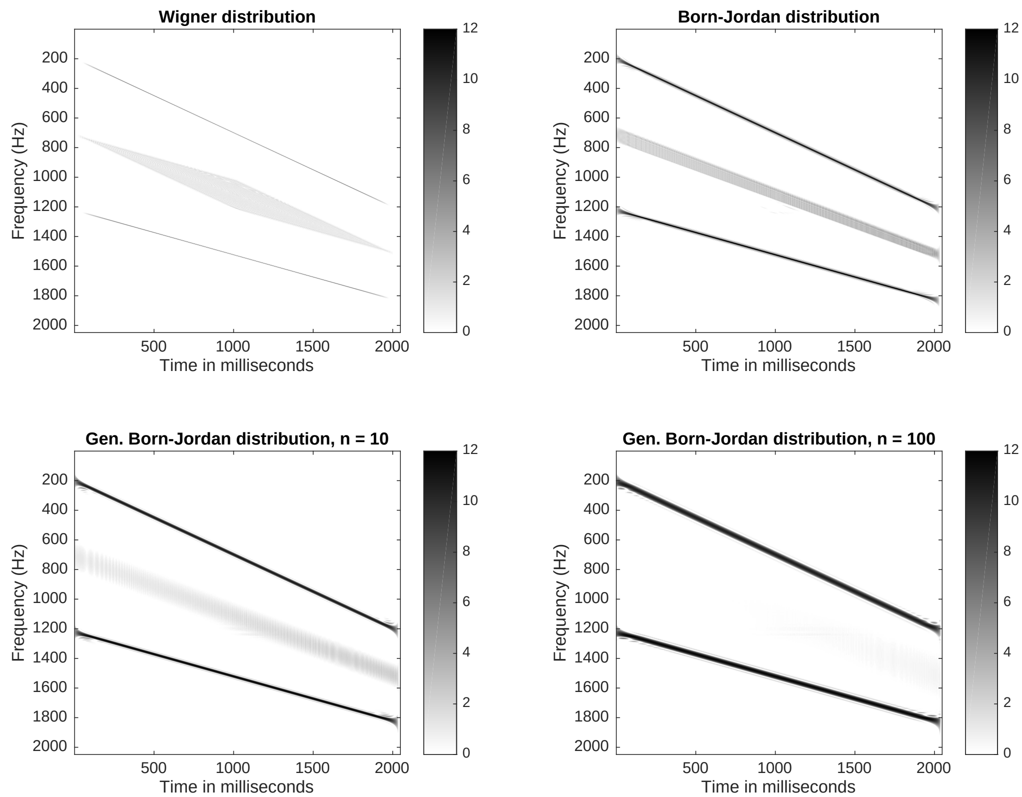

FIGURE 2. Two linear chirps: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

FIGURE 3. Bat call signal: Comparison of Wigner distribution, Born-Jordan and generalized Born-Jordan distribution

Observe that choosing $n = 0$, it reduces to the Weyl quantization. We denote by $\hbar$ a positive parameter; in physics it is viewed as the reduced Planck constant.

Definition 1.8. *For $n \in \mathbb{N}$, the Born–Jordan quantization of order n is the mapping*

$$a \in \mathcal{S}'(\mathbb{R}^{2d}) \mapsto \widehat{A}_{\mathrm{BJ},n} = \mathrm{Op}_{BJ,n}(a) = \left(\tfrac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^{2d}} (\mathcal{F}_\sigma a)(z)\Theta^n(z)\widehat{T}(z)dz, \tag{1.14}$$

where $\widehat{T}(z) = e^{-i\sigma(\widehat{z},z)/\hbar}$ *is the Heisenberg operator and* σ *the standard symplectic form (see the notation below).*

The case $n = 0$ *(*$\Theta^0 \equiv 1$*) is the well-known Weyl quantization.*

In the sequel we shall set $\hbar = 1/2\pi$, as is customary in time-frequency analysis. Hence the constant in front of the integrals in (1.14) disappears.

2. Preliminaries

2.1. **Notation.** We use the notation $x\omega = x \cdot \omega = x_1\omega_1 + \ldots + x_d\omega_d$ for the scalar product in $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbb{R}^d)$ and for the duality pairing between Schwartz functions and temperate distributions (it is antilinear in the second argument by convention). Given functions f, g, we write $f \lesssim g$ if $f(x) \leq Cg(x)$ for every x and some constant $C > 0$, and similarly for $\gtrsim$. The notation $f \asymp g$ means $f \lesssim g$ and $f \gtrsim g$.

We write $\mathcal{C}_c^\infty(\mathbb{R}^d)$ for the class of smooth functions on $\mathbb{R}^d$ with compact support.

We denote by σ the standard symplectic form on the phase space $\mathbb{R}^{2d} \equiv \mathbb{R}^d \times \mathbb{R}^d$; the phase space variable is denoted $z = (x, \omega)$ and the dual variable by $\zeta = (\zeta_1, \zeta_2)$. By definition $\sigma(z, \zeta) = Jz \cdot \zeta = \omega \cdot \zeta_1 - x \cdot \zeta_2$, where

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}.$$

The Fourier transform of a function $f(x)$ in $\mathbb{R}^d$ is

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x)\, dx,$$

and the symplectic Fourier transform of a function $F(z)$ in the phase space $\mathbb{R}^{2d}$ is defined by

$$\mathcal{F}_\sigma F(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta,z)} F(z)\, dz.$$

The symplectic Fourier transform is an involution, i.e., $\mathcal{F}_\sigma(\mathcal{F}_\sigma F) = F$. Moreover, $\mathcal{F}_\sigma F(\zeta) = \mathcal{F}F(J\zeta)$.

Observe that $\Theta^n(J(\zeta_1, \zeta_2)) = \Theta^n(\zeta_1, \zeta_2)$ so that

$$\mathcal{F}_\sigma(\Theta^n) = \mathcal{F}(\Theta^n), \quad \forall n \in \mathbb{N}_+. \tag{2.1}$$

For $s \in \mathbb{R}$ the L^2-based Sobolev space $H^s(\mathbb{R}^d)$ is constituted by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^s} := \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^2} < \infty. \tag{2.2}$$

2.2. **Time-frequency representations and main properties.**

2.2.1. *Wigner distribution and ambiguity function* [26, 32]. We already defined in the Introduction, see (1.1), the Wigner distribution Wf of a signal $f \in \mathcal{S}'(\mathbb{R}^d)$. In general, we have $Wf \in \mathcal{S}'(\mathbb{R}^{2d})$. When $f \in L^2(\mathbb{R}^d)$ we have $Wf \in L^2(\mathbb{R}^{2d})$ and in fact it turns out

$$\|Wf\|_{L^2(\mathbb{R}^{2d})} = \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

In the sequel we will encounter several times the symplectic Fourier transform of Wf, which is known as Woodward's (radar) *ambiguity function* Af. We have the formula

$$Af(\zeta_1,\zeta_2) = \mathcal{F}_\sigma Wf(\zeta_1,\zeta_2) = \int_{\mathbb{R}^d} f\big(y+\frac{1}{2}\zeta_1\big)\overline{f\big(y-\frac{1}{2}\zeta_1\big)}e^{-2\pi i\zeta_2 y}\,dy. \tag{2.4}$$

We refer to [26, Chapter 9] and [29] for more details.

2.2.2. *Marginal properties of Q^n.* The members of the Cohen class are also called *pseudo-density functions* since they are supposed to indicate how the signal density is distributed over time and frequency. The terminology *pseudo-density* comes from the fact that such distributions in general are not positive functions and can take not only negative but even complex values. In order for Q^n to be a pseudo-density function, it must satisfy certain requirements. In particular, the marginal densities

$$\int_{\mathbb{R}^d} Q^n f(x,\omega)d\omega = |f(x)|^2, \quad \int_{\mathbb{R}^d} Q^n f(x,\omega)dx = |\hat{f}(\omega)|^2, \tag{2.5}$$

for every f in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. It can be shown (see [37] or [29, Proposition 97]) that those conditions are equivalent to the requirements

$$\mathcal{F}(\Theta^n)(x,0) = 1,\ \forall x \in \mathbb{R}^d, \quad \mathcal{F}(\Theta^n)(0,\omega) = 1,\ \forall \omega \in \mathbb{R}^d. \tag{2.6}$$

In this case, using (2.1), (1.8) and (1.7), one sees that are trivially satisfied, since $\mathrm{sinc}^n(0) = 1$, for every $n \in \mathbb{N}$.

2.2.3. *The Moyal identity is not satisfied.* A quite convenient property of Cohen's kernel (1.2) is Moyal's identity ([15, Theorem 14.2 and 27.15])

$$\langle Q(f_1,g_1), Q(f_2,g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1,f_2\rangle_{L^2(\mathbb{R}^d)}\overline{\langle g_1,g_2\rangle}_{L^2(\mathbb{R}^d)}, \quad f_1,f_2,g_1,g_2 \in L^2(\mathbb{R}^d). \tag{2.7}$$

It plays an essential role in quantum mechanics (but perhaps not in signal analysis, as already observed by Janssen in [37]). While the Wigner distribution, the STFT and the ambiguity function satisfy (2.7), *the BJDn Q^n, does not for $n \in \mathbb{N}_+$.* To prove this, we will use the following characterization (*cf.* [37, Section 3] and [28]):

Proposition 2.1. *A member of the Cohen class, cf.* (1.2)*, satisfies Moyal's identity* (2.7) *if and only if*

$$|\theta(x,\omega)| = 1, \quad \textit{for all} \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{2.8}$$

Choosing $Q = Q^n$, $n \in \mathbb{N}_+$, we have $\theta(x,\omega) = \mathrm{sinc}^n(x\omega)$, so that condition (2.8) is not satisfied for any $n \in \mathbb{N}_+$. Observe that for $n = 0$ (the Wigner distribution) the previous conditions holds, as expected.

2.3. **Modulation spaces** [26, 20, 21, 22, 32]**.** Modulation spaces are used in Time-Frequency Analysis to measure the time-frequency concentration of a signal. As already observed in the introduction, the construction of these functional spaces relies on the notion of short-time (or windowed) Fourier transform defined in (1.10).

Let now $s \in \mathbb{R}$, $1 \le p,q \le \infty$. The *modulation space* $M^{p,q}_s(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_s} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \langle\omega\rangle^{sp}\, dx\right)^{q/p} d\omega\right)^{1/q} < \infty \tag{2.9}$$

(with obvious changes for $p = \infty$ or $q = \infty$). When $s = 0$ we write $M^{p,q}(\mathbb{R}^d)$ instead of $M^{p,q}_0(\mathbb{R}^d)$. We will also use the shorthand notation $M^p_s(\mathbb{R}^d)$ for $M^{p,p}_s(\mathbb{R}^d)$. The spaces $M^{p,q}_s(\mathbb{R}^d)$ are Banach spaces for any $1 \le p,q \le \infty$, and every non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ yields an equivalent norm in (2.9).

Modulation spaces generalize and include as special cases several function spaces arising in Harmonic Analysis. In particular for $p = q = 2$ we have

$$M^2_s(\mathbb{R}^d) = H^s(\mathbb{R}^d),$$

whereas $M^1(\mathbb{R}^d)$ coincides with the Segal algebra $S_0(\mathbb{R}^d)$ (cf. [18]), and $M^{\infty,1}(\mathbb{R}^d)$ is the so-called Sjöstrand class [33].

For members of $M^{p,q}_s$ the exponent p is a measure of average decay at infinity in the scale of spaces ℓ^p, whereas the exponent q is a measure of smoothness in the scale $\mathcal{F}L^q$. The number s is a further regularity index, completely analogous to that appearing in the Sobolev spaces $H^s(\mathbb{R}^d)$.

Other modulation spaces, also known as *Wiener amalgam spaces*, are obtained by exchanging the order of integration in (2.9). Precisely, the modulation spaces $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, for $p,q \in [1,+\infty)$, is given by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p,L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\, d\omega\right)^{q/p} dx\right)^{1/q} < \infty$$

(with obvious changes for $p = \infty$ or $q = \infty$). Using Parseval's identity in (1.10), we can write the so-called fundamental identity of time-frequency analysis

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega,-x),$$

hence

$$|V_g f(x,\omega)| = |V_{\hat{g}} \hat{f}(\omega,-x)| = |\mathcal{F}(\hat{f}\, T_\omega \overline{\hat{g}})(-x)|$$

so that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f}\; T_\omega \overline{\hat{g}}\|^q_{\mathcal{F}L^p}\, d\omega\right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L^p, L^q)}.$$

This means that Wiener amalgam spaces can be viewed as the images by a Fourier transform of modulation spaces: $\mathcal{F}(M^{p,q}) = W(\mathcal{F}L^p, L^q)$.

We will frequently use the following product property of Wiener amalgam spaces ([20, Theorem 1 (v)]): for $1 \le p, q \le \infty$,

$$\textit{if } f \in W(\mathcal{F}L^1, L^\infty) \textit{ and } g \in W(\mathcal{F}L^p, L^q) \textit{ then } fg \in W(\mathcal{F}L^p, L^q). \tag{2.10}$$

Taking $p = 1, q = \infty$, we see that $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ is an algebra under pointwise multiplication.

Proposition 2.2. *Let $1 \le p, q \le \infty$ and $A \in GL(d, \mathbb{R})$. Then, for every $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,*

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \le C|\det A|^{(1/p-1/q-1)}(\det(I + A^*A))^{1/2}\|f\|_{W(\mathcal{F}L^p,L^q)}. \tag{2.11}$$

In particular, for $A = \lambda I$, $\lambda > 0$,

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \le C\lambda^{d\left(\frac{1}{p}-\frac{1}{q}-1\right)}(\lambda^2+1)^{d/2}\|f\|_{W(\mathcal{F}L^p,L^q)}. \tag{2.12}$$

In the proof of Theorem 1.7 we will use the following dilation properties of Gaussians (first proved in [48, Lemma 1.8], see also [10, Lemma 3.2]):

Lemma 2.3. *Let $\varphi(x) = e^{-\pi|x|^2}$ and $\lambda > 0$. For $1 \le p, q \le \infty$,*

$$\|\varphi(\lambda\,\cdot)\|_{M^{p,q}} \asymp \lambda^{-d/q'} \quad \text{as } \lambda \to +\infty,$$

where q' is the conjugate exponent of q, that is $1/q + 1/q' = 1$.

2.4. **Wave-front set for Fourier-Lebesgue spaces** [36, 42]**.** The notion of H^s wave-front set allows to quantify the regularity of a function or distribution in the Sobolev scale at any given point and direction. This is done by microlocalizing the definition of the H^s norm in (2.2) as follows (*cf.* [36, Chapter XIII]).

Given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ we define its wave-front set $WF_{H^s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, as the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \neq 0$, for which the following condition is *not* satisfied: for some cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x_0) \neq 0$ and some open conic neighborhood of $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0 we have

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^2(\Gamma)} < \infty.$$

More generally one can start from the Fourier–Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \le q \le \infty$, which is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm in (1.11) is finite. Arguing exactly as above (with the space L^2 replaced by L^q) one then arrives in a natural way to a corresponding notion of wave-front set $WF_{\mathcal{F}L^q_s}(f)$ as we anticipated in Introduction (see (1.12)).

We need to recall some basic results about the action of constant coefficient linear partial differential operators on such wave-front set (cf. [42]). Given the operator

$$P = \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha, \quad c_\alpha \in \mathbb{C};$$

it is straightforward to see that, for $1 \leq q \leq \infty$, $s \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$WF_{\mathcal{F}L^q_s}(Pf) \subset WF_{\mathcal{F}L^q_{s+m}}(f).$$

Consider now the inverse inclusion. We say that $\zeta \in \mathbb{R}^d$, $\zeta \neq 0$, is non characteristic for the operator P if

$$\sum_{|\alpha|=m} c_\alpha \zeta^\alpha \neq 0$$

i.e. the operator P is elliptic in the direction ζ. The following result is a microlocal version of the classical regularity result of elliptic operators (see [42, Corollary 1 (2)]):

Proposition 2.4. *Let $1 \leq q \leq \infty$, $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Let $z \in \mathbb{R}^d$ and suppose that $\zeta \in \mathbb{R}^d \setminus \{0\}$ is non characteristic for P. Then, if $(z, \zeta) \notin WF_{\mathcal{F}L^q_s}(Pf)$ we have $(z, \zeta) \notin WF_{\mathcal{F}L^q_{s+m}}(f)$.*

3. Generalized Born–Jordan Kernels for Monomials

Let $\mathbb{C}[x, \omega]$ be the commutative ring of polynomials generated by x and ω; it consists of all finite sums $a(x, \omega) = \sum \lambda_{m\ell} a_{m\ell}(x, \omega)$ ($\lambda_{m\ell} \in \mathbb{C}$) where $a_{m\ell}(x, \omega) = \omega^m x^\ell$ with $(m, \ell) \in \mathbb{N}^2$. We identify $\mathbb{C}[x, \omega]$ with the ring of polynomial functions in the variables $(x, \omega) \in \mathbb{R}^2$. We denote by $\mathbb{C}[\widehat{x}, \widehat{\omega}]$ the corresponding Weyl algebra; it is realized as the non-commutative unital algebra generated by the two operators $\widehat{x}$ and $\widehat{\omega}$ satisfying $[\widehat{x}, \widehat{\omega}] = (i/2\pi)I_{\rm d}$. These operators are realized as the unbounded operators defined on $L^2(\mathbb{R})$ by $\widehat{x}f = xf$ and $\widehat{\omega}f = -(i/2\pi)\partial_x f$. We will call *quantization of* $\mathbb{C}[x, \omega]$ any continuous linear mapping $\mathrm{Op} : \mathbb{C}[x, \omega] \longrightarrow \mathbb{C}[\widehat{x}, \widehat{\omega}]$ having the following properties:

(Q1) Triviality: $\mathrm{Op}(1) = I_{\rm d}$, $\mathrm{Op}(x) = \widehat{x}$, and $\mathrm{Op}(\omega) = \widehat{\omega}$;

(Q2) Dirac's restricted rule:

$$[x, \mathrm{Op}(a_{m\ell})] = (i/2\pi)\,\mathrm{Op}(\{x, a_{m\ell}\}) \quad , \quad [\omega, \mathrm{Op}(a_{m\ell})] = (i/2\pi)\,\mathrm{Op}(\{\omega, a_{m\ell}\});$$

(Q3) Self-adjointness: If $a \in \mathbb{C}[x, \omega]$ then $\mathrm{Op}(a)$ is self-adjoint on its domain.

One shows [16] (also see [8]) that for every quantization of $\mathbb{C}[x, \omega]$ there exists [8, 16] a function f with $f(0) = 1$ and $e^{-it/2}f$ real such that

$$\mathrm{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} j! \binom{m}{j} \binom{\ell}{j} f^{(j)}(0)(2\pi)^{-j} \widehat{\omega}^{m-j} \widehat{x}^{\ell-j}. \tag{3.1}$$

Let $(a_{m\ell})_\sigma = \mathcal{F}_\sigma a_{m\ell}$ be the symplectic Fourier transform of $a_{m\ell}$ and $\widehat{T}(z) = e^{-2\pi i\sigma(\widehat{z},z)}$ the Heisenberg operator.

Proposition 3.1. *Let* $\mathrm{Op} : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ *be a quantization having the properties (Q1), (Q2), (Q3). (i) The restriction of* Op *to* $\mathbb{C}[x,\omega]$ *is then given by*

$$\mathrm{Op}(a_{m\ell}) = \int (a_{m\ell})_\sigma(x,\omega)\Phi(2\pi x\omega)\widehat{T}(x,\omega)d\omega dx \tag{3.2}$$

where $\Phi(t) = e^{-it/2}f(t)$. *(ii) The Cohen kernel* θ *of* Op *thus has symplectic Fourier transform* $\mathcal{F}_\sigma\theta$ *given by*

$$\mathcal{F}_\sigma\theta(x,\omega) = \Phi(2\pi x\omega). \tag{3.3}$$

Proof. A detailed proof is given in Domingo and Galapon [16] (formulas (10) and (14)). Notice that formula (3.2) readily follows from (3.1) using the elementary formula

$$\mathcal{F}(\omega^m \otimes x^\ell) = (i/2\pi)^{m+\ell}\delta^{(m)}(\omega) \otimes \delta^{(\ell)}(x).$$

Formula (3.3) follows since (3.2) is the Weyl representation of the operator with twisted symbol $(a_{m\ell})_\sigma\Phi$ [the twisted symbol is the symplectic Fourier transform of the usual symbol]. ∎

Remark 3.2. *This result shows that if one limits oneself to pseudo-differential calculi satisfying the Dirac conditions (Q2) then the Cohen kernel is of a very particular type: its Fourier transform only depends on the product* ωx. *In particular, the associated quasidistribution* $Q\psi = W\psi*\theta$ *satisfies the marginal conditions since* $\mathcal{F}_\sigma\theta(0) = \Phi(0) = 1$ *(see* [28], *formula (7.29), p. 107).*

We now focus on the case where the symplectic Fourier transform of the Cohen kernel is given by

$$\mathcal{F}_\sigma\theta(x,\omega) = \operatorname{sinc}^n(\pi x\omega)\ ,\ n \in \mathbb{N} = \{0,1,2,...\}.$$

With the notation above we thus have $\Phi(\pi x\omega) = \operatorname{sinc}^n(\pi x\omega)$ so that $\Phi(t) = \operatorname{sinc}^n(t/2)$ and hence $f(t) = e^{it/2}\operatorname{sinc}^n(t/2)$. Suppose first $n = 0$; then $f^{(j)}(0) = (i/2)^j$ hence formula (3.1) yields

$$\mathrm{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} \binom{m}{j}\binom{\ell}{j} j! \left(\frac{i}{4\pi}\right)^j \widehat{\omega}^{m-j}\widehat{x}^{\ell-j}$$

so that $\mathrm{Op}(a_{m\ell}) = \mathrm{Op}^{\mathrm{W}}(a_{m\ell}) = \mathrm{Op}_{BJ,0}(a_{m\ell})$ (see (1.14)) is just the Weyl ordering of the monomial $a_{m\ell}$ ([16] and [28], p.34). Suppose next $n = 1$. Then $f^{(j)}(0) =$

$i^j/(j+1)$ and

$$\mathrm{Op}(a_{m\ell}) = \sum_{j=0}^{\min(m,\ell)} \binom{m}{j}\binom{\ell}{j}\frac{j!}{j+1}\left(\frac{i}{2\pi}\right)^j \widehat{\omega}^{m-j}\widehat{x}^{\ell-j};$$

here $\mathrm{Op}(a_{m\ell}) = \mathrm{Op}_{BJ,1}(a_{m\ell})$ is the Born-Jordan ordering ([16] and [28, page 34]).

In the case of a general n we have, by Leibniz's formula,

$$f^{(j)}(0) = \sum_{k=0}^{j} \binom{j}{k}\left(\frac{i}{2}\right)^{j-k}\left(\frac{1}{2}\right)^k \left(\frac{d^k}{dt^k}\operatorname{sinc}^n\right)(0). \tag{3.4}$$

The derivatives of sinc^n at $t=0$ can be calculated using Faà di Bruno's formula [17] for the derivatives of the composition of two functions

$$(g\circ h)^{(k)}(t) = \sum_{\kappa\cdot\alpha=k}\binom{k}{\alpha} g^{(|\alpha|)}(h(t))\Pi_\alpha(t) \tag{3.5}$$

where $\kappa = (1,2,...,k)$, $\alpha = (\alpha_1,\alpha_2,...,\alpha_k)\in\mathbb{N}^k$ and

$$\Pi_\alpha(t) = \left(\frac{1}{1!}h^{(1)}(t)\right)^{\alpha_1}\left(\frac{1}{2!}h^{(2)}(t)\right)^{\alpha_2}\cdots\left(\frac{1}{k!}h^{(k)}(t)\right)^{\alpha_k}.$$

Choosing $g(t) = x^n$ and $h(t) = \operatorname{sinc}(t/2)$ this formula yields

$$\frac{d^k}{dt^k}\operatorname{sinc}^n(0) = \sum_{\substack{\kappa\cdot\alpha=k\\ |\alpha|\le n}}\binom{k}{\alpha}\binom{n}{|\alpha|}|\alpha|!\Pi_\alpha(0);$$

since $\operatorname{sinc}^{(2m+1)}(0)=0$ and $\operatorname{sinc}^{(2m)}(0) = (-1)^m/(2m+1)$ we have

$$\Pi_\alpha(0) = \frac{1}{1!(\alpha_1+1)^{\alpha_1}2!(\alpha_2+1)^{\alpha_2}\cdots k!(\alpha_k+1)^{\alpha_k}}.$$

4. Time-frequency Analysis of the n^{th} Born-Jordan kernel

The Born–Jordan kernel Θ^1 in (1.3) belongs to the space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, as proved in [13]:

Proposition 4.1. *The function Θ^1 in* (1.3) *belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

The previous property is true for any Θ^n, $n\in\mathbb{N}_+$, as shown below.

Proposition 4.2. *For $n\in\mathbb{N}_+$, the function Θ^n defined in* (1.7) *belongs to the Wiener algebra $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

Proof. The result is attained by induction on n. We know that $\Theta^1 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ by Proposition 4.1. If we assume $\Theta^n \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, for a certain integer $n > 1$, we obtain

$$\Theta^{n+1} = \Theta^n \cdot \Theta^1 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \cdot W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \hookrightarrow W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}),$$

since the Banach space $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ is an algebra by pointwise product. This gives the claim. ∎

We now have the tools we need to prove Theorem 1.3 stated in the Introduction.

Proof of Theorem 1.3. We need to show that $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$. Taking the symplectic Fourier transform in (1.4) we are reduced to prove that

$$\Theta^n \mathcal{F}_\sigma(Wf) = \Theta^n Af \in W(\mathcal{F}L^p, L^q)$$

where $\mathcal{F}_\sigma(Wf) = Af$ is the ambiguity function of f in (2.4). The claim is proven using the product property (2.10): by Proposition 4.2, the function Θ^n is in $W(\mathcal{F}L^1, L^\infty)$ and in view of the assumption $Wf \in M^{p,q}(\mathbb{R}^{2d})$ so that $\mathcal{F}(Wf) \in W(\mathcal{F}L^p, L^q)$. Therefore $\mathcal{F}_\sigma(Wf)(\zeta) = \mathcal{F}(Wf)(J\zeta) \in W(\mathcal{F}L^p, L^q)$ by Proposition 2.2 and we are done. ∎

An alternative proof relies on the continuity of the mapping

$$A : \, a \longmapsto a * \Theta^1_\sigma, \tag{4.1}$$

which was shown to be bounded on $M^{p,q}(\mathbb{R}^{2d})$ in [12, Proposition 5.1], see also the subsequent work [31]. By induction it then follows that the same continuity property holds for $\mathcal{Q}^n$ in (1.8), with $a = Wf$, and Theorem 1.2 is thus proved.

Actually, the previous issue is a special case of the general result for members of the Cohen class stated in Theorem 1.4 (recall that, if $\Theta^n \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$, then $\mathcal{F}_\sigma\Theta^n \in M^{1,\infty}(\mathbb{R}^{2d})$), which can be proved as follows.

Proof of Theorem 1.4. It is a consequence of the convolution relations for modulation spaces (cf. [9]):

$$M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d}),$$

for any $1 \leq p, q \leq \infty$. ∎

In [13] the following property for the chirp function was proven:

Proposition 4.3. *The function $F(\zeta_1, \zeta_2) = e^{2\pi i \zeta_1 \zeta_2}$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

Since $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ can be characterized as the space of pointwise multipliers on the Feichtinger algebra $W(\mathcal{F}L^1, L^1)(\mathbb{R}^{2d})$ [23, Corollary 3.2.10], the result in Proposition 4.3 could also be deduced from general results about the action of second order characters on the Feichtinger algebra, cf. [18, 46].

By Proposition 4.3 and by the dilation properties for Wiener amalgam spaces (2.11) we can state:

Corollary 4.4. *For $\zeta = (\zeta_1, \zeta_2)$, consider the function $F_J(\zeta) = F(J\zeta) = e^{-2\pi i \zeta_1 \zeta_2}$. Then $F_J \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

5. Smoothness of the Born-Jordan distribution of order n

In the present section we compare the smoothness of the Born–Jordan distribution of order n with that of the Wigner distribution. In particular we will prove Theorem 1.5.

We begin by stating and proving the following global result.

Theorem 5.1. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal such that $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \le p, q \le \infty$. Then*

$$Q^n f \in M^{p,q}(\mathbb{R}^{2d})$$

and moreover

$$(\nabla_x \cdot \nabla_\omega)^n Q^n f \in M^{p,q}(\mathbb{R}^{2d}). \tag{5.1}$$

Here we used the notation

$$\nabla_x \cdot \nabla_\omega = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j \partial \omega_j}.$$

Proof. The property $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$ is proven in Theorem 1.3.

Let us now prove (5.1). Taking the symplectic Fourier transform we see that it is sufficient to prove that

$$(\zeta_1\zeta_2)^n \operatorname{sinc}^n(\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \frac{1}{\pi^n} \sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

We have

$$\sin(\pi\zeta_1\zeta_2) = \frac{e^{\pi i \zeta_1\zeta_2} - e^{-\pi i \zeta_1\zeta_2}}{2i} \in W(\mathcal{F}L^1, L^\infty), \tag{5.2}$$

by Proposition 4.3, Corollary 4.4 and Proposition 2.2, with the scaling $\lambda = 1/\sqrt{2}$.

Hence, for $n = 1$,

$$\frac{1}{\pi} \sin(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q)$$

by the product property (2.10). Assume now that, for a certain $n \in \mathbb{N}_+$,

$$\frac{1}{\pi^n} \sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

Then

$$\frac{1}{\pi^{n+1}}\sin^{n+1}(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \underbrace{\frac{1}{\pi}\sin(\pi\zeta_1\zeta_2)}_{\in W(\mathcal{F}L^1,L^\infty)} \cdot \underbrace{\frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf}_{\in W(\mathcal{F}L^p,L^q)} \in W(\mathcal{F}L^p,L^q),$$

by (5.2) and the product property (2.10) again. By induction we attain the result. ∎

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Consider $n \in \mathbb{N}_+$. We will apply Proposition 2.4 to the $2n$-th order operator P^n, where $P = \nabla_x \cdot \nabla_\omega$ in $\mathbb{R}^{2d}$. The non characteristic directions for P^n are given by the vectors $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^d \times \mathbb{R}^d$, satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. By (5.1) (with $p = \infty$) we have

$$WF_{\mathcal{F}L^q}(P^nQ^nf) = \emptyset,$$

because $\varphi F \in \mathcal{F}L^q$ if $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and $F \in M^{\infty,q}(\mathbb{R}^{2d})$ (here $F = P^nQ^nf$). Hence we obtain

$$(z,\zeta) \notin WF_{\mathcal{F}L^q}(P^nQ^nf), \quad \forall (z,\zeta) \text{ such that } \zeta = (\zeta_1,\zeta_2),\ \zeta_1\cdot\zeta_2 \neq 0.$$

Since ζ is non characteristic for the operator P^n, by Proposition 2.4 we infer

$$(z,\zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^nf)$$

for every $z \in \mathbb{R}^{2d}$. ∎

Proof of Corollary 1.6. Apply Theorem 1.5 with $q = 2$. Indeed, for $f \in L^2(\mathbb{R}^d)$ Moyal's identity gives $Wf \in L^2(\mathbb{R}^{2d}) = M^{2,2}(\mathbb{R}^d) \subset M^{\infty,2}(\mathbb{R}^{2d})$ (*cf.* (2.3)). Observe that the $\mathcal{F}L^2_{2n}$ wave-front set coincides with the H^{2n} wave-front set. ∎

The proof of Theorem 1.7 requires Lemma 5.1 in [13]:

Lemma 5.2. *Let $\chi \in C_c^\infty(\mathbb{R})$. Then the function $\chi(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

As announced in the introduction, the smoothing properties of the Q^n distributions do not hold in the whole phase space. We do not have any gain in the directions $\zeta_1 \cdot \zeta_2 = 0$ as it comes up clearly from the proof of the following issue.

Proof of Theorem 1.7. The pattern is similar to that of Theorem 1.4 in [13]. We detail the main steps for sake of clarity. The idea is to test the estimate (1.13) using rescaled Gaussian functions $f(x) = \varphi(\lambda x)$, with $\lambda > 0$ large parameter. We shall prove that, restricting to a neighbourhood of $\zeta_1 \cdot \zeta_2 = 0$, the constrain $q_1 \geq q_2$ must be satisfied.

An easy computation (see e.g. [32, Formula (4.20)]) yields

$$W(\varphi(\lambda\,\cdot))(x,\omega)=2^{d/2}\lambda^{-d}\varphi(\sqrt{2}\lambda\,x)\varphi(\sqrt{2}\lambda^{-1}\,\omega). \tag{5.3}$$

For every $1\leq p,q\leq\infty$, the above formula gives

$$\|W(\varphi(\lambda\,\cdot))\|_{M^{p,q}}=2^{d/2}\lambda^{-d}\|\varphi(\sqrt{2}\lambda\,\cdot)\|_{M^{p,q}}\|\varphi(\sqrt{2}\lambda^{-1}\,\cdot)\|_{M^{p,q}}.$$

By the dilation properties of Gaussians in Lemma 2.3

$$\|W(\varphi(\lambda\,\cdot))\|_{M^{p,q}}\asymp\lambda^{-2d+d/q+d/p}\quad\text{as }\lambda\to+\infty. \tag{5.4}$$

We now study the $M^{p,q}$-norm of the BJDn $Q^n(\varphi(\lambda\,\cdot))$. The idea is to estimate this norm from below obtaining the same expansion as in (5.4).

$$\|Q^n(\varphi(\lambda\,\cdot))\|_{M^{p,q}}=\|\mathcal{F}_\sigma(\Theta^n)*W(\varphi(\lambda\,\cdot))\|_{M^{p,q}}.$$

By taking the symplectic Fourier transform and using Lemma 5.2 and the product property (2.10) we have

$$\begin{aligned}\|\mathcal{F}_\sigma(\Theta^n)*W(\varphi(\lambda\,\cdot))\|_{M^{p,q}}&\asymp\|\Theta^n\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}\\&\gtrsim\|\Theta^n(\zeta_1,\zeta_2)\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}\end{aligned}$$

for any $\chi\in C_c^\infty(\mathbb{R})$ and $n\in\mathbb{N}_+$. Choosing χ supported in the interval $[-1/4,1/4]$ and $\chi\equiv1$ in the interval $[-1/8,1/8]$ (the latter condition will be used later), we write

$$\chi(\zeta_1\zeta_2)=\chi(\zeta_1\zeta_2)\Theta^n(\zeta_1,\zeta_2)\Theta^{-n}(\zeta_1,\zeta_2)\tilde{\chi}(\zeta_1\zeta_2),$$

with $\tilde{\chi}\in C_c^\infty(\mathbb{R})$ supported in $[-1/2,1/2]$ and $\tilde{\chi}=1$ on $[-1/4,1/4]$, therefore on the support of χ. Since by Lemma 5.2 the function $\Theta^{-n}(\zeta_1,\zeta_2)\tilde{\chi}(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1,L^\infty)$, by the product property the last expression can be estimated from below as

$$\gtrsim\|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}.$$

We end up with the same object that was already estimated in the proof of Theorem 1.4 in [13], were it was shown that

$$\|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}\gtrsim\lambda^{-2d+d/p+d/q}\quad\text{as }\lambda\to+\infty. \tag{5.5}$$

Comparing (5.5) with (5.4) we obtain the desired conclusion. ■

6. Pseudodifferential Calculus

The Weyl quantization was introduced by Weyl in [50] and is the $n=0$ case of the Born–Jordan quantization of order n in (1.14):

$$a\in\mathcal{S}'(\mathbb{R}^{2d})\mapsto\widehat{A}_{\rm W}=\mathrm{Op}_{\rm W}(a)=\left(\tfrac{1}{2\pi\hbar}\right)^d\int_{\mathbb{R}^{2d}}\mathcal{F}_\sigma a(z)\widehat{T}(z)dz.$$

Comparing with (1.14), we infer the symbol relation

$$\mathcal{F}_\sigma a_{BJ,n}\Theta^n=\mathcal{F}_\sigma a_W$$

(observe that $a_{BJ,n}$ denotes the symbol of $\widehat{A}_{\mathrm{BJ},n}$ whereas a_W is the Weyl symbol) that is

$$a_{BJ,n} * \mathcal{F}_\sigma(\Theta^n) = a_W. \tag{6.1}$$

Using the weak definition for Weyl operators via the Wigner distribution

$$\langle Op_W(a)f, g\rangle = \langle a, W(g,f)\rangle, \quad a \in \mathcal{S}'(\mathbb{R}^{2d}),\ f, g \in \mathcal{S}(\mathbb{R}^d)$$

and the convolution property (whenever is well-defined)

$$\langle F * G, H\rangle = \langle F, H * G\rangle$$

we can also define, for $n \in \mathbb{N}$, the n-th Born-Jordan pseudodifferential operator with symbol $a \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle Op_{BJ,n}(a)f, g\rangle = \langle a, Q^n(g,f)\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{6.2}$$

(Observe that $n = 1$ is the standard Born–Jordan operator, whereas $n = 0$ gives the Weyl operator).

We aim at studying continuity properties of such operators and of the related distributions on modulation spaces.

First, we analyze the quadratic representations Q^n.

Theorem 6.1. *Assume $s \geq 0$, $p_1, q_1, p, q \in [1, \infty]$ such that*

$$2\min\{\frac{1}{p_1}, \frac{1}{q_1}\} \geq \frac{1}{p} + \frac{1}{q}. \tag{6.3}$$

If $f \in M^{p_1,q_1}_{v_s}(\mathbb{R}^d)$ the Cohen distribution $Q^n f$, $n \in \mathbb{N}_+$, is in $M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})$, with

$$\|Q^n f\|_{M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})} \lesssim \|\Theta^n\|_{W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})} \|f\|^2_{M^{p_1,q_1}_{v_s}(\mathbb{R}^d)}. \tag{6.4}$$

Proof. In [14, Theorem 1.2] two of us proved that, if the Cohen kernel θ, defined in (1.2), is in $M^{1,\infty}(\mathbb{R}^{2d})$, then the related Cohen distribution Qf satisfies

$$\|Q^n f\|_{M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})} \lesssim \|\theta\|_{M^{1,\infty}(\mathbb{R}^{2d})} \|f\|^2_{M^{p_1,q_1}_{v_s}(\mathbb{R}^d)}$$

where the indices $p_1, q_1, p, q \in [1, \infty]$ are related by condition (6.3).

By Proposition 4.2, the function Θ^n is in $W(\mathcal{F}L^1, L^\infty)$, so that the BJ kernel $\mathcal{F}_\sigma(\Theta^n)$ is in $M^{1,\infty}(\mathbb{R}^{2d})$ with $\|\mathcal{F}_\sigma(\Theta^n)\|_{M^{1,\infty}} \asymp \|\Theta^n\|_{W(\mathcal{F}L^1, L^\infty)}$ and the thesis follows. ∎

We write q' for the conjugate exponent of $q \in [1, \infty]$ (it is defined by $1/q + 1/q' =$). The n-th Born-Jordan operator enjoys the same continuity properties as for the $n = 1$ case, proved in [12, Theorem 1.1]. Indeed, we can state:

Theorem 6.2. *Consider* $1 \le p, q, r_1, r_2 \le \infty$, *such that*

$$p \le q' \tag{6.5}$$

and

$$q \le \min\{r_1, r_2, r_1', r_2'\}. \tag{6.6}$$

Then the Born-Jordan operator $Op_{BJ,n}(a)$, *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$, *having symbol* $a \in M^{p,q}(\mathbb{R}^{2d})$, *extends uniquely to a bounded operator on* $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, *with the estimate*

$$\|Op_{BJ,n}(a)f\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|a\|_{M^{p,q}}\|f\|_{\mathcal{M}^{r_1,r_2}}, \quad f \in \mathcal{M}^{r_1,r_2}. \tag{6.7}$$

Conversely, if this conclusion holds true, the constraints (6.5) *are satisfied and it must hold*

$$\max\left\{\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_1'}, \frac{1}{r_2'}\right\} \le \frac{1}{q} + \frac{1}{p}, \tag{6.8}$$

that is (6.6) *for* $p = \infty$.

Proof. The sufficient conditions are proved by induction on n. The result holds true for $n = 1$ by Theorem [12, Theorem 1.1]. Assume now that the result is true for a certain $n \in \mathbb{N}_+$ and observe, by definition (1.14), that

$$Op_{BJ,n+1}(a) = Op_{BJ,n}(b), \quad \text{with} \quad a = b * \mathcal{F}_\sigma\Theta.$$

The claim follows from the convolution relation $M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d})$.

The necessary conditions are obtained arguing exactly as in the case $n = 1$, for details we refer to the proof of Theorem 1.1 given in [12]. ∎

Technical notes

The figures in the introduction were produced using LTFAT (The Large Time-Frequency Analysis Toolbox), cf. [45] as well as the Time-Frequency Toolbox (TFTB), distributed under the terms of the GNU Public Licence:

http://tftb.nongnu.org/

The bat sonar signal in Figure 3 was recorded as a .mat file in the latter toolbox.

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Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy

Email address: elena.cordero@unito.it

University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1 A-1090 Wien, Austria

Email address: maurice.de.gosson@univie.ac.at

University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1 A-1090 Wien, Austria

Email address: monika.doerfler@univie.ac.at

Dipartimento di Scienze Matematiche, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy

Email address: fabio.nicola@polito.it