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Abstract
Here we discuss the calculation of an integral containing the Bessel function J0(r) and the modified Bessel function of the first kind I1(r). The calculus is based on a function of J0(r), I1(r) and of their derivatives, having a Wronskian form. The method here described could be useful for training the students in the manipulation of such integrals.

Article body

An integral containing a Bessel Function and a Modified Bessel Function of the First Kind

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Here we discuss the calculation of an integral containing the Bessel function $J_0(r)$ and the modified Bessel function of the first kind $I_1(r)$. The calculus is based on a function of $J_0(r)$, $I_1(r)$ and of their derivatives, having a Wronskian form. The method here described could be useful for training the students in the manipulation of such integrals.

Here we are proposing a method for the calculus of an integral of the form $\int r^2 J_0(r) I_1(r)\,dr$. The method is based on some suitable functions, which are the products of the Bessel and modified Bessel functions and their derivatives. The approach, which is coming from the author's experience, could be useful for training the students in the manipulation of integrals containing Bessel functions. The author's experience was concerning a calculus required by a method for determining the thermal diffusivity, based on the measurement of the thermal expansion of a cylindrical sample [1-4]. For this method, I had to consider the problem of the thermoelasticity.
In particular, I had to test if the thermal expansion of a cylindrical sample, assumed as the integral of temperature $\theta(r,z,t)$, that is $\Delta=\beta\int_0^l \theta(r,z,t)dz$, ($\beta$ is the coefficient of the expansion), was equal to the component u_z of the displacement field $\vec{u}$ found by means of the thermoelasticity equations. These equations are [5]:

$$\mu\,\nabla^2\vec{u}+(\lambda+\mu)\,grad\ div(\vec{u})-(3\lambda+2\mu)\beta\,grad\,\theta=\rho\,\ddot{\vec{u}}$$
$$k\nabla^2\theta-c_v\dot{\theta}-\frac{(c_p-c_v)}{\beta}div(\dot{\vec{u}})=0$$

λ and μ are the Lamè coefficients. c_p and c_v are the specific heats at constant pressure and volume, k is the thermal conductivity and ρ the density. In the case of a solid material, the second equation reduces to $k\nabla^2\theta-c_v\dot{\theta}=0$. The solution of these equations for a cylindrical sample is requiring hyperbolic sines and cosines and Bessel functions $J_0(r)$ and $I_1(r)$, and the related integrals. Here we consider and discuss how to calculate one of these integrals, that of the form: $\int r^2 I_1(r) J_0(r)\,dr$.
Let us start from the equation of the Bessel function and of the modified Bessel function of the first kind (the variable r is here assumed as dimensionless):

$$r^2 J_0''(r)+r\,J_0'(r)+r^2 J_0(r)=0 \qquad (1)$$
$$r^2 I_1''(r)+r\,I_1'(r)-(r^2+1)\,I_1(r)=0 \qquad (2)$$

We can multiply (1) by $I_1(r)$ and (2) by $J_0(r)$ and then subtract the results as follow:

$$r^2 J_0''(r)I_1(r)+r\,J_0'(r)I_1(r)+r^2 J_0(r)I_1(r)-r^2 I_1''(r)J_0(r)-r\,I_1'(r)J_0(r)$$
$$+(r^2+1)J_0(r)I_1(r)=0$$

$$r^2[J_0''(r)I_1(r)-I_1''(r)J_0(r)]+r\,[J_0'(r)I_1(r)-I_1'(r)J_0(r)]+(2r^2+1)J_0(r)I_1(r)=0 \qquad (3)$$

Let us consider a Wronskian function W and its derivative, defined in the following manner:

$$W=J_0'(r)I_1(r)-J_0(r)I_1'(r)$$

$$W'=J_0''(r)I_1(r)+J_0'(r)I_1'(r)-J_0'(r)I_1'(r)-J_0(r)I_1''(r)$$
$$=J_0''(r)I_1(r)-J_0(r)I_1''(r)$$

$$r^2W'+rW+(2r^2+1)J_0(r)I_1(r)=0$$

Therefore, from (3), we have:

$$r^2W'+2rW=rW-(2r^2+1)J_0(r)I_1(r)$$

A simple integration gives:

$$r^2W=\int rW\,dr-2\int r^2 J_0(r)I_1(r)\,dr-\int J_0(r)I_1(r)\,dr \qquad (4)$$

Moreover, we have the derivatives of the Bessel functions [6]:

$$J_0'(r)=-J_1(r)\quad;\quad J_1'(r)=J_0(r)-\frac{J_1(r)}{r}$$
$$I_0'(r)=I_1(r)\quad;\quad I_1'(r)=I_0(r)-\frac{I_1(r)}{r} \qquad (5)$$

From (4), using (5):

$$-r^2W=-\int rW\,dr+2\int r^2 J_0(r)I_1(r)\,dr+\int J_0(r)I_1(r)\,dr$$
$$=\int r\,[J_1(r)I_1(r)+J_0(r)I_0(r)-J_0(r)I_1(r)/r]\,dr+2\int r^2 J_0(r)I_1(r)\,dr+\int J_0(r)I_1(r)\,dr$$

$$-r^2W=\int r\,[J_1(r)I_1(r)+J_0(r)I_0(r)]\,dr+2\int r^2 J_0(r)I_1(r)\,dr$$

Therefore;

$$\int r^2 J_0(r)I_1(r)\,dr=-\frac{1}{2}r^2W-\frac{1}{2}\int r\,[J_1(r)I_1(r)+J_0(r)I_0(r)]\,dr \qquad (6)$$

Then, to evaluate the integral in the left side of (6), we need the integrals $\int r\,J_1(r)I_1(r)\,dr$ and $\int rJ_0(r)I_0(r)\,dr$. These integrals are easy to calculate. Let us consider the two functions $\Pi_1=r\,J_0(r)I_1(r);\Pi_2=r\,J_1(r)I_0(r)$. Let us evaluate the following:

$$\Pi_1'=J_0(r)I_1(r)-r\,J_1(r)I_1(r)+r\,J_0(r)\,I_0(r)-r\,J_0(r)I_1(r)/r$$
$$=-r\,J_1(r)I_1(r)+r\,J_0(r)\,I_0(r)$$

$$\Pi_2'=J_1(r)I_0(r)+r\,J_0(r)I_0(r)-r\,J_1(r)I_0(r)/r+r\,J_1(r)I_1(r)$$
$$=r\,J_0(r)I_0(r)+r\,J_1(r)I_1(r)$$

Adding Π_1',Π_2', we obtain:

$$\Pi_1'+\Pi_2'=-r\,J_1(r)I_1(r)+r\,J_0(r)\,I_0(r)+r\,J_0(r)I_0(r)+r\,J_1(r)I_1(r)$$
$$=2r\,J_0(r)I_0(r)$$

As a consequence (after an integration):

$$2\int rJ_0(r)I_0(r)\,dr=\Pi_1+\Pi_2\rightarrow\int r\,J_0(r)I_0(r)\,dr=\frac{r}{2}[J_0(r)I_1(r)+J_1(r)I_0(r)] \qquad (7)$$

Subtracting Π_1',Π_2', we obtain:

$$\Pi_1'-\Pi_2'=-r\,J_1(r)I_1(r)+r\,J_0(r)\,I_0(r)-r\,J_0(r)I_0(r)-r\,J_1(r)I_1(r)$$
$$=-2r\,J_1(r)I_1(r)$$

Therefore, we have:

$$2\int rJ_1(r)I_1(r)\,dr=-\Pi_1+\Pi_2\rightarrow\int r\,J_1(r)I_1(r)\,dr=\frac{r}{2}[-J_0(r)I_1(r)+J_1(r)I_0(r)] \qquad (8)$$

Using the results (7) and (8):

$$\int r^2 J_0(r)I_1(r)\,dr=-\frac{1}{2}r^2[J_0'(r)I_1(r)-J_0(r)I_1'(r)]$$
$$+\frac{r}{4}[J_0(r)I_1(r)-J_1(r)I_0(r)]-\frac{r}{4}[J_0(r)I_1(r)+J_1(r)I_0(r)]$$
$$=-\frac{1}{2}r^2[-J_1(r)I_1(r)-J_0(r)I_0(r)+J_0(r)I_1(r)/r]-\frac{r}{2}J_1(r)I_0(r)$$
$$=\frac{1}{2}r^2[J_0(r)I_0(r)+J_1(r)I_1(r)]-\frac{r}{2}[J_0(r)I_1(r)+J_1(r)I_0(r)]$$

As we have seen in the calculation, to find the integral $\int r^2 J_0(r)I_1(r)\,dr$ we used functions W,Π_1,Π_2 and their derivatives. The search for these functions could be intriguing to students, and for this reason, it could stimulate their interest during the study of the integrals of Bessel functions.

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