Graphs of q-exponentials and q-trigonometric functions

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Amelia Carolina Sparavigna
Politecnico di Torino

Abstract Here we will show the behavior of some of q-functions. In particular we plot the q-exponential and the q-trigonometric functions. Since these functions are not generally included as software routines, a Fortran program was necessary to give them.

Keywords q-calculus, q-exponential, q-trigonometric functions, Tsallis q-functions.

Introduction Frank Hilton Jackson (1870-1960) was an English clergyman and mathematician who systematically studied what today is known as the q-calculus. In particular, he studied some q-functions and the q-analogs of derivative and integral [1]. Actually, Jackson started his studies introducing the q-difference operator, an operator the origins of which can be tracked back to Eduard Heine and Leonhard Euler [2]. For this reason, the q-difference operator is also known as the Euler-Heine-Jackson operator [3]. Also Carl Friedrich Gauss was involved in the q-calculus since he proposed some relations such as the q-analog of binomials [4].
As told in [5], the q-calculus has various "dialects," and for this reason it is known as "quantum calculus," "time-scale calculus" or "calculus of partitions" [5]. Here we will use the notation given in the book by Kac and Cheung [6]. This book is discussing two modified versions of a quantum calculus, defined as the "h-calculus" and the "q-calculus." The letter "h" indicates the Planck's constant and the letter "q" stands for quantum. Let us note that these two versions of calculus reduce to Newton's calculus in the limit where $h \to 0$ or $q \to 1$.
Here, in the framework of the q-calculus of [6], we will show the behaviors of some of the q-functions. In particular we will plot the q-exponential and of the q-trigonometric functions. We will see that, in the limit where $q \to 1$, these functions become the functions commonly used. Since these q-functions are not generally included as software routines, we prepared a Fortran program for giving them. Before showing the results, let us discuss shortly the q-difference operator and the related q-derivative.

The quantum difference operator. In the q-calculus, the q-difference is simply given by:

$$d_q f = f(qx) - f(x)$$

From this difference, the q-derivative is:

$$D_q f = \frac{f(qx) - f(x)}{qx - x}$$

The q-derivative reduces to the Newton's derivative as $q \to 1$. We have also the h-derivative:

$$D_h f = \frac{f(x+h) - f(x)}{h}$$

In the limit as $h \to 0$, this reduces to the usual derivative.

Taking $h=(q-1)x$, we may see that:

$$D_h f=\frac{f(x+h)-f(x)}{h}=\frac{f(x+(q-1)x)-f(x)}{qx-x}=\frac{f(qx)-f(x)}{qx-x}=D_q f$$

From the two formulas for $D_q f$ and $D_h f$, we see that $D_h f$ concerns how the function $f(x)$ changes when a quantity *h* is added to *x*, whereas $D_q f$ considers how it changes when the variable *x* is multiplied by a factor *q*.
Let us consider the function $f(x)=x^n$. If we calculate its q-derivative, we obtain:

$$(1)\ \ D_q x^n=\frac{(qx)^n-x^n}{qx-x}=\frac{q^n-1}{q-1}x^{n-1}$$

Comparing the ordinary calculus, giving $(x^n)'=nx^{n-1}$, to Equation (1), we can define the "q-integer" $[n]$ by:

$$[n]=\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-1}$$

Therefore Equation (1) turns out to be:

$$D_q x^n=[n]x^{n-1}$$

As a consequence, a *n*-th q-derivative of $f(x)=x^n$, which is obtained by reperting *n* times the q-derivative, generates the q-factorial:

$$[n]!=[n][n-1]\ldots[3][2][1]$$

Form the q-factorials, we can define q-binomial coefficients:

$$\frac{[n]!}{[m]![n-m]!}$$

This means that we can use the usual Taylor formula, replacing the derivatives by the q-derivatives and the factorials by q-factorials.

The q-exponentials As given in [6], the q-analog of the exponential function is defined as

$$e_q^x=\sum_{n=0}^{\infty}\frac{x^n}{[n]!}$$

Since we have that [6]:

$$\sum_{n=0}^{\infty} \frac{x^n}{(1-q^n)(1-q^{n-1})\cdots(1-q)} = \sum_{n=0}^{\infty} \frac{x^n (1-q)^{-n}}{[n]!}$$

The q-exponential has the following expression too:

$$e_q^x = \sum_{n=0}^{\infty} x^n \frac{(1-q)^n}{(1-q^n)(1-q^{n-1})\cdots(1-q)}$$

Another q-analog of the exponential function is [6]:

$$E_q^x = \sum_{n=0}^{\infty} x^n \frac{\left(1-\frac{1}{q}\right)^n}{\left(1-\left(\frac{1}{q}\right)^n\right)\left(1-\left(\frac{1}{q}\right)^{n-1}\right)\cdots\left(1-\frac{1}{q}\right)} = e_{1/q}^x$$

A property of the q-exponential functions is

$$e_q^x e_q^y = e_q^{x+y} \quad if \quad yx = qxy$$

Due to this commutation relation, *x* and *y* are not symmetric and therefore:

$$e_q^x e_q^y \neq e_q^y e_q^x$$

When q → 1, the q-exponentials become the usual exponential. We can see this in the following Figure 1.

The q-trigonometric functions As given in [6], the q-analog of the trigonometric functions are defined as:

$$\sin_q x = \frac{e_q^{ix} - e_q^{-ix}}{2i}; \cos_q x = \frac{e_q^{ix} + e_q^{-ix}}{2}$$

$$\mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i}; \mathrm{Cos}_q x = \frac{E_q^{ix} + E_q^{-ix}}{2}$$

We have that:

$$(2) \quad \cos_q x \, \mathrm{Cos}_q x + \sin_q x \, \mathrm{Sin}_q x = 1 .$$

In the Figure 2, we can see the graphs of the q-trigonometric functions for two different values of q. When $q \to 1$, these functions become the usual ones. To check the calculation, in the Figure 2, it is shown also the value of $\cos_q x \, \mathrm{Cos}_q x + \sin_q x \, \mathrm{Sin}_q x$. It is equal to 1, as it must be because given by (2). From the plots, we see that the behavior of the q-trigonometric functions become more different from that of the trigonometric functions, when the value of x is large. Let us also note that when q is close to 1, the q-functions are close to cos x and sin x.

Other q-functions It is necessary to note that other q-functions exist, which are linked to the formulation of the entropy given by Constantino Tsallis. In 1948 [7], Claude Shannon defined the entropy S of a discrete random variable Ξ as the expected value of the information content: $S = \sum_i p_i \, I_i = -\sum_i p_i \, log_b p_i$ [8]. In this expression, I is the information content of Ξ, the probability of i-event is p_i and b is the base of the used logarithm. Common values of the base are 2, Euler's number e, and 10. Tsallis generalized the Shannon entropy in the following manner [9]:

$$S_q = \frac{1}{q-1}\left(1 - \sum_{\mathrm{i}} p_i^q\right)$$

Let us note that, if we consider the q-derivative of the quantity $\sum_i p_i^x$, we have:

$$D_q \sum_i p_i^x = \frac{\sum_i p_i^{qx} - \sum_i p_i^x}{qx - x}$$

Therefore, when $x \to 1$:

$$\lim_{x\to 1} D_q \sum_i p_i^x = \lim_{x\to 1} \frac{\sum_i p_i^{qx} - \sum_i p_i^x}{qx - x} = \frac{\sum_i p_i^q - 1}{q-1} = -S_q$$

Then, the Tsallis entropy is linked to the q-derivative. But, in the framework of the Tsallis approach to statistics, we have a deformation of the exponential function, the Tsallis q-exponential function, given by [10]:

$$(3) \;\; e_q(x) = \begin{cases} \exp(x) & if \; q = 1 \\ [1 + (1-q)x]^{1/(1-q)} & if \; q \neq 1 \; and \; 1 + (1-q)x > 0 \end{cases}$$

This is different from the q-exponential previously discussed.
If the function in (3) is expanded in Taylor series, we have [11]:

$$(4) \;\; e_q(x) = 1 + \sum_{n=1}^{\infty} \frac{Q_{n-1}}{n!} x^n$$

In (4), we have $Q_n(q) = 1 \cdot q(2q-1)(3q-2)\cdots[nq-(n-1)]$.
From the expression (4) of the Tsallis q-exponential, given for complex ix, we can obtain the Tsallis trigonometric functions:

$$\cos_q x = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j Q_{2j-1}}{(2j)!} x^{2j} \; ; \quad \sin_q x = \sum_{j=0}^{\infty} \frac{(-1)^j Q_{2j}}{(2j+1)!} x^{2j+1}$$

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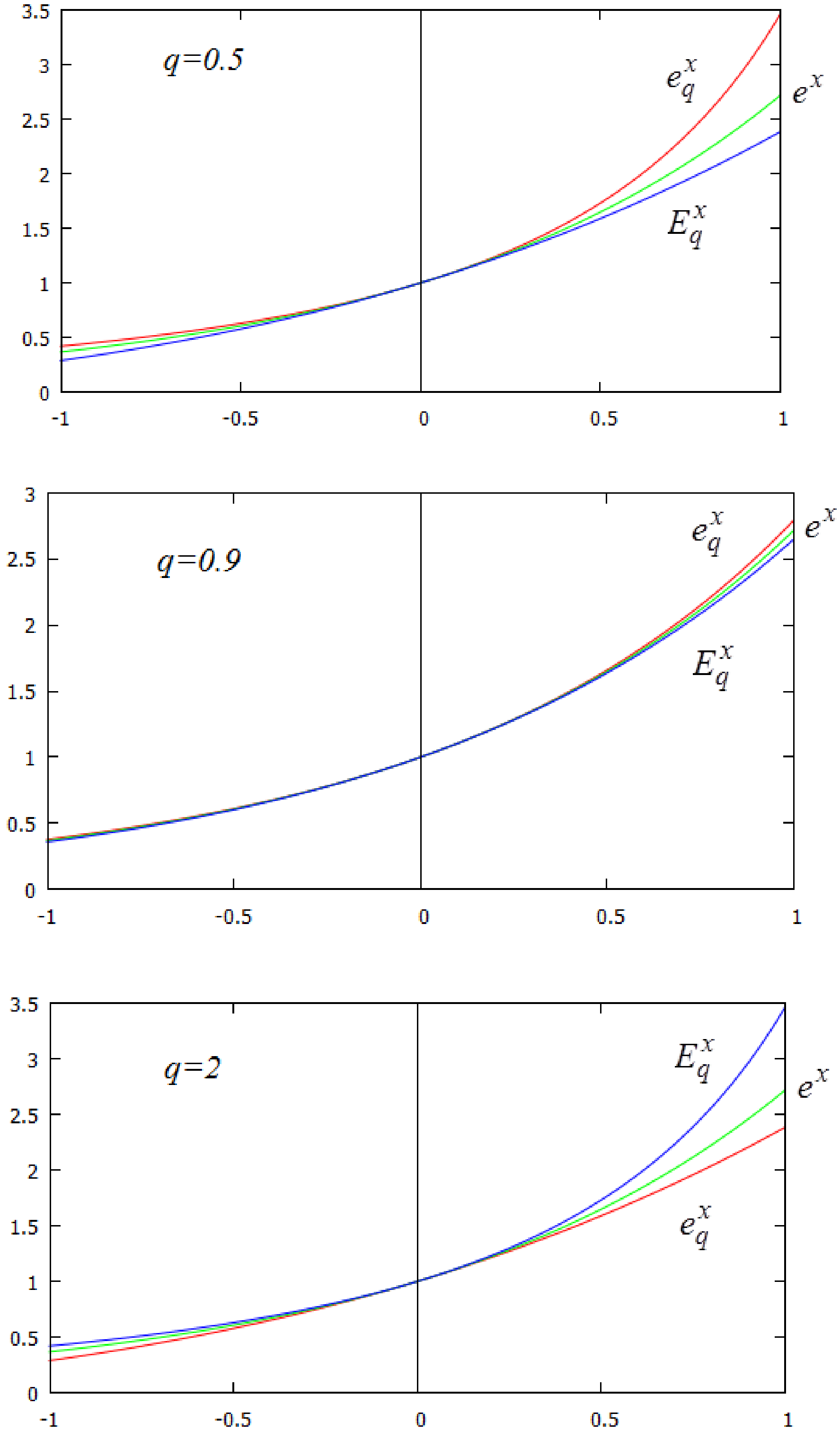

Figure 1: Behavior of e_q^x, e^x, E_q^x for different values of q.

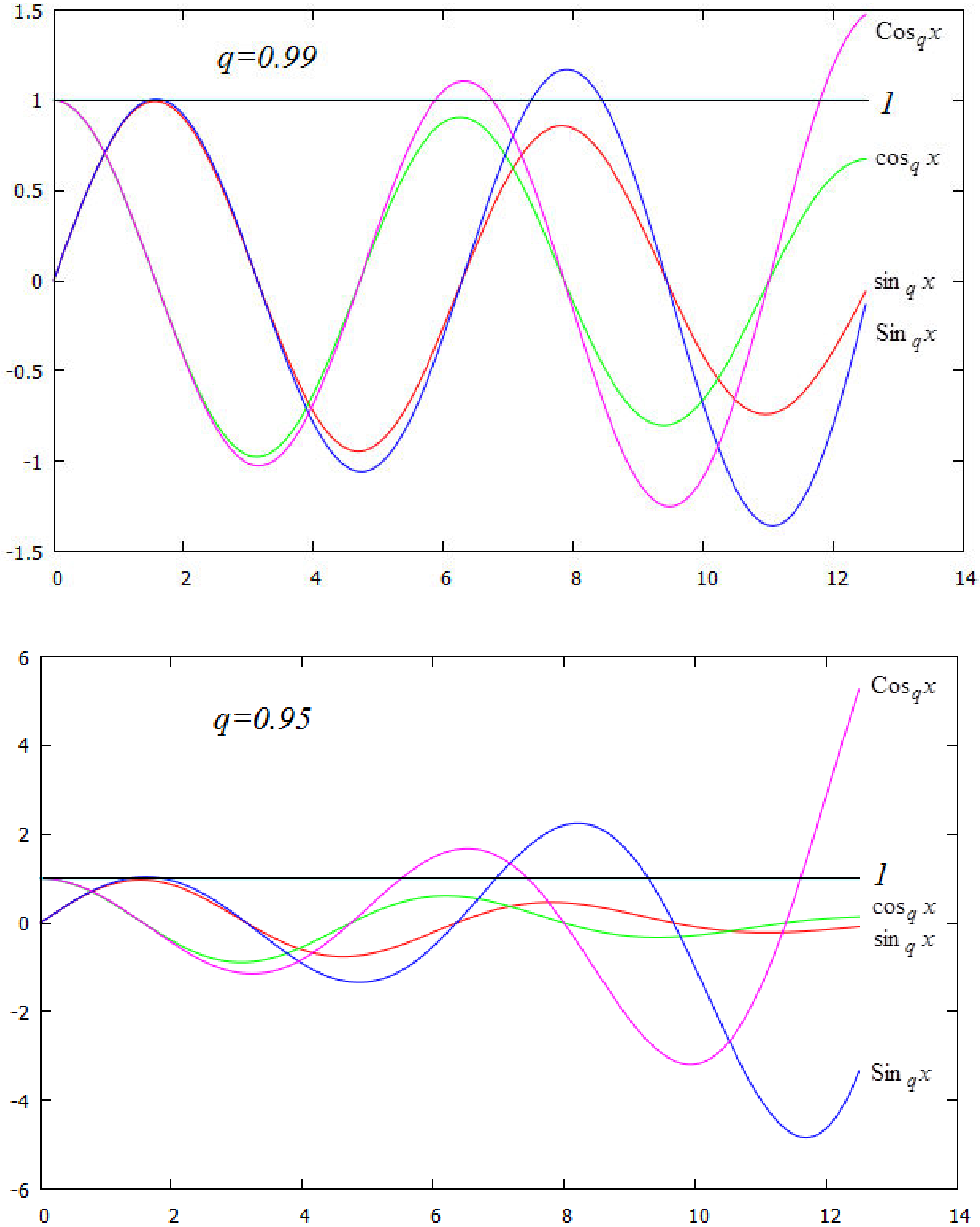

Figure 2: q-trigonometric functions for two different values of q.