## POLITECNICO DI TORINO

## Repository ISTITUZIONALE

A characterization of modulation spaces by symplectic rotations

Original
A characterization of modulation spaces by symplectic rotations / Cordero, Elena; De Gosson, Maurice; Nicola, Fabio. In: JOURNAL OF FUNCTIONAL ANALYSIS. - ISSN 0022-1236. - STAMPA. - 278:11(2020). [10.1016/j.jfa.2020.108474]

## Availability:

This version is available at: 11583/2853971 since: 2020-11-27T11:29:54Z
Publisher:
Elsevier

Published
DOI:10.1016/j.jfa.2020.108474

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright
(Article begins on next page)

# A CHARACTERIZATION OF MODULATION SPACES BY SYMPLECTIC ROTATIONS 

ELENA CORDERO, MAURICE DE GOSSON, AND FABIO NICOLA


#### Abstract

This note contains a new characterization of modulation spaces $M_{m}^{p}\left(\mathbb{R}^{n}\right), 1 \leq p \leq \infty$, by symplectic rotations. Precisely, instead to measure the time-frequency content of a function by using translations and modulations of a fixed window as building blocks, we use translations and metaplectic operators corresponding to symplectic rotations. Technically, this amounts to replace, in the computation of the $M_{m}^{p}\left(\mathbb{R}^{n}\right)$-norm, the integral in the timefrequency plane with an integral on $\mathbb{R}^{n} \times U(2 n, \mathbb{R})$ with respect to a suitable measure, $U(2 n, \mathbb{R})$ being the group of symplectic rotations. More conceptually, we are considering a sort of polar coordinates in the time-frequency plane. To have invariance under symplectic rotations we choose a Gaussian as suitable window function. We also provide a similar (and easier) characterization with the group $U(2 n, \mathbb{R})$ being reduced to the $n$-dimensional torus $\mathbb{T}^{n}$.


## 1. Introduction

The objective of this study is to find a new characterization of modulation spaces using symplectic rotations. Precisely, we are interested in those metaplectic operators $\widehat{S} \in M p(n, \mathbb{R})$, such that the corresponding projection $S:=\pi(\widehat{S})$ onto the symplectic group $S p(n, \mathbb{R})$ is a symplectic rotation. Let us recall that the symplectic group $S p(n, \mathbb{R})$ is the subgroup of $2 n \times 2 n$ invertible matrices $G L(2 n, \mathbb{R})$, defined by

$$
\begin{equation*}
S p(n, \mathbb{R})=\left\{S \in G L(2 n, \mathbb{R}): S J S^{T}=J\right\} \tag{1}
\end{equation*}
$$

where $J$ is the orthogonal matrix

$$
J=\left(\begin{array}{cc}
0_{n} & I_{n} \\
-I_{n} & 0_{n}
\end{array}\right)
$$

( $I_{n}, 0_{n}$ are the $n \times n$ identity matrix and null matrix, respectively). Here we consider the subgroup

$$
U(2 n, \mathbb{R}):=S p(n, \mathbb{R}) \cap O(2 n, \mathbb{R}) \simeq U(n)
$$

[^0]of symplectic rotations (cf., e.g. [15, Section 2.3]), namely
\[

U(2 n, \mathbb{R})=\left\{\left($$
\begin{array}{cc}
A & -B  \tag{2}\\
B & A
\end{array}
$$\right): A A^{T}+B B^{T}=I_{n}, A B^{T}=B^{T} A\right\} \subset \operatorname{Sp}(n, \mathbb{R})
\]

endowed with the normalized Haar measure $d S$ (the group $U(2 n, \mathbb{R})$, being compact, is unimodular).

In the 80 's H. Feichtinger [16] introduced modulation spaces to measure the time-frequency concentration of a function/distribution on the time-frequency space (or phase space) $\mathbb{R}^{2 n}$. They are nowadays become popular among mathematicians and engineers because they have found numerous applications in signal processing [6, 19, 20], pseudodifferential and Fourier integral operators [7, 8, 9, 28, 29], partial differential equations $[1,2,3,4,10,13,11,11,32,33,34]$ and quantum mechanics $[12,15]$.

To recall their definition, we need a few time-frequency tools. First, the translation $T_{x}$ and modulation $M_{\xi}$ operators are defined by

$$
T_{x} f(t)=f(t-x), \quad M_{\xi} f(t)=e^{2 \pi i t \cdot \xi} f(t), \quad t, x, \xi \in \mathbb{R}^{n}
$$

for any function $f$ on $\mathbb{R}^{n}$.
The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ with respect to a function $g \in \mathcal{S}\left(\mathbb{R}^{n}\right) \backslash\{0\}$ (so-called window), given by

$$
\begin{equation*}
V_{g} f(x, \xi)=\left\langle f, M_{\xi} T_{x} g\right\rangle=\int_{\mathbb{R}^{n}} f(t) \overline{g(t-x)} e^{-2 \pi i t \cdot \xi} d t, \quad x, \xi \in \mathbb{R}^{n} \tag{3}
\end{equation*}
$$

The short-time Fourier transform is well-defined whenever the bracket $\langle\cdot, \cdot\rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f \in$ $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right), g \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, or for $f, g \in L^{2}\left(\mathbb{R}^{n}\right)$.

Let $m(x, \xi)$ be a continuous weight, $v$-moderate for some submultiplicative weight $v$ (see [22, Section 11.1] for details - we will not use explicitly these properties). We also assume that $m$ has at most polynomial growth.

Definition 1.1 (Modulation spaces). Given $g \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, and $1 \leq p \leq \infty$, the modulation space $M_{m}^{p}\left(\mathbb{R}^{n}\right)$ consists of all tempered distributions $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $V_{g} f \in L_{m}^{p}\left(\mathbb{R}^{2 n}\right)$. The norm on $M_{m}^{p}\left(\mathbb{R}^{n}\right)$ is

$$
\begin{align*}
\|f\|_{M_{m}^{p}}=\left\|V_{g} f\right\|_{L_{m}^{p}} & =\left(\int_{\mathbb{R}^{2 n}}\left|V_{g} f(x, \xi)\right|^{p} m(x, \xi)^{p} d x d \xi\right)^{1 / p}  \tag{4}\\
& =\left(\int_{\mathbb{R}^{2 n}}\left|\left\langle f, M_{\xi} T_{x} g\right\rangle\right|^{p} m(x, \xi)^{p} d x d \xi\right)^{1 / p}
\end{align*}
$$

(with obvious modifications for $p=\infty$ ).

The spaces $M_{m}^{p}\left(\mathbb{R}^{n}\right)$ are Banach spaces, and every nonzero $g \in M_{v}^{1}\left(\mathbb{R}^{n}\right)$ yields an equivalent norm in (4), so that their definition is independent of the choice of $g \in M_{v}^{1}\left(\mathbb{R}^{n}\right)($ see $[16,22])$.

We now provide an equivalent norm to (4) by using translations $T_{x}$ (or modulations $\left.M_{\xi}\right)$ and the operators $\widehat{S}$, with $S \in U(2 n, \mathbb{R})$ as follows.

Theorem 1.2. Consider the Gaussian function $\varphi(t)=2^{d / 4} e^{-\pi|t|^{2}}$.
(i) For $1 \leq p<\infty$ and $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right)$, we have

$$
\begin{equation*}
\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times U(2 n, \mathbb{R})}|x|^{n}\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S\right)^{\frac{1}{p}} \tag{5}
\end{equation*}
$$

where $d x$ is the Lebesgue measure on $\mathbb{R}^{n}$ and $d S$ the Haar measure on $U(2 n, \mathbb{R})$.
Similarly,

$$
\begin{equation*}
\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times U(2 n, \mathbb{R})}|\xi|^{n}\left|\left\langle f, \widehat{S} M_{\xi} \varphi\right\rangle\right|^{p} m\left(S(0, \xi)^{T}\right)^{p} d \xi d S\right)^{\frac{1}{p}} \tag{6}
\end{equation*}
$$

with $d \xi$ being the Lebesgue measure on $\mathbb{R}^{n}$ and $d S$ the Haar measure on $U(2 n, \mathbb{R})$.
(ii) For $p=\infty, f \in M_{m}^{\infty}\left(\mathbb{R}^{n}\right)$, it occurs

$$
\begin{equation*}
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{S \in U(2 n, \mathbb{R})} \sup _{x \in \mathbb{R}^{n}}\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right| m\left(S(x, 0)^{T}\right) \tag{7}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{S \in U(2 n, \mathbb{R})} \sup _{\xi \in \mathbb{R}^{n}}\left|\left\langle f, \widehat{S} M_{\xi} \varphi\right\rangle\right| m\left(S(0, \xi)^{T}\right) \tag{8}
\end{equation*}
$$

The interpretation of the integral (5) above is as follows. The metaplectic operator $\widehat{S}$ produces a time-frequency rotation of the shifted Gaussian $T_{x} \varphi$. In this way, the operator

$$
f \mapsto\left\langle f, \widehat{S} T_{x} \varphi\right\rangle
$$

detects the time-frequency content of $f$ in an oblique strip, see Figure 1. All the contributions are then added together with a weight $|x|^{n}$ which takes into account the underlapping of the strips as $|x| \rightarrow+\infty$ and the overlapping as $x \rightarrow 0$.

Formulas (6), (7) and (8) have similar meanings.
Observe that in dimension $n=1, U(2, \mathbb{R}) \simeq U(1)$ and the above formula is essentially a transition to polar coordinates with $|x|$ being the Jacobian.

Comparing (4) and (5) we observe that in (5) the modulation operator $M_{\xi}$ is replaced by the metaplectic operator $\widehat{S}$ and the integral on the phase space $\mathbb{R}^{2 n}$ has become an integral on the cartesian product $\mathbb{R}^{n} \times U(2 n, \mathbb{R})$. The integration parameters $(x, \xi)$ of (4) live in $\mathbb{R}^{2 n}$, with $\operatorname{dim} \mathbb{R}^{2 n}=2 n$, whereas the parameters $(x, S)$ of (5) live in $\mathbb{R}^{n} \times U(2 n, \mathbb{R})$. Recall that $\operatorname{dim} U(2 n, \mathbb{R})=n^{2}$ [15]; this suggests that a formula similar to (5) should hold when $U(2 n, \mathbb{R})$ is reduced to


Figure 1. The time-frequency content of $f$ in the oblique strip is detected by the operator $f \mapsto\left\langle f, \widehat{S} T_{x} \varphi\right\rangle$.
a suitable subgroup $K \subset U(2 n, \mathbb{R})$ of dimension $n$. This is indeed the case (and easier to see), as shown in the subsequent Theorem 1.3.

Consider the $n$-dimensional torus

$$
\mathbb{T}^{n}=\left\{S=\left(\begin{array}{ccc}
e^{i \theta_{1}} & &  \tag{9}\\
& \ddots & \\
& & e^{i \theta_{n}}
\end{array}\right): \theta_{1}, \ldots, \theta_{n} \in \mathbb{R}\right\} \subset U(n)
$$

with the Haar measure $d S=d \theta_{1} \ldots d \theta_{n}$. The torus is isomorphic to a subgroup $K \subset U(2 n, \mathbb{R}$ ), via the isomorphism $\iota$ in formula (16) below (see the subsequent section).

We exhibit the following characterization for $M^{p}$-spaces.
Theorem 1.3. Let $\varphi$ be the Gaussian of Theorem 1.2.
(i) For $1 \leq p<\infty$, $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right)$, we have

$$
\begin{equation*}
\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times \mathbb{T}^{n}}\left|x_{1} \ldots x_{n}\right|\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S\right)^{\frac{1}{p}} \tag{10}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times \mathbb{T}^{n}}\left|\xi_{1} \ldots \xi_{n} \|\left\langle f, \widehat{S} M_{\xi} \varphi\right\rangle\right|^{p} m\left(S(0, \xi)^{T}\right)^{p} d \xi d S\right)^{\frac{1}{p}} \tag{11}
\end{equation*}
$$

(ii) For $p=\infty$,

$$
\begin{equation*}
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{S \in \mathbb{T}^{n}} \sup _{x \in \mathbb{R}^{n}}\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right| m\left(S(x, 0)^{T}\right) \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{S \in \mathbb{T}^{n}} \sup _{\xi \in \mathbb{R}^{n}}\left|\left\langle f, \widehat{S} M_{\xi} \varphi\right\rangle\right| m\left(S(0, \xi)^{T}\right) \tag{13}
\end{equation*}
$$

The above results for the groups $U(2 n, \mathbb{R})$ and $\mathbb{T}^{n}$ can be interpreted, in a sense, as two extreme cases, and it would be interesting to find, more generally, for which compact subgroups $K \subset U(2 n, \mathbb{R})$ similar characterizations hold. We conjecture that they should be precisely the subgroups $K \subset U(2 n, \mathbb{R})$ such that every orbit for their action on $\mathbb{R}^{2 n}$ intersects $\{0\} \times \mathbb{R}^{n}$ (up to subsets of measure zero), with a corresponding weighted measure on $\mathbb{R}^{n} \times K$ to be determined.

Another problem which is worth investigating is the study of discrete versions of the above characterizations via coorbit theory [17].

The paper is organized as follows: in Section 2 we collected some preliminary results, whereas Section 3 is devoted to the proof of Theorems 1.2 and 1.3. In Section 4 we rephrase more explicitly Theorem 1.3 in terms of the partial fractional Fourier transform.

## 2. Notation and Preliminaries

Notation. We write $x \cdot y$ for the scalar product on $\mathbb{R}^{n}$ and $|t|^{2}=t \cdot t$, for $t, x, y \in \mathbb{R}^{n}$. For expressions $A, B \geq 0$, we use the notation $A \lesssim B$ to represent the inequality $A \leq c B$ for a suitable constant $c>0$, and $A \asymp B$ for the equivalence $c^{-1} B \leq A \leq c B$.

The Schwartz class is denoted by $\mathcal{S}\left(\mathbb{R}^{n}\right)$, the space of tempered distributions by $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \times \mathcal{S}\left(\mathbb{R}^{n}\right)$ of the inner product $\langle f, g\rangle=\int f(t) \overline{g(t)} d t$ on $\mathcal{S}\left(\mathbb{R}^{n}\right)$.

Metaplectic Operators. The metaplectic representation $\mu$ of $M p(n, \mathbb{R})$, the two-sheeted cover of the symplectic group $\operatorname{Sp}(n, \mathbb{R})$, defined in (1) arises as intertwining operator between the standard Schrödinger representation $\rho$ of the Heisenberg group $\mathbb{H}^{d}$ and the representation that is obtained from it by composing $\rho$ with the action of $S p(n, \mathbb{R})$ by automorphisms on $\mathbb{H}^{d}$ (see, e.g., $[15,21,23])$. Let us recall the main points of a direct construction.

The symplectic group $S p(n, \mathbb{R})$ is generated by the so-called free symplectic matrices

$$
S=\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right) \in S p(n, \mathbb{R}), \quad \operatorname{det} B \neq 0
$$

To each such a matrix the associated generating function is defined by

$$
W\left(x, x^{\prime}\right)=\frac{1}{2} D B^{-1} x \cdot x-B^{-1} x \cdot x^{\prime}+\frac{1}{2} B^{-1} A x^{\prime} \cdot x^{\prime} .
$$

Conversely, to every polynomial of the type

$$
W\left(x, x^{\prime}\right)=\frac{1}{2} P x \cdot x-L x \cdot x^{\prime}+\frac{1}{2} Q x^{\prime} \cdot x^{\prime}
$$

with

$$
P=P^{T}, Q=Q^{T}
$$

and

$$
\operatorname{det} L \neq 0
$$

it can be associated a free symplectic matrix, namely

$$
S_{W}=\left(\begin{array}{cc}
L^{-1} Q & L^{-1} \\
P L^{-1} Q-L^{T} & P L^{-1}
\end{array}\right)
$$

Given $S_{W}$ as above and $m \in \mathbb{Z}$ such that

$$
m \pi \equiv \arg \operatorname{det} L \quad \bmod 2 \pi
$$

the related operator $\widehat{S}_{W, m}$ is defined by setting, for $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\widehat{S}_{W, m} \psi(x)=\frac{1}{i^{n / 2}} \Delta(W) \int_{\mathbb{R}^{n}} e^{2 \pi i W\left(x, x^{\prime}\right)} \psi\left(x^{\prime}\right) d x^{\prime} \tag{14}
\end{equation*}
$$

(with $i^{n / 2}=e^{i \pi n / 4}$ ) where

$$
\Delta(W)=i^{m} \sqrt{|\operatorname{det} L|}
$$

The operator $\widehat{S}_{W, m}$ is named quadratic Fourier transform associated to the free symplectic matrix $S_{W}$ (as a remark, for integral representations of metaplectic operators that do not arise from free symplectic matrices see [14, 24]). The class modulo 4 of the integer $m$ is called Maslov index of $\widehat{S}_{W, m}$. Observe that if $m$ is one choice of Maslov index, then $m+2$ is another equally good choice: hence to each function $W$ we associate two operators, namely $\widehat{S}_{W, m}$ and $\widehat{S}_{W, m+2}=-\widehat{S}_{W, m}$.

The quadratic Fourier transform corresponding to the choices $S_{W}=J$ and $m=0$ is denoted by $\widehat{J}$. The generating function of $J$ is simply $W\left(x, x^{\prime}\right)=-x \cdot x^{\prime}$. It follows that

$$
\begin{equation*}
\widehat{J} \psi(x)=\frac{1}{i^{n / 2}} \int_{\mathbb{R}^{n}} e^{-2 \pi i x \cdot x^{\prime}} \psi\left(x^{\prime}\right) d x^{\prime}=\frac{1}{i^{n / 2}} \mathcal{F} \psi(x) \tag{15}
\end{equation*}
$$

for $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, where $\mathcal{F}$ is the usual unitary Fourier transform.
The quadratic Fourier transforms $\widehat{S}_{W, m}$ form a subset of the group $\mathcal{U}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ of unitary operators acting on $L^{2}\left(\mathbb{R}^{n}\right)$, which is mapped into itself by the operation of inversion and they generate a subgroup of $\mathcal{U}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ which is, by definition, the metaplectic group $M p(n, \mathbb{R})$. The elements of $M p(n, \mathbb{R})$ are called metaplectic operators.

Hence, every $\widehat{S} \in M p(n, \mathbb{R})$ is, by definition, a product

$$
\widehat{S}_{W_{1}, m_{1}} \ldots \widehat{S}_{W_{k}, m_{k}}
$$

of metaplectic operators associated to free symplectic matrices.
Indeed, it can be proved that every $\widehat{S} \in M p(n, \mathbb{R})$ can be written as a product of exactly two quadratic Fourier transforms: $\widehat{S}=\widehat{S}_{W, m} \widehat{S}_{W^{\prime}, m^{\prime}}$. Now, it can be shown that the mapping $\widehat{S}_{W, m} \longmapsto S_{W}$ extends to a group homomorphism $\pi: M p(n, \mathbb{R}) \rightarrow S p(n, \mathbb{R})$, which is in fact a double covering.

We also observe that each metaplectic operator is, by construction, a unitary operator in $L^{2}\left(\mathbb{R}^{n}\right)$, but also an automorphism of $\mathcal{S}\left(\mathbb{R}^{n}\right)$ and of $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$.

We are interested in its restriction $\widehat{S}=\pi(S)$, with $S \in U(2 n, \mathbb{R})$, the symplectic rotations in (2).

Observe that $U(n):=U(n, \mathbb{C})$, the complex unitary group the group of $n \times n$ invertible complex matrices $V$ satisfying $V V^{*}=V^{*} V=I_{n}$ ) is isomorphic to $U(2 n, \mathbb{R})$. The isomorphism $\iota$ is the mapping $\iota: U(n) \rightarrow U(2 n, \mathbb{R})$ given by

$$
\iota(A+i B)=\left(\begin{array}{cc}
A & -B  \tag{16}\\
B & A
\end{array}\right)
$$

for details see [15, Chapter 2.3].
We present here some results related to the group $U(2 n, \mathbb{R})$, which will be used in the sequel to attain the characterization of Theorem 1.2. First, we recall a well-known result, see for instance [22, Lemma 9.4.3]:

Lemma 2.1. For $f, g \in L^{2}\left(\mathbb{R}^{n}\right)$ and $S \in S p(n, \mathbb{R})$, the $S T F T V_{g} f$ satisfies

$$
\begin{equation*}
\left|V_{\widehat{S} g}(\widehat{S} f)(x, \xi)\right|=\left|V_{g} f\left(S^{-1}(x, \xi)\right)\right|, \quad(x, \xi) \in \mathbb{R}^{2 n} \tag{17}
\end{equation*}
$$

This second issue is contained in [5], we sketch the proof for the sake of consistency.
Lemma 2.2. For $\varphi, \psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and $S \in U(2 n, \mathbb{R})$, the $\operatorname{STFT} V_{\varphi}(\widehat{S} \psi)$ is a Schwartz function, with seminorms uniformly bounded when $S \in U(2 n, \mathbb{R})$.

Proof. Since $\varphi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, the $\operatorname{STFT} V_{\varphi}$ is a continuous mapping from $\mathcal{S}\left(\mathbb{R}^{n}\right)$ into $\mathcal{S}\left(\mathbb{R}^{2 n}\right)$ (see [16]). Hence, it is enough to show that

$$
\{\hat{S} \varphi: S \in U(2 n, \mathbb{R})\}
$$

is a bounded subset of the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right)$, i.e., every Schwartz seminorm is bounded on it. Since the group $U(2 n, \mathbb{R})$ is compact, it is sufficient to show that every seminorm is locally bounded, that is, we can limit ourselves to consider $S$ in a sufficiently small neighbourhood for any fixed $S_{0} \in U(2 n, \mathbb{R})$. Equivalently, we can consider $S$ of the form $S=S_{1} J^{-1} S_{0}$ where $S_{1}$ belongs to a enough small neighbourhood of $J$ in $U(2 n, \mathbb{R})$. Using the representation of metaplectic operators recalled at the beginning of this section, we can write

$$
\begin{aligned}
\hat{S} \varphi(x) & = \pm \widehat{S}_{1}\left[\widehat{J}^{-1} \widehat{S}_{0} \varphi\right](x) \\
& =c \sqrt{|\operatorname{det} L|} \int_{\mathbb{R}^{n}} e^{2 \pi i\left(\frac{1}{2} P x \cdot x-L x \cdot y+\frac{1}{2} Q y \cdot y\right)}[\underbrace{\left.\widehat{J}^{-1} \widehat{S}_{0} \varphi\right]}_{\in \mathcal{S}\left(\mathbb{R}^{n}\right)}(y) d y
\end{aligned}
$$

where $|c|=1$ and, we might say, $\|P\|<\epsilon,\|Q\|<\epsilon,\|L-I\|<\epsilon$. If $\epsilon<1$, it is straightforward to check that $\hat{S} \varphi$ belongs to a bounded subset of $\mathcal{S}\left(\mathbb{R}^{n}\right)$, as desired.

Lemma 2.3. Let $B=\left(b_{i, j}\right)_{i, j=1, \ldots . n}$ be the $n \times n$ submatrix in (2). The subset $\Sigma \subset U(2 n, \mathbb{R})$ obtained by setting $b_{i, 1}=0, i=1, \ldots n$ (i.e., the first column of $B$ is set to zero), is a submanifold of codimension $n$.

Proof. We have to verify that the coordinates $b_{1,1}, \ldots, b_{n, 1}$ are independent on the subset $\Sigma$, namely the projection

$$
\left(b_{1,1}, \ldots, b_{n, 1}\right): U(2 n, \mathbb{R}) \rightarrow \mathbb{R}^{n}
$$

has rank $n$ on $\Sigma$.
Let us first show that for every $S_{0} \in \Sigma$ there exists a $U(2 n, \mathbb{R})$-valued smooth function $S\left(b_{1}, \ldots, b_{n}\right)$, defined in a neighbourhood of $0 \in \mathbb{R}^{n}$, such that $S(0)=S_{0}$ and the first column "of its submatrix $B$ " is precisely $\left(b_{1}, \ldots, b_{n}\right)^{T}$.

Let $S_{0}=A+i B=\left(V_{1}, \ldots, V_{n}\right) \in \Sigma$, with $V_{j}$ being a $n \times 1$ complex vector, $j=1, \ldots, n$, so that by assumption $\left(b_{i, 1}\right)_{i=1, \ldots, n}=\operatorname{Im} V_{1}=0$. We consider any smooth function $V_{1}\left(b_{1}, \ldots, b_{n}\right)$, defined in a neighbourhood of $0 \in \mathbb{R}^{n}$, valued in the unit sphere of $\mathbb{C}^{n}$, such that

$$
\operatorname{Im} V_{1}\left(b_{1}, \ldots, b_{n}\right)=\left(b_{1}, \ldots, b_{n}\right)^{T}, \quad V_{1}(0)=V_{1} .
$$

Then, we apply the Gram-Schmidt orthonormalization procedure in $\mathbb{C}^{n}$ to the set of vectors $\left(V_{1}\left(b_{1}, \ldots, b_{n}\right), V_{2}, \ldots, V_{n}\right)$. This provides the desired $U(n)$-valued function $S\left(b_{1}, \ldots, b_{n}\right)$. In particular $S(0)=S_{0}$.

Now, the composition of the mapping

$$
\left(b_{1}, \ldots, b_{n}\right) \mapsto S\left(b_{1}, \ldots, b_{n}\right)
$$

followed by the projection $\left(b_{1,1}, \ldots, b_{n, 1}\right): U(2 n, \mathbb{R}) \rightarrow \mathbb{R}^{n}$ is therefore the identity mapping in a neighbourhood of 0 and has rank $n$. Hence the same is true for the projection $\left(b_{1,1}, \ldots, b_{n, 1}\right): U(2 n, \mathbb{R}) \rightarrow \mathbb{R}^{n}$ at $S_{0}$.

Lemma 2.4. For every $\epsilon>0$, define the ( $x$-independent) function

$$
\begin{equation*}
\chi_{\epsilon}(x, \xi)=\frac{1}{\epsilon^{n}} \mathbb{1}_{Q}\left(\frac{\xi}{\epsilon}\right), \tag{18}
\end{equation*}
$$

where

$$
Q=\left[-\frac{1}{2}, \frac{1}{2}\right]^{n} \subset \mathbb{R}^{n} \quad \text { and } \quad \mathbb{1}_{Q}= \begin{cases}1, & \xi \in Q \\ 0, & \xi \notin Q\end{cases}
$$

and

$$
\begin{equation*}
\tilde{\chi}_{\epsilon}(z)=\frac{\chi_{\epsilon}(z)}{\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(S z) d S}, \quad z \in \mathbb{R}^{2 n} . \tag{19}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\int_{U(2 n, \mathbb{R})} \tilde{\chi}_{\epsilon}(S z) d S=1, \quad \forall z \in \mathbb{R}^{2 n} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0^{+}} \int_{\mathbb{R}^{2 n}} \tilde{\chi}_{\epsilon}(x, \xi) \Phi(x, \xi) d x d \xi=C \int_{\mathbb{R}^{n}}|x|^{n} \Phi(x, 0) d x \tag{21}
\end{equation*}
$$

for some $C>0$ and for every continuous function $\Phi$ on $\mathbb{R}^{2 n}$ with a rapid decay at infinity.
Proof. We will show in a moment that, for $z=(x, \xi) \in \mathbb{R}^{2 n}$,

$$
\begin{equation*}
\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(S z) d S \gtrsim \min \left\{\epsilon^{-n},|z|^{-n}\right\} \tag{22}
\end{equation*}
$$

(with the convention, at $z=0$, that $\min \left\{\epsilon^{-n},+\infty\right\}=\epsilon^{-n}$ ). In particular, $\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(S z) d S \neq 0$, for every $z \in \mathbb{R}^{2 n}$. Formula (20) then follows, because

$$
\begin{aligned}
\int_{U(2 n, \mathbb{R})} \tilde{\chi}_{\epsilon}(S z) d S & =\int_{U(2 n, \mathbb{R})} \frac{\chi_{\epsilon}(S z)}{\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(U S z) d U} d S \\
& =\int_{U(2 n, \mathbb{R})} \frac{\chi_{\epsilon}(S z)}{\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(U z) d U} d S=1
\end{aligned}
$$

for every $z \in \mathbb{R}^{2 n}$, since the Haar measure is right invariant.
Let us now prove (22). For $z=0$ we have

$$
\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(S z) d S=\frac{1}{\epsilon^{n}} \int_{U(2 n, \mathbb{R})} d S=\frac{C_{0}}{\epsilon^{n}},
$$

with $C_{0}=\operatorname{meas}(U(2 n, \mathbb{R}))>0$. Consider now $z \neq 0$. Observe that the function

$$
\Psi_{\epsilon}(z):=\int_{U(2 n, \mathbb{R})} \chi_{\epsilon}(S z) d S
$$

is constant on the orbits of $U(2 n, \mathbb{R})$ in $\mathbb{R}^{2 n}$, so that we can suppose

$$
z=(x, 0), \quad x=\left(x_{1}, 0, \ldots, 0\right), \quad x_{1}=|x|=|z|>0
$$

Now, by the definition of $\chi_{\epsilon}$ and $\Psi_{\epsilon}$,

$$
\Psi_{\epsilon}(z)=\epsilon^{-n} \text { meas }\left\{S=\left(\begin{array}{cc}
A & -B  \tag{23}\\
B & A
\end{array}\right) \in U(2 n, \mathbb{R}):\left|b_{i, 1}\right|<\frac{\epsilon}{2|z|}, i=1, \ldots, n\right\}
$$

where $\left(b_{i, 1}\right)_{i=1, \ldots, n}$, is the first column of the matrix $B=\left(b_{i, j}\right)_{i, j=1, \ldots n}$.
Define, for $\mu>0$,

$$
f(\mu)=\text { meas }\left\{S=\left(\begin{array}{cc}
A & -B \\
B & A
\end{array}\right) \in U(2 n, \mathbb{R}):\left|b_{i, 1}\right|<\mu, i=1, \ldots, n\right\}
$$

Observe that $f(\mu)$ is non-decreasing and constant for $\mu \geq 1$. Moreover, from Lemma 2.3 we know that by setting $b_{i, 1}=0, i=1, \ldots, n$, in $U(2 n, \mathbb{R})$, we get a submanifold $\Sigma$ of codimension $n$, and the function $f(\mu)$ is the measure
of a tubular neighbourhood of $\Sigma$ in $U(2 n, \mathbb{R})$. Hence we have the asymptotic behaviour

$$
\begin{equation*}
\mu^{-n} f(\mu) \rightarrow C_{0}>0, \quad \text { as } \quad \mu \rightarrow 0^{+} \tag{24}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
f(\mu) \gtrsim \min \left\{1, \mu^{n}\right\} . \tag{25}
\end{equation*}
$$

We then infer

$$
\begin{equation*}
\Psi_{\epsilon}(z)=\epsilon^{-n} f\left(\frac{\epsilon}{2|z|}\right) \rightarrow \frac{C_{1}}{|z|^{n}}, \quad \text { as } \epsilon \rightarrow 0^{+} \tag{26}
\end{equation*}
$$

locally uniformly in $\mathbb{R}^{2 n} \backslash\{0\}$, with $C_{1}=2^{-n} C_{0}$, and

$$
\begin{equation*}
\Psi_{\epsilon}(z) \gtrsim \epsilon^{-n} \min \left\{1,\left(\frac{\epsilon}{|z|}\right)^{n}\right\}=\min \left\{\epsilon^{-n},|z|^{-n}\right\} \tag{27}
\end{equation*}
$$

which is (22).
Let us finally prove (21). We are interested in the limit $\epsilon \rightarrow 0^{+}$, so we can assume $\epsilon \leq 1$. Consider a continuous function $\Phi$ on $\mathbb{R}^{2 n}$ with rapid decay at infinity. By definition of $\tilde{\chi}_{\epsilon}(z)$ in (19) we have

$$
\tilde{\chi}_{\epsilon}(x, \xi)=\frac{\epsilon^{-n}}{\Psi_{\epsilon}(x, \xi)} \mathbb{1}_{[-\epsilon / 2, \epsilon / 2]^{n}}(\xi)
$$

so that, by (27),

$$
\left|\tilde{\chi}_{\epsilon}(x, \xi) \Phi(x, \xi)\right| \lesssim \epsilon^{-n}\left(1+|x|^{n}\right) \mathbb{1}_{[-\epsilon / 2, \epsilon / 2]^{n}}(\xi)|\Phi(x, \xi)| \in L^{1}\left(\mathbb{R}^{2 n}\right)
$$

for $0<\epsilon \leq 1$. Fubini's Theorem then allows one to look at the first integral in (21) as an iterated integral

$$
I_{\epsilon}:=\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n}} \tilde{\chi}_{\epsilon}(x, \xi) \Phi(x, \xi) d \xi\right) d x
$$

and we apply the dominated convergence theorem to the integral with respect to the $x$ variable as follows. Setting

$$
\Upsilon_{\epsilon}(x):=\int_{\mathbb{R}^{n}} \tilde{\chi}_{\epsilon}(x, \xi) \Phi(x, \xi) d \xi=\epsilon^{-n} \int_{[-\epsilon / 2, \epsilon / 2]^{n}} \frac{1}{\Psi_{\epsilon}(x, \xi)} \Phi(x, \xi) d \xi
$$

by (26) we have, for every fixed $x \neq 0$,

$$
\Upsilon_{\epsilon}(x) \rightarrow C|x|^{n} \Phi(x, 0) ;
$$

for some constant $C>0$. On the other hand $\Upsilon_{\epsilon}(x)$ is dominated, using (27), by

$$
(1+|x|)^{n} \sup _{\xi \in \mathbb{R}^{n}}|\Phi(x, \xi)| \in L^{1}\left(\mathbb{R}^{n}\right)
$$

Hence

$$
\lim _{\epsilon \rightarrow 0^{+}} I_{\epsilon}=\int_{\mathbb{R}^{n}} \lim _{\epsilon \rightarrow 0^{+}} \Upsilon_{\epsilon}(x) d x=C \int_{\mathbb{R}^{n}}|x|^{n} \Phi(x, 0) d x
$$

This concludes the proof.
Remark 2.5. Observe that there are no conditions on the derivatives of the function $\Phi$ in (21).

## 3. Proofs of the main results

In what follows we prove Theorems 1.2 and 1.3.
Proof of Theorem 1.2. (i) First Step. Let us start with showing that formula (5) is true for any function $\psi$ in the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right) \subset M^{p}\left(\mathbb{R}^{n}\right), 1 \leq p<\infty$. Using the Gaussian $\varphi(t)=2^{d / 4} e^{-\pi|t|^{2}}$ as window function, we compute the $M_{m}^{p}-$ norm of $\psi$ as in (4) and then use Lemma 2.4 so that

$$
\begin{aligned}
\|\psi\|_{M_{m}^{p}}^{p} & =\int_{\mathbb{R}^{2 n}}\left|V_{\varphi} \psi(z)\right|^{p} m(z)^{p} d z=\int_{\mathbb{R}^{2 n}} \int_{U(2 n, \mathbb{R})} \tilde{\chi}_{\epsilon}(S z)\left|V_{\varphi} \psi(z)\right|^{p} m(z)^{p} d S d z \\
& =\int_{\mathbb{R}^{2 n}} \int_{U(2 n, \mathbb{R})} \tilde{\chi}_{\epsilon}(z)\left|V_{\varphi} \psi\left(S^{-1} z\right)\right|^{p} m\left(S^{-1} z\right)^{p} d S d z \\
& =\int_{\mathbb{R}^{2 n}} \int_{U(2 n, \mathbb{R})} \tilde{\chi}_{\epsilon}(z)\left|V_{\widehat{S} \varphi} \widehat{S} \psi(z)\right|^{p} m\left(S^{-1} z\right)^{p} d S d z
\end{aligned}
$$

where in the last equality we used Lemma 2.1. Observe that, since $S$ is unitary and $\varphi$ is a Gaussian, $\widehat{S} \varphi=c \varphi$, for some phase factor $c \in \mathbb{C}$, with $|c|=1$ (see [15, Proposition 252]) and this phase factor is killed by the modulus obtaining $\left|V_{\widehat{S} \varphi} \widehat{S} \psi(z)\right|=\left|V_{\varphi} \widehat{S} \psi(z)\right|$. Continuing the above computation we infer

$$
\|\psi\|_{M_{m}^{p}}^{p}=\int_{\mathbb{R}^{2 n}} \tilde{\chi}_{\epsilon}(z) \int_{U(2 n, \mathbb{R})}\left|V_{\varphi} \widehat{S} \psi(z)\right|^{p} m\left(S^{-1} z\right)^{p} d S d z
$$

Set

$$
\Phi(z)=\int_{U(2 n, \mathbb{R})}\left|V_{\varphi} \widehat{S} \psi(z)\right|^{p} m\left(S^{-1} z\right)^{p} d S
$$

The dominated convergence theorem guarantees that $\Phi$ is continuous on $\mathbb{R}^{2 n}$ and moreover $\Phi$ has rapid decay at infinity. This follows from Lemma 2.2 (recall that $m$ is continuous and has at most polynomial growth).

Letting $\epsilon \rightarrow 0^{+}$and using (21) we obtain

$$
\begin{aligned}
\|\psi\|_{M_{m}^{p}}^{p} & =C \int_{\mathbb{R}^{n}}|x|^{n} \int_{U(2 n, \mathbb{R})}\left|V_{\varphi} \widehat{S} \psi(x, 0)\right|^{p} m\left(S^{-1}(x, 0)^{T}\right)^{p} d S d x \\
& =C \int_{\mathbb{R}^{n}}|x|^{n} \int_{U(2 n, \mathbb{R})}\left|\left\langle\widehat{S} \psi, T_{x} \varphi\right\rangle\right|^{p} m\left(S^{-1}(x, 0)^{T}\right)^{p} d S d x \\
& =C \int_{\mathbb{R}^{n}}|x|^{n} \int_{U(2 n, \mathbb{R})}\left|\left\langle\psi, \widehat{S} T_{x} \varphi\right\rangle\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d S d x .
\end{aligned}
$$

The last equality is due to $\left\langle\widehat{S} \psi, T_{x} \varphi\right\rangle=\left\langle\psi, \widehat{S}^{-1} T_{x} \varphi\right\rangle$ and the invariance of the Haar measure of $U(2 n, \mathbb{R})$ with respect to the change of variable $S \rightarrow S^{-1}$.

Second Step. Consider $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right), 1 \leq p<\infty$. Using the density of the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right)$ in $M_{m}^{p}\left(\mathbb{R}^{n}\right)$ (cf. e.g., [22, Chapter 12]), there exists a sequence $\left\{\psi_{k}\right\}_{k} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that $\psi_{k} \rightarrow f$ in $M_{m}^{p}\left(\mathbb{R}^{n}\right)$. This implies that $\psi_{k} \rightarrow f$ in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and

$$
\left\langle\psi_{k}, \widehat{S} T_{x} \varphi\right\rangle \rightarrow\left\langle\psi, \widehat{S} T_{x} \varphi\right\rangle
$$

pointwise for every $x \in \mathbb{R}^{n}, S \in U(2 n, \mathbb{R})$. Let us define, for every $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\left|\|f \mid\|=\left(\int_{\mathbb{R}^{n} \times U(2 n, \mathbb{R})}|x|^{n}\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S\right)^{\frac{1}{p}}\right. \tag{28}
\end{equation*}
$$

By Fatou's Lemma, for any $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right)$ :

$$
\begin{equation*}
\left\|\|f\|^{p} \leq \liminf _{k \rightarrow \infty}\right\|\left\|\psi_{k}\right\|\left\|^{p} \lesssim \liminf _{k \rightarrow \infty}\right\| \psi_{k}\left\|_{M_{m}^{p}}^{p}=\right\| f \|_{M_{m}^{p}}^{p} \tag{29}
\end{equation*}
$$

It is easy to check that $\|\|f\|\|$ is a seminorm on $M_{m}^{p}\left(\mathbb{R}^{n}\right)$. Applying (29) to the difference $f-\psi_{k}$ we obtain $\left\|\left\|f-\psi_{k} \mid\right\| \rightarrow 0\right.$ and hence $\| \mid \psi_{k}\| \| \rightarrow\|f\| \|$. By assumption we also have $\left\|\psi_{k}\right\|_{M_{m}^{p}} \rightarrow\|f\|_{M_{m}^{p}}$, and the desired norm equivalence in (5) then extents from $\mathcal{S}\left(\mathbb{R}^{n}\right)$ to $M_{m}^{p}\left(\mathbb{R}^{n}\right)$.

Third Step. We will show that (6) easily follows from (5). By the definition of the symplectic group (1), for any $S \in U(2 n, \mathbb{R})$,

$$
J^{-1} S=\left(S^{T}\right)^{-1} J^{-1}=S J^{-1}
$$

for $S^{-1}=S^{T}$. On the other hand, for any $f \in M_{m}^{p}\left(\mathbb{R}^{n}\right),\|f\|_{M_{m}^{p}} \asymp\|\hat{f}\|_{M_{\tilde{m}}^{p}}$, with $\tilde{m}(z)=m\left(J^{-1} z\right)$; see [16]. Using (15),

$$
\begin{aligned}
\left|\left\langle\hat{f}, \widehat{S} T_{x} \varphi\right\rangle\right| & =\left|\left\langle f, \widehat{J^{-1}} \widehat{S} T_{x} \varphi\right\rangle\right|=\left|\left\langle f, \widehat{S} \mathcal{F}^{-1} T_{x} \varphi\right\rangle\right| \\
& =\left|\left\langle f, \widehat{S} M_{x} \mathcal{F}^{-1} \varphi\right\rangle\right|=\left|\left\langle f, \widehat{S} M_{x} \varphi\right\rangle\right|
\end{aligned}
$$

since the Gaussian is an eigenvector of $\mathcal{F}^{-1}$ with eigenvalue equal to 1 . Moreover

$$
\tilde{m}\left(S(x, 0)^{T}\right)=m\left(J^{-1} S(x, 0)^{T}\right)=m\left(S J^{-1}(x, 0)^{T}\right)=m\left(S(0, x)^{T}\right)
$$

Hence (6) follows from (5).
(ii) Case $p=\infty$. Observe that any $z \in \mathbb{R}^{2 n}$ can be written as

$$
z=S^{-1}(x, 0)^{T}
$$

for some $x \in \mathbb{R}^{n}, S \in U(2 n, \mathbb{R})$, so that, for any $f \in M_{m}^{\infty}\left(\mathbb{R}^{n}\right)$,

$$
\begin{aligned}
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} & =\sup _{z \in \mathbb{R}^{2 n}}\left|V_{\varphi} f(z)\right| m(z) \asymp \sup _{S \in U(2 n, \mathbb{R})} \sup _{x \in \mathbb{R}^{n}}\left|V_{\varphi} f\left(S^{-1}(x, 0)^{T}\right)\right| m\left(S^{-1}(x, 0)^{T}\right) \\
& =\sup _{S \in U(2 n, \mathbb{R})} \sup _{x \in \mathbb{R}^{n}}\left|V_{\varphi}(\widehat{S} f)(x, 0)\right| m\left(S^{-1}(x, 0)^{T}\right) \\
& =\sup _{S \in U(2 n, \mathbb{R})} \sup _{x \in \mathbb{R}^{n}}\left|\left\langle\widehat{S} f, T_{x} \varphi\right\rangle\right| m\left(S^{-1}(x, 0)^{T}\right) \\
& =\sup _{S \in U(2 n, \mathbb{R})} \sup _{x \in \mathbb{R}^{n}}\left|\left\langle f, \widehat{S} T_{x} \varphi\right\rangle\right| m\left(S(x, 0)^{T}\right)
\end{aligned}
$$

which gives (7). Formula (8) follows as above.
We now prove the similar result, with the group $U(2 n, \mathbb{R})$ replaced by the subgroup $\mathbb{T}^{n}$ (up to isomorphisms).

Proof of Theorem 1.3. (i) We could follow a similar pattern to the proof of Theorem 1.2 , replacing the group $U(2 n, \mathbb{R})$ by $\mathbb{T}^{n}$. The preparation of Lemma 2.3 would be no longer necessary. Lemma 2.4 would require some small adjustments. On the other hand a more direct argument can be given. Namely, writing $z_{j}=\left(x_{j}, \xi_{j}\right)$ in complex notation as $r_{j} e^{i \theta_{j}}$, and setting $r=\left(r_{1}, \ldots, r_{n}\right)$, $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right)$ we have

$$
\begin{aligned}
\|f\|_{M_{m}^{p}}^{p} & =\int_{\mathbb{R}^{2 n}}\left|V_{\varphi} f(z)\right|^{p} m(z)^{p} d z \\
& =\int_{\mathbb{R}_{+}^{n} \times[0,2 \pi]^{n}} r_{1} \cdots r_{n}\left|V_{\varphi} f\left(r_{1} e^{i \theta_{1}}, \ldots, r_{n} e^{i \theta_{n}}\right)\right|^{p} m\left(r_{1} e^{i \theta_{1}}, \ldots, r_{n} e^{i \theta_{n}}\right)^{p} d r d \theta .
\end{aligned}
$$

With $S$ as in (9) and using Lemma 2.1, therefore we have

$$
\begin{aligned}
\|f\|_{M_{m}^{p}}^{p} & \asymp \int_{\mathbb{R}^{n} \times \mathbb{T}^{n}}\left|x_{1} \cdots x_{n}\right|\left|V_{\varphi} f\left(S(x, 0)^{T}\right)\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S \\
& =\int_{\mathbb{R}^{n} \times \mathbb{T}^{n}}\left|x_{1} \cdots x_{n}\right|\left|V_{\varphi}\left(\widehat{S}^{-1} f\right)(x, 0)\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S \\
& =\int_{\mathbb{R}^{n} \times \mathbb{T}^{n}}\left|x_{1} \cdots x_{n}\right|\left|\left\langle\widehat{S}^{-1} f, T_{x} \varphi\right\rangle\right|^{p} m\left(S(x, 0)^{T}\right)^{p} d x d S,
\end{aligned}
$$

which is (10). The characterization (11) has the same proof as the corresponding formula (6).
(ii) The $M^{\infty}$ case uses the same argument as in the proofs of (7) and (8), with the group $U(2 n, \mathbb{R})$ replaced by $\mathbb{T}^{n}$.

## 4. Integral representations for the torus in terms of the fractional Fourier Transform

Observe that the symplectic matrix in $U(2 n, \mathbb{R})$ corresponding to the complex matrix $S \in \mathbb{T}^{n}$ in (9) via the isomorphism $\iota$ in (16) is given by

$$
\iota(S)=\left(\begin{array}{cc}
A & -B \\
B & A
\end{array}\right)
$$

with

$$
A=\operatorname{diag}\left[\cos \theta_{1}, \ldots, \cos \theta_{n}\right] \quad B=\operatorname{diag}\left[\sin \theta_{1}, \ldots, \sin \theta_{n}\right] .
$$

Consider the case $\theta_{i} \neq k \pi, k \in \mathbb{Z}, i=1, \ldots, n$. The matrix $\iota(S)$ is a free symplectic matrix and the related metaplectic operator possesses the integral representation (14). Since

$$
A B^{-1}=B^{-1} A=\operatorname{diag}\left[\frac{\cos \theta_{1}}{\sin \theta_{1}}, \ldots, \frac{\cos \theta_{n}}{\sin \theta_{n}}\right]
$$

the polynomial $W\left(x, x^{\prime}\right)$ becomes

$$
\begin{equation*}
W\left(x_{1}, \ldots, x_{n}, x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right)=\sum_{i=1}^{n} \frac{1}{2 \sin \theta_{i}}\left(\cos \theta_{i} x_{i}^{2}-2 x_{i} x_{i}^{\prime}+\cos \theta_{i} x_{i}^{\prime 2}\right) \tag{30}
\end{equation*}
$$

and

$$
\Delta(W)=\frac{c}{\sqrt{\left|\sin \theta_{1} \cdots \sin \theta_{n}\right|}} .
$$

for some phase factor $c \in \mathbb{C}$, with $|c|=1$. Hence we obtain, for $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\widehat{\iota(S)} \psi(x)=\frac{c}{\sqrt{\left|\sin \theta_{1} \cdots \sin \theta_{n}\right|}} \int_{\mathbb{R}^{n}} e^{2 \pi i W\left(x, x^{\prime}\right)} \psi\left(x^{\prime}\right) d x^{\prime} \tag{31}
\end{equation*}
$$

with $W\left(x, x^{\prime}\right)$ in (30). From (31) we deduce that $\widehat{\iota(S)}$ can be written as the composition of the operators

$$
\begin{equation*}
\widehat{\iota(S)}= \pm \widehat{\iota\left(S_{1}\right)} \cdots \widehat{\iota\left(S_{n}\right)}, \tag{32}
\end{equation*}
$$

where, for some phase factor $c$,

$$
\widehat{\iota\left(S_{i}\right)} \psi(x)=\frac{c}{\sqrt{\left|\sin \theta_{i}\right|}} \int_{\mathbb{R}} e^{\frac{\pi i}{\sin \theta_{i}}\left(\cos \theta_{i} x_{i}^{2}-2 x_{i} x_{i}^{\prime}+\cos \theta_{i} x_{i}^{\prime 2}\right)} \psi\left(x_{1}^{\prime}, \ldots, x_{i}^{\prime}, \ldots, x_{n}^{\prime}\right) d x_{i}^{\prime} .
$$

Indeed if $\theta_{i}=\pi / 2$, then $\widehat{\iota\left(S_{i}\right)}= \pm \widehat{J}$ is the Fourier transform with respect to the variable $x_{i}$. Otherwise, $\widehat{\iota\left(S_{i}\right)}= \pm \mathcal{F}_{\theta_{i}}$, the $\theta_{i}$-angle partial fractional Fourier transform (again referred to the variable $x_{i}$ ).

Alternatively, the same conclusion (32) can be drawn by writing

$$
\begin{equation*}
S=\operatorname{diag}\left[e^{i \theta_{1}} \ldots, e^{i \theta_{n}}\right]=\operatorname{diag}\left[e^{i \theta_{1}}, 1, \ldots 1\right] \cdots \operatorname{diag}\left[1, \ldots 1, e^{i \theta_{n}}\right] \tag{33}
\end{equation*}
$$

that is

$$
S=S_{1} \cdots S_{i} \cdots S_{n}
$$

with

$$
S_{i}=\operatorname{diag}\left[1, \ldots 1, e^{i \theta_{i}}, 1, \ldots 1\right], \quad i=1, \ldots, n
$$

so that

$$
\widehat{\iota(S)}=\iota\left(S_{1} \widehat{) \ldots \iota}\left(S_{1}\right)= \pm \widehat{\iota\left(S_{1}\right)} \cdots \widehat{\iota\left(S_{n}\right)}\right.
$$

If $\theta_{i}=2 k \pi$ for some $k \in \mathbb{Z}, \widehat{\iota\left(S_{i}\right)}= \pm I$ with $I$ the identity operator. If $\theta_{i}=$ $(2 k+1) \pi$ for some $k \in \mathbb{Z}, \widehat{\iota\left(S_{i}\right)} \psi(x)= \pm \psi\left(x_{1}, \ldots,-x_{i}, \ldots, x_{n}\right)$.

Hence using the $\theta_{i}$-angle partial fractional Fourier transform $\mathcal{F}_{\theta_{i}}= \pm \widehat{\iota\left(S_{i}\right)}$ we can rephrase Theorem 1.3 as follows.

Theorem 4.1. Let $\varphi$ be the Gaussian of Theorem 1.2.
(i) For $1 \leq p<\infty, f \in M_{m}^{p}\left(\mathbb{R}^{n}\right)$, we have
$\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times[0,2 \pi]^{n}}\left|x_{1} \ldots x_{n}\right|\left|\left\langle f, \mathcal{F}_{\theta_{1}} \ldots \mathcal{F}_{\theta_{n}} T_{x} \varphi\right\rangle\right|^{p} m\left(x_{1} e^{i \theta_{1}}, \ldots, x_{n} e^{i \theta_{n}}\right)^{p} d x d \theta\right)^{\frac{1}{p}}$,
and
$\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)} \asymp\left(\int_{\mathbb{R}^{n} \times[0,2 \pi]^{n}}\left|\xi_{1} \ldots \xi_{n} \|\left\langle f, \mathcal{F}_{\theta_{1}} \ldots \mathcal{F}_{\theta_{n}} M_{\xi} \varphi\right\rangle\right|^{p} m\left(\xi_{1} e^{i \theta_{1}}, \ldots, \xi_{n} e^{i \theta_{n}}\right)^{p} d \xi d \theta\right)^{\frac{1}{p}}$.
(ii) For $p=\infty$,

$$
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{\theta \in[0,2 \pi]^{n}} \sup _{x \in \mathbb{R}^{n}}\left|\left\langle f, \mathcal{F}_{\theta_{1}} \ldots \mathcal{F}_{\theta_{n}} T_{x} \varphi\right\rangle\right| m\left(x_{1} e^{i \theta_{1}}, \ldots, x_{n} e^{i \theta_{n}}\right)
$$

and

$$
\|f\|_{M_{m}^{\infty}\left(\mathbb{R}^{n}\right)} \asymp \sup _{\theta \in[0,2 \pi]^{n}} \sup _{\xi \in \mathbb{R}^{n}}\left|\left\langle f, \mathcal{F}_{\theta_{1}} \ldots \mathcal{F}_{\theta_{n}} M_{\xi} \varphi\right\rangle\right| m\left(\xi_{1} e^{i \theta_{1}}, \ldots, \xi_{n} e^{i \theta_{n}}\right)
$$

We observe that this result could also be obtained by writing $\|f\|_{M_{m}^{p}\left(\mathbb{R}^{n}\right)}$ in terms of the weighted $L^{p}$ norm of the Bargmann transform of $f$ and using the covariance property of the Bargmann transform; the papers [18, 30] and specially [31] are relevant in this connection.

## Acknowledgment

We would like to thank the referees for interesting remarks and suggestions which lead in particular to a cleaner proof of Theorem 1.3.

The first and the third author have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). MdG has been financed by the Austrian Research Foundation FWF grant P27773.

## References

[1] A. Bényi, K. Gröchenig, K.A. Okoudjou and L.G. Rogers. Unimodular Fourier multipliers for modulation spaces. J. Funct. Anal., 246(2): 366-384, 2007.
[2] A. Bényi and K.A. Okoudjou. Time-frequency estimates for pseudodifferential operators. Contemporary Math., Amer. Math. Soc., 428:13-22, 2007.
[3] A. Bényi and K.A. Okoudjou. Local well-posedness of nonlinear dispersive equations on modulation spaces. Bull. Lond. Math. Soc., 41(3):549-558, 2009.
[4] A. Bényi, O. Tadahiro and O. Pocovnicu. On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^{d}, d \geq 3$. Trans. Amer. Math. Soc. Ser. B, 2:1-50, 2015.
[5] A. Cauli, F. Nicola and A. Tabacco. Strichartz estimates for the metaplectic representation. Rev. Mat. Iberoam., 35(7): 2079-2092, 2019.
[6] O. Christensen, H.G. Feichtinger and S. Paukner. Gabor analysis for imaging. Handbook of mathematical methods in imaging. Vol. 1, 2, 3, 1717-1757, Springer, New York, 2015.
[7] F. Concetti and J. Toft. Trace ideals for Fourier integral operators with non-smooth symbols, "Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis", Fields Inst. Commun., Amer. Math. Soc., 52:255-264, 2007.
[8] F. Concetti, G. Garello and J. Toft. Trace ideals for Fourier integral operators with nonsmooth symbols II. Osaka J. Math., 47(3):739-786, 2010.
[9] E. Cordero, K. Gröchenig, F. Nicola and L. Rodino. Wiener algebras of Fourier integral operators. J. Math. Pures Appl., 99:219-233, 2013.
[10] E. Cordero and F. Nicola. Boundedness of Schrödinger type propagators on modulation spaces. J. Fourier Anal. Appl. 16(3):311-339, 2010.
[11] E. Cordero and F. Nicola. Schrödinger equations with bounded perturbations. J. PseudoDiffer. Op. and Appl., 5(3):319-341, 2014.
[12] E. Cordero, K. Gröchenig, F. Nicola and L. Rodino. Generalized metaplectic operators and the Schrödinger equation with a potential in the Sjöstrand class. J. Math. Phys., $55(8)$, art. no. 081506, 2014.
[13] E. Cordero, F. Nicola and L. Rodino. Sparsity of Gabor representation of Schrödinger propagators. Appl. Comput. Harmon. Anal., 26(3):357-370, 2009.
[14] E. Cordero, F. Nicola and L. Rodino. Integral representations for the class of generalized metaplectic operators. J. Fourier Anal. Appl., 21:694-714, 2015.
[15] M. A. de Gosson. Symplectic Methods in Harmonic Analysis and in Mathematical Physics, volume 7 of Pseudo-Differential Operators. Theory and Applications. Birkhäuser/Springer Basel AG, Basel, 2011.
[16] H. G. Feichtinger. Modulation spaces on locally compact abelian groups, Technical Report, University Vienna, 1983, and also in Wavelets and Their Applications, M. Krishna, R. Radha, S. Thangavelu, editors, Allied Publishers, 99-140, 2003.
[17] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. J. Funct. Anal., 86(2):307-340, 1989.
[18] H. G. Feichtinger K. Gröchenig and D. Walnut. Wilson bases and modulation spaces. Math. Nach., 155:7-17, 1992.
[19] H. G. Feichtinger. Choosing function spaces in harmonic analysis, Excursions in harmonic analysis. Vol. 4, Appl. Numer. Harmon. Anal., 65-101, Birkhäuser/Springer, Cham, 2015.
[20] H. G. Feichtinger. Modulation spaces: looking back and ahead. Sampl. Theory Signal Image Process., 5(2):109-140, 2006
[21] G. B. Folland. Harmonic Analysis in Phase Space. Princeton Univ. Press, Princeton, NJ, 1989.
[22] K. Gröchenig. Foundations of Time-frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.
[23] Leray, Jean Lagrangian Analysis and Quantum Mechanics. A mathematical structure related to asymptotic expansions and the Maslov index, translated from the French by Carolyn Schroeder, MIT Press, Cambridge, Mass.-London, (1981).
[24] H. Morsche and P.J. Oonincx. Integral representations of affine transformations in phase space with an application to energy localization problems. CWI report, Amsterdam, 1999.
[25] M. Sugimoto and N. Tomita. Boundedness properties of pseudo-differential operators and Calderòn-Zygmund operators on modulation spaces. J. Fourier Anal. Appl., 14(1):124143, 2008.
[26] M. Ruzhansky, B. Wang and H. Zhang. Global well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data in modulation and Sobolev spaces, J. Math. Pures Appl. (9),105(1):31-65, 2016.
[27] M. Ruzhansky, M. Sugimoto and B. Wang. Modulation spaces and nonlinear evolution equations, Evolution equations of hyperbolic and Schrödinger type, Progr. Math., 301:267283, Birkhäuser/Springer Basel AG, Basel, 2012.
[28] J. Toft. Continuity properties for modulation spaces, with applications to pseudodifferential calculus. I. J. Funct. Anal., 207(2):399-429, 2004.
[29] J. Toft. Continuity properties for modulation spaces, with applications to pseudodifferential calculus. II. Ann. Global Anal. Geom., 26(1):73-106, 2004.
[30] J. Toft, The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators. J. Pseudo-Differ. Oper. Appl., 3:145-227, 2012.
[31] J. Toft, Images of function and distribution spaces under the Bargmann transform. J. Pseudo-Differ. Oper. Appl., 8:83-139, 2017.
[32] B. Wang, Z. Huo, C. Hao and Z. Guo. Harmonic Analysis Method for Nonlinear Evolution Equations. I, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[33] B. Wang, Z. Lifeng and G. Boling. Isometric decomposition operators, function spaces $E_{p, q}^{\lambda}$ and applications to nonlinear evolution equations. J. Funct. Anal., 233(1):1-39, 2006.
[34] B. Wang and H. Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. J. Differential Equations, 232(1):36-73, 2007.

Dipartimento di Matematica, Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy

E-mail address: elena.cordero@unito.it
University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1
A-1090 Wien, Austria
E-mail address: maurice.de.gosson@univie.ac.at
Dipartimento di Scienze Matematiche, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy

E-mail address: fabio.nicola@polito.it


[^0]:    2010 Mathematics Subject Classification. 42B35,22C05.
    Key words and phrases. modulation spaces, metaplectic operators, symplectic group, unitary group, short-time Fourier transform.

