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# A CHARACTERIZATION OF MODULATION SPACES BY SYMPLECTIC ROTATIONS

ELENA CORDERO, MAURICE DE GOSSON, AND FABIO NICOLA

ABSTRACT. This note contains a new characterization of modulation spaces  $M_m^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , by symplectic rotations. Precisely, instead to measure the time-frequency content of a function by using translations and modulations of a fixed window as building blocks, we use translations and metaplectic operators corresponding to symplectic rotations. Technically, this amounts to replace, in the computation of the  $M_m^p(\mathbb{R}^n)$ -norm, the integral in the time-frequency plane with an integral on  $\mathbb{R}^n \times U(2n, \mathbb{R})$  with respect to a suitable measure,  $U(2n, \mathbb{R})$  being the group of symplectic rotations. More conceptually, we are considering a sort of polar coordinates in the time-frequency plane. To have invariance under symplectic rotations we choose a Gaussian as suitable window function. We also provide a similar (and easier) characterization with the group  $U(2n, \mathbb{R})$  being reduced to the  $n$ -dimensional torus  $\mathbb{T}^n$ .

## 1. INTRODUCTION

The objective of this study is to find a new characterization of modulation spaces using symplectic rotations. Precisely, we are interested in those metaplectic operators  $\widehat{S} \in Mp(n, \mathbb{R})$ , such that the corresponding projection  $S := \pi(\widehat{S})$  onto the symplectic group  $Sp(n, \mathbb{R})$  is a symplectic rotation. Let us recall that the symplectic group  $Sp(n, \mathbb{R})$  is the subgroup of  $2n \times 2n$  invertible matrices  $GL(2n, \mathbb{R})$ , defined by

$$(1) \quad Sp(n, \mathbb{R}) = \{S \in GL(2n, \mathbb{R}) : SJS^T = J\},$$

where  $J$  is the orthogonal matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

( $I_n, 0_n$  are the  $n \times n$  identity matrix and null matrix, respectively). Here we consider the subgroup

$$U(2n, \mathbb{R}) := Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R}) \simeq U(n)$$

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of symplectic rotations (cf., e.g. [15, Section 2.3]), namely

$$(2) \quad U(2n, \mathbb{R}) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : AA^T + BB^T = I_n, AB^T = B^T A \right\} \subset Sp(2n, \mathbb{R}),$$

endowed with the normalized Haar measure  $dS$  (the group  $U(2n, \mathbb{R})$ , being compact, is unimodular).

In the 80's H. Feichtinger [16] introduced modulation spaces to measure the time-frequency concentration of a function/distribution on the time-frequency space (or phase space)  $\mathbb{R}^{2n}$ . They are nowadays become popular among mathematicians and engineers because they have found numerous applications in signal processing [6, 19, 20], pseudodifferential and Fourier integral operators [7, 8, 9, 28, 29], partial differential equations [1, 2, 3, 4, 10, 13, 11, 11, 32, 33, 34] and quantum mechanics [12, 15].

To recall their definition, we need a few time-frequency tools. First, the translation  $T_x$  and modulation  $M_\xi$  operators are defined by

$$T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t), \quad t, x, \xi \in \mathbb{R}^n,$$

for any function  $f$  on  $\mathbb{R}^n$ .

The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution  $f \in \mathcal{S}'(\mathbb{R}^n)$  with respect to a function  $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$  (so-called window), given by

$$(3) \quad V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^n} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \xi} dt, \quad x, \xi \in \mathbb{R}^n.$$

The short-time Fourier transform is well-defined whenever the bracket  $\langle \cdot, \cdot \rangle$  makes sense for dual pairs of function or distribution spaces, in particular for  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,  $g \in \mathcal{S}(\mathbb{R}^n)$ , or for  $f, g \in L^2(\mathbb{R}^n)$ .

Let  $m(x, \xi)$  be a continuous weight,  $v$ -moderate for some submultiplicative weight  $v$  (see [22, Section 11.1] for details - we will not use explicitly these properties). We also assume that  $m$  has at most polynomial growth.

**Definition 1.1 (Modulation spaces).** *Given  $g \in \mathcal{S}(\mathbb{R}^n)$ , and  $1 \leq p \leq \infty$ , the modulation space  $M_m^p(\mathbb{R}^n)$  consists of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that  $V_g f \in L_m^p(\mathbb{R}^{2n})$ . The norm on  $M_m^p(\mathbb{R}^n)$  is*

$$(4) \quad \begin{aligned} \|f\|_{M_m^p} &= \|V_g f\|_{L_m^p} = \left( \int_{\mathbb{R}^{2n}} |V_g f(x, \xi)|^p m(x, \xi)^p dx d\xi \right)^{1/p} \\ &= \left( \int_{\mathbb{R}^{2n}} |\langle f, M_\xi T_x g \rangle|^p m(x, \xi)^p dx d\xi \right)^{1/p} \end{aligned}$$

(with obvious modifications for  $p = \infty$ ).

The spaces  $M_m^p(\mathbb{R}^n)$  are Banach spaces, and every nonzero  $g \in M_v^1(\mathbb{R}^n)$  yields an equivalent norm in (4), so that their definition is independent of the choice of  $g \in M_v^1(\mathbb{R}^n)$  (see [16, 22]).

We now provide an equivalent norm to (4) by using translations  $T_x$  (or modulations  $M_\xi$ ) and the operators  $\widehat{S}$ , with  $S \in U(2n, \mathbb{R})$  as follows.

**Theorem 1.2.** *Consider the Gaussian function  $\varphi(t) = 2^{d/4}e^{-\pi|t|^2}$ .*

(i) *For  $1 \leq p < \infty$  and  $f \in M_m^p(\mathbb{R}^n)$ , we have*

$$(5) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |x|^n |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}},$$

where  $dx$  is the Lebesgue measure on  $\mathbb{R}^n$  and  $dS$  the Haar measure on  $U(2n, \mathbb{R})$ . Similarly,

$$(6) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |\xi|^n |\langle f, \widehat{S}M_\xi\varphi \rangle|^p m(S(0, \xi)^T)^p d\xi dS \right)^{\frac{1}{p}},$$

with  $d\xi$  being the Lebesgue measure on  $\mathbb{R}^n$  and  $dS$  the Haar measure on  $U(2n, \mathbb{R})$ .

(ii) *For  $p = \infty$ ,  $f \in M_m^\infty(\mathbb{R}^n)$ , it occurs*

$$(7) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x\varphi \rangle| m(S(x, 0)^T)$$

and, similarly,

$$(8) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S}M_\xi\varphi \rangle| m(S(0, \xi)^T).$$

The interpretation of the integral (5) above is as follows. The metaplectic operator  $\widehat{S}$  produces a time-frequency rotation of the shifted Gaussian  $T_x\varphi$ . In this way, the operator

$$f \mapsto \langle f, \widehat{S}T_x\varphi \rangle$$

detects the time-frequency content of  $f$  in an oblique strip, see Figure 1. All the contributions are then added together with a weight  $|x|^n$  which takes into account the underlapping of the strips as  $|x| \rightarrow +\infty$  and the overlapping as  $x \rightarrow 0$ .

Formulas (6), (7) and (8) have similar meanings.

Observe that in dimension  $n = 1$ ,  $U(2, \mathbb{R}) \simeq U(1)$  and the above formula is essentially a transition to polar coordinates with  $|x|$  being the Jacobian.

Comparing (4) and (5) we observe that in (5) the modulation operator  $M_\xi$  is replaced by the metaplectic operator  $\widehat{S}$  and the integral on the phase space  $\mathbb{R}^{2n}$  has become an integral on the cartesian product  $\mathbb{R}^n \times U(2n, \mathbb{R})$ . The integration parameters  $(x, \xi)$  of (4) live in  $\mathbb{R}^{2n}$ , with  $\dim \mathbb{R}^{2n} = 2n$ , whereas the parameters  $(x, S)$  of (5) live in  $\mathbb{R}^n \times U(2n, \mathbb{R})$ . Recall that  $\dim U(2n, \mathbb{R}) = n^2$  [15]; this suggests that a formula similar to (5) should hold when  $U(2n, \mathbb{R})$  is reduced to

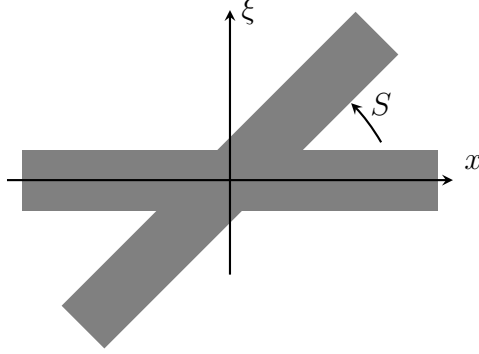


FIGURE 1. The time-frequency content of  $f$  in the oblique strip is detected by the operator  $f \mapsto \langle f, \widehat{S}T_x\varphi \rangle$ .

a suitable subgroup  $K \subset U(2n, \mathbb{R})$  of dimension  $n$ . This is indeed the case (and easier to see), as shown in the subsequent Theorem 1.3.

Consider the  $n$ -dimensional torus

$$(9) \quad \mathbb{T}^n = \left\{ S = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \theta_1, \dots, \theta_n \in \mathbb{R} \right\} \subset U(n)$$

with the Haar measure  $dS = d\theta_1 \dots d\theta_n$ . The torus is isomorphic to a subgroup  $K \subset U(2n, \mathbb{R})$ , via the isomorphism  $\iota$  in formula (16) below (see the subsequent section).

We exhibit the following characterization for  $M^p$ -spaces.

**Theorem 1.3.** *Let  $\varphi$  be the Gaussian of Theorem 1.2.*

(i) *For  $1 \leq p < \infty$ ,  $f \in M_m^p(\mathbb{R}^n)$ , we have*

$$(10) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \dots x_n| |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}},$$

and, similarly,

$$(11) \quad \|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times \mathbb{T}^n} |\xi_1 \dots \xi_n| |\langle f, \widehat{S}M_\xi\varphi \rangle|^p m(S(0, \xi)^T)^p d\xi dS \right)^{\frac{1}{p}}.$$

(ii) *For  $p = \infty$ ,*

$$(12) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x\varphi \rangle| m(S(x, 0)^T)$$

and

$$(13) \quad \|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{S \in \mathbb{T}^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \widehat{S}M_\xi\varphi \rangle| m(S(0, \xi)^T).$$

The above results for the groups  $U(2n, \mathbb{R})$  and  $\mathbb{T}^n$  can be interpreted, in a sense, as two extreme cases, and it would be interesting to find, more generally, for which compact subgroups  $K \subset U(2n, \mathbb{R})$  similar characterizations hold. We conjecture that they should be precisely the subgroups  $K \subset U(2n, \mathbb{R})$  such that every orbit for their action on  $\mathbb{R}^{2n}$  intersects  $\{0\} \times \mathbb{R}^n$  (up to subsets of measure zero), with a corresponding weighted measure on  $\mathbb{R}^n \times K$  to be determined.

Another problem which is worth investigating is the study of discrete versions of the above characterizations via coorbit theory [17].

The paper is organized as follows: in Section 2 we collected some preliminary results, whereas Section 3 is devoted to the proof of Theorems 1.2 and 1.3. In Section 4 we rephrase more explicitly Theorem 1.3 in terms of the partial fractional Fourier transform.

## 2. NOTATION AND PRELIMINARIES

**Notation.** We write  $x \cdot y$  for the scalar product on  $\mathbb{R}^n$  and  $|t|^2 = t \cdot t$ , for  $t, x, y \in \mathbb{R}^n$ . For expressions  $A, B \geq 0$ , we use the notation  $A \lesssim B$  to represent the inequality  $A \leq cB$  for a suitable constant  $c > 0$ , and  $A \asymp B$  for the equivalence  $c^{-1}B \leq A \leq cB$ .

The Schwartz class is denoted by  $\mathcal{S}(\mathbb{R}^n)$ , the space of tempered distributions by  $\mathcal{S}'(\mathbb{R}^n)$ . We use the brackets  $\langle f, g \rangle$  to denote the extension to  $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$  of the inner product  $\langle f, g \rangle = \int f(t)\overline{g(t)}dt$  on  $\mathcal{S}(\mathbb{R}^n)$ .

**Metaplectic Operators.** The metaplectic representation  $\mu$  of  $Mp(n, \mathbb{R})$ , the two-sheeted cover of the symplectic group  $Sp(n, \mathbb{R})$ , defined in (1) arises as intertwining operator between the standard Schrödinger representation  $\rho$  of the Heisenberg group  $\mathbb{H}^d$  and the representation that is obtained from it by composing  $\rho$  with the action of  $Sp(n, \mathbb{R})$  by automorphisms on  $\mathbb{H}^d$  (see, e.g., [15, 21, 23]). Let us recall the main points of a direct construction.

The symplectic group  $Sp(n, \mathbb{R})$  is generated by the so-called free symplectic matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), \quad \det B \neq 0.$$

To each such a matrix the associated generating function is defined by

$$W(x, x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'.$$

Conversely, to every polynomial of the type

$$W(x, x') = \frac{1}{2}Px \cdot x - Lx \cdot x' + \frac{1}{2}Qx' \cdot x'$$

with

$$P = P^T, Q = Q^T$$

and

$$\det L \neq 0$$

it can be associated a free symplectic matrix, namely

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}.$$

Given  $S_W$  as above and  $m \in \mathbb{Z}$  such that

$$m\pi \equiv \arg \det L \pmod{2\pi},$$

the related operator  $\widehat{S}_{W,m}$  is defined by setting, for  $\psi \in \mathcal{S}(\mathbb{R}^n)$ ,

$$(14) \quad \widehat{S}_{W,m}\psi(x) = \frac{1}{i^{n/2}} \Delta(W) \int_{\mathbb{R}^n} e^{2\pi i W(x,x')} \psi(x') dx'$$

(with  $i^{n/2} = e^{i\pi n/4}$ ) where

$$\Delta(W) = i^m \sqrt{|\det L|}.$$

The operator  $\widehat{S}_{W,m}$  is named *quadratic Fourier transform* associated to the free symplectic matrix  $S_W$  (as a remark, for integral representations of metaplectic operators that do not arise from free symplectic matrices see [14, 24]). The class modulo 4 of the integer  $m$  is called *Maslov index* of  $\widehat{S}_{W,m}$ . Observe that if  $m$  is one choice of Maslov index, then  $m + 2$  is another equally good choice: hence to each function  $W$  we associate two operators, namely  $\widehat{S}_{W,m}$  and  $\widehat{S}_{W,m+2} = -\widehat{S}_{W,m}$ .

The quadratic Fourier transform corresponding to the choices  $S_W = J$  and  $m = 0$  is denoted by  $\widehat{J}$ . The generating function of  $J$  is simply  $W(x, x') = -x \cdot x'$ . It follows that

$$(15) \quad \widehat{J}\psi(x) = \frac{1}{i^{n/2}} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot x'} \psi(x') dx' = \frac{1}{i^{n/2}} \mathcal{F}\psi(x)$$

for  $\psi \in \mathcal{S}(\mathbb{R}^n)$ , where  $\mathcal{F}$  is the usual unitary Fourier transform.

The quadratic Fourier transforms  $\widehat{S}_{W,m}$  form a subset of the group  $\mathcal{U}(L^2(\mathbb{R}^n))$  of unitary operators acting on  $L^2(\mathbb{R}^n)$ , which is mapped into itself by the operation of inversion and they generate a subgroup of  $\mathcal{U}(L^2(\mathbb{R}^n))$  which is, by definition, the metaplectic group  $Mp(n, \mathbb{R})$ . The elements of  $Mp(n, \mathbb{R})$  are called metaplectic operators.

Hence, every  $\widehat{S} \in Mp(n, \mathbb{R})$  is, by definition, a product

$$\widehat{S}_{W_1, m_1} \cdots \widehat{S}_{W_k, m_k}$$

of metaplectic operators associated to free symplectic matrices.

Indeed, it can be proved that every  $\widehat{S} \in Mp(n, \mathbb{R})$  can be written as a product of exactly two quadratic Fourier transforms:  $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$ . Now, it can be shown that the mapping  $\widehat{S}_{W,m} \mapsto S_W$  extends to a group homomorphism  $\pi : Mp(n, \mathbb{R}) \rightarrow Sp(n, \mathbb{R})$ , which is in fact a double covering.

We also observe that each metaplectic operator is, by construction, a unitary operator in  $L^2(\mathbb{R}^n)$ , but also an automorphism of  $\mathcal{S}(\mathbb{R}^n)$  and of  $\mathcal{S}'(\mathbb{R}^n)$ .

We are interested in its restriction  $\widehat{S} = \pi(S)$ , with  $S \in U(2n, \mathbb{R})$ , the symplectic rotations in (2).

Observe that  $U(n) := U(n, \mathbb{C})$ , the complex unitary group (the group of  $n \times n$  invertible complex matrices  $V$  satisfying  $VV^* = V^*V = I_n$ ) is isomorphic to  $U(2n, \mathbb{R})$ . The isomorphism  $\iota$  is the mapping  $\iota : U(n) \rightarrow U(2n, \mathbb{R})$  given by

$$(16) \quad \iota(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

for details see [15, Chapter 2.3].

We present here some results related to the group  $U(2n, \mathbb{R})$ , which will be used in the sequel to attain the characterization of Theorem 1.2. First, we recall a well-known result, see for instance [22, Lemma 9.4.3]:

**Lemma 2.1.** *For  $f, g \in L^2(\mathbb{R}^n)$  and  $S \in Sp(n, \mathbb{R})$ , the STFT  $V_g f$  satisfies*

$$(17) \quad |V_{\widehat{S}g}(\widehat{S}f)(x, \xi)| = |V_g f(S^{-1}(x, \xi))|, \quad (x, \xi) \in \mathbb{R}^{2n}.$$

This second issue is contained in [5], we sketch the proof for the sake of consistency.

**Lemma 2.2.** *For  $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$  and  $S \in U(2n, \mathbb{R})$ , the STFT  $V_\varphi(\widehat{S}\psi)$  is a Schwartz function, with seminorms uniformly bounded when  $S \in U(2n, \mathbb{R})$ .*

*Proof.* Since  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , the STFT  $V_\varphi$  is a continuous mapping from  $\mathcal{S}(\mathbb{R}^n)$  into  $\mathcal{S}(\mathbb{R}^{2n})$  (see [16]). Hence, it is enough to show that

$$\{\widehat{S}\varphi : S \in U(2n, \mathbb{R})\}$$

is a bounded subset of the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ , i.e., every Schwartz seminorm is bounded on it. Since the group  $U(2n, \mathbb{R})$  is compact, it is sufficient to show that every seminorm is locally bounded, that is, we can limit ourselves to consider  $S$  in a sufficiently small neighbourhood for any fixed  $S_0 \in U(2n, \mathbb{R})$ . Equivalently, we can consider  $S$  of the form  $S = S_1 J^{-1} S_0$  where  $S_1$  belongs to a enough small neighbourhood of  $J$  in  $U(2n, \mathbb{R})$ . Using the representation of metaplectic operators recalled at the beginning of this section, we can write

$$\begin{aligned} \widehat{S}\varphi(x) &= \pm \widehat{S}_1[\widehat{J}^{-1}\widehat{S}_0\varphi](x) \\ &= c\sqrt{|\det L|} \int_{\mathbb{R}^n} e^{2\pi i(\frac{1}{2}Px \cdot x - Lx \cdot y + \frac{1}{2}Qy \cdot y)} \underbrace{[\widehat{J}^{-1}\widehat{S}_0\varphi]}_{\in \mathcal{S}(\mathbb{R}^n)}(y) dy \end{aligned}$$

where  $|c| = 1$  and, we might say,  $\|P\| < \epsilon$ ,  $\|Q\| < \epsilon$ ,  $\|L - I\| < \epsilon$ . If  $\epsilon < 1$ , it is straightforward to check that  $\widehat{S}\varphi$  belongs to a bounded subset of  $\mathcal{S}(\mathbb{R}^n)$ , as desired.  $\square$



**Lemma 2.3.** *Let  $B = (b_{i,j})_{i,j=1,\dots,n}$  be the  $n \times n$  submatrix in (2). The subset  $\Sigma \subset U(2n, \mathbb{R})$  obtained by setting  $b_{i,1} = 0$ ,  $i = 1, \dots, n$  (i.e., the first column of  $B$  is set to zero), is a submanifold of codimension  $n$ .*

*Proof.* We have to verify that the coordinates  $b_{1,1}, \dots, b_{n,1}$  are independent on the subset  $\Sigma$ , namely the projection

$$(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$$

has rank  $n$  on  $\Sigma$ .

Let us first show that for every  $S_0 \in \Sigma$  there exists a  $U(2n, \mathbb{R})$ -valued smooth function  $S(b_1, \dots, b_n)$ , defined in a neighbourhood of  $0 \in \mathbb{R}^n$ , such that  $S(0) = S_0$  and the first column “of its submatrix  $B$ ” is precisely  $(b_1, \dots, b_n)^T$ .

Let  $S_0 = A + iB = (V_1, \dots, V_n) \in \Sigma$ , with  $V_j$  being a  $n \times 1$  complex vector,  $j = 1, \dots, n$ , so that by assumption  $(b_{i,1})_{i=1,\dots,n} = \text{Im } V_1 = 0$ . We consider any smooth function  $V_1(b_1, \dots, b_n)$ , defined in a neighbourhood of  $0 \in \mathbb{R}^n$ , valued in the unit sphere of  $\mathbb{C}^n$ , such that

$$\text{Im } V_1(b_1, \dots, b_n) = (b_1, \dots, b_n)^T, \quad V_1(0) = V_1.$$

Then, we apply the Gram-Schmidt orthonormalization procedure in  $\mathbb{C}^n$  to the set of vectors  $(V_1(b_1, \dots, b_n), V_2, \dots, V_n)$ . This provides the desired  $U(n)$ -valued function  $S(b_1, \dots, b_n)$ . In particular  $S(0) = S_0$ .

Now, the composition of the mapping

$$(b_1, \dots, b_n) \mapsto S(b_1, \dots, b_n)$$

followed by the projection  $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$  is therefore the identity mapping in a neighbourhood of  $0$  and has rank  $n$ . Hence the same is true for the projection  $(b_{1,1}, \dots, b_{n,1}) : U(2n, \mathbb{R}) \rightarrow \mathbb{R}^n$  at  $S_0$ .  $\square$

**Lemma 2.4.** *For every  $\epsilon > 0$ , define the ( $x$ -independent) function*

$$(18) \quad \chi_\epsilon(x, \xi) = \frac{1}{\epsilon^n} \mathbb{1}_Q \left( \begin{pmatrix} \xi \\ \epsilon \end{pmatrix} \right),$$

where

$$Q = \left[ -\frac{1}{2}, \frac{1}{2} \right]^n \subset \mathbb{R}^n \quad \text{and} \quad \mathbb{1}_Q = \begin{cases} 1, & \xi \in Q \\ 0, & \xi \notin Q \end{cases}$$

and

$$(19) \quad \tilde{\chi}_\epsilon(z) = \frac{\chi_\epsilon(z)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS}, \quad z \in \mathbb{R}^{2n}.$$

Then we have

$$(20) \quad \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) dS = 1, \quad \forall z \in \mathbb{R}^{2n}$$

and

$$(21) \quad \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(x, \xi) \Phi(x, \xi) dx d\xi = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx,$$

for some  $C > 0$  and for every continuous function  $\Phi$  on  $\mathbb{R}^{2n}$  with a rapid decay at infinity.

*Proof.* We will show in a moment that, for  $z = (x, \xi) \in \mathbb{R}^{2n}$ ,

$$(22) \quad \int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \gtrsim \min\{\epsilon^{-n}, |z|^{-n}\}$$

(with the convention, at  $z = 0$ , that  $\min\{\epsilon^{-n}, +\infty\} = \epsilon^{-n}$ ). In particular,  $\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \neq 0$ , for every  $z \in \mathbb{R}^{2n}$ . Formula (20) then follows, because

$$\begin{aligned} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) dS &= \int_{U(2n, \mathbb{R})} \frac{\chi_\epsilon(Sz)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(USz) dU} dS \\ &= \int_{U(2n, \mathbb{R})} \frac{\chi_\epsilon(Sz)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(Uz) dU} dS = 1 \end{aligned}$$

for every  $z \in \mathbb{R}^{2n}$ , since the Haar measure is right invariant.

Let us now prove (22). For  $z = 0$  we have

$$\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS = \frac{1}{\epsilon^n} \int_{U(2n, \mathbb{R})} dS = \frac{C_0}{\epsilon^n},$$

with  $C_0 = \text{meas}(U(2n, \mathbb{R})) > 0$ . Consider now  $z \neq 0$ . Observe that the function

$$\Psi_\epsilon(z) := \int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS$$

is constant on the orbits of  $U(2n, \mathbb{R})$  in  $\mathbb{R}^{2n}$ , so that we can suppose

$$z = (x, 0), \quad x = (x_1, 0, \dots, 0), \quad x_1 = |x| = |z| > 0.$$

Now, by the definition of  $\chi_\epsilon$  and  $\Psi_\epsilon$ ,

$$(23) \quad \Psi_\epsilon(z) = \epsilon^{-n} \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \frac{\epsilon}{2|x|}, \quad i = 1, \dots, n \right\},$$

where  $(b_{i,1})_{i=1, \dots, n}$ , is the first column of the matrix  $B = (b_{i,j})_{i,j=1, \dots, n}$ .

Define, for  $\mu > 0$ ,

$$f(\mu) = \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \mu, \quad i = 1, \dots, n \right\}.$$

Observe that  $f(\mu)$  is non-decreasing and constant for  $\mu \geq 1$ . Moreover, from Lemma 2.3 we know that by setting  $b_{i,1} = 0$ ,  $i = 1, \dots, n$ , in  $U(2n, \mathbb{R})$ , we get a submanifold  $\Sigma$  of codimension  $n$ , and the function  $f(\mu)$  is the measure

of a tubular neighbourhood of  $\Sigma$  in  $U(2n, \mathbb{R})$ . Hence we have the asymptotic behaviour

$$(24) \quad \mu^{-n} f(\mu) \rightarrow C_0 > 0, \quad \text{as } \mu \rightarrow 0^+$$

and in particular

$$(25) \quad f(\mu) \gtrsim \min\{1, \mu^n\}.$$

We then infer

$$(26) \quad \Psi_\epsilon(z) = \epsilon^{-n} f\left(\frac{\epsilon}{2|z|}\right) \rightarrow \frac{C_1}{|z|^n}, \quad \text{as } \epsilon \rightarrow 0^+$$

locally uniformly in  $\mathbb{R}^{2n} \setminus \{0\}$ , with  $C_1 = 2^{-n}C_0$ , and

$$(27) \quad \Psi_\epsilon(z) \gtrsim \epsilon^{-n} \min\left\{1, \left(\frac{\epsilon}{|z|}\right)^n\right\} = \min\{\epsilon^{-n}, |z|^{-n}\},$$

which is (22).

Let us finally prove (21). We are interested in the limit  $\epsilon \rightarrow 0^+$ , so we can assume  $\epsilon \leq 1$ . Consider a continuous function  $\Phi$  on  $\mathbb{R}^{2n}$  with rapid decay at infinity. By definition of  $\tilde{\chi}_\epsilon(z)$  in (19) we have

$$\tilde{\chi}_\epsilon(x, \xi) = \frac{\epsilon^{-n}}{\Psi_\epsilon(x, \xi)} \mathbb{1}_{[-\epsilon/2, \epsilon/2]^n}(\xi)$$

so that, by (27),

$$|\tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi)| \lesssim \epsilon^{-n}(1 + |x|^n) \mathbb{1}_{[-\epsilon/2, \epsilon/2]^n}(\xi) |\Phi(x, \xi)| \in L^1(\mathbb{R}^{2n})$$

for  $0 < \epsilon \leq 1$ . Fubini's Theorem then allows one to look at the first integral in (21) as an iterated integral

$$I_\epsilon := \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi \right) dx$$

and we apply the dominated convergence theorem to the integral with respect to the  $x$  variable as follows. Setting

$$\Upsilon_\epsilon(x) := \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi = \epsilon^{-n} \int_{[-\epsilon/2, \epsilon/2]^n} \frac{1}{\Psi_\epsilon(x, \xi)} \Phi(x, \xi) d\xi,$$

by (26) we have, for every fixed  $x \neq 0$ ,

$$\Upsilon_\epsilon(x) \rightarrow C|x|^n\Phi(x, 0);$$

for some constant  $C > 0$ . On the other hand  $\Upsilon_\epsilon(x)$  is dominated, using (27), by

$$(1 + |x|)^n \sup_{\xi \in \mathbb{R}^n} |\Phi(x, \xi)| \in L^1(\mathbb{R}^n).$$

Hence

$$\lim_{\epsilon \rightarrow 0^+} I_\epsilon = \int_{\mathbb{R}^n} \lim_{\epsilon \rightarrow 0^+} \Upsilon_\epsilon(x) dx = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx.$$

This concludes the proof.  $\square$

**Remark 2.5.** *Observe that there are no conditions on the derivatives of the function  $\Phi$  in (21).*

### 3. PROOFS OF THE MAIN RESULTS

In what follows we prove Theorems 1.2 and 1.3.

*Proof of Theorem 1.2.* (i) **First Step.** Let us start with showing that formula (5) is true for any function  $\psi$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ . Using the Gaussian  $\varphi(t) = 2^{d/4}e^{-\pi|t|^2}$  as window function, we compute the  $M_m^p$ -norm of  $\psi$  as in (4) and then use Lemma 2.4 so that

$$\begin{aligned} \|\psi\|_{M_m^p}^p &= \int_{\mathbb{R}^{2n}} |V_\varphi\psi(z)|^p m(z)^p dz = \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) |V_\varphi\psi(z)|^p m(z)^p dSdz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(z) |V_\varphi\psi(S^{-1}z)|^p m(S^{-1}z)^p dSdz \\ &= \int_{\mathbb{R}^{2n}} \int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(z) |V_{\widehat{S}\varphi}\widehat{S}\psi(z)|^p m(S^{-1}z)^p dSdz \end{aligned}$$

where in the last equality we used Lemma 2.1. Observe that, since  $S$  is unitary and  $\varphi$  is a Gaussian,  $\widehat{S}\varphi = c\varphi$ , for some phase factor  $c \in \mathbb{C}$ , with  $|c| = 1$  (see [15, Proposition 252]) and this phase factor is killed by the modulus obtaining  $|V_{\widehat{S}\varphi}\widehat{S}\psi(z)| = |V_\varphi\widehat{S}\psi(z)|$ . Continuing the above computation we infer

$$\|\psi\|_{M_m^p}^p = \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(z) \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(z)|^p m(S^{-1}z)^p dSdz.$$

Set

$$\Phi(z) = \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(z)|^p m(S^{-1}z)^p dS.$$

The dominated convergence theorem guarantees that  $\Phi$  is continuous on  $\mathbb{R}^{2n}$  and moreover  $\Phi$  has rapid decay at infinity. This follows from Lemma 2.2 (recall that  $m$  is continuous and has at most polynomial growth).

Letting  $\epsilon \rightarrow 0^+$  and using (21) we obtain

$$\begin{aligned} \|\psi\|_{M_m^p}^p &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |V_\varphi\widehat{S}\psi(x, 0)|^p m(S^{-1}(x, 0)^T)^p dSdx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |\langle \widehat{S}\psi, T_x\varphi \rangle|^p m(S^{-1}(x, 0)^T)^p dSdx \\ &= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n, \mathbb{R})} |\langle \psi, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dSdx. \end{aligned}$$

The last equality is due to  $\langle \widehat{S}\psi, T_x\varphi \rangle = \langle \psi, \widehat{S}^{-1}T_x\varphi \rangle$  and the invariance of the Haar measure of  $U(2n, \mathbb{R})$  with respect to the change of variable  $S \rightarrow S^{-1}$ .

**Second Step.** Consider  $f \in M_m^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ . Using the density of the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  in  $M_m^p(\mathbb{R}^n)$  (cf. e.g., [22, Chapter 12]), there exists a sequence  $\{\psi_k\}_k \in \mathcal{S}(\mathbb{R}^n)$  such that  $\psi_k \rightarrow f$  in  $M_m^p(\mathbb{R}^n)$ . This implies that  $\psi_k \rightarrow f$  in  $\mathcal{S}'(\mathbb{R}^n)$  and

$$\langle \psi_k, \widehat{S}T_x\varphi \rangle \rightarrow \langle \psi, \widehat{S}T_x\varphi \rangle$$

pointwise for every  $x \in \mathbb{R}^n$ ,  $S \in U(2n, \mathbb{R})$ . Let us define, for every  $f \in M_m^p(\mathbb{R}^n)$ ,

$$(28) \quad |||f||| = \left( \int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |x|^n |\langle f, \widehat{S}T_x\varphi \rangle|^p m(S(x, 0)^T)^p dx dS \right)^{\frac{1}{p}}.$$

By Fatou's Lemma, for any  $f \in M_m^p(\mathbb{R}^n)$ :

$$(29) \quad |||f|||^p \leq \liminf_{k \rightarrow \infty} |||\psi_k|||^p \lesssim \liminf_{k \rightarrow \infty} \|\psi_k\|_{M_m^p}^p = \|f\|_{M_m^p}^p.$$

It is easy to check that  $|||f|||$  is a seminorm on  $M_m^p(\mathbb{R}^n)$ . Applying (29) to the difference  $f - \psi_k$  we obtain  $|||f - \psi_k||| \rightarrow 0$  and hence  $|||\psi_k||| \rightarrow |||f|||$ . By assumption we also have  $\|\psi_k\|_{M_m^p} \rightarrow \|f\|_{M_m^p}$ , and the desired norm equivalence in (5) then extends from  $\mathcal{S}(\mathbb{R}^n)$  to  $M_m^p(\mathbb{R}^n)$ .

**Third Step.** We will show that (6) easily follows from (5). By the definition of the symplectic group (1), for any  $S \in U(2n, \mathbb{R})$ ,

$$J^{-1}S = (S^T)^{-1}J^{-1} = SJ^{-1}$$

for  $S^{-1} = S^T$ . On the other hand, for any  $f \in M_m^p(\mathbb{R}^n)$ ,  $\|f\|_{M_m^p} \asymp \|\widehat{f}\|_{M_m^p}$ , with  $\widehat{m}(z) = m(J^{-1}z)$ ; see [16]. Using (15),

$$\begin{aligned} |\langle \widehat{f}, \widehat{S}T_x\varphi \rangle| &= |\langle f, \widehat{J^{-1}}\widehat{S}T_x\varphi \rangle| = |\langle f, \widehat{S}\mathcal{F}^{-1}T_x\varphi \rangle| \\ &= |\langle f, \widehat{S}M_x\mathcal{F}^{-1}\varphi \rangle| = |\langle f, \widehat{S}M_x\varphi \rangle| \end{aligned}$$

since the Gaussian is an eigenvector of  $\mathcal{F}^{-1}$  with eigenvalue equal to 1. Moreover

$$\widehat{m}(S(x, 0)^T) = m(J^{-1}S(x, 0)^T) = m(SJ^{-1}(x, 0)^T) = m(S(0, x)^T).$$

Hence (6) follows from (5).

(ii) Case  $p = \infty$ . Observe that any  $z \in \mathbb{R}^{2n}$  can be written as

$$z = S^{-1}(x, 0)^T,$$

for some  $x \in \mathbb{R}^n$ ,  $S \in U(2n, \mathbb{R})$ , so that, for any  $f \in M_m^\infty(\mathbb{R}^n)$ ,

$$\begin{aligned}
 \|f\|_{M_m^\infty(\mathbb{R}^n)} &= \sup_{z \in \mathbb{R}^{2n}} |V_\varphi f(z)| m(z) \asymp \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |V_\varphi f(S^{-1}(x, 0)^T)| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |V_\varphi(\widehat{S}f)(x, 0)| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle \widehat{S}f, T_x \varphi \rangle| m(S^{-1}(x, 0)^T) \\
 &= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \widehat{S}T_x \varphi \rangle| m(S(x, 0)^T),
 \end{aligned}$$

which gives (7). Formula (8) follows as above.  $\square$

We now prove the similar result, with the group  $U(2n, \mathbb{R})$  replaced by the subgroup  $\mathbb{T}^n$  (up to isomorphisms).

*Proof of Theorem 1.3.* (i) We could follow a similar pattern to the proof of Theorem 1.2, replacing the group  $U(2n, \mathbb{R})$  by  $\mathbb{T}^n$ . The preparation of Lemma 2.3 would be no longer necessary. Lemma 2.4 would require some small adjustments. On the other hand a more direct argument can be given. Namely, writing  $z_j = (x_j, \xi_j)$  in complex notation as  $r_j e^{i\theta_j}$ , and setting  $r = (r_1, \dots, r_n)$ ,  $\theta = (\theta_1, \dots, \theta_n)$  we have

$$\begin{aligned}
 \|f\|_{M_m^p}^p &= \int_{\mathbb{R}^{2n}} |V_\varphi f(z)|^p m(z)^p dz \\
 &= \int_{\mathbb{R}_+^n \times [0, 2\pi]^n} r_1 \cdots r_n |V_\varphi f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p m(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})^p dr d\theta.
 \end{aligned}$$

With  $S$  as in (9) and using Lemma 2.1, therefore we have

$$\begin{aligned}
 \|f\|_{M_m^p}^p &\asymp \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi f(S(x, 0)^T)|^p m(S(x, 0)^T)^p dx dS \\
 &= \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |V_\varphi(\widehat{S}^{-1}f)(x, 0)|^p m(S(x, 0)^T)^p dx dS \\
 &= \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \cdots x_n| |\langle \widehat{S}^{-1}f, T_x \varphi \rangle|^p m(S(x, 0)^T)^p dx dS,
 \end{aligned}$$

which is (10). The characterization (11) has the same proof as the corresponding formula (6).

(ii) The  $M^\infty$  case uses the same argument as in the proofs of (7) and (8), with the group  $U(2n, \mathbb{R})$  replaced by  $\mathbb{T}^n$ .  $\square$

4. INTEGRAL REPRESENTATIONS FOR THE TORUS IN TERMS OF THE  
FRACTIONAL FOURIER TRANSFORM

Observe that the symplectic matrix in  $U(2n, \mathbb{R})$  corresponding to the complex matrix  $S \in \mathbb{T}^n$  in (9) via the isomorphism  $\iota$  in (16) is given by

$$\iota(S) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with

$$A = \text{diag}[\cos \theta_1, \dots, \cos \theta_n] \quad B = \text{diag}[\sin \theta_1, \dots, \sin \theta_n].$$

Consider the case  $\theta_i \neq k\pi$ ,  $k \in \mathbb{Z}$ ,  $i = 1, \dots, n$ . The matrix  $\iota(S)$  is a free symplectic matrix and the related metaplectic operator possesses the integral representation (14). Since

$$AB^{-1} = B^{-1}A = \text{diag}\left[\frac{\cos \theta_1}{\sin \theta_1}, \dots, \frac{\cos \theta_n}{\sin \theta_n}\right],$$

the polynomial  $W(x, x')$  becomes

$$(30) \quad W(x_1, \dots, x_n, x'_1, \dots, x'_n) = \sum_{i=1}^n \frac{1}{2 \sin \theta_i} (\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x_i'^2)$$

and

$$\Delta(W) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}}.$$

for some phase factor  $c \in \mathbb{C}$ , with  $|c| = 1$ . Hence we obtain, for  $\psi \in \mathcal{S}(\mathbb{R}^n)$ ,

$$(31) \quad \widehat{\iota(S)\psi}(x) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}} \int_{\mathbb{R}^n} e^{2\pi i W(x, x')} \psi(x') dx',$$

with  $W(x, x')$  in (30). From (31) we deduce that  $\widehat{\iota(S)}$  can be written as the composition of the operators

$$(32) \quad \widehat{\iota(S)} = \pm \widehat{\iota(S_1)} \cdots \widehat{\iota(S_n)},$$

where, for some phase factor  $c$ ,

$$\widehat{\iota(S_i)}\psi(x) = \frac{c}{\sqrt{|\sin \theta_i|}} \int_{\mathbb{R}} e^{\frac{\pi i}{\sin \theta_i} (\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x_i'^2)} \psi(x'_1, \dots, x'_i, \dots, x'_n) dx'_i.$$

Indeed if  $\theta_i = \pi/2$ , then  $\widehat{\iota(S_i)} = \pm \widehat{J}$  is the Fourier transform with respect to the variable  $x_i$ . Otherwise,  $\widehat{\iota(S_i)} = \pm \mathcal{F}_{\theta_i}$ , the  $\theta_i$ -angle partial fractional Fourier transform (again referred to the variable  $x_i$ ).

Alternatively, the same conclusion (32) can be drawn by writing

$$(33) \quad S = \text{diag}[e^{i\theta_1}, \dots, e^{i\theta_n}] = \text{diag}[e^{i\theta_1}, 1, \dots, 1] \cdots \text{diag}[1, \dots, 1, e^{i\theta_n}]$$

that is

$$S = S_1 \cdots S_i \cdots S_n,$$

with

$$S_i = \text{diag}[1, \dots, 1, e^{i\theta_i}, 1, \dots, 1], \quad i = 1, \dots, n$$

so that

$$\widehat{\iota(S)} = \iota(S_1) \widehat{\iota(S_1)} = \pm \iota(S_1) \cdots \widehat{\iota(S_n)}.$$

If  $\theta_i = 2k\pi$  for some  $k \in \mathbb{Z}$ ,  $\widehat{\iota(S_i)} = \pm I$  with  $I$  the identity operator. If  $\theta_i = (2k+1)\pi$  for some  $k \in \mathbb{Z}$ ,  $\widehat{\iota(S_i)}\psi(x) = \pm\psi(x_1, \dots, -x_i, \dots, x_n)$ .

Hence using the  $\theta_i$ -angle partial fractional Fourier transform  $\mathcal{F}_{\theta_i} = \widehat{\iota(S_i)}$  we can rephrase Theorem 1.3 as follows.

**Theorem 4.1.** *Let  $\varphi$  be the Gaussian of Theorem 1.2.*

(i) *For  $1 \leq p < \infty$ ,  $f \in M_m^p(\mathbb{R}^n)$ , we have*

$$\|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times [0, 2\pi]^n} |x_1 \dots x_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi \rangle|^p m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})^p dx d\theta \right)^{\frac{1}{p}},$$

and

$$\|f\|_{M_m^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times [0, 2\pi]^n} |\xi_1 \dots \xi_n| |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi \rangle|^p m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n})^p d\xi d\theta \right)^{\frac{1}{p}}.$$

(ii) *For  $p = \infty$ ,*

$$\|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{\theta \in [0, 2\pi]^n} \sup_{x \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} T_x \varphi \rangle| m(x_1 e^{i\theta_1}, \dots, x_n e^{i\theta_n})$$

and

$$\|f\|_{M_m^\infty(\mathbb{R}^n)} \asymp \sup_{\theta \in [0, 2\pi]^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \dots \mathcal{F}_{\theta_n} M_\xi \varphi \rangle| m(\xi_1 e^{i\theta_1}, \dots, \xi_n e^{i\theta_n}).$$

We observe that this result could also be obtained by writing  $\|f\|_{M_m^p(\mathbb{R}^n)}$  in terms of the weighted  $L^p$  norm of the Bargmann transform of  $f$  and using the covariance property of the Bargmann transform; the papers [18, 30] and specially [31] are relevant in this connection.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY

*E-mail address:* elena.cordero@unito.it

UNIVERSITY OF VIENNA, FACULTY OF MATHEMATICS, OSKAR-MORGENSTERN-PLATZ 1 A-1090 WIEN, AUSTRIA

*E-mail address:* maurice.de.gosson@univie.ac.at

DIPARTIMENTO DI SCIENZE MATEMATICHE, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY

*E-mail address:* fabio.nicola@polito.it