

A characterization of modulation spaces by symplectic rotations

Original

A characterization of modulation spaces by symplectic rotations / Cordero, E., De Gosson, M., Nicola, F.. - In: JOURNAL OF FUNCTIONAL ANALYSIS. - ISSN 0022-1236. - STAMPA. - 278:11(2020). [10.1016/j.jfa.2020.108474]

Availability:

This version is available at: 11583/2853971 since: 2020-11-27T11:29:54Z

Publisher:

Elsevier

Published

DOI:10.1016/j.jfa.2020.108474

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

A CHARACTERIZATION OF MODULATION SPACES BY SYMPLECTIC ROTATIONS

ELENA CORDERO, MAURICE DE GOSSON, AND FABIO NICOLA

ABSTRACT. This note contains a new characterization of modulation spaces $M_m^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, by symplectic rotations. Precisely, instead to measure the time-frequency content of a function by using translations and modulations of a fixed window as building blocks, we use translations and metaplectic operators corresponding to symplectic rotations. Technically, this amounts to replace, in the computation of the $M_m^p(\mathbb{R}^n)$ -norm, the integral in the time-frequency plane with an integral on $\mathbb{R}^n \times U(2n, \mathbb{R})$ with respect to a suitable measure, $U(2n, \mathbb{R})$ being the group of symplectic rotations. More conceptually, we are considering a sort of polar coordinates in the time-frequency plane. To have invariance under symplectic rotations we choose a Gaussian as suitable window function. We also provide a similar (and easier) characterization with the group $U(2n, \mathbb{R})$ being reduced to the n -dimensional torus \mathbb{T}^n .

1. INTRODUCTION

The objective of this study is to find a new characterization of modulation spaces using symplectic rotations. Precisely, we are interested in those metaplectic operators $\widehat{S} \in Mp(n, \mathbb{R})$, such that the corresponding projection $S := \pi(\widehat{S})$ onto the symplectic group $Sp(n, \mathbb{R})$ is a symplectic rotation. Let us recall that the symplectic group $Sp(n, \mathbb{R})$ is the subgroup of $2n \times 2n$ invertible matrices $GL(2n, \mathbb{R})$, defined by

$$(1) \quad Sp(n, \mathbb{R}) = \{S \in GL(2n, \mathbb{R}) : SJS^T = J\},$$

where J is the orthogonal matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

($I_n, 0_n$ are the $n \times n$ identity matrix and null matrix, respectively). Here we consider the subgroup

$$U(2n, \mathbb{R}) := Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R}) \simeq U(n)$$

2010 *Mathematics Subject Classification.* 42B35, 22C05.

Key words and phrases. modulation spaces, metaplectic operators, symplectic group, unitary group, short-time Fourier transform.

of symplectic rotations (cf., e.g. [15, Section 2.3]), namely

$$(2) \quad U(2n, \mathbb{R}) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : AA^T + BB^T = I_n, AB^T = B^T A \right\} \subset Sp(2n, \mathbb{R}),$$

endowed with the normalized Haar measure dS (the group $U(2n, \mathbb{R})$, being compact, is unimodular).

In the 80's H. Feichtinger [16] introduced modulation spaces to measure the time-frequency concentration of a function/distribution on the time-frequency space (or phase space) \mathbb{R}^{2n} . They are nowadays become popular among mathematicians and engineers because they have found numerous applications in signal processing [6, 19, 20], pseudodifferential and Fourier integral operators [7, 8, 9, 28, 29], partial differential equations [1, 2, 3, 4, 10, 13, 11, 11, 32, 33, 34] and quantum mechanics [12, 15].

To recall their definition, we need a few time-frequency tools. First, the translation T_x and modulation M_ξ operators are defined by

$$T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t), \quad t, x, \xi \in \mathbb{R}^n,$$

for any function f on \mathbb{R}^n .

The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to a function $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (so-called window), given by

$$(3) \quad V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^n} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \xi} dt, \quad x, \xi \in \mathbb{R}^n.$$

The short-time Fourier transform is well-defined whenever the bracket $\langle \cdot, \cdot \rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f \in \mathcal{S}'(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$, or for $f, g \in L^2(\mathbb{R}^n)$.

Let $m(x, \xi)$ be a continuous weight, v -moderate for some submultiplicative weight v (see [22, Section 11.1] for details - we will not use explicitly these properties). We also assume that m has at most polynomial growth.

Definition 1.1 (Modulation spaces). *Given $g \in \mathcal{S}(\mathbb{R}^n)$, and $1 \leq p \leq \infty$, the modulation space $M_m^p(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $V_g f \in L_m^p(\mathbb{R}^{2n})$. The norm on $M_m^p(\mathbb{R}^n)$ is*

$$(4) \quad \begin{aligned} \|f\|_{M_m^p} &= \|V_g f\|_{L_m^p} = \left(\int_{\mathbb{R}^{2n}} |V_g f(x, \xi)|^p m(x, \xi)^p dx d\xi \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^{2n}} |\langle f, M_\xi T_x g \rangle|^p m(x, \xi)^p dx d\xi \right)^{1/p} \end{aligned}$$

(with obvious modifications for $p = \infty$).

The spaces $M_m^p(\mathbb{R}^n)$ are Banach spaces, and every nonzero $g \in M_v^1(\mathbb{R}^n)$ yields an equivalent norm in (4), so that their definition is independent of the choice of $g \in M_v^1(\mathbb{R}^n)$ (see [16, 22]).

We now provide an equivalent norm to (4) by using translations T_x (or modulations M_ξ) and the operators \widehat{S} , with $S \in U(2n, \mathbb{R})$ as follows.

Theorem 1.2. *Consider the Gaussian function $\varphi(t) = 2^{d/4}e^{-\pi|t|^2}$.*

(i) *For $1 \leq p < \infty$ and $f \in M_m^p(\mathbb{R}^n)$, we have*

$$(5) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |x|^n |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}},$$

where dx is the Lebesgue measure on \mathbb{R}^n and dS the Haar measure on $U(2n, \mathbb{R})$. Similarly,

$$(6) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |\xi|^n |\langle f, \widehat{S}M_\xi\varphi \rangle|^p m(S(0, \xi)^T)^p d\xi dS \right)^{\frac{1}{p}},$$

with $d\xi$ being the Lebesgue measure on \mathbb{R}^n and dS the Haar measure on $U(2n, \mathbb{R})$.

(ii) *For $p = \infty$, $f \in M_m^\infty(\mathbb{R}^n)$, it occurs*

$$(7) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x\varphi \rangle| m(S(x, 0)^T)$$

and, similarly,

$$(8) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S}M_\xi\varphi \rangle| m(S(0, \xi)^T).$$

The interpretation of the integral (5) above is as follows. The metaplectic operator \widehat{S} produces a time-frequency rotation of the shifted Gaussian $T_x\varphi$. In this way, the operator

$$f \mapsto \langle f, \widehat{S}T_x\varphi \rangle$$

detects the time-frequency content of f in an oblique strip, see Figure 1. All the contributions are then added together with a weight $|x|^n$ which takes into account the underlapping of the strips as $|x| \rightarrow +\infty$ and the overlapping as $x \rightarrow 0$.

Formulas (6), (7) and (8) have similar meanings.

Observe that in dimension $n = 1$, $U(2, \mathbb{R}) \simeq U(1)$ and the above formula is essentially a transition to polar coordinates with $|x|$ being the Jacobian.

Comparing (4) and (5) we observe that in (5) the modulation operator M_ξ is replaced by the metaplectic operator \widehat{S} and the integral on the phase space \mathbb{R}^{2n} has become an integral on the cartesian product $\mathbb{R}^n \times U(2n, \mathbb{R})$. The integration parameters (x, ξ) of (4) live in \mathbb{R}^{2n} , with $\dim \mathbb{R}^{2n} = 2n$, whereas the parameters (x, S) of (5) live in $\mathbb{R}^n \times U(2n, \mathbb{R})$. Recall that $\dim U(2n, \mathbb{R}) = n^2$ [15]; this suggests that a formula similar to (5) should hold when $U(2n, \mathbb{R})$ is reduced to

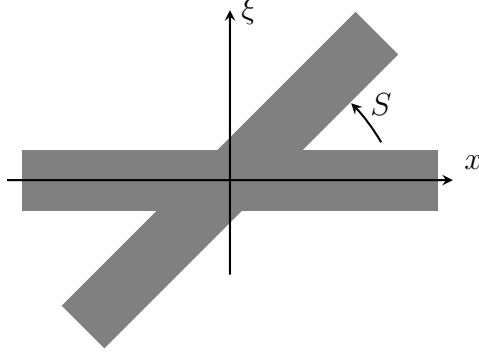


FIGURE 1. The time-frequency content of f in the oblique strip is detected by the operator $f \mapsto \langle f, \widehat{S}T_x\varphi \rangle$.

a suitable subgroup $K \subset U(2n, \mathbb{R})$ of dimension n . This is indeed the case (and easier to see), as shown in the subsequent Theorem 1.3.

Consider the n -dimensional torus

$$(9) \quad \mathbb{T}^n = \left\{ S = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \theta_1, \dots, \theta_n \in \mathbb{R} \right\} \subset U(n)$$

with the Haar measure $dS = d\theta_1 \dots d\theta_n$. The torus is isomorphic to a subgroup $K \subset U(2n, \mathbb{R})$, via the isomorphism ι in formula (16) below (see the subsequent section).

We exhibit the following characterization for M^p -spaces.

Theorem 1.3. *Let φ be the Gaussian of Theorem 1.2.*

(i) *For $1 \leq p < \infty$, $f \in M_m^p(\mathbb{R}^n)$, we have*

$$(10) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \dots x_n| |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}},$$

and, similarly,

$$(11) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times \mathbb{T}^n} |\xi_1 \dots \xi_n| |\langle f, \widehat{S}M_\xi\varphi \rangle|^p m(S(0, \xi)^T)^p d\xi dS \right)^{\frac{1}{p}}.$$

(ii) *For $p = \infty$,*

$$(12) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x\varphi \rangle| m(S(x, 0)^T)$$

and

$$(13) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S}M_\xi\varphi \rangle| m(S(0, \xi)^T).$$

The above results for the groups $U(2n, \mathbb{R})$ and \mathbb{T}^n can be interpreted, in a sense, as two extreme cases, and it would be interesting to find, more generally, for which compact subgroups $K \subset U(2n, \mathbb{R})$ similar characterizations hold. We conjecture that they should be precisely the subgroups $K \subset U(2n, \mathbb{R})$ such that every orbit for their action on \mathbb{R}^{2n} intersects $\{0\} \times \mathbb{R}^n$ (up to subsets of measure zero), with a corresponding weighted measure on $\mathbb{R}^n \times K$ to be determined.

Another problem which is worth investigating is the study of discrete versions of the above characterizations via coorbit theory [17].

The paper is organized as follows: in Section 2 we collected some preliminary results, whereas Section 3 is devoted to the proof of Theorems 1.2 and 1.3. In Section 4 we rephrase more explicitly Theorem 1.3 in terms of the partial fractional Fourier transform.

2. NOTATION AND PRELIMINARIES

Notation. We write $x \cdot y$ for the scalar product on \mathbb{R}^n and $|t|^2 = t \cdot t$, for $t, x, y \in \mathbb{R}^n$. For expressions $A, B \geq 0$, we use the notation $A \lesssim B$ to represent the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^n)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ of the inner product $\langle f, g \rangle = \int f(t)\overline{g(t)}dt$ on $\mathcal{S}(\mathbb{R}^n)$.

Metaplectic Operators. The metaplectic representation μ of $Mp(n, \mathbb{R})$, the two-sheeted cover of the symplectic group $Sp(n, \mathbb{R})$, defined in (1) arises as intertwining operator between the standard Schrödinger representation ρ of the Heisenberg group \mathbb{H}^d and the representation that is obtained from it by composing ρ with the action of $Sp(n, \mathbb{R})$ by automorphisms on \mathbb{H}^d (see, e.g., [15, 21, 23]). Let us recall the main points of a direct construction.

The symplectic group $Sp(n, \mathbb{R})$ is generated by the so-called free symplectic matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), \quad \det B \neq 0.$$

To each such a matrix the associated generating function is defined by

$$W(x, x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'.$$

Conversely, to every polynomial of the type

$$W(x, x') = \frac{1}{2}Px \cdot x - Lx \cdot x' + \frac{1}{2}Qx' \cdot x'$$

with

$$P = P^T, Q = Q^T$$

and

$$\det L \neq 0$$

it can be associated a free symplectic matrix, namely

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}.$$

Given S_W as above and $m \in \mathbb{Z}$ such that

$$m\pi \equiv \arg \det L \pmod{2\pi},$$

the related operator $\widehat{S}_{W,m}$ is defined by setting, for $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$(14) \quad \widehat{S}_{W,m}\psi(x) = \frac{1}{i^{n/2}} \Delta(W) \int_{\mathbb{R}^n} e^{2\pi i W(x,x')} \psi(x') dx'$$

(with $i^{n/2} = e^{i\pi n/4}$) where

$$\Delta(W) = i^m \sqrt{|\det L|}.$$

The operator $\widehat{S}_{W,m}$ is named *quadratic Fourier transform* associated to the free symplectic matrix S_W (as a remark, for integral representations of metaplectic operators that do not arise from free symplectic matrices see [14, 24]). The class modulo 4 of the integer m is called *Maslov index* of $\widehat{S}_{W,m}$. Observe that if m is one choice of Maslov index, then $m + 2$ is another equally good choice: hence to each function W we associate two operators, namely $\widehat{S}_{W,m}$ and $\widehat{S}_{W,m+2} = -\widehat{S}_{W,m}$.

The quadratic Fourier transform corresponding to the choices $S_W = J$ and $m = 0$ is denoted by \widehat{J} . The generating function of J is simply $W(x, x') = -x \cdot x'$. It follows that

$$(15) \quad \widehat{J}\psi(x) = \frac{1}{i^{n/2}} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot x'} \psi(x') dx' = \frac{1}{i^{n/2}} \mathcal{F}\psi(x)$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$, where \mathcal{F} is the usual unitary Fourier transform.

The quadratic Fourier transforms $\widehat{S}_{W,m}$ form a subset of the group $\mathcal{U}(L^2(\mathbb{R}^n))$ of unitary operators acting on $L^2(\mathbb{R}^n)$, which is mapped into itself by the operation of inversion and they generate a subgroup of $\mathcal{U}(L^2(\mathbb{R}^n))$ which is, by definition, the metaplectic group $Mp(n, \mathbb{R})$. The elements of $Mp(n, \mathbb{R})$ are called metaplectic operators.

Hence, every $\widehat{S} \in Mp(n, \mathbb{R})$ is, by definition, a product

$$\widehat{S}_{W_1, m_1} \cdots \widehat{S}_{W_k, m_k}$$

of metaplectic operators associated to free symplectic matrices.

Indeed, it can be proved that every $\widehat{S} \in Mp(n, \mathbb{R})$ can be written as a product of exactly two quadratic Fourier transforms: $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$. Now, it can be shown that the mapping $\widehat{S}_{W,m} \mapsto S_W$ extends to a group homomorphism $\pi : Mp(n, \mathbb{R}) \rightarrow Sp(n, \mathbb{R})$, which is in fact a double covering.

We also observe that each metaplectic operator is, by construction, a unitary operator in $L^2(\mathbb{R}^n)$, but also an automorphism of $\mathcal{S}(\mathbb{R}^n)$ and of $\mathcal{S}'(\mathbb{R}^n)$.

We are interested in its restriction $\widehat{S} = \pi(S)$, with $S \in U(2n, \mathbb{R})$, the symplectic rotations in (2).

Observe that $U(n) := U(n, \mathbb{C})$, the complex unitary group (the group of $n \times n$ invertible complex matrices V satisfying $VV^* = V^*V = I_n$) is isomorphic to $U(2n, \mathbb{R})$. The isomorphism ι is the mapping $\iota : U(n) \rightarrow U(2n, \mathbb{R})$ given by

$$(16) \quad \iota(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

for details see [15, Chapter 2.3].

We present here some results related to the group $U(2n, \mathbb{R})$, which will be used in the sequel to attain the characterization of Theorem 1.2. First, we recall a well-known result, see for instance [22, Lemma 9.4.3]:

Lemma 2.1. *For $f, g \in L^2(\mathbb{R}^n)$ and $S \in Sp(n, \mathbb{R})$, the STFT $V_g f$ satisfies*

$$(17) \quad |V_{\widehat{S}g}(\widehat{S}f)(x, \xi)| = |V_g f(S^{-1}(x, \xi))|, \quad (x, \xi) \in \mathbb{R}^{2n}.$$

This second issue is contained in [5], we sketch the proof for the sake of consistency.

Lemma 2.2. *For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $S \in U(2n, \mathbb{R})$, the STFT $V_\varphi(\widehat{S}\psi)$ is a Schwartz function, with seminorms uniformly bounded when $S \in U(2n, \mathbb{R})$.*

Proof. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the STFT V_φ is a continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$ (see [16]). Hence, it is enough to show that

$$\{\widehat{S}\varphi : S \in U(2n, \mathbb{R})\}$$

is a bounded subset of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, i.e., every Schwartz seminorm is bounded on it. Since the group $U(2n, \mathbb{R})$ is compact, it is sufficient to show that every seminorm is locally bounded, that is, we can limit ourselves to consider S in a sufficiently small neighbourhood for any fixed $S_0 \in U(2n, \mathbb{R})$. Equivalently, we can consider S of the form $S = S_1 J^{-1} S_0$ where S_1 belongs to a enough small neighbourhood of J in $U(2n, \mathbb{R})$. Using the representation of metaplectic operators recalled at the beginning of this section, we can write

$$\begin{aligned} \widehat{S}\varphi(x) &= \pm \widehat{S}_1[\widehat{J}^{-1}\widehat{S}_0\varphi](x) \\ &= c\sqrt{|\det L|} \int_{\mathbb{R}^n} e^{2\pi i(\frac{1}{2}Px \cdot x - Lx \cdot y + \frac{1}{2}Qy \cdot y)} \underbrace{[\widehat{J}^{-1}\widehat{S}_0\varphi]}_{\in \mathcal{S}(\mathbb{R}^n)}(y) dy \end{aligned}$$

where $|c| = 1$ and, we might say, $\|P\| < \epsilon$, $\|Q\| < \epsilon$, $\|L - I\| < \epsilon$. If $\epsilon < 1$, it is straightforward to check that $\widehat{S}\varphi$ belongs to a bounded subset of $\mathcal{S}(\mathbb{R}^n)$, as desired. \square

Lemma 2.3. *Let $B = (b_{i,j})_{i,j=1,\dots,n}$ be the $n \times n$ submatrix in (2). The subset $\Sigma \subset U(2n, \mathbb{R})$ obtained by setting $b_{i,1} = 0$, $i = 1, \dots, n$ (i.e., the first column of B is set to zero), is a submanifold of codimension n .*

Proof. We have to verify that the coordinates $b_{1,1}, \dots, b_{n,1}$ are independent on the subset Σ , namely the projection

$$(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$$

has rank n on Σ .

Let us first show that for every $S_0 \in \Sigma$ there exists a $U(2n, \mathbb{R})$ -valued smooth function $S(b_1, \dots, b_n)$, defined in a neighbourhood of $0 \in \mathbb{R}^n$, such that $S(0) = S_0$ and the first column “of its submatrix B ” is precisely $(b_1, \dots, b_n)^T$.

Let $S_0 = A + iB = (V_1, \dots, V_n) \in \Sigma$, with V_j being a $n \times 1$ complex vector, $j = 1, \dots, n$, so that by assumption $(b_{i,1})_{i=1,\dots,n} = \text{Im } V_1 = 0$. We consider any smooth function $V_1(b_1, \dots, b_n)$, defined in a neighbourhood of $0 \in \mathbb{R}^n$, valued in the unit sphere of \mathbb{C}^n , such that

$$\text{Im } V_1(b_1, \dots, b_n) = (b_1, \dots, b_n)^T, \quad V_1(0) = V_1.$$

Then, we apply the Gram-Schmidt orthonormalization procedure in \mathbb{C}^n to the set of vectors $(V_1(b_1, \dots, b_n), V_2, \dots, V_n)$. This provides the desired $U(n)$ -valued function $S(b_1, \dots, b_n)$. In particular $S(0) = S_0$.

Now, the composition of the mapping

$$(b_1, \dots, b_n) \mapsto S(b_1, \dots, b_n)$$

followed by the projection $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$ is therefore the identity mapping in a neighbourhood of 0 and has rank n . Hence the same is true for the projection $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$ at S_0 . \square

Lemma 2.4. *For every $\epsilon > 0$, define the (x -independent) function*

$$(18) \quad \chi_\epsilon(x, \xi) = \frac{1}{\epsilon^n} \mathbb{1}_Q \left(\begin{pmatrix} \xi \\ \epsilon \end{pmatrix} \right),$$

where

$$Q = \left[-\frac{1}{2}, \frac{1}{2} \right]^n \subset \mathbb{R}^n \quad \text{and} \quad \mathbb{1}_Q = \begin{cases} 1, & \xi \in Q \\ 0, & \xi \notin Q \end{cases}$$

and

$$(19) \quad \tilde{\chi}_\epsilon(z) = \frac{\chi_\epsilon(z)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS}, \quad z \in \mathbb{R}^{2n}.$$

Then we have

$$(20) \quad \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) dS = 1, \quad \forall z \in \mathbb{R}^{2n}$$

and

$$(21) \quad \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(x, \xi) \Phi(x, \xi) dx d\xi = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx,$$

for some $C > 0$ and for every continuous function Φ on \mathbb{R}^{2n} with a rapid decay at infinity.

Proof. We will show in a moment that, for $z = (x, \xi) \in \mathbb{R}^{2n}$,

$$(22) \quad \int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \gtrsim \min\{\epsilon^{-n}, |z|^{-n}\}$$

(with the convention, at $z = 0$, that $\min\{\epsilon^{-n}, +\infty\} = \epsilon^{-n}$). In particular, $\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \neq 0$, for every $z \in \mathbb{R}^{2n}$. Formula (20) then follows, because

$$\begin{aligned} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) dS &= \int_{U(2n, \mathbb{R})} \frac{\chi_\epsilon(Sz)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(USz) dU} dS \\ &= \int_{U(2n, \mathbb{R})} \frac{\chi_\epsilon(Sz)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(Uz) dU} dS = 1 \end{aligned}$$

for every $z \in \mathbb{R}^{2n}$, since the Haar measure is right invariant.

Let us now prove (22). For $z = 0$ we have

$$\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS = \frac{1}{\epsilon^n} \int_{U(2n, \mathbb{R})} dS = \frac{C_0}{\epsilon^n},$$

with $C_0 = \text{meas}(U(2n, \mathbb{R})) > 0$. Consider now $z \neq 0$. Observe that the function

$$\Psi_\epsilon(z) := \int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS$$

is constant on the orbits of $U(2n, \mathbb{R})$ in \mathbb{R}^{2n} , so that we can suppose

$$z = (x, 0), \quad x = (x_1, 0, \dots, 0), \quad x_1 = |x| = |z| > 0.$$

Now, by the definition of χ_ϵ and Ψ_ϵ ,

$$(23) \quad \Psi_\epsilon(z) = \epsilon^{-n} \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \frac{\epsilon}{2|x|}, i = 1, \dots, n \right\},$$

where $(b_{i,1})_{i=1, \dots, n}$, is the first column of the matrix $B = (b_{i,j})_{i,j=1, \dots, n}$.

Define, for $\mu > 0$,

$$f(\mu) = \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \mu, i = 1, \dots, n \right\}.$$

Observe that $f(\mu)$ is non-decreasing and constant for $\mu \geq 1$. Moreover, from Lemma 2.3 we know that by setting $b_{i,1} = 0$, $i = 1, \dots, n$, in $U(2n, \mathbb{R})$, we get a submanifold Σ of codimension n , and the function $f(\mu)$ is the measure

of a tubular neighbourhood of Σ in $U(2n, \mathbb{R})$. Hence we have the asymptotic behaviour

$$(24) \quad \mu^{-n} f(\mu) \rightarrow C_0 > 0, \quad \text{as } \mu \rightarrow 0^+$$

and in particular

$$(25) \quad f(\mu) \gtrsim \min\{1, \mu^n\}.$$

We then infer

$$(26) \quad \Psi_\epsilon(z) = \epsilon^{-n} f\left(\frac{\epsilon}{2|z|}\right) \rightarrow \frac{C_1}{|z|^n}, \quad \text{as } \epsilon \rightarrow 0^+$$

locally uniformly in $\mathbb{R}^{2n} \setminus \{0\}$, with $C_1 = 2^{-n}C_0$, and

$$(27) \quad \Psi_\epsilon(z) \gtrsim \epsilon^{-n} \min\left\{1, \left(\frac{\epsilon}{|z|}\right)^n\right\} = \min\{\epsilon^{-n}, |z|^{-n}\},$$

which is (22).

Let us finally prove (21). We are interested in the limit $\epsilon \rightarrow 0^+$, so we can assume $\epsilon \leq 1$. Consider a continuous function Φ on \mathbb{R}^{2n} with rapid decay at infinity. By definition of $\tilde{\chi}_\epsilon(z)$ in (19) we have

$$\tilde{\chi}_\epsilon(x, \xi) = \frac{\epsilon^{-n}}{\Psi_\epsilon(x, \xi)} \mathbb{1}_{[-\epsilon/2, \epsilon/2]^n}(\xi)$$

so that, by (27),

$$|\tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi)| \lesssim \epsilon^{-n}(1 + |x|^n) \mathbb{1}_{[-\epsilon/2, \epsilon/2]^n}(\xi) |\Phi(x, \xi)| \in L^1(\mathbb{R}^{2n})$$

for $0 < \epsilon \leq 1$. Fubini's Theorem then allows one to look at the first integral in (21) as an iterated integral

$$I_\epsilon := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi \right) dx$$

and we apply the dominated convergence theorem to the integral with respect to the x variable as follows. Setting

$$\Upsilon_\epsilon(x) := \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi = \epsilon^{-n} \int_{[-\epsilon/2, \epsilon/2]^n} \frac{1}{\Psi_\epsilon(x, \xi)} \Phi(x, \xi) d\xi,$$

by (26) we have, for every fixed $x \neq 0$,

$$\Upsilon_\epsilon(x) \rightarrow C|x|^n\Phi(x, 0);$$

for some constant $C > 0$. On the other hand $\Upsilon_\epsilon(x)$ is dominated, using (27), by

$$(1 + |x|)^n \sup_{\xi \in \mathbb{R}^n} |\Phi(x, \xi)| \in L^1(\mathbb{R}^n).$$

Hence

$$\lim_{\epsilon \rightarrow 0^+} I_\epsilon = \int_{\mathbb{R}^n} \lim_{\epsilon \rightarrow 0^+} \Upsilon_\epsilon(x) dx = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx.$$

This concludes the proof. \square

Remark 2.5. *Observe that there are no conditions on the derivatives of the function Φ in (21).*

3. PROOFS OF THE MAIN RESULTS

In what follows we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. (i) **First Step.** Let us start with showing that formula (5) is true for any function ψ in the Schwartz class $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Using the Gaussian $\varphi(t) = 2^{d/4}e^{-\pi|t|^2}$ as window function, we compute the M_m^p -norm of ψ as in (4) and then use Lemma 2.4 so that

$$\begin{aligned} \|\psi\|_{M_m^p}^p &= \int_{\mathbb{R}^{2n}} |V_\varphi\psi(z)|^p m(z)^p dz = \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) |V_\varphi\psi(z)|^p m(z)^p dSdz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(z) |V_\varphi\psi(S^{-1}z)|^p m(S^{-1}z)^p dSdz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(z) |V_{\widehat{S}\varphi}\widehat{S}\psi(z)|^p m(S^{-1}z)^p dSdz \end{aligned}$$

where in the last equality we used Lemma 2.1. Observe that, since S is unitary and φ is a Gaussian, $\widehat{S}\varphi = c\varphi$, for some phase factor $c \in \mathbb{C}$, with $|c| = 1$ (see [15, Proposition 252]) and this phase factor is killed by the modulus obtaining $|V_{\widehat{S}\varphi}\widehat{S}\psi(z)| = |V_\varphi\widehat{S}\psi(z)|$. Continuing the above computation we infer

$$\|\psi\|_{M_m^p}^p = \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(z) \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(z)|^p m(S^{-1}z)^p dSdz.$$

Set

$$\Phi(z) = \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(z)|^p m(S^{-1}z)^p dS.$$

The dominated convergence theorem guarantees that Φ is continuous on \mathbb{R}^{2n} and moreover Φ has rapid decay at infinity. This follows from Lemma 2.2 (recall that m is continuous and has at most polynomial growth).

Letting $\epsilon \rightarrow 0^+$ and using (21) we obtain

$$\begin{aligned} \|\psi\|_{M_m^p}^p &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(x, 0)|^p m(S^{-1}(x, 0)^T)^p dSdx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |\langle \widehat{S}\psi, T_x\varphi \rangle|^p m(S^{-1}(x, 0)^T)^p dSdx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |\langle \psi, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dSdx. \end{aligned}$$

The last equality is due to $\langle \widehat{S}\psi, T_x\varphi \rangle = \langle \psi, \widehat{S}^{-1}T_x\varphi \rangle$ and the invariance of the Haar measure of $U(2n, \mathbb{R})$ with respect to the change of variable $S \rightarrow S^{-1}$.

Second Step. Consider $f \in M_m^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Using the density of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in $M_m^p(\mathbb{R}^n)$ (cf. e.g., [22, Chapter 12]), there exists a sequence $\{\psi_k\}_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\psi_k \rightarrow f$ in $M_m^p(\mathbb{R}^n)$. This implies that $\psi_k \rightarrow f$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\langle \psi_k, \widehat{S}T_x\varphi \rangle \rightarrow \langle \psi, \widehat{S}T_x\varphi \rangle$$

pointwise for every $x \in \mathbb{R}^n$, $S \in U(2n, \mathbb{R})$. Let us define, for every $f \in M_m^p(\mathbb{R}^n)$,

$$(28) \quad |||f||| = \left(\int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |x|^n |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}}.$$

By Fatou's Lemma, for any $f \in M_m^p(\mathbb{R}^n)$:

$$(29) \quad |||f|||^p \leq \liminf_{k \rightarrow \infty} |||\psi_k|||^p \lesssim \liminf_{k \rightarrow \infty} \|\psi_k\|_{M_m^p}^p = \|f\|_{M_m^p}^p.$$

It is easy to check that $|||f|||$ is a seminorm on $M_m^p(\mathbb{R}^n)$. Applying (29) to the difference $f - \psi_k$ we obtain $|||f - \psi_k||| \rightarrow 0$ and hence $|||\psi_k||| \rightarrow |||f|||$. By assumption we also have $\|\psi_k\|_{M_m^p} \rightarrow \|f\|_{M_m^p}$, and the desired norm equivalence in (5) then extends from $\mathcal{S}(\mathbb{R}^n)$ to $M_m^p(\mathbb{R}^n)$.

Third Step. We will show that (6) easily follows from (5). By the definition of the symplectic group (1), for any $S \in U(2n, \mathbb{R})$,

$$J^{-1}S = (S^T)^{-1}J^{-1} = SJ^{-1}$$

for $S^{-1} = S^T$. On the other hand, for any $f \in M_m^p(\mathbb{R}^n)$, $\|f\|_{M_m^p} \asymp \|\widehat{f}\|_{M_m^p}$, with $\widehat{m}(z) = m(J^{-1}z)$; see [16]. Using (15),

$$\begin{aligned} |\langle \widehat{f}, \widehat{S}T_x\varphi \rangle| &= |\langle f, \widehat{J^{-1}}\widehat{S}T_x\varphi \rangle| = |\langle f, \widehat{S}\mathcal{F}^{-1}T_x\varphi \rangle| \\ &= |\langle f, \widehat{S}M_x\mathcal{F}^{-1}\varphi \rangle| = |\langle f, \widehat{S}M_x\varphi \rangle| \end{aligned}$$

since the Gaussian is an eigenvector of \mathcal{F}^{-1} with eigenvalue equal to 1. Moreover

$$\widehat{m}(S(x, 0)^T) = m(J^{-1}S(x, 0)^T) = m(SJ^{-1}(x, 0)^T) = m(S(0, x)^T).$$

Hence (6) follows from (5).

(ii) Case $p = \infty$. Observe that any $z \in \mathbb{R}^{2n}$ can be written as

$$z = S^{-1}(x, 0)^T,$$

for some $x \in \mathbb{R}^n$, $S \in U(2n, \mathbb{R})$, so that, for any $f \in M_m^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
 \|f\|_{M_m^\infty(\mathbb{R}^n)} &= \sup_{z \in \mathbb{R}^{2n}} |V_\varphi f(z)| m(z) \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |V_\varphi f(S^{-1}(x, 0)^T)| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |V_\varphi(\widehat{S}f)(x, 0)| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle \widehat{S}f, T_x \varphi \rangle| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x \varphi \rangle| m(S(x, 0)^T),
 \end{aligned}$$

which gives (7). Formula (8) follows as above. \square

We now prove the similar result, with the group $U(2n, \mathbb{R})$ replaced by the subgroup \mathbb{T}^n (up to isomorphisms).

Proof of Theorem 1.3. (i) We could follow a similar pattern to the proof of Theorem 1.2, replacing the group $U(2n, \mathbb{R})$ by \mathbb{T}^n . The preparation of Lemma 2.3 would be no longer necessary. Lemma 2.4 would require some small adjustments. On the other hand a more direct argument can be given. Namely, writing $z_j = (x_j, \xi_j)$ in complex notation as $r_j e^{i\theta_j}$, and setting $r = (r_1, \dots, r_n)$, $\theta = (\theta_1, \dots, \theta_n)$ we have

$$\begin{aligned}
 \|f\|_{M_m^p}^p &= \int_{\mathbb{R}^{2n}} |V_\varphi f(z)|^p m(z)^p dz \\
 &= \int_{\mathbb{R}_+^n \times [0, 2\pi]^n} r_1 \cdots r_n |V_\varphi f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p m(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})^p dr d\theta.
 \end{aligned}$$

With S as in (9) and using Lemma 2.1, therefore we have

$$\begin{aligned}
 \|f\|_{M_m^p}^p &\asymp \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi f(S(x, 0)^T)|^p m(S(x, 0)^T)^p dx dS \\
 &= \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi(\widehat{S}^{-1}f)(x, 0)|^p m(S(x, 0)^T)^p dx dS \\
 &= \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |\langle \widehat{S}^{-1}f, T_x \varphi \rangle|^p m(S(x, 0)^T)^p dx dS,
 \end{aligned}$$

which is (10). The characterization (11) has the same proof as the corresponding formula (6).

(ii) The M^∞ case uses the same argument as in the proofs of (7) and (8), with the group $U(2n, \mathbb{R})$ replaced by \mathbb{T}^n . \square

4. INTEGRAL REPRESENTATIONS FOR THE TORUS IN TERMS OF THE
FRACTIONAL FOURIER TRANSFORM

Observe that the symplectic matrix in $U(2n, \mathbb{R})$ corresponding to the complex matrix $S \in \mathbb{T}^n$ in (9) via the isomorphism ι in (16) is given by

$$\iota(S) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with

$$A = \text{diag}[\cos \theta_1, \dots, \cos \theta_n] \quad B = \text{diag}[\sin \theta_1, \dots, \sin \theta_n].$$

Consider the case $\theta_i \neq k\pi$, $k \in \mathbb{Z}$, $i = 1, \dots, n$. The matrix $\iota(S)$ is a free symplectic matrix and the related metaplectic operator possesses the integral representation (14). Since

$$AB^{-1} = B^{-1}A = \text{diag}\left[\frac{\cos \theta_1}{\sin \theta_1}, \dots, \frac{\cos \theta_n}{\sin \theta_n}\right],$$

the polynomial $W(x, x')$ becomes

$$(30) \quad W(x_1, \dots, x_n, x'_1, \dots, x'_n) = \sum_{i=1}^n \frac{1}{2 \sin \theta_i} (\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x_i'^2)$$

and

$$\Delta(W) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}}.$$

for some phase factor $c \in \mathbb{C}$, with $|c| = 1$. Hence we obtain, for $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$(31) \quad \widehat{\iota(S)\psi}(x) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}} \int_{\mathbb{R}^n} e^{2\pi i W(x, x')} \psi(x') dx',$$

with $W(x, x')$ in (30). From (31) we deduce that $\widehat{\iota(S)}$ can be written as the composition of the operators

$$(32) \quad \widehat{\iota(S)} = \pm \widehat{\iota(S_1)} \cdots \widehat{\iota(S_n)},$$

where, for some phase factor c ,

$$\widehat{\iota(S_i)}\psi(x) = \frac{c}{\sqrt{|\sin \theta_i|}} \int_{\mathbb{R}} e^{\frac{\pi i}{\sin \theta_i} (\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x_i'^2)} \psi(x'_1, \dots, x'_i, \dots, x'_n) dx'_i.$$

Indeed if $\theta_i = \pi/2$, then $\widehat{\iota(S_i)} = \pm \widehat{J}$ is the Fourier transform with respect to the variable x_i . Otherwise, $\widehat{\iota(S_i)} = \pm \mathcal{F}_{\theta_i}$, the θ_i -angle partial fractional Fourier transform (again referred to the variable x_i).

Alternatively, the same conclusion (32) can be drawn by writing

$$(33) \quad S = \text{diag}[e^{i\theta_1}, \dots, e^{i\theta_n}] = \text{diag}[e^{i\theta_1}, 1, \dots, 1] \cdots \text{diag}[1, \dots, 1, e^{i\theta_n}]$$

that is

$$S = S_1 \cdots S_i \cdots S_n,$$

with

$$S_i = \text{diag} [1, \dots, 1, e^{i\theta_i}, 1, \dots, 1], \quad i = 1, \dots, n$$

so that

$$\widehat{\iota(S)} = \iota(S_1) \widehat{\dots \iota(S_1)} = \pm \iota(S_1) \dots \iota(S_n).$$

If $\theta_i = 2k\pi$ for some $k \in \mathbb{Z}$, $\widehat{\iota(S_i)} = \pm I$ with I the identity operator. If $\theta_i = (2k+1)\pi$ for some $k \in \mathbb{Z}$, $\widehat{\iota(S_i)}\psi(x) = \pm\psi(x_1, \dots, -x_i, \dots, x_n)$.

Hence using the θ_i -angle partial fractional Fourier transform $\mathcal{F}_{\theta_i} = \widehat{\iota(S_i)}$ we can rephrase Theorem 1.3 as follows.

Theorem 4.1. *Let φ be the Gaussian of Theorem 1.2.*

(i) *For $1 \leq p < \infty$, $f \in M_m^p(\mathbb{R}^n)$, we have*

$$\|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times [0, 2\pi]^n} |x_1 \dots x_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi \rangle|^p m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})^p dx d\theta \right)^{\frac{1}{p}},$$

and

$$\|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left(\int_{\mathbb{R}^n \times [0, 2\pi]^n} |\xi_1 \dots \xi_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi \rangle|^p m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n})^p d\xi d\theta \right)^{\frac{1}{p}}.$$

(ii) *For $p = \infty$,*

$$\|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{\theta \in [0, 2\pi]^n} \sup_{x \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi \rangle| m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})$$

and

$$\|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{\theta \in [0, 2\pi]^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi \rangle| m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n}).$$

We observe that this result could also be obtained by writing $\|f\|_{M_m^p(\mathbb{R}^n)}$ in terms of the weighted L^p norm of the Bargmann transform of f and using the covariance property of the Bargmann transform; the papers [18, 30] and specially [31] are relevant in this connection.

ACKNOWLEDGMENT

We would like to thank the referees for interesting remarks and suggestions which lead in particular to a cleaner proof of Theorem 1.3.

The first and the third author have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). MdG has been financed by the Austrian Research Foundation FWF grant P27773.

REFERENCES

- [1] A. Bényi, K. Gröchenig, K.A. Okoudjou and L.G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2): 366–384, 2007.
- [2] A. Bényi and K.A. Okoudjou. Time-frequency estimates for pseudodifferential operators. *Contemporary Math.*, Amer. Math. Soc., 428:13–22, 2007.
- [3] A. Bényi and K.A. Okoudjou. Local well-posedness of nonlinear dispersive equations on modulation spaces. *Bull. Lond. Math. Soc.*, 41(3):549–558, 2009.
- [4] A. Bényi, O. Tadahiro and O. Pocovnicu. On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on \mathbb{R}^d , $d \geq 3$. *Trans. Amer. Math. Soc. Ser. B*, 2:1–50, 2015.
- [5] A. Cauli, F. Nicola and A. Tabacco. Strichartz estimates for the metaplectic representation. *Rev. Mat. Iberoam.*, 35(7): 2079–2092, 2019.
- [6] O. Christensen, H.G. Feichtinger and S. Paukner. *Gabor analysis for imaging. Handbook of mathematical methods in imaging. Vol. 1, 2, 3*, 1717–1757, Springer, New York, 2015.
- [7] F. Concetti and J. Toft. Trace ideals for Fourier integral operators with non-smooth symbols, “*Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis*”, Fields Inst. Commun., Amer. Math. Soc., 52:255–264, 2007.
- [8] F. Concetti, G. Garello and J. Toft. Trace ideals for Fourier integral operators with non-smooth symbols II. *Osaka J. Math.*, 47(3):739–786, 2010.
- [9] E. Cordero, K. Gröchenig, F. Nicola and L. Rodino. Wiener algebras of Fourier integral operators. *J. Math. Pures Appl.*, 99:219–233, 2013.
- [10] E. Cordero and F. Nicola. Boundedness of Schrödinger type propagators on modulation spaces. *J. Fourier Anal. Appl.* 16(3):311–339, 2010.
- [11] E. Cordero and F. Nicola. Schrödinger equations with bounded perturbations. *J. Pseudo-Differ. Op. and Appl.*, 5(3):319–341, 2014.
- [12] E. Cordero, K. Gröchenig, F. Nicola and L. Rodino. Generalized metaplectic operators and the Schrödinger equation with a potential in the Sjöstrand class. *J. Math. Phys.*, 55(8), art. no. 081506, 2014.
- [13] E. Cordero, F. Nicola and L. Rodino. Sparsity of Gabor representation of Schrödinger propagators. *Appl. Comput. Harmon. Anal.*, 26(3):357–370, 2009.
- [14] E. Cordero, F. Nicola and L. Rodino. Integral representations for the class of generalized metaplectic operators. *J. Fourier Anal. Appl.*, 21:694–714, 2015.
- [15] M. A. de Gosson. *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, volume 7 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.
- [16] H. G. Feichtinger. Modulation spaces on locally compact abelian groups, *Technical Report, University Vienna, 1983*, and also in *Wavelets and Their Applications*, M. Krishna, R. Radha, S. Thangavelu, editors, Allied Publishers, 99–140, 2003.
- [17] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.
- [18] H. G. Feichtinger K. Gröchenig and D. Walnut. Wilson bases and modulation spaces. *Math. Nach.*, 155:7–17, 1992.
- [19] H. G. Feichtinger. *Choosing function spaces in harmonic analysis, Excursions in harmonic analysis. Vol. 4*, Appl. Numer. Harmon. Anal., 65–101, Birkhäuser/Springer, Cham, 2015.
- [20] H. G. Feichtinger. Modulation spaces: looking back and ahead. *Sampl. Theory Signal Image Process.*, 5(2):109–140, 2006
- [21] G. B. Folland. *Harmonic Analysis in Phase Space*. Princeton Univ. Press, Princeton, NJ, 1989.

- [22] K. Gröchenig. *Foundations of Time-frequency Analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.
- [23] Leray, Jean *Lagrangian Analysis and Quantum Mechanics. A mathematical structure related to asymptotic expansions and the Maslov index*, translated from the French by Carolyn Schroeder, MIT Press, Cambridge, Mass.-London, (1981).
- [24] H. Morsche and P.J. Oonincx. Integral representations of affine transformations in phase space with an application to energy localization problems. CWI report, Amsterdam, 1999.
- [25] M. Sugimoto and N. Tomita. Boundedness properties of pseudo-differential operators and Calderón-Zygmund operators on modulation spaces. *J. Fourier Anal. Appl.*, 14(1):124–143, 2008.
- [26] M. Ruzhansky, B. Wang and H. Zhang. Global well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data in modulation and Sobolev spaces, *J. Math. Pures Appl. (9)*,105(1):31–65, 2016.
- [27] M. Ruzhansky, M. Sugimoto and B. Wang. *Modulation spaces and nonlinear evolution equations, Evolution equations of hyperbolic and Schrödinger type*, Progr. Math., 301:267–283, Birkhäuser/Springer Basel AG, Basel, 2012.
- [28] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.*, 207(2):399–429, 2004.
- [29] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. *Ann. Global Anal. Geom.*, 26(1):73–106, 2004.
- [30] J. Toft, The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators. *J. Pseudo-Differ. Oper. Appl.*, 3:145–227, 2012.
- [31] J. Toft, Images of function and distribution spaces under the Bargmann transform. *J. Pseudo-Differ. Oper. Appl.*, 8:83–139, 2017.
- [32] B. Wang, Z. Huo, C. Hao and Z. Guo. *Harmonic Analysis Method for Nonlinear Evolution Equations. I*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [33] B. Wang, Z. Lifeng and G. Boling. Isometric decomposition operators, function spaces $E_{p,q}^\lambda$ and applications to nonlinear evolution equations. *J. Funct. Anal.*, 233(1):1–39, 2006.
- [34] B. Wang and H. Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. *J. Differential Equations*, 232(1):36–73, 2007.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY

E-mail address: elena.cordero@unito.it

UNIVERSITY OF VIENNA, FACULTY OF MATHEMATICS, OSKAR-MORGENSTERN-PLATZ 1 A-1090 WIEN, AUSTRIA

E-mail address: maurice.de.gosson@univie.ac.at

DIPARTIMENTO DI SCIENZE MATEMATICHE, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY

E-mail address: fabio.nicola@polito.it