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Detecting metrologically useful asymmetry and entanglement by a few local measurements

Chao Zhang,1,2 Benjamin Yadim,3 Zhi-Bo Hou,1,2 Huan Cao,1,2 Bi-Heng Liu,1,2 Yun-Feng Huang,1,2,4 Reevu Maity,3 Vlatko Vedral,1,3,4 Chuan-Feng Li1,2,5 Guang-Can Guo1,2 and Davide Girolami1,3,4

1Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei, 230026, China
2Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, 230026, People’s Republic of China
3Department of Atomic and Laser Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom
4Centre for Quantum Technologies, National University of Singapore, 117543, Singapore

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Important properties of a quantum system are not directly measurable, but they can be disclosed by how fast the system changes under controlled perturbations. In particular, asymmetry and entanglement can be verified by reconstructing the state of a quantum system. Yet, this usually requires experimental and computational resources which increase exponentially with the system size. Here we show how to detect metrologically useful asymmetry and entanglement by a limited number of measurements. This is achieved by studying how they affect the speed of evolution of a system under a unitary transformation. We show that the speed of multipartite systems can be evaluated by measuring a set of local observables, providing exponential advantage with respect to state tomography. Indeed, the presented method requires neither the knowledge of the state and the parameter-encoding Hamiltonian nor global measurements performed on all the constituent subsystems. We implement the detection scheme in an all-optical experiment.

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I. INTRODUCTION

Quantum coherence and entanglement can generate non-classical speedup in information processing [1]. Yet, their experimental verification is challenging. Being not directly observable, their detection usually implies reconstructing the full state of the system, which requires a number of measurements growing exponentially with the system size [2]. Also, verifying their presence is necessary, but not sufficient to guarantee a computational advantage.

Here we show how to detect useful coherence and entanglement in systems of arbitrary dimension by a limited sequence of measurements. We propose an experimentally friendly measure of the speed of a quantum system, i.e., how fast its state changes under a generic channel, which for n-qubit systems is a function of a linearly scaling \[ O(n) \] number of observables. The speed of a quantum system determines its computational power [3–6]. Quantum speed limits of open systems also provide information about the environment structure [7–9], helping develop efficient control strategies [10–13], and investigate phase transitions of condensed matter systems [14,15]. We prove a quantitative link between our speed measure, when undertaking a unitary dynamics, and metrological quantum resources. In Sec. II, we relate speed to asymmetry, i.e., the coherence with respect to a Hamiltonian eigenbasis. Asymmetry underpins the usefulness of a probe to phase estimation and reference frame alignment [16–19]. Moreover, a superlinear scaling of the speed of multipartite systems certifies an advantage in metrology powered by phase estimation and reference frame alignment [16–19].

II. RELATING ASYMMETRY TO OBSERVABLES

The sensitivity of a quantum system to a quantum operation described by a parametrized channel \( \Phi, [1] \), where \( t \) is the time, is determined by how fast its state \( \rho_t := \Phi(\rho_0) \) evolves. We quantify the system speed over an interval \( 0 \leq t \leq \tau \) by the average rate of change of the state, which is given by mean values of quantum operators \( \langle \sigma \rangle_{\rho} = \text{Tr}(\rho \sigma) \):

\[
s_t(\rho_t) := \frac{||\rho_t - \rho_0||_2}{\tau} = \frac{\langle (\rho_t)_{\rho_0} + (\rho_0)_{\rho_0} - 2(\rho_t)_{\rho_0} \rangle^{1/2}}{(\tau)},
\]

where the Euclidean distance is employed. Measuring the swap operator on two system copies is sufficient to quantify state overlaps, \( \langle \sigma \rangle_{\rho} = \langle V(\phi_1) \otimes \phi_2) | V(\phi_1) \otimes \phi_2) \rangle = |\phi_2) \otimes |\phi_1) \rangle \). The global swap is the product of local swaps, \( V_S = \otimes_{i=1}^{n} V_S \). Then, for n-qubit systems \( S = \{S_i, i = 1, \ldots, n\} \), a state overlap \( \langle \sigma \rangle_{\rho} \) is obtained by evaluating \( O(n) \) observables, one for each pair of subsystem \( S_i \) copies [20–22]. Each local swap can be recast in terms of projections on the Bell singlet \( V_S = I_x - 2T \Psi^0_{S_1} - \Psi^0_{S_2} = \langle \Psi^- | \Psi^- \rangle = (1/\sqrt{2})(|01) - |10)). A standard routine of quantum information processing, e.g., in bosonic lattices. Bell state projections.
are implemented by $n$ beam splitters interfering each pair of $S_i$ copies, and coincidence detection on the correlated pairs. Hence the speed of an $n$-qubit system is evaluated by networks whose size scales linearly with the number of subsystems, employing $O(n)$ two-qubit gates and detectors. Note that tomography demands to prepare $O(2^n)$ system copies and perform a measurement on each of them [2]. It is also possible to extract the swap value by single qubit interferometry [23–25]. The two copies of the system are correlated with an ancillary qubit by a controlled-swap gate.

The mean value of the swap is then encoded in the ancilla polarization. Yet, the implementation of a controlled-swap gate is currently a serious challenge [26].

Crucial properties of quantum systems can be determined by measuring the speed defined in Eq. (1), without further data. Performing a quantum computation $U_i \rho U_i^\dagger, U_i = e^{-iH_i}$, relies on the coherences in the Hamiltonian $H$ eigenbasis, a property called $[U(1)]$ asymmetry [16–19]. In fact, incoherent states in such a basis do not evolve. Asymmetry is operationally defined as the system ability to break a symmetry generated by the Hamiltonian. Asymmetry measures are defined as nonincreasing functions in symmetry-preserving dynamics, which are modeled by transformations $\Phi$ commuting with the Hamiltonian evolution, $[\Phi, U_i] = 0$.

Experimentally measuring coherence, and in particular asymmetry, is hard [27,28]. One cannot discriminate with certainty coherent states from incoherent mixtures, without full state reconstruction. We show how to evaluate the asymmetry of a system by its speed (full details and proofs in Sec. V).

To quantify the sensitivity of a probe state of a system by its speed (full details and proofs in Sec. V).

Note that the SLDF is one of the many quantum extensions to the unitary transformation, $[\Phi, U_i]$ asymmetry. Asymmetry measures are defined as nonincreasing functions in symmetry-preserving dynamics, which are modeled by transformations $\Phi$ commuting with the Hamiltonian evolution, $[\Phi, U_i] = 0$.

The SLDF is an appealing strategy to verify an advantage given by entanglement asymptotically enables up to a quadratic improvement $I_{\Phi}(\rho, H_n) = O(n^2), n \to \infty$. Specifically, with the adopted normalization, the relation $I_{\Phi}(\rho, H_n) > n/4$, i.e., superlinear asymmetry with respect to an additive observable, witnesses entanglement [32]. Given Eq. (4), entanglement-enhanced precision in estimating a phase shift $\tau$ is verified if

$$I_{\Phi}(\rho, H_n) > n/4. \tag{5}$$

The overlap detection network for $n$-qubit systems and additive Hamiltonians is depicted in Fig. 1. Evaluating the SLDF is an appealing strategy to verify an advantage given by entanglement, rather than just detecting quantum correlations [22,33–38]. The SLDF of thermal states can be extracted by measuring the system dynamic susceptibility [39], while lower bounds are obtained by two-time detections of a global observable [40,41]. Also, collective observables can witness entanglement in highly symmetric states [42]. Our proposal has two peculiar advantages. First, it is applicable to any probe state $\rho$ without a priori information and assumptions, e.g., invariance under permutation of the subsystems. Second, only local pairwise interactions and detections are needed. This means that distant laboratories can verify quantum speedup due to entanglement in a shared system $S$ by local operations and classical communication [1], providing each laboratory with two copies of a subsystem $S_i$. Note that quadratic speed

$$S_i(\rho, H_n) > n/4.$$
in Eq. (4), is extracted by
generated by the Hamiltonians
Each pair of subsystem $S_i$ copies, in the state $\rho_h^i \otimes \rho_h^2$, enters a two-arm channel (blue and green). The unitaries $U_{\pi/2} = e^{-i h_{\pi/2}}$ are applied to the second copy of each pair. Leaving both copies unperturbed, the network measures the state purity. The measurement apparatus (red) interferes each pair of subsystem copies by $O(n)$ beam splitter gates $U_{BS}$ [20]. The overlap, and therefore the speed function in Eq. (4), is extracted by $O(n)$ local detections.

scaling certifies the probe optimization, $S_e(\rho, H_n) = O(n^2) \Rightarrow T_e(\rho, H_n) = O(n^2)$.

IV. EXPERIMENTAL ASYMMETRY AND ENTANGLEMENT DETECTION

A. Implementation

We experimentally extract a lower bound to metrologically useful asymmetry and entanglement of a two-qubit system $AB$ in an optical setup, by measuring its speed during a unitary evolution. While employing state tomography would require fifteen measurements, we verify that the proposed protocol needs six. The system is prepared in a mixture of Bell states, $\rho_{p,AB} = p\langle\phi^+\vert\langle\phi^+| + (1-p)\langle\phi^-\vert\langle\phi^-|, \langle\phi^\pm\vert = 1/\sqrt{2}(\vert0\rangle \pm \vert1\rangle)$, $p \in [0,1]$. We implement transformations generated by the Hamiltonians $H_2 = \sum_{i=1}^n h_i$, $h = \alpha, \sigma, \tau$, where $\alpha, \gamma, \tau$ are the spin-1/2 Pauli matrices, for equally stepped values of the mixing parameter, $p = 0.01, 0.1, 0.2, \ldots, 0.91$, over an interval $\tau = \pi/6$. The squared speed function $S_{\pi/6}(\rho, H_2)$ is evaluated from purity and overlap measurements.

Each run of the experiment implements the scheme in Fig. 2. We prepare two copies (Copy 1,2) of a maximally entangled two-qubit state $\vert\phi^+\rangle = \frac{1}{\sqrt{2}}(\vert HH\rangle + \vert VV\rangle)$, where $H, V$ label horizontal and vertical photon polarizations, from three spontaneous parametric down-conversion sources (SPDC Source 1,2,3). They are generated by ultrafast 90 mW pump pulses from a mode-locked Ti:sapphire laser, with a central wavelength of 780 nm, a pulse duration of 140 fs, and a repetition rate of 76 MHz. Copy 1 (photons 1,2) is obtained from Source 1, by employing a sandwichlike beta-barium borate (BBO) crystal [43]. Copy 2 is prepared from Source 2,3. Two photon pairs (photons 3−6) are generated via single BBO crystals (beamlike type-II phase matching). By detecting photons 5,6, a product state encoded in photons 3,4 is triggered. Photons 3−4 polarizations are rotated via half-wave plates (HWPs). They are then interfered by a polarizing beam splitter (PBS) for parity check measurements. We then simulate the preparation of the state $\rho_\phi^1 \otimes \rho_\phi^2 = p^2 \rho_{12}^\phi \rho_\phi^2 + p(1-p)\rho_\phi^1 \rho_{12}^\phi \rho_\phi^2 + (1-p)^2 \rho_{12}^\phi \rho_\phi^2$, $\sigma^\phi_{12} = [\sigma^\phi_{1}, B_1] \otimes [\sigma^\phi_{2}, B_2]$. Classical mixing is obtained by applying quarter-wave plates (QWP1,QWP2) to each system copy. A 90° rotated QWP swaps the Bell states, $\vert\phi^\pm\rangle \rightarrow \vert\phi^\mp\rangle$, generating a $\pi$ phase shift between $H, V$ polarizations. The four terms of the mixture are obtained in separate runs by engineering the rotation sequences (QWP1,QWP2) = [0°,90°],(0°,90°),(90°,0°),(90°,90°)], with a duration proportional to $(p^2, p(1-p), p(1-p), (1-p)^2)$, respectively. The collected data from the four cases are then identical to the ones obtained from direct preparation of the mixture.

We quantify the speed by measuring the purity $S_{\pi/6}(\rho_\phi^1, \rho_\phi^2)$ and the overlap $S_{\pi/6}(\rho_\phi^1, \rho_\phi^2)$. The unitary gate $U_{\pi/6} = U_{\pi/6} = U_{\pi/6,A_1} \otimes U_{\pi/6,B_1} = e^{-i h_{\pi/6} U_{\pi/6}}$ is applied to the second system copy by a sequence of one HWP sandwiched by two QWPs. The sequences of gates implementing each
Table I. Angles of the wave plates implementing the unitary gates.

<table>
<thead>
<tr>
<th>Angles</th>
<th>$I$</th>
<th>$U_x$</th>
<th>$U_y$</th>
<th>$U_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$\frac{\pi}{4}$</td>
<td>$-\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{3}$</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{6}$</td>
</tr>
</tbody>
</table>

Hamiltonian are obtained as follows. Single qubit unitary gates implement SU(2) group transformations. We parametrize the rotations by the Euler angles $(\xi, \eta, \zeta)$:

$$u(\xi, \eta, \zeta) := \exp\left(-\frac{i}{2} \xi \sigma_z\right) \exp\left(-\frac{i}{2} \eta \sigma_y\right) \exp\left(-\frac{i}{2} \zeta \sigma_z\right).$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. One can engineer arbitrary single qubit gates by a $\theta$-rotated HWP implementing the transformation $H_\theta$, sandwiched by two rotated QWPs (transformations $Q_\theta$):

$$u(\xi, \eta, \zeta) = Q_\theta H_\theta Q_\theta,$$

where $\theta_{1,2,3}$ are the rotation angles to apply to each plate [44]. In particular, any unitary transformation is prepared by a gate sequence of the form

$$\theta_1 = \frac{\pi}{4} - \frac{\xi}{2} \mod \pi,$$

$$\theta_2 = -\frac{\pi}{4} + \frac{(\xi + \eta - \zeta)}{4} \mod \pi,$$

$$\theta_3 = \frac{\pi}{4} + \frac{\xi}{2} \mod \pi.$$

The phase shift angles characterizing the Hamiltonian evolutions studied in our experiment are shown in Table I.

The mean value of the swap operator is extracted by local and bilocal projections on the Bell singlet: $V_{12} = I_{12} - 2I_1 \otimes I_2 - 2I_1 \otimes I_2^\dagger + 4I_1 \otimes I_2^\dagger \phi^-$. That is, three projections are required for evaluating purity and overlap, respectively. Note that for $n$ qubits $O(2^n)$ projections are required, still having exponential advantage with respect to full tomography. The projections are obtained via Bell state measurement (BSM) schemes applied to each subsystem pair. The BSMs consist of PBSs, HWPs, and photon detectors. We place a 45° HWP in the input ports of the PBS corresponding to the $A_1, B_1$ subsystems to deterministically project into the Bell singlet [45]. All the photons pass through single mode fibers for spatial mode selection. For spectral mode selection, photons 1–4 (5,6) pass through 3 nm (8 nm) bandwidth filters.

The theoretical values to be extracted are given in Table II. The experimental results are reported in Fig. 3. For each Hamiltonian, we reconstruct the speed function $S_{\pi/\phi}(\rho_H, H_2)$ from purity and overlap measurements, and compare it against the values obtained by state tomography of the two system copies. By Eq. (5), entanglement is detected by superlinear speed scaling $S_{\pi/\phi}(\rho_H, H_2) \geq 1/2$. We observe that speed values above the threshold detect entanglement yielding nonclassical precision in phase estimation, not just nonseparability of the density matrix (the state $\rho_H$ is entangled for $p \neq 1/2$).

B. Diagnostic of the experimental setup

1. Error sources

We discuss the efficiency of the experimental setup. The four photons interfering into the BSMs form a closed-loop network (Fig. 2). This poses the problem to rule out the case of BSMs measuring two photon pairs emitted by a single SPDC source [46]. We guarantee that the two system copies from different sources by preparing Copy 2 from two photon pair sources by postselection. Single source double down conversion can also occur because of high-order emission noise, which has been minimized by setting a low pump power. The coincidences have been counted by a multichannel unit, with a 50 h rate for about 6 h in each experiment run. Here the main error source is the imperfection of the three Hang-Ou-Mandel interferometers (one for the PBS and each BSM), which have a visibility of 0.91. This is due to the temporal distinguishability between the interfering photons, determined by the pulse duration. The 3 nm and 8 nm narrow-band filters were placed in front of each detector to increase the photon overlap.

2. Tomography of the input Bell state copies

We perform full state reconstruction of the two copies (Copy 1,2) of the Bell states $\phi^\pm_{12}$ obtained by SPDC sources. The fidelity of the input states are respectively 0.9889 ($\phi^+_{12}$), 0.9901 ($\phi^-_{12}$), 0.9279 ($\phi^+_{12}$), and 0.9319 ($\phi^-_{12}$). We recall that Copy 1 (subsystems $A_1, B_1$) is generated by the sandwichlike Source 1 (photons 1,2), while Copy 2 ($A_2, B_2$) is triggered by Sources 2,3 via parity check gate and postselection applied to two product states (photons 3–6). The counting rate for the Copy 1 photon pair is 32 000 s, while for the four photons of Copy 2 is 110 s. We use the maximum likelihood estimation.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\sigma_i$</th>
<th>$\sigma_y$</th>
<th>$\sigma_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1(\rho_H, H_2)$</td>
<td>$p$</td>
<td>$(1 - p)</td>
<td>(1 - 2p)^2$</td>
</tr>
<tr>
<td>$S_1(\rho_H, H_2)$</td>
<td>$(p \sin \tau/4\tau)^2$</td>
<td>$[(1 - p) \sin \tau/4\tau]^2$</td>
<td>$[(1 - 2p) \sin \tau/4\tau]^2$</td>
</tr>
<tr>
<td>$I_1(\rho_H, H_2) &gt; 0.5$</td>
<td>$p &gt; 0.5$</td>
<td>$p &lt; 0.5$</td>
<td>$p &lt; 0.147, p &gt; 0.853$</td>
</tr>
<tr>
<td>$S_1(\rho_H, H_2) &gt; 0.5$</td>
<td>$p &gt; 0.741$</td>
<td>$p &lt; 0.259$</td>
<td>$p &lt; 0.129, p &gt; 0.870$</td>
</tr>
</tbody>
</table>
The interferometry visibility in our setting is 0.91. Two BSM (1,2) are required to evaluate purity and overlap by measurements. The estimated Bell state projections \( \rho_p \) of the form 
\[
\phi_i^\pm = \begin{pmatrix}
0.5146 + 0.0000i \\
-0.0158 + 0.0031i \\
0.0058 - 0.0029i \\
0.4923 - 0.0071i
\end{pmatrix}, \\
\begin{pmatrix}
-0.0158 - 0.0031i \\
0.0039 + 0.0000i \\
-0.0003 - 0.0026i \\
-0.0173 + 0.0021i
\end{pmatrix}, \\
\begin{pmatrix}
0.0058 + 0.0029i \\
-0.0003 + 0.0026i \\
0.0029 + 0.0000i \\
0.0029 - 0.0043i
\end{pmatrix}, \\
\begin{pmatrix}
-0.0058 - 0.0029i \\
0.0003 - 0.0026i \\
0.0029 - 0.0000i \\
0.4787 + 0.0000i
\end{pmatrix}, \\
\begin{pmatrix}
0.5072 + 0.0000i \\
-0.0065 + 0.0008i \\
0.0007 + 0.0021i \\
-0.0056 + 0.0034i
\end{pmatrix}, \\
\begin{pmatrix}
-0.0065 - 0.0008i \\
0.0030 + 0.0000i \\
0.0006 + 0.0016i \\
0.4869 + 0.0000i
\end{pmatrix}, \\
\begin{pmatrix}
0.4931 + 0.0090i \\
-0.0065 - 0.0016i \\
0.0056 - 0.0034i \\
0.4869 + 0.0000i
\end{pmatrix}, \\
\begin{pmatrix}
0.4881 + 0.0000i \\
-0.0108 + 0.0041i \\
0.0063 + 0.0091i \\
0.4486 + 0.0509i
\end{pmatrix}, \\
\begin{pmatrix}
-0.0108 - 0.0041i \\
0.0216 + 0.0000i \\
-0.0029 - 0.0066i \\
-0.0140 - 0.0068i
\end{pmatrix}, \\
\begin{pmatrix}
0.0063 - 0.0091i \\
-0.0029 + 0.0066i \\
0.0198 + 0.0000i \\
0.0044 - 0.0073i
\end{pmatrix}, \\
\begin{pmatrix}
0.4486 - 0.0509i \\
-0.0140 + 0.0068i \\
0.0044 + 0.0073i \\
0.4706 + 0.0000i
\end{pmatrix}, \\
\begin{pmatrix}
0.4911 + 0.0000i \\
0.0041 - 0.0184i \\
0.0058 + 0.0075i \\
-0.4502 - 0.0462i
\end{pmatrix}, \\
\begin{pmatrix}
0.0041 + 0.0184i \\
0.0155 + 0.0000i \\
0.0005 + 0.0080i \\
0.0041 - 0.0089i
\end{pmatrix}, \\
\begin{pmatrix}
0.0058 - 0.0075i \\
0.0005 - 0.0080i \\
0.0209 + 0.0000i \\
-0.0085 + 0.0182i
\end{pmatrix}, \\
\begin{pmatrix}
-0.4502 + 0.0462i \\
0.0041 + 0.0089i \\
-0.0085 - 0.0182i \\
0.4724 + 0.0000i
\end{pmatrix},
\]

The error bars are determined by Monte Carlo simulation with Poisson-distributed error (1000 samples for each point). For comparison, the two green dashed lines depict the speed function computed from the reconstructed states of copy 1,2 (the density matrices are reported in the main text), respectively. Superlinear scaling due to entanglement is detected for values above the horizontal, black dotted line.

### 3. Tomography of the Bell state measurements

We analyze the efficiency of the measurement apparatus. A BSM consists of Hang-Ou-Mandel (HOM) interferometers and coincidence counts. The BSM is only partially deterministic, discriminating two of the four Bell states \((|\phi^+\rangle \text{ or } |\psi^+\rangle)\) at a time. The interferometry visibility in our setting is 0.91. Two BSM (1,2) are required to evaluate purity and overlap by measurements on two system copies. This requires the indistinguishability of the four interfering photons 1–4, including their arriving time, spatial mode, and frequency. As explained, our three source scheme ensures that, postselecting sixfold coincidences, each detected photon pair is emitted by a different source. We test our measurement hardware by performing BSM tomography. The probe states are chosen of the form \(|\{H,V,D,A,R,L\}\rangle \otimes |\{H,V,D,A,R,L\}\rangle\), where the labels identify the following photon polarizations: horizontal \(H\), vertical \(V\), diagonal \(D = (H + V)/\sqrt{2}\), antidiagonal \(A = (H - V)/\sqrt{2}\), right circular \(R = (H + iV)/\sqrt{2}\), and left circular \(L = (H - iV)/\sqrt{2}\). The measurement results for all the possible outcomes are recorded accordingly. An iterative maximum likelihood estimation algorithm yields the estimation of what projection is performed in each run [47]. The average fidelities of BSM1 and BSM2 are 0.9389 ± 0.0030 and 0.9360 ± 0.0034, being the standard deviation calculated from 100 runs, by assuming Poisson statistics. The estimated Bell state projections \(\Pi_{1,2}^{(12)} = |1⟩⟨A_1(R_1)A_2(R_2)|\), \(x = |\phi^\pm, \psi^\pm\rangle\), reconstructed from
BSM1 (detecting on subsystems $A_1 A_2$) and BSM2 (detecting on $B_1 B_2$), are given by

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.5142 & 0.0096 - 0.0012i & 0.0043 - 0.0055i & 0.4443 - 0.0088i \\
0.0096 + 0.0102i & 0.0024 & -0.0005 + 0.0007i & -0.0037 + 0.0018i \\
0.0043 + 0.0055i & -0.0005 - 0.0007i & 0.0052 & 0.0003 + 0.0110i \\
0.4443 + 0.0088i & -0.0037 - 0.0018i & 0.0003 - 0.0110i & 0.4863 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.4816 & -0.0088 + 0.0057i & -0.0081 + 0.0039i & -0.4481 + 0.0048i \\
-0.0088 - 0.0057i & 0.0031 & 0.0013 + 0.0019i & 0.0136 - 0.0096i \\
-0.0081 - 0.0039i & 0.0013 - 0.0019i & 0.0018 & -0.0001 - 0.0055i \\
-0.4481 - 0.0048i & 0.0136 + 0.0096i & -0.0001 + 0.0055i & 0.5033 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.0014 & -0.0000 - 0.0083i & 0.0100 - 0.0010i & 0.0006 + 0.0006i \\
-0.0000 + 0.0083i & 0.4954 & 0.4382 - 0.0059i & -0.0136 + 0.0147i \\
0.0100 + 0.0010i & 0.4382 + 0.0059i & 0.5059 & -0.0057 + 0.0143i \\
0.0006 - 0.0006i & -0.0136 - 0.0147i & -0.0057 - 0.0143i & 0.0014 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.0027 & -0.0008 + 0.0128i & -0.0062 + 0.0026i & 0.0032 + 0.0033i \\
-0.0008 - 0.0128i & 0.4991 & -0.4390 + 0.0033i & 0.0038 - 0.0068i \\
-0.0062 - 0.0026i & -0.4390 - 0.0033i & 0.4871 & 0.0054 - 0.0198i \\
0.0032 - 0.0033i & 0.0038 + 0.0068i & 0.0054 + 0.0198i & 0.0090 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.4893 & 0.0043 - 0.0223i & 0.0064 - 0.0182i & 0.4397 - 0.0667i \\
0.0043 + 0.0223i & 0.0017 & 0.0008 - 0.0004i & 0.0003 + 0.0159i \\
0.0064 + 0.0182i & 0.0008 + 0.0004i & 0.0012 & 0.0123 + 0.0107i \\
0.4397 + 0.0667i & 0.0003 - 0.0159i & 0.0123 - 0.0107i & 0.4942 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.5036 & 0.0050 - 0.0021i & -0.0015 + 0.0040i & -0.4413 + 0.0636i \\
0.0050 + 0.0021i & 0.0023 & -0.0011 + 0.0008i & 0.0091 - 0.0072i \\
-0.0015 - 0.0040i & -0.0011 - 0.0008i & 0.0011 & -0.0069 + 0.0007i \\
-0.4413 - 0.0636i & 0.0091 + 0.0072i & -0.0069 - 0.0007i & 0.4987 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.0032 & -0.0098 + 0.0070i & -0.0140 + 0.0192i & 0.0018 + 0.0016i \\
-0.0098 - 0.0070i & 0.4919 & 0.4375 + 0.0446i & -0.0101 - 0.0085i \\
-0.0140 - 0.0192i & 0.4375 - 0.0446i & 0.5012 & -0.0059 - 0.0061i \\
0.0018 - 0.0016i & -0.0101 + 0.0085i & -0.0059 + 0.0061i & 0.0050 \\
\end{pmatrix},
\]

\[
\Pi_i^{\pm} = \begin{pmatrix}
0.0039 & 0.0005 + 0.0173i & 0.0091 - 0.0049i & -0.0001 + 0.0014i \\
0.0005 - 0.0173i & 0.5041 & -0.4371 - 0.0451i & 0.0007 - 0.0002i \\
0.0091 + 0.0049i & -0.4371 + 0.0451i & 0.4965 & 0.0004 - 0.0052i \\
-0.0001 - 0.0014i & 0.0007 + 0.0002i & 0.0004 + 0.0052i & 0.0021 \\
\end{pmatrix}.
\]

V. THEORY BACKGROUND AND FULL PROOFS

A. Quantum Fisher information as measures of state sensitivity

Quantum information geometry studies quantum states and channels as geometric objects. The Hilbert space of a finite $d$-dimensional quantum system admits a Riemannian structure; thus it is possible to apply differential geometry concepts and tools to characterize quantum processes. For an introduction to the subject, see Refs. [48,49].

The information about a $d$-dimensional physical system is encoded in states represented by $d \times d$ complex Hermitian matrices $\rho \succeq 0$, $\text{Tr}(\rho) = 1$, $\rho = \rho^\dagger$, in the system Hilbert space $\mathcal{H}$. Each subset of rank $k$ states is a smooth manifold $\mathcal{M}^k(\mathcal{H})$ of dimension $2dk - k^2 - 1$ [50]. The set of all states $\mathcal{M}(\mathcal{H}) = \bigcup_{k=1}^d \mathcal{M}^k(\mathcal{H})$ forms a stratified manifold, where the stratification is induced by the rank $k$. The boundary of the manifold is given by the density matrices satisfying the condition $\det \rho = 0$.

State transformations are represented on $\mathcal{M}(\mathcal{H})$ as piecewise smooth curves $\rho : t \rightarrow \rho_t$, where $\rho_t$ represents the quantum state of the system at time $t \in \mathbb{R}$. By employing differential geometry techniques, it is possible to study the space of quantum states $\mathcal{M}(\mathcal{H})$ as a Riemannian structure.
The length of a path \( \rho_t, t \in [0, \tau] \), on the manifold is given by the integral of the line element

\[
 l_\rho = \int_0^\tau ds = \int_0^\tau \|d\rho_t\| \, dt, \tag{9}
\]

where the norm is induced by equipping \( \mathcal{M}(\mathcal{H}) \) with a symmetric, semipositive definite metric. The path length is invariant under monotone reparametrizations of the coordinate \( t \). The definition of a metric function yields the notion of distance \( d(\rho, \sigma) \) between two quantum states \( \rho, \sigma \). The choice of the metric is arbitrary. However, Morozov, Chentsov, and Petz identified a class of functions, the quantum Fisher information, which extends the contractivity of the classical Fisher-Rao metric under noisy operations to quantum information, which extends the contractivity of the class.

\[
 ||d_\rho(t)||^2_{\rho} = \sum_{i,j} (\rho f(i,j)) \frac{2(\rho f(i,j))}{(\rho f(j))(\rho f(i))} + \sum_{i<j} (\rho f(i,j,\lambda))(\rho f(i,j,\lambda)) = \sum_{i}(d_{\lambda_i}(t))^2/4\lambda_i(t)
\]

The length of a path \( \rho_t, t \in [0, \tau] \), on the manifold is given by the integral of the line element

\[
 l_\rho = \int_0^\tau ds = \int_0^\tau \|d\rho_t\| \, dt, \tag{9}
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\]

The dynamics of the quantum Fisher information for closed and open quantum systems has been studied in Ref. [7].

All such metrics reduce to the classical Fisher-Rao metric

\[
 |\langle i|f(\rho, H)|j\rangle|^2 = \frac{1}{4} |\langle i|f(\rho, H)|j\rangle|^2.
\]

For pure states, one has

\[
 |\langle i|f(\rho, H)|j\rangle|^2 = \frac{1}{4} \sum_{\lambda_i, \lambda_j} (\lambda_i - \lambda_j)^2 |\langle i|H|j\rangle|^2,
\]

where each term in the sum is taken to be zero whenever \( \lambda_i = \lambda_j \).

### B. Proofs of theoretical results

**1. Proof that any quantum Fisher information is an ensemble asymmetry monotone, extending the result in Eq. (3)**

We prove two preliminary results upon which the result will be demonstrated.

(i) For any set of states \( \rho_\mu \) and normalized probabilities \( p_\mu \), and an orthonormal set \( \{\ket{\mu}\} \),

\[
 \mathcal{I}_f \left( \sum_\mu p_\mu \rho_\mu \otimes \ket{\mu}\bra{\mu} \otimes I \right) = \sum_\mu p_\mu \mathcal{I}_f(\rho_\mu, H), \forall f.
\]
Let each $\rho_\mu$ have a spectral decomposition $\rho_\mu = \sum_i \lambda_{i,\mu} |\psi_{i,\mu}\rangle \langle \psi_{i,\mu}|$. Recalling Eq. (2), and defining $\lambda_{\mu,i} := p_\mu \lambda_{i,\mu}$, one has

$$\mathcal{I}_f \left( \sum_\mu p_\mu \rho_\mu \otimes |\mu\rangle \langle \mu|, H \otimes I \right) = \frac{1}{4} \sum_{\mu,\nu,i,j} \frac{(\lambda_{\mu,i} - \lambda_{\nu,j})^2}{\lambda_{\nu,j}} f(\lambda_{\mu,i}/\lambda_{\nu,j}) \left( |\psi_{\mu,i}(H \otimes I)\psi_{\nu,j}|^2 \right)$$

$$= \frac{1}{4} \sum_{\mu,i,j} \frac{(\lambda_{\mu,i} - \lambda_{\mu,i})^2}{\lambda_{\mu,i}} f(\lambda_{\mu,i}/\lambda_{\mu,i}) \left( |\psi_{\mu,i}|^2 \right)$$

$$= \frac{1}{4} \sum_{\mu,i,j} p_\mu^2 (\lambda_{\mu,i} - \lambda_{\mu,i})^2 f(\lambda_{\mu,i}/\lambda_{\mu,i}) \left( |\psi_{\mu,i}|^2 \right) = \sum_\mu p_\mu \mathcal{I}_f(p_\mu, H).$$

(ii) $\mathcal{I}_f(\rho, H)$ is convex in $\rho$. This follows from (i), by tracing out the ancillary system, as $\mathcal{I}_f$ is monotonically decreasing under partial trace:

$$\sum_\mu p_\mu \mathcal{I}_f(p_\mu, H) = \mathcal{I}_f \left( \sum_\mu p_\mu \rho_\mu \otimes |\mu\rangle \langle \mu| \right)$$

$$\geq \mathcal{I}_f \left( \sum_\mu p_\mu \rho_\mu, H \right).$$

We are now ready to prove deterministic monotonicity. Recall that a $U(1)$-covariant channel, i.e., a symmetric operation, $\Phi$ is defined to be such that $[\Phi, U_i] = 0$, where $U_i(\rho) := e^{-iHt} \rho e^{iHt}$. Noting that $-i[H, \rho] = d U_i(\rho)|_{t=0}$, we have $\mathcal{I}_f(\rho, H) = \frac{1}{2} \|d U_i(\rho)|_0\|^2$. The linearity of $\Phi$ and the monotonicity property then give

$$\|d U_i(\Phi(\rho))\|_f = \|d U_i(\Phi(U_i(\rho)))\|_f$$

$$= \|d U_i(U_i(\rho))\|_f \leq \|d U_i(\rho)\|_f,$$

so that $\mathcal{I}_f(\Phi(\rho), A) \leq \mathcal{I}_f(\rho, A), \forall f$.

To prove the ensemble monotonicity, we introduce a quantum instrument as a set of covariant maps $\{\Phi_\mu\}$ which are not necessarily trace preserving, while the sum $\sum_\mu \Phi_\mu$ is. For every quantum instrument, one can construct a trace-preserving operation by including in the output an ancilla that records which outcome was obtained via a set of orthonormal states $\{|\mu\rangle\}$, $\Phi'(\rho) := \sum_\mu \Phi_\mu(\rho) \otimes |\mu\rangle \langle \mu|$. Tracing out the ancilla results in the channel $\sum_\mu \Phi_\mu$. It is clear that $\Phi'$ is covariant whenever each of the $\Phi_\mu$ is. Writing $\Phi'(\rho) = \sum_\mu p_\mu \rho_\mu \otimes |\mu\rangle \langle \mu|$, result (i) and deterministic monotonicity imply

$$\sum_\mu p_\mu \mathcal{I}_f(\rho_\mu, H) = \mathcal{I}_f \left( \sum_\mu p_\mu \rho_\mu \otimes |\mu\rangle \langle \mu|, H \otimes I \right)$$

$$= \mathcal{I}_f(\Phi'(\rho), H \otimes I)$$

$$\leq \mathcal{I}_f(\rho, H).$$

2. Proof that the speed bounds any quantum Fisher information, generalizing Eq. (4)

It is possible to express the system speed in terms of the Hilbert-Schmidt distance $D_{HS}(\rho, \sigma) = \sqrt{\text{Tr}((\rho - \sigma)^2)}$ and the related norm,

$$S_r(\rho, H) = D_{HS}(\rho, U_r \rho U_r^\dagger)/(2\tau^2) = ||U_r \rho U_r^\dagger - \rho||_2^2/(2\tau^2).$$

The zero shift limit is given by

$$S_0(\rho, H) := \lim_{\tau \to 0} S_r(\rho, H) = -1/2 \text{Tr}(\rho H^2).$$

By expanding the quantity in terms of the state spectrum and eigenbasis, one has $S_0(\rho, H) = \sum_{\mu,\nu} (\lambda_\mu - \lambda_\nu)^2/2(\langle i|H|j\rangle)^2$.

We recall the norm inequality chain $f(0)/2 \|A\|_f \leq 1/4 |\langle i|A|j\rangle|_f$, $\forall f, A$, which, for unitary transformations $e^{-iHt} \rho e^{iHt}$, implies the topological equivalence of the quantum Fisher information:

$$2 f(0)\mathcal{I}_f(\rho, H) \leq \mathcal{I}_f(\rho, H) \leq \mathcal{I}_f(\rho, H), \forall f, \rho, H,$$

where $F$ labels the SLDF [53]. We note that its expansion for unitary transformations reads $\mathcal{I}_f(\rho, H) = \sum_{\mu,\nu} (\lambda_\mu - \lambda_\nu)^2/2(\langle i|H|j\rangle)^2$. Since $\lambda_\mu + \lambda_j \leq 1$, $\forall i, j$, it follows that

$$S_0(\rho, H) \leq \mathcal{I}_f(\rho, H), \forall \rho, H.$$

Any distance between two states is defined as the length of the shortest path between them. By recalling the von Neumann equation $\partial_t \rho_t = i[H, \rho_t]$, and integrating over the unitary evolution $U_t$, one obtains

$$D_{HS}(\rho, U_t \rho U_t^\dagger) \leq \int_{\rho=\rho_t} \|\partial_t \rho_t\|_2^2 dt$$

$$\leq \int_{\rho=\rho_t} \|\partial_t \rho_t\|_2^2 (-\text{Tr}(\rho H^2))^{1/2} dt$$

$$= \int_{\rho=\rho_t} \|\partial_t \rho_t\|_2^2 [2S_0(\rho, H)]^{1/2} dt$$

$$\leq 2S_0(\rho, H)^{1/2} \tau$$

$$\leq 2\mathcal{I}_f(\rho, H)^{1/2} \tau, \forall f.$$

Hence the bound is proven. The inequality is saturated for pure states in the limit $\tau \to 0$.

3. Bonus: Determining the scaling of the SLDF from speed measurements for pure states mixed with white noise

Suppose we are given the state

$$\rho_\epsilon = (1 - \epsilon)\rho + \epsilon I_d.$$
where $I_d$ is the identity of dimension $d$, while $\rho_{\epsilon}$ is an arbitrary pure state and $\epsilon$ is unknown. By convexity, one has
\[
I_F(\rho, H) \leq (1 - \epsilon)I_F(\langle \psi \rangle \langle \psi | , H) \\
\leq (1 - \epsilon)S_d(\langle \psi \rangle \langle \psi | , H), \quad \forall H,
\]
since $I_d/d$ is an incoherent state in any basis. By Eq. (2), few algebra steps give
\[
S_d(\rho_{\epsilon}, H) \leq I_F(\rho_{\epsilon}, H) \leq \sqrt{d - 1 \over d \operatorname{Tr}(\rho_{\epsilon}^2)} - 1 \leq S_d(\rho_{\epsilon}, H).
\]

By Taylor expansion about $\tau = 0$, one has $S_r(\rho, H) = S_0(\rho, H) + O(\tau^2), \forall \rho, H$. Thus measuring the speed function $S_r(\rho_{\epsilon})$ and the state purity determines both upper and lower bounds to the SDLF, and consequently to any quantum Fisher information, up to an experimentally controllable error due to the selected time shift.

VI. CONCLUSION

We showed how to extract quantitative bounds to metrologically useful asymmetry and entanglement in multipartite systems from a limited number of measurements, demonstrating the method in an all-optical experiment. The scalability of the scheme may make possible to certify quantum speedup in large scale registers [11,11,38], and to study critical properties of many-body systems [14,15,39], by limited laboratory resources. On this hand, we remark that we here compared our method with state tomography, as the two approaches share the common assumption that no a priori knowledge about the input state and the Hamiltonian is given. An interesting followup work would test the efficiency of our entanglement witness against two-time measurements of the classical Fisher information, when local measurements on the subsystems are only available. A further development would be to investigate macroscopic quantum effects via speed detection, as they have been linked to quadratic precision scaling in phase estimation [$I_F(\rho, H_n) = O(n^2)$] [20,31,40].

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