Zenodo, 27 October 2020, DOI: 10.5281/zenodo.4139216

Infinite product expansions from Euler's Introductio in Analysin Infinitorum

Amelia Carolina Sparavigna
Department of Applied Science and Technology, Politecnico di Torino, Torino, Italy

Here we show some of the infinite product expansions that we can find in Leonhard Euler's book entitled Introductio in analysin infinitorum, 1797. It is a book that can be read with comparative ease by contemporary scholars. Then, only some small changes in notation are required to propose the formulas.

Torino, 27 October 2020, DOI: 10.5281/zenodo.4139216

Here we show some infinite product expansions deduced by Leonhard Euler in his Introductio in analysin infinitorum, Volume 1, 1797. This book, written in Latin, is freely available among the Google Books, but it is also possible to find an annotated translation by Ian Bruce. We can find Bruce's work at the following link
www.17centurymaths.com/contents/euler/introductiontoanalysisvolone/ch9vol1.pdf
Euler's book, a book that can be read with comparative ease by contemporary scholars. Only some small changes in notation are required to propose the formulas.

Let us consider the formulas in Euler's book, Chapter 9.

n.156

$$e^x - e^{-x} = 2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\left(1+\frac{x^2}{4^2\pi^2}\right)\left(1+\frac{x^2}{5^2\pi^2}\right)\ldots$$

$$e^x - e^{-x} = 2x\prod_{j=1}^{\infty}\left(1+\frac{x^2}{j^2\pi^2}\right)$$

We used this formula in [1], concerning the κ-calculus [2-4]. In the Figure 1, the behaviour of the function and the product.

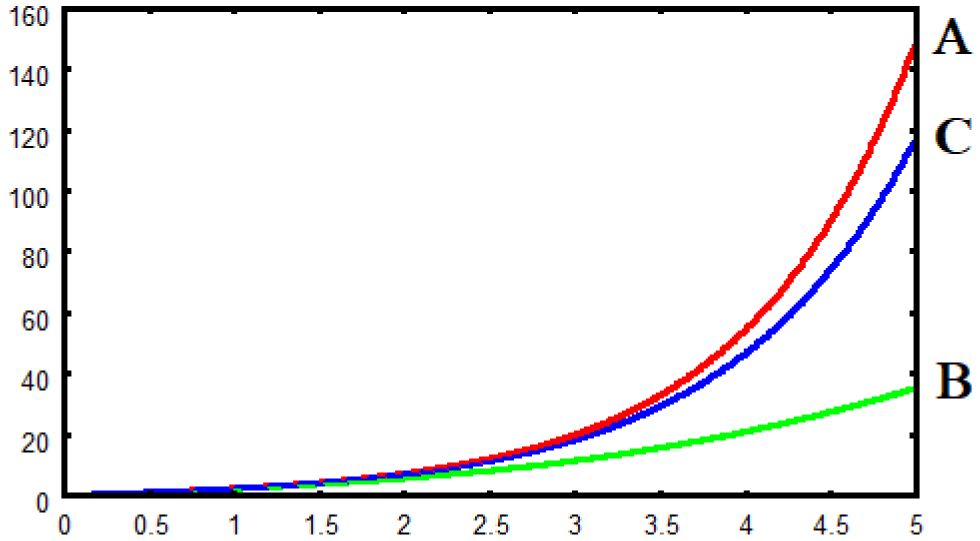

Figure 1: Curve A shows the behaviour of function $e^x - e^{-x}$, curve B of $2x(1+\frac{x^2}{\pi^2})$ that is when we consider only the first factor in the product, and curve C , $2x\prod_{j=1}^{10}(1+\frac{x^2}{j^2\pi^2})$, with ten factors.

n.157

$$e^x + e^{-x} = 2(1+\frac{4x^2}{\pi^2})(1+\frac{4x^2}{3^2\pi^2})(1+\frac{4x^2}{5^2\pi^2})(1+\frac{4x^2}{7^2\pi^2})...$$

$$e^x + e^{-x} = 2\prod_{j=0}^{\infty}(1+\frac{4x^2}{(2j+1)^2\pi^2}) = 2\prod_{j=1}^{\infty}(1+\frac{4x^2}{(2j-1)^2\pi^2})$$

Also this formula has been used in [1].

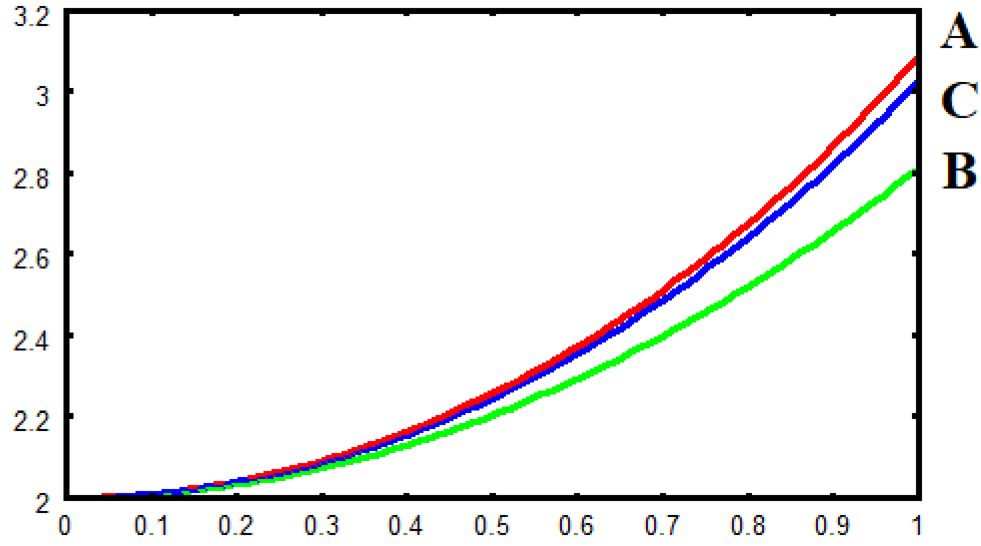

Figure 2: Curve A shows the behaviour of function e^x+e^{-x}, curve B of $2(1+\frac{4x^2}{\pi^2})$ that is when we consider only the first facto in the product, and curve C the function $2\prod_{j=1}^{5}(1+\frac{4x^2}{(2j-1)^2\pi^2})$ with five factors.

n.158

Using the previous results Euler determines the infinite product expansion for sine and cosine.

158a: $$\sin z = z(1-\frac{z}{\pi})(1+\frac{z}{\pi})(1-\frac{z}{2\pi})(1+\frac{z}{2\pi})(1-\frac{z}{3\pi})(1+\frac{z}{3\pi})\ldots$$

$$\sin z = z\prod_{j=1}^{\infty}(1+\frac{z}{j\pi})(1-\frac{z}{j\pi})$$

158b: $$\cos z = (1-\frac{2z}{\pi})(1+\frac{2z}{\pi})(1-\frac{2z}{3\pi})(1+\frac{2z}{3\pi})(1-\frac{2z}{5\pi})(1+\frac{2z}{5\pi})\ldots$$

$$\cos z = \prod_{j=1}^{\infty}(1+\frac{2z}{(2j-1)\pi})(1-\frac{2z}{(2j-1)\pi})$$

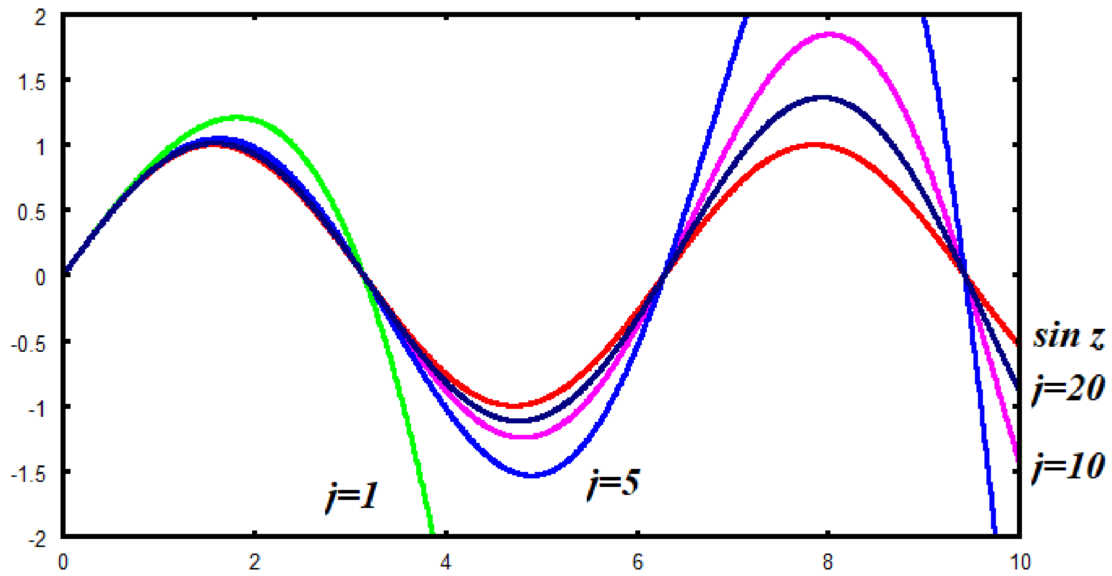

Figure 3 : Function sin z and its approximations by means of different numbers of factors.

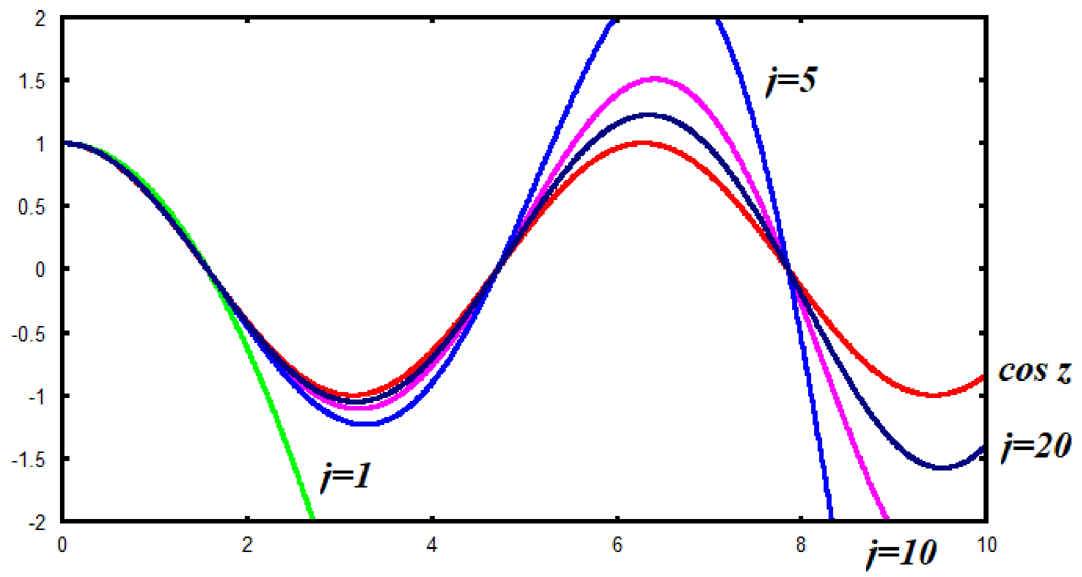

Figure 4 : Function cos z and its approximations by means of different numbers of factors.

n.159

159a: $$\frac{e^x-2\cos\gamma+e^{-x}}{2(1-\cos\gamma)}=(1+\frac{xx}{\gamma\gamma})(1+\frac{xx}{(2\pi-\gamma)^2})(1+\frac{xx}{(2\pi+\gamma)^2})$$

$$(1+\frac{xx}{(4\pi-\gamma)^2})(1+\frac{xx}{(4\pi+\gamma)^2})(1+\frac{xx}{(6\pi-\gamma)^2})(1+\frac{xx}{(6\pi+\gamma)^2})\ldots$$

This is the expression as written by Euler. And in the same manner we find written the expressions previously given in Euler's book. In contemporary notation:

$$\frac{e^x-2\cos\gamma+e^{-x}}{2(1-\cos\gamma)} = (1+\frac{x^2}{\gamma^2})\prod_{j=1}^{\infty}(1+\frac{x^2}{(2\pi j-\gamma)^2})(1+\frac{x^2}{(2\pi j+\gamma)^2})$$

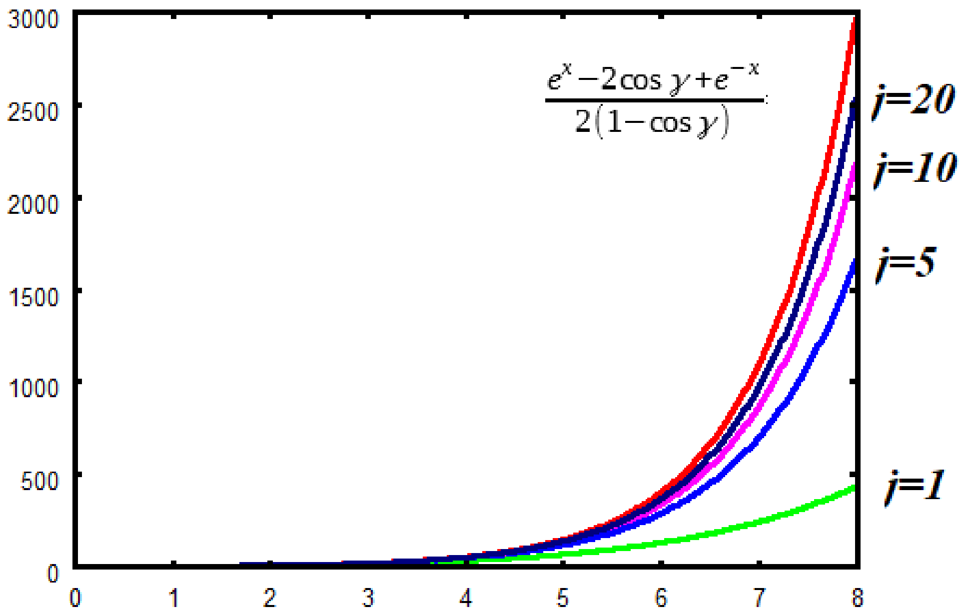

Figure 5 : Function $\frac{e^x-2\cos\gamma+e^{-x}}{2(1-\cos\gamma)}$ and its approximations by means of different numbers of factors (1, 5, 10, 20). $\gamma = \pi/3$.

159b: $$\frac{\cos\zeta-\cos\gamma}{1-\cos\gamma}=(1-\frac{\zeta}{\gamma})(1+\frac{\zeta}{\gamma})(1-\frac{\zeta}{(2\pi-\gamma)})(1+\frac{\zeta}{(2\pi-\gamma)})$$

$$(1-\frac{\zeta}{(2\pi+\gamma)})(1+\frac{\zeta}{(2\pi+\gamma)})(1-\frac{\zeta}{(4\pi-\gamma)})(1+\frac{\zeta}{(4\pi-\gamma)})\ldots =$$

$$(1-\frac{\zeta}{\gamma})(1+\frac{\zeta}{\gamma})\prod_{j=1}^{\infty}(1-\frac{\zeta}{(2\pi j-\gamma)})(1+\frac{\zeta}{(2\pi j-\gamma)})(1-\frac{\zeta}{(2\pi j+\gamma)})(1+\frac{\zeta}{(2\pi j+\gamma)})$$

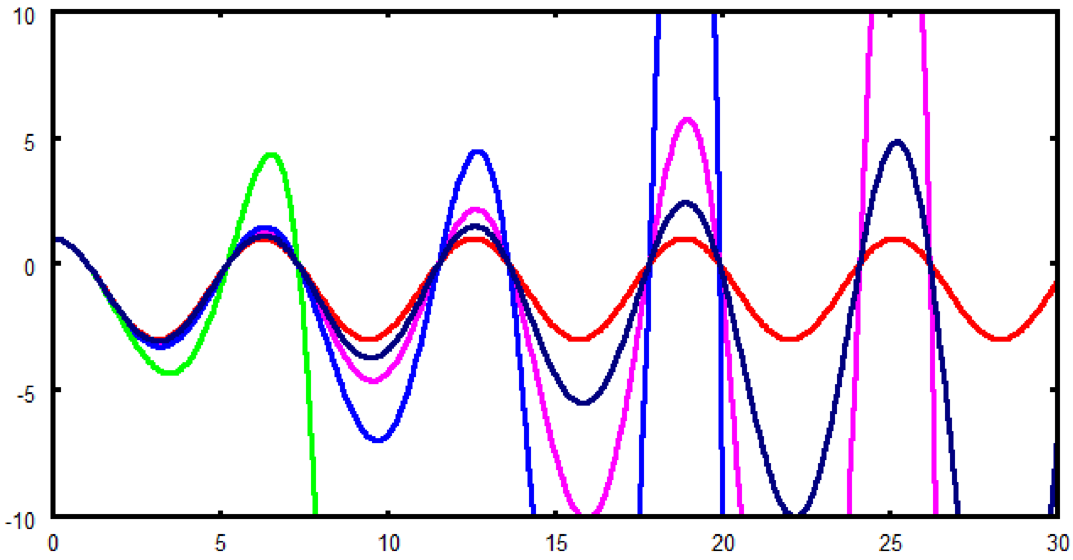

Figure 6 : Function $\frac{\cos\zeta-\cos\gamma}{1-\cos\gamma}$ and its approximations by means of different numbers of factors (1, 5, 10, 20). $\gamma = \pi/3$.

n.161

$$\frac{e^{b+x}+e^{c-x}}{e^b+e^c}=(1+\frac{4(b-c)x+4xx}{\pi\pi+(b-c)^2})(1+\frac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2})(1+\frac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2})\ldots$$

$$= \left(1+\frac{4(b-c)x+4x^2}{\pi^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{(3\pi)^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{(5\pi)^2+(b-c)^2}\right)\ldots$$

$$\frac{e^{b+x}+e^{c-x}}{e^b+e^c}=\prod_{j=1}^{\infty}\left(1+\frac{4(b-c)x+4x^2}{[(2j-1)\pi]^2+(b-c)^2}\right)$$

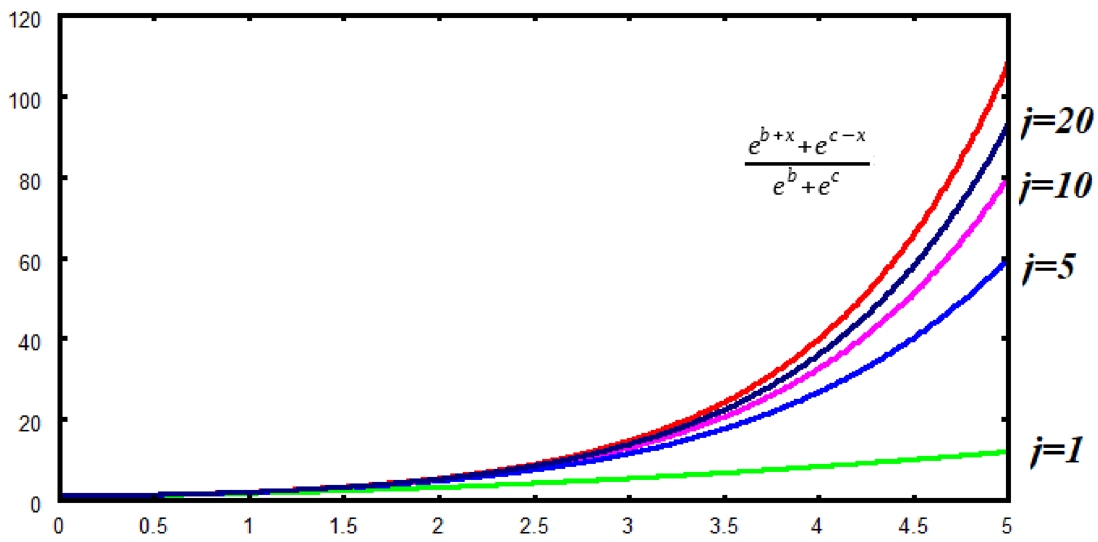

Figure 7 : The function and its approximations by means of different numbers of factors (1, 5, 10, 20). $b = 2$, $c = 1$,

After the above-mentioned expression, we find also the following:

$$\frac{e^{b+x}-e^{c-x}}{e^b-e^c}=\left(1+\frac{2x}{b-c}\right)\left(1+\frac{4(b-c)x+4x^2}{4\pi^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{16\pi^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{36\pi^2+(b-c)^2}\right)\ldots$$

$$= \left(1+\frac{2x}{b-c}\right)\left(1+\frac{4(b-c)x+4x^2}{2^2\pi^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{4^2\pi^2+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4x^2}{6^2\pi^2+(b-c)^2}\right)\ldots$$

$$\frac{e^{b+x}-e^{c-x}}{e^b-e^c}=\left(1+\frac{2x}{b-c}\right)\prod_{j=1}^{\infty}\left(1+\frac{4(b-c)x+4x^2}{(2j)^2\pi^2+(b-c)^2}\right)$$

Several other infinite product expansions are given in the Euler's book and in the Ian Bruce's translation.
Here we have analysed a few of them to investigate their numerical convergence. Plots have been proposed accordingly. Let us note that expansions 156, 157, and 158 are given in [5], without mentioning Euler.

References

[1] Sparavigna, A.C. (2020). Entropy and logarithm of Kaniadakis calculus expressed by means of an Euler infinite product expansion, Zenodo, 10.5281/zenodo.4088214

[2] Kaniadakis, G. (2001). Non-linear kinetics underlying generalized statistics. Physica A: Statistical mechanics and its applications, 296(3-4), 405-425.

[3] Kaniadakis, G. (2002). Statistical mechanics in the context of special relativity. Physical review E, 66(5), 056125.

[4] Kaniadakis, G. (2013). Theoretical foundations and mathematical formalism of the power-law tailed statistical distributions. Entropy, 15(10), 3983-4010.

[5] Abramowitz, M., Stegun, I. A., & Romer, R. H. (1988). Handbook of mathematical functions with formulas, graphs, and mathematical tables. Available at the following link http://people.math.sfu.ca/~cbm/aands/