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NOISE ROBUSTNESS CONDITION FOR CHAOTIC MAPS WITH PIECEWISE CONSTANT INVARIANT DENSITY

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ABSTRACT

Chaotic maps represent an effective method for generating random-like sequences, that combines the benefits of relying on simple, causal models with good unpredictability. Regrettably such positive features are counterbalanced by the fact that statistics of true-implemented chaotic maps are generally strongly dependent on implementation errors and external perturbations. Here we study the effect of an external, additive, map-independent noise perturbation in the map model, and present a technique to guarantee, for a quite large class of maps, independence of the first-order statistics of the noise features.

1. INTRODUCTION

Random-like sources are fundamental for many information technology applications including communications [1], power conversion [2][3], signal coding and watermarking [4], electronic equipment testing [5], and many other tasks. Recently, very good results have been achieved in the generation of random sequences by using chaotic maps [6], thanks to their aperiodic and strong unpredictable behavior. Additionally, being simple 1-D autonomous nonlinear discrete-time systems, chaotic maps can easily be implemented with analog hardware.

Here analog implementation is a fundamental issue since no digital system can exhibit chaotic behavior. Regrettably, implementations errors, which are unavoidable in any analog circuit implementation, as well as perturbation during operation, may prevent the statistics of the signals generated by the map to align the desired ones, or even block the possibility to achieve the desired chaotic behavior [7][8]. As a consequence, map statistical robustness is of great practical concern.

In this paper we discuss on the robustness of 1-D discrete-time chaotic map when their behavior is perturbed by an external additive noise [6, chap. 10][8][9] [10]. One of the typical problems related to this setting is that the presence of an additive noise superimposed to the state of the map, could drive it out of its definition set. To cope with this, a classical solution [9] is to add suitably defined "hooks" to the map to assure that the state is reinjected into its nominal definition set. This tecnique does not affect *a priori* any statistical property of the map since these hooks are normally not reachable; however due to the presence of the noise they can influence statistical features of the signals generated by the map. The main aim of this contribution is present a criterion for constructing the hooks, i.e. to expand a map out from its nominal definition set, in order to make the first order statistic of the generated signals independent of the noise perturbation.

The paper is organized as follows. In section 2 we present a simple mathematical model to describe a chaotic map with an additive noise perturbation. In section 3 we discuss a mathematical condition to assure statistical robustness to noise for a particular class of chaotic maps, i.e. maps with piecewise constant invariant density, such as the so-called Piecewise Affine Markov (PWAM) maps [11]. Finally, in section 4 we specialize this condition to some concrete maps and present some practical example.

2. MATHEMATICAL MODEL

In the following we consider the 1-D chaotic map defined as

$$x_{k+1} = \varphi(x_k) \;, \quad \varphi : I \to I \tag{1}$$

where the interval $I = \left[I^-, I^+\right] \subset \mathbb{R}$ and φ is an appropriate non-invertible function. Let us assume that φ is *exact* and indicate with $\mathbf{P}$ its Perron-Frobenius operator (PFO), defined as [6]

$$\mathbf{P}f(x) = \int_I \delta(\varphi(\xi) - x)\, f(\xi)\, d\xi$$

where $\delta(\cdot)$ is the Dirac's generalized functions. With this, if we assume that an initial condition x_0 is randomly drawn according to a probability density function (pdf) $\rho_0 : I \mapsto \mathbb{R}^+$, we know that, for large k, the k-th iterate of the map $\varphi^k(x_0)$ approximately distributes according to $\overline{\rho}$ independently of the density ρ_0, where $\overline{\rho}$ is the *invariant density* of the map since it satisfies the condition $\overline{\rho} = \mathbf{P}\overline{\rho}$ [6].

Let us then consider a map-independent noise, modeled as a discrete-time stochastic process with samples $\nu_k \in N$, $\forall k$, where $N = \left[N^-, N^+\right] \ni 0 \subset \mathbb{R}$, and distributed according to the pdf ρ_ν. By referring to additive perturbation, the system model (1), will be changed in

$$x_{k+1} = \varphi_E(x_k + \nu_k) \;, \quad \varphi_E : J \to I \tag{2}$$

where $J = \left[I^- + N^-, I^+ + N^+\right]$, and where φ_E is an extension on φ in J, i.e. $\varphi_E(x) = \varphi(x)$, $\forall x \in I$. System (2) cannot be studied with chaotic map theory; for example, while the behavior of (1) is assured by the exactness of φ, certainly exist φ_E and ρ_ν such that (2) has not a chaotic behaviour. A treatment on these systems can be found in [6, cap. 10]. For our purposes it is only necessary that, with the considered noise density ρ_ν, φ_E is such

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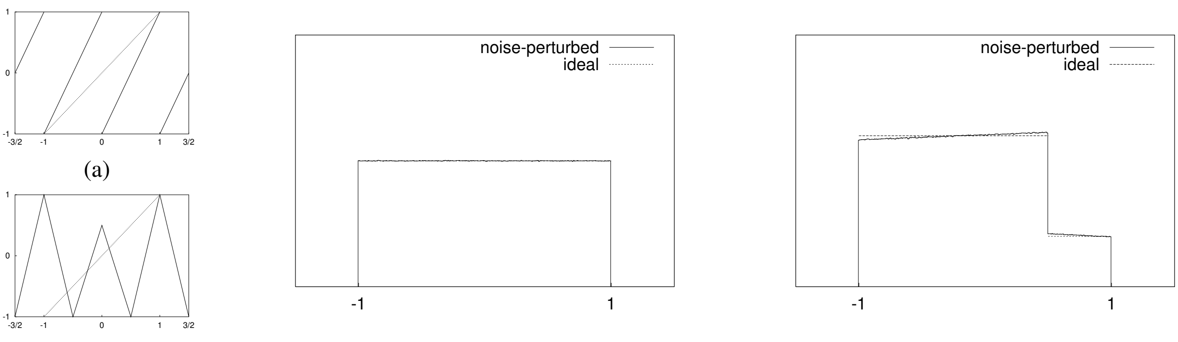

(a) (b) (c) (d)

Fig. 1. A possible extension of Bernoulli shift map (a) and W-map (b) and their first order density (c), (d) obtained through numerical simulation in the perturbed and unperturbed case, with $I = [-1,\, 1]$ and with noise uniformly distributed in $N = \left[-\frac{1}{2},\, \frac{1}{2}\right]$

that system (2) can be studied through densities. Indicating with $\mathbf{P}_E$ the PFO associated to φ_E [1] and cosidering that the pdf of the sum of two independent random variabiles is the convolution of their respective pdfs, we require that for large k symbols x_k are distributed according to the density $\tilde{\rho}$, which is the unique density that solves

$$\tilde{\rho}\,(x) = \mathbf{P}_E\,[\tilde{\rho} * \rho_\nu]\,(x) \qquad (3)$$

and where $*$ is the usual convolution operator. Since this condition, certainly true for a quite large set of ρ_ν and φ_E, it is very difficult to deal with, it will be taken as an assumption.

We say that the extended map φ_E is *noise robust* when $\tilde{\rho} = \overline{\rho}$ for any choice of ρ_ν, i.e. when its invariant density is independent of the noise density and, of course, is equal to the unperturbed system invariant density.

Figures 1 (a) and (b) show a possible extension for the Bernoulli shift and W-map respectively [11], while (c) and (d) report the corresponding simulated invariant densities in the ideal and perturbed case, for an additive uniformly distributed noise . As shown, it can be seen that both maps with such a perturbation maintain a chaotic behaviour with a stationary invariant density, which is for the Bernoulli map equal to the the ideal invariant density $\overline{\rho}$. It is also worthwhile to notice that the proposed extended Bernoulli shift map is noise robust for $N = \left[-\frac{1}{2},\, \frac{1}{2}\right]$. In fact, if we define $\chi_{[a,\,b]}$ as the indicator function of interval $[a,\, b]$, $\mathbf{P}_E$ can be expressed as

$$\mathbf{P}_E f(x) = \left[f\left(\frac{x-3}{2}\right)\chi_{[0,\,1]}(x) + f\left(\frac{x-1}{2}\right)\chi_{[-1,\,1]}(x) + f\left(\frac{x+1}{2}\right)\chi_{[-1,\,1]}(x) + f\left(\frac{x+3}{2}\right)\chi_{[-1,\,0]}(x)\right]$$

and with it we can verify that (3) is solved by $\overline{\rho}\,(x) = \frac{1}{2}\chi_{[-1,\,1]}(x)$

$$\mathbf{P}_E[\overline{\rho}*\rho_\nu](x) = \frac{1}{4}\left[\int_{\frac{x-5}{2}}^{\frac{x-1}{2}} \rho_\nu\,(\xi)d\xi\,\chi_{[0,1]}(x) + \int_{\frac{x-3}{2}}^{\frac{x+1}{2}} \rho_\nu\,(\xi)\,d\xi\,\chi_{[-1,1]}(x) + \int_{\frac{x-1}{2}}^{\frac{x+3}{2}} \rho_\nu\,(\xi)\,d\xi\,\chi_{[-1,\,1]}(x) + \int_{\frac{x+1}{2}}^{\frac{x+5}{2}} \rho_\nu\,(\xi)\,d\xi\,\chi_{[-1,\,0]}(x)\right]$$

$$= \frac{1}{4}\left[\int_{\frac{x-5}{2}}^{\frac{x+3}{2}} \rho_\nu\,(\xi)d\xi + \int_{\frac{x-3}{2}}^{\frac{x+1}{2}} \rho_\nu\,(\xi)\,d\xi\right]\chi_{[0,1]}(x) + \frac{1}{4}\left[\int_{\frac{x-3}{2}}^{\frac{x+5}{2}} \rho_\nu\,(\xi)\,d\xi + \int_{\frac{x-1}{2}}^{\frac{x+3}{2}} \rho_\nu\,(\xi)\,d\xi\right]\chi_{[-1,\,0]}(x)$$

By simply noting that for every point in the corresponding indicator function all the last integrals are computed over intervals greater than N, we have that $\tilde{\rho} = \overline{\rho}$, $\forall \rho_\nu$, i.e. the proposed map is noise robust.

3. NOISE ROBUSTNESS CONDITION

In order to introduce a general noise robustness condition, we need to prove the following

Lemma 1. *Let $\varphi : I \to I$ be a chaotic map, and φ_E its extension in J; let also be $\overline{\rho}$ the unperturbed system invariant density and $\Delta\overline{\rho}^{\varepsilon}(x) = \overline{\rho}\,(x) = \overline{\rho}\,(x-\varepsilon)$ the difference between shifted and non-shifted invariant density. The extended map φ_E is robust to any noise if and only if*

$$\mathbf{P}_E \Delta\overline{\rho}_T^{\varepsilon} = 0,\ \forall \varepsilon \in N \qquad (4)$$

Proof. For the necessary condition, it is enough to consider the linearity of $\mathbf{P}_E$ and a constant noise perturbation, i.e. a delta-like noise density ρ_ν. For the sufficient, if we introduce the shifted invariant density $\overline{\rho}_T^{\varepsilon} = \overline{\rho}\,(x-\varepsilon)$, we know from the linearity of PFO that $\mathbf{P}_E\overline{\rho} = \mathbf{P}_E\overline{\rho}_T^{\varepsilon}\ \forall\varepsilon \in N$, and then, since $\mathbf{P}_E\overline{\rho} = \overline{\rho}$, (3) is verified by $\overline{\rho}$, namely

$$\mathbf{P}_E\,[\overline{\rho} * \rho_\nu]\,(x) = \mathbf{P}_E\left[\int_N \rho_\nu\,(\xi)\,\overline{\rho}\,(x-\xi)\ d\xi\right] = \int_N \rho_\nu\,(\xi)\,\mathbf{P}_E\overline{\rho}_T^{\xi}\,(x)\ d\xi = \int_N \rho_\nu\,(\xi)\ d\xi\,\overline{\rho}\,(x) = \overline{\rho}\,(x)$$

□

Although of general applicability, condition (4) is relatively difficult to use in practical, since it does not explicitly depend on the map structure. To obtain simple robustness conditions we will here restrict to the class of map with piecewise constant $\overline{\rho}$. Although particular, this class is of great practical concern since it

[1] It is easy to see that $\mathbf{P}$ is the restriction of $\mathbf{P}_E$ to densities whose support is I. Also, like $\mathbf{P}$, $\mathbf{P}_E$ is a linear operator

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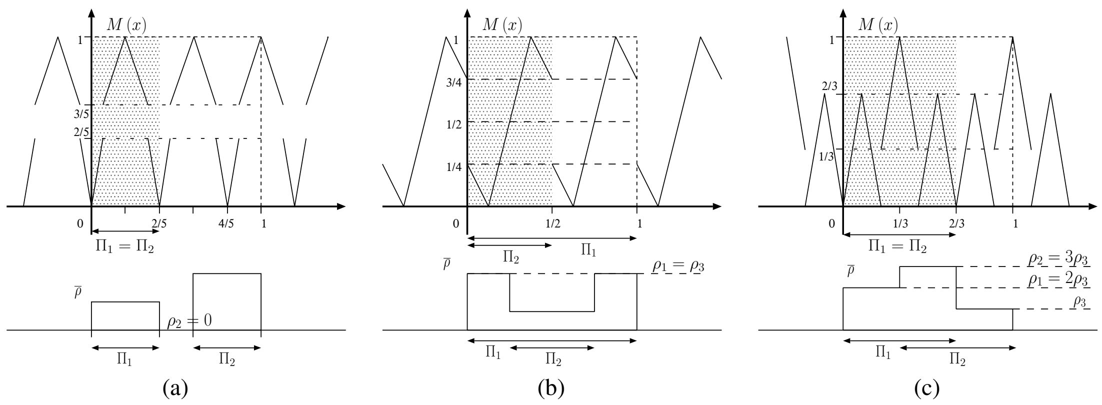

Fig. 2. Examples of 3-constant piece invariant density robust maps

include the PWAM maps family, that has already been widely employed in several practical application (see [7][11] and references therein). Our main result is stated in the following

Theorem 1. *Let $\varphi : I \to I$ be a chaotic map with a piecewise constant invariant density that can be expressed as*

$$\overline{\rho} = \sum_{k=1}^{m} \beta_k \chi_{\left[I_k^-,\, I_k^+\right]}(x) \tag{5}$$

where, for $k = 1, \ldots, m$, $\left[I_k^-, I_k^+\right]$ is an arbitrary partition of I and β_k are suitable coefficients. The extended map φ_E is noise robust in N if and only if

$$\sum_{k=1}^{m+1} (\beta_k - \beta_{k-1}) \int_{[0,\,\varepsilon]} \delta\left(\varphi_E\left(I_k^- + \xi\right) - x\right)\, d\xi = 0,\ \forall \varepsilon \in N \tag{6}$$

where $\beta_0 = \beta_{m+1} = 0$ and $I_{m+1}^- = I_m^+$.

Proof. From (5) we can write

$$\Delta\overline{\rho}^{\varepsilon}(x) = \sum_{k=1}^{m+1} (\beta_k - \beta_{k-1})\, \chi_{\left[I_k^-,\, I_k^- + \varepsilon\right]}(x)$$

where we have assumed that $\varepsilon > 0$ (the case $\varepsilon < 0$ can be treated similarly). Hence, (4) can be rewritten

$$\int_J \delta\left(\varphi_E(\xi) - x\right) \sum_{k=1}^{m+1} (\beta_k - \beta_{k-1})\, \chi_{\left[I_k^-,\, I_k^- + \varepsilon\right]}(\xi)\, d\xi =$$
$$\sum_{k=1}^{m+1} (\beta_k - \beta_{k-1}) \int_{\left[I_k^-,\, I_k^- + \varepsilon\right]} \delta\left(\varphi_E(\xi) - x\right)\, d\xi = 0$$

which yields to (6) with a simple change of variable. □

Let us briefly comment on the significance of the above theorem. Though a necessary condition simpler than (6) in the general case cannot be found, a sufficient condition can easily be obtained. If $\beta_k = \beta_{k-1}$ for some k, the corresponding integral terms in the sum of (6) gives no contribution. Note that this correspond to compute (5) with respect to the coarsest partition, i.e. the partition with the smallest number of elements. Being the simplest case, we will make such an assumption from now on. Then, supposing that $\beta_k \neq \beta_{k-1}\ \forall k$, if exist p, q for which φ_E is a periodic function with period $\Pi = I_p^- - I_q^-$, i.e. $\varphi_E\left(I_p^- + x\right) = \varphi_E\left(I_q^- + x\right)$, the complessive contribute of the two corrispondig integral terms in (6) will vanish if $\beta_p - \beta_{p-1} = \beta_{q-1} - \beta_q$. More generally, if we can group all integral terms of (6) in sets which vanish through map periodicity and coefficients' compensation, (6) is verified.

The simplest possible case is when all integral terms coincide, which yields to

$$\begin{cases} \varphi\left(I_p^- + x\right) = \varphi\left(I_q^- + x\right)\ , \quad \forall p, q = 1 \ldots m \\ \sum_{k=1}^{m+1} (\beta_k - \beta_{k-1}) = 0 \end{cases}$$

The second equation is obviously an identity, while the first one requires that the measure of all intervals $\left[I_p^-, I_q^-\right]$ must be a period for φ_E. If relations exist between the β_k, such a strong periodicity constraint can be be relaxed. Although it is possible to enumerate all possible cases (i.e. to take into account all the possible combinations of the integral terms in (6)), a complete treatment is out of the scope of this work. We will therefore limit ourselves to the simple examples presented in the following section.

4. SOME EXAMPLES OF ROBUST MAPS

As the simplest example, consider a map with uniform invariant density in $I = \left[I^-, I^+\right]$, such as the Bernoulli shift considered in section 2. In this case $\overline{\rho}(x) = \beta\chi_{\left[I^-,\, I^+\right]}(x)$ and condition (6) can be expressed as

$$\beta \int_{[0,\,\varepsilon]} \left[\delta\left(\varphi_E\left(I^- + \xi\right) - x\right) - \delta\left(\varphi_E\left(I^+ + \xi\right) - x\right)\right]\, d\xi = 0$$

for all $\varepsilon \in N$. In this simple case it can be seen that requiring map periodicity with $\Pi = I^+ - I^-$ is a necessary and sufficient condition for noise robustness. The proposed robust extension of the Bernoulli map satisfies this constraint.

For a map with an invariant density like the W-map in figure 1 (b), this method requires a periodicity of $\Pi = 1/2$, and it is not suitable for the W-map since not even the original map possesses this periodicity.

For a more complex example let us refer to a map with 3-constant pieces invariant density:

$$\overline{\rho}(x) = \beta_1\chi_{\left[I_1^-,\, I_1^+\right]}(x) + \beta_2\chi_{\left[I_2^-,\, I_2^+\right]}(x) + \beta_3\chi_{\left[I_3^-,\, I_3^+\right]}(x)$$

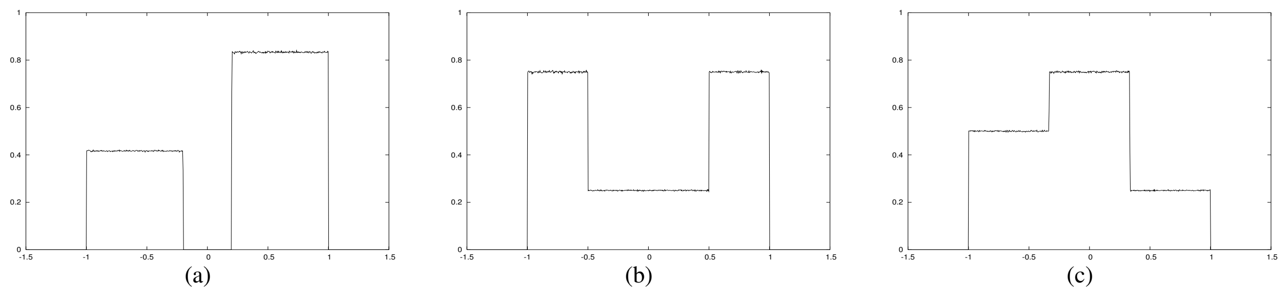

Fig. 3. Simulation of maps of figure 2, where $I = [-1,\ 1]$ and noise is uniformly distributed in $N = \left[-\frac{1}{2},\ \frac{1}{2}\right]$

so that the robustness condition (6) becomes

$$\int_{[0,\,\varepsilon]} \Big[\beta_1 \delta\left(\varphi_E\left(I_1^- + \xi\right) - x\right) + (\beta_2 - \beta_1)\,\delta\left(\varphi_E\left(I_2^- + \xi\right) - x\right) + \\ + (\beta_3 - \beta_2)\,\delta\left(\varphi_E\left(I_3^- + \xi\right) - x\right) - \beta_3 \delta\left(\varphi_E\left(I_4^- + \xi\right) - x\right)\Big]\, d\xi = 0$$

We have four possible solutions

$$i) \qquad \begin{cases} \varphi_E\left(I_1^- + \xi\right) = \varphi_E\left(I_2^- + \xi\right) \\ \varphi_E\left(I_3^- + \xi\right) = \varphi_E\left(I_4^- + \xi\right) \\ \beta_2 = 0 \end{cases}$$

$$ii) \qquad \begin{cases} \varphi_E\left(I_1^- + \xi\right) = \varphi_E\left(I_4^- + \xi\right) \\ \varphi_E\left(I_2^- + \xi\right) = \varphi_E\left(I_3^- + \xi\right) \\ \beta_1 - \beta_3 = 0 \end{cases}$$

$$iii) \qquad \begin{cases} \varphi_E\left(I_1^- + \xi\right) = \varphi_E\left(I_3^- + \xi\right) \\ \varphi_E\left(I_2^- + \xi\right) = \varphi_E\left(I_4^- + \xi\right) \\ \beta_1 - \beta_2 + \beta_3 = 0 \end{cases}$$

$$iv) \qquad \varphi_E\left(I_1^- + \xi\right) = \varphi_E\left(I_2^- + \xi\right) = \varphi_E\left(I_3^- + \xi\right) = \varphi_E\left(I_4^- + \xi\right)$$

Case $iv)$ correspond to the absence of particular relations between the coefficients β_k and it give rise to the most restrictive periodicity constraints on φ_E. The other three cases introduce constraints among the coefficients and require weaker periodicity conditions. An example of maps satisfying the robustness condition for the cases $i) - iii)$ is reported in figure 2 where the respective invariant densities in the unperturbed case are also reported. Figure 3 shows the corresponding invariant pdf computed trough numerical simulation. As can be seen all maps are, as expected, noise robust.

As a final remark, it is interesting to notice that the four case are not mutually exclusive. We already noticed that case $iv)$ is compatible with all other cases. Additionally, maps exist with an invariant density where both $\beta_1 - \beta_3 = 0$ and $\beta_1 - \beta_2 + \beta_3 = 0$. For these maps, both $ii)$ and $iii)$ are admissible and can produce different robustness constraints on the map structure. In figure 4 is represented an example of this. The two maps present the same invariant density, and both are noise robust, but they have a completely different periodicity.

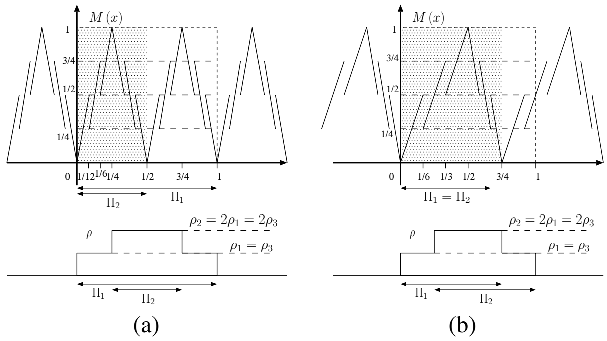

Fig. 4. Possible cases are not mutually exclusive

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