Entropy and logarithm of Kaniadakis calculus expressed by means of an Euler infinite product expansion

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Here we show how an Euler infinite product expansion can be used to display easily the link of nonadditive Kaniadakis entropy to Shannon entropy. At the same time we can see how the natural logarithm is linked to the expression of κ-logarithm.

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In 2001, Giorgio Kaniadakis proposed a formulation of entropy based on his new approach to calculation which is today known as κ-calculus [1-3]. This entropy is a nonadditive composable one [4]. In [5], we have consider some further properties of this entropy, with the aim of applying it to the analysis of images [6] and theory of numbers [7], among the many applications of this calculus. Here we aim to show how an Euler infinite product expansion can be used to display easily the link of this entropy to Shannon entropy. At the same time we can see how the natural logarithm is linked to the expression of the κ-logarithm.

Kaniadakis entropy has the discrete form:

$$S_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}$$

Integer *i* enumerates cases with probability p_i and κ is a real number.

Now, let us consider a term in the sum:

$$\frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa} = p_i \frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa}$$

Let us use Euler number and logarithm:

$$\frac{p_i}{2\kappa}(p_i^{\kappa}-p_i^{-\kappa})=\frac{p_i}{2\kappa}(e^{\kappa\ln p_i}-e^{-\kappa\ln p_i})=\frac{p_i}{2\kappa}(e^{u_i}-e^{-u_i})$$

where we define $u_i=\kappa\ln p_i$.

In [8] we can find mentioned a useful formula [9], which is an Euler infinite product expansion:

$$e^u-e^{-u}=2u(1+\frac{u^2}{\pi^2})(1+\frac{u^2}{2^2\pi^2})(1+\frac{u^2}{3^2\pi^2})...=2u\prod_{j=1}^{\infty}(1+\frac{u^2}{j^2\pi^2})$$

Then the entropy can be written as:

$$S_\kappa=-\sum_i\frac{p_i^{1+\kappa}-p_i^{1-\kappa}}{2\kappa}=-\frac{1}{2\kappa}\sum_i 2p_i u_i\prod_{j=1}^{\infty}(1+\frac{u_i^2}{j^2\pi^2})$$

$$= -\frac{1}{2\kappa}\sum_i 2p_i\kappa(\ln p_i)\prod_{j=1}^{\infty}(1+\frac{(\kappa\ln p_i)^2}{j^2\pi^2})=-\sum_i(p_i\ln p_i)\prod_{j=1}^{\infty}(1+\frac{(\kappa\ln p_i)^2}{j^2\pi^2})$$

So we can write:

$$S_\kappa=-\sum_i(p_i\ln p_i)\prod_{j=1}^{\infty}(1+\frac{(\kappa\ln p_i)^2}{j^2\pi^2})$$

And here we can see clearly that Kaniadakis entropy becomes Shannon entropy for $\kappa\to 0$:

$$S_{Shannon}=-\sum_i p_i\ln p_i$$

Kaniadakis proposed a form of logarithm in the κ-calculus [1-3] so that:

$$S_\kappa=-\sum_i p_i\ln_\kappa p_i$$

where

$$\ln_\kappa p_i=\frac{p_i^{\kappa}-p_i^{-\kappa}}{2\kappa}$$

As a consequence, using Euler function we can tell that:

$$\ln_\kappa p_i = (\ln p_i) \prod_{j=1}^{\infty} \left(1 + \frac{(\kappa \ln p_i)^2}{j^2 \pi^2}\right)$$

In the generalized additivity of Kaniadakis entropy, it appears another function [5]:

$$I_\kappa = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

In [8] we find another Euler formula [10]:

$$e^u + e^{-u} = 2\left(1 + \frac{4u^2}{\pi^2}\right)\left(1 + \frac{4u^2}{3^2 \pi^2}\right)\left(1 + \frac{4u^2}{5^2 \pi^2}\right) \ldots = 2 \prod_{j=0}^{\infty} \left(1 + \frac{4u^2}{(2j+1)^2 \pi^2}\right)$$

And then:

$$I_\kappa = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2} = \sum_i \frac{p_i}{2} \left[2 \prod_{j=0}^{\infty} \left(1 + \frac{4(\kappa \ln p_i)^2}{(2j+1)^2 \pi^2}\right)\right]$$

In the case that $\kappa \to 0$, we have $I_\kappa = 1$.

Let is remember that this function appears in the generalized additivity of Kaniadakis entropy. In the case of two independent systems *A* and *B*:

$$S_\kappa^{A \cup B} = S_\kappa^A I_\kappa^B + S_\kappa^B I_\kappa^A$$

When $\kappa \to 0$, we have :

$$S_\kappa^{A \cup B} = S^A + S^B$$

where the entropies are Shannon entropies.

As a conclusion, using two Euler infinite product expansions we can rewrite Kaniadakis entropy and discuss, in a different manner, its properties with respect to Shannon entropy.

References

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