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Generalized Markovian Quantity Distribution Systems: Social Science Applications

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Abstract: We propose a model of Markovian quantity flows on connected networks that relaxes several properties of the standard compartmental Markov process. The motivation of our generalization are social science applications of the standard model that do not comport with its steady state predictions. The proposed generalization relaxes the predictions that every node belonging to the same nontrivial strong component of a network must acquire the same fraction of its members' initial quantities and that the sink component(s) of the network must absorb all of the system's available initial quantity. For example, when applied to refugee flows from a nation in chaos to other nations on a network with one or more sink nations, the standard model predicts that all the refugees will be eventually located in the sink(s) of the network and none that will permanently locate themselves in the nations along the paths to the sink(s). We illustrate this and several other social science applications to which our proposed model is applicable.

Keywords: Markov chains; compartmental systems; social science; networks; quantity flows

THE Markov chain model has been a fertile platform of important normal science developments and interdisciplinary applications for more than 100 years (Kemeny and Snell 1976; Seneta 2006; Meyn and Tweedie 2012). The dynamics of a homogeneous (or time-stationary) chain with n states is uniquely characterized by its matrix of transitional probabilities $\mathbf{W}$ or, equivalently, the behavior of the deterministic linear discrete-time system

$$\mathbf{x}(k+1)^\top = \mathbf{x}(k)^\top \mathbf{W} = \cdots = \mathbf{x}(0)^\top \mathbf{W}^{k+1}, \quad k = 0, 1, 2, \ldots. \tag{1}$$

The $n \times n$ matrix $\mathbf{W} = [w_{ij}]$ has to be row-stochastic, that is, $w_{ij} \geq 0$ and $\sum_{j=1}^n w_{ij} = 1$ for all $i = 1, \ldots, n$. Its entry w_{ij} stands for the (time-invariant) probability of transition from state i to state j. The $\mathbf{x}(0) = [x_i(0)]$ is a column vector probability distribution with $x_i(0) \geq 0$ and $\sum_{i=1}^n x_i(0) = 1$ whose element $x_i(0)$ stands for the probability that the chain starts at state i. Similarly, $x_i(k)$ is the probability that the chain visits state i at period k. Equation (1) admits an alternative interpretation, which is broadly applied in biology, chemistry, and economics to describe compartmental systems, that is, systems distributing some quantity over a network with n nodes (compartments) (Walter and Contreras 1999). In such applications, the vector $\mathbf{x}(k)$ stands for the distribution of quantity at period k and the $\sum_{i=1}^n x_i(k) = c > 0$ $\forall k$, where c is the total amount of quantity in the system. The w_{ij} is interpreted as the proportion of quantity that node i transfers to node j in the weighted directed network $\mathcal{D}$ associated with $\mathbf{W}$ at each stage of quantity redistribution.

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Network Topology

In the generic situation, the matrix $\mathbf{W}$ is aperiodic, that is, the limits $\mathbf{V} = \lim_{k\to\infty} \mathbf{W}^k$ and $\mathbf{x}(\infty)^\top = \lim_{k\to\infty} \mathbf{x}(k)^\top = \mathbf{x}(0)^\top \mathbf{V}$ exist. The structure of $\mathbf{V}$ and $\mathbf{x}(\infty)$ are substantially constrained by the topology of the network $\mathcal{D}$ that is associated with $\mathbf{W}$ and, in particular, by its *sink* component(s). As usual, arc $i \to j$ of the network $\mathcal{D}$ corresponds to the positive entry $w_{ij} > 0$, that is, the flow of quantity from node i to node j. Positive diagonal entries $w_{ii} > 0$ stand for self-arcs. The network $\mathcal{D}$ is strongly connected (or strong) if every two nodes i and $j \neq i$ are connected by a path. A network that is not strong contains several strong components. Two nodes i and j are in the same strong component if there is path from i to j and a path from j to i. A strong component can consist of a single node; such components are called trivial; otherwise the component is nontrivial. For aperiodicity (the existence of matrix $\mathbf{V}$), it suffices that each nontrivial strong component contains some node with a self-loop ($w_{ii} > 0$). If the network is strong, then it has only one strong component (which coincides with $\mathcal{D}$). Otherwise, at least two strong components exist. It can be shown that at least one of them is a *sink* component, that is, has no outcoming arcs.

Without loss of generality, henceforth we assume that the matrix $\mathbf{W} + \mathbf{W}^\top$ corresponds to a connected undirected graph, and the network $\mathcal{D}$ is said to be (weakly) connected, or weak. Otherwise, the Equation (1) system splits into several independent subsystems, and each of them can be studied separately. Three substantially different cases are possible in a connected network $\mathcal{D}$: (a) the network is strongly connected; (b) the network has several strong components, and only one of them is a sink (a one-sink network); (c) the network has two or more sink components (a multiple-sink network). In the cases (a) and (b) the behavior of the Equation (1) system is simple in the sense that the final distribution $\mathbf{x}(\infty)$ is independent of the initial distribution $\mathbf{x}(0)$. In the case (c), the behavior is more complex, and $\mathbf{x}(\infty)$ depends on $\mathbf{x}(0)$.

Peculiar Restrictions

In its social science applications on quantity flows, the Markovian compartmental dynamics has several peculiar restrictions that are not realistic and will be relaxed. The first principal restriction is the concentration of the total quantity in the sink strong components of the network (in probability theory, sink components are known as classes of communicating recurrent states of the Markov chain, and the latter property means that the chain eventually leaves all nonrecurrent states and visits only recurrent ones). Consider the Figure 1 network $\mathcal{D}$ of refugee flows in which each node is a nation state. This is a network with two trivial sink components (14 and 17). Let node 1 be a nation in chaos that generates a large number of refugees from it. Let nodes 2 to 17 be nations that accept refugees. The $\mathbf{W}$ that is associated with this $\mathcal{D}$ must have $w_{ii} = 1$ for the two sink nodes (14 and 17). Each of the remaining 15 nodes (which may have no self-loops) has two arcs with $0 < w_{ij} < 1$ to other nodes. The Equation (1) Markov flow process on this network $\mathcal{D}$ will absorb all the refugees from nation 1 in the two sink nations (14 and 17), and no refugees will permanently locate themselves in the nations along the paths to the two sink

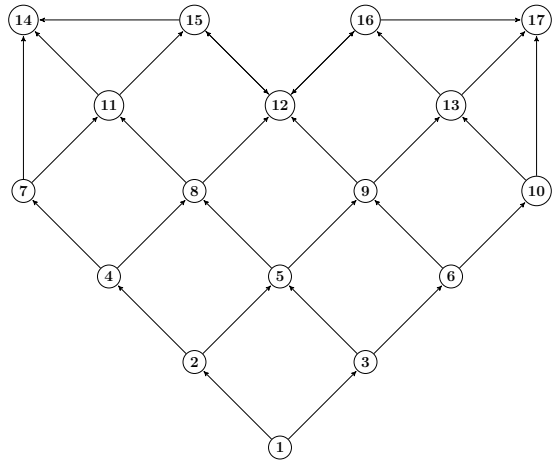

Figure 1: Refugee flow network among nations ($w_{ii} > 0$ loops are not displayed).

nations. Our generalized model allows a settlement of positive heterogeneous fractions of the refugees in every nation 2 to 17. We will illustrate this and several other social science applications to which our proposed generalization is applicable.

The second restriction is homogeneity of quantity redistribution within strong components. Namely, each strong component $\mathcal{C}$ corresponds to the submatrix of $\mathbf{V}(\mathcal{C})$ that either is a null matrix or is stochastic with identical rows. In other words, if i, j, k are nodes of $\mathcal{C}$, then $v_{ij} = v_{kj}$. As will be discussed, the entry v_{ij} stands for the proportion of initial quantity $x_j(0)$ that is redistributed to node i. Hence, each i belonging to the component acquires the same proportion of the initial quantity of each j belonging to the component. This proportion can be zero, as in the case of the strong component (12, 15, 16) in Figure 1 and any other nonsink strong component.

The first and the second restriction imply the third geometric restriction on the set of possible outcomes $\mathbf{x}(\infty)$. It can be shown that if the number of sink components in $\mathcal{D}$ is s, then each $x_i(\infty)$ is a convex combination of s vectors, uniquely determined by matrix $\mathbf{W}$ and total quantity c. Except for the degenerate case $\mathbf{W} = \mathbf{I}_n$ (the quantity does not circulate), we always have $s < n$ and, typically, $s \ll n$ for large-scale networks. The convex polytope spanned by s vectors is a "very thin" subset of the simplex of all possible distributions $\Delta_c = \{\mathbf{x} : x_i \geq 0, \sum_i x_i = c\}$ (mathematically, it has a lower affine dimension than Δ_c and, in particular, has zero Lebesgue measure in $\mathbb{R}^n$). In the case of $s = 1$, the set of possible $\mathbf{x}(\infty)$ is a singleton, for $s = 2$ (as in Figure 1) it is a line segment, for $s = 3$ it is a triangle, and so on.

Contribution

We will introduce a generalized Markov flow process that allows a larger set of outcomes and is free of the aforementioned three limitations. In particular, it allows distributions that do not concentrate quantity in one or multiple sink components. Our article is organized as follows. First, we describe and illustrate the aforementioned restrictions, caused by the structure of matrix $\mathbf{V}$ and its submatrices $\mathbf{V}(\mathcal{C})$, corresponding to strong components (all rows should be equal). Second, we intro-

duce a generalization of the Markov process that circumvents these limitations, in particular, allowing accumulation of quantity in nonsink strong components and enabling all final distributions $\mathbf{x}(\infty) \in \Delta_c$. Third, we illustrate the generalization in four social science applications.

Classic Markov Model

A matrix $\mathbf{W}$ is called *SIA* (stochastic, indecomposable, aperiodic) (Wolfowitz 1963) or *fully regular* if the limit $\mathbf{V} = \lim_{k\to\infty} \mathbf{W}^k$ exists and the rows of $\mathbf{V}$ are identical (that is, $\mathbf{V}$ is a rank-one stochastic matrix):

$$\mathbf{V} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & \dots & \vdots \\ v_1 & v_2 & \dots & v_n \end{bmatrix} = \mathbf{1}_n \mathbf{v}^\top. \tag{2}$$

Here $\mathbf{1}_n$ stands for the column vector of ones of dimension n. An equivalent algebraic condition is that $\mathbf{W}$ has no eigenvalues beyond the open disk $\{z \in \mathbf{C} : |z| < 1\}$ except for $z = 1$, and the eigenvalue $z = 1$ is simple. For an SIA matrix, the row vector $\mathbf{v}^\top = (v_1, \dots, v_n)$ is the unique left eigenvector of $\mathbf{W}$ such that

$$\mathbf{v}^\top \mathbf{W} = \mathbf{v}^\top, \quad \sum_j v_j = 1,$$

and therefore

$$\begin{aligned} \mathbf{x}(\infty)^\top &= \mathbf{x}(0)^\top \lim_{k\to\infty} \mathbf{W}^k = \mathbf{x}(0)^\top \mathbf{V} = c\mathbf{v}^\top, \\ \mathbf{x}_j(\infty) &= \sum_{i=1}^n \mathbf{x}_i(0) v_j = c v_j, \end{aligned} \tag{3}$$

where c is the total quantity in the system. The second equation in (3) can be restated as follows: each node j with a positive v_j gets an equal proportion of each initial quantity $\mathbf{x}_i(0)$. The final distribution does not depend on the initial distribution $\mathbf{x}(0)$ and is determined by the eigenvector $\mathbf{v}$ and the total amount of quantity c.

In general, the rule that applies to all possible connected network topologies (being aperiodic, so that matrix $\mathbf{V}$ is well defined) is that the submatrices of $\mathbf{V}$ corresponding to nontrivial strong components are rank-one matrices with equal rows that are either stochastic or null matrices. This property means that each member i of the component acquires the same fraction of the initial quantity from each j belonging to the *same* component.

Strongly Connected Systems

If the connected network $\mathcal{D}$ is strongly connected (one nontrivial strong component), then the matrix $\mathbf{W}$ is said to be *irreducible*. For irreducible matrices, the SIA property is equivalent to primitivity: for a sufficiently large $k > 0$, the matrix $\mathbf{W}^k$ has strictly positive entries. This holds, for instance, if at least one diagonal entry is positive $w_{ii} > 0$. An important property of irreducible matrices is that, according to

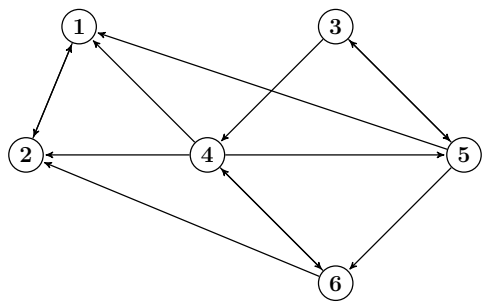

Figure 2: Unilaterally connected $\mathcal{D}$ ($w_{ii} > 0$ loops are not displayed).

the Perron–Frobenius theorem, the vector $\mathbf{v}$ is strictly positive, that is, each node accumulates some amount of quantity. In this case, the $\mathbf{V}$ matrix for this system will have identical rows of positive v_{ij} values.

Networks with One Sink Component

Irreducibility is in fact not necessary for the SIA property. A reducible matrix $\mathbf{W}$ is SIA if and only if two conditions hold: (1) $\mathcal{D}$ has a unique sink strong component and (2) the irreducible submatrix $\mathbf{W}(\mathcal{C})$ corresponding to this component is SIA (for instance, has a positive diagonal entry). In this situation, Equation (3) remains valid. However, unlike strongly connected networks, the row $\mathbf{v}^\top$ of matrix $\mathbf{V}$ includes null elements, namely, $v_i > 0$ for the nodes of the unique sink component $\mathcal{C}$ and $v_i = 0$ for all other nodes. Thus, the quantity is concentrated in the unique sink component. The uniqueness of a sink component is well known for unilaterally connected networks where for each two nodes i and j either a path from i to j or a path from j to i exists. A weakly connected network may also have a single sink component, for instance, the network $1 \to 2 \leftrightarrow 3 \leftarrow 4$ with the nontrivial sink component $(2, 3)$.

Consider the unilateral $\mathcal{D}$ in Figure 2. Its unique sink is a nontrivial strong component $\mathcal{C}_1 = \{1, 2\}$, and the other nodes also belong to a nontrivial strong component $\mathcal{C}_2 = \{3, 4, 5, 6\}$. The $\mathbf{V}(\mathcal{C}_1)$ submatrix will have identical rows of positive v_{ij} values. The $\mathbf{V}(\mathcal{C}_2)$ submatrix will have identical rows of null v_{ij} values.

Let the $\mathbf{W}$ for this network be

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.7630 & 0.2370 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4053 & 0.4068 & 0.1880 & 0 \\ 0.0915 & 0.2155 & 0 & 0 & 0.2703 & 0.4227 \\ 0.2120 & 0 & 0.4560 & 0 & 0 & 0.3320 \\ 0 & 0.2874 & 0 & 0.7126 & 0 & 0 \end{bmatrix} \Longrightarrow \mathbf{V} = \left[\begin{array}{cccccc} 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \\ 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \\ \hline 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \\ 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \\ 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \\ 0.4328 & 0.5672 & 0 & 0 & 0 & 0 \end{array}\right].$$

All quantity circulating in the network is accumulated in the sink component. Node 1 accumulates the same fraction (0.4328) of the initial quantity from each node $i = 1, 2, \ldots, n$. Node 2 accumulates the same fraction (0.5672) of the initial quantity from each node $i = 1, 2, \ldots, n$.

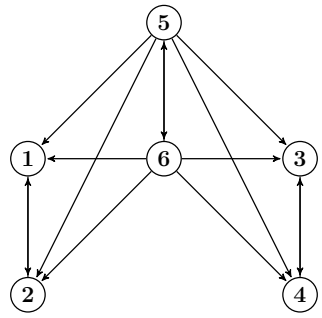

Figure 3: Weakly connected $\mathcal{D}$ ($w_{ii} > 0$ loops are not displayed).

The generalized quantity flow model that we develop allows a $\mathbf{V}(\mathcal{C}_1)$ submatrix with positive nonidentical rows of v_{ij} values and a $\mathbf{V}(\mathcal{C}_2)$ submatrix with zero or positive nonidentical rows of v_{ij} values.

Networks with Multiple Sink Components

Without loss of generality, for networks with $s > 1$ strong components, we can assume that such a network is weakly connected, or weak (that is, the network corresponding to the matrix $\mathbf{W} + \mathbf{W}^\top$ is connected). A network that is not weak can be decomposed into several disconnected weak networks. The limit distribution $\mathbf{x}(\infty)$ and matrix $\mathbf{V}$ are well defined if all sink components $\mathcal{C}_1, \ldots, \mathcal{C}_s$ are aperiodic (e.g., contain self-loops) or, equivalently, the corresponding submatrices $\mathbf{W}(\mathcal{C}_i)$ are SIA. Algebraically, this means that $\mathbf{W}$ has no eigenvalues on the circle $\{z \in \mathbf{C} : |z| = 1\}$ except for $z = 1$, which eigenvalue, however, can be multiple. Its multiplicity s coincides with the number of sink components in the network $\mathcal{D}$. In matrix $\mathbf{V}$, the sink components correspond to stochastic rank-one submatrices as in Equation (2) $\mathbf{V}(\mathcal{C}_i)$; all other entries of $\mathbf{V}$ are zero. In particular, any strong component that is not a sink is associated with a zero submatrix.

Carefully consider Figure 3. The nodes 1 and 2 belong to $\mathcal{C}_1$, nodes 3 and 4 belong to $\mathcal{C}_2$, and nodes 5 and 6 belong to $\mathcal{C}_3$. The two sink components are $\mathcal{C}_1$ and $\mathcal{C}_2$. The corresponding submatrices of $\mathbf{V}$ have identical rows. As has been discussed, this means that each member i of the component acquires the same fraction of the initial quantity from each j belonging to the component. The submatrix of the $\mathcal{C}_3$ component is a null matrix, and the initial quantities $[x_5(0)\ x_6(0)]$ are distributed unequally to the two sink components.

$$\mathbf{W} = \begin{bmatrix} 0.25 & 0.75 & 0 & 0 & 0 & 0 \\ 0.65 & 0.35 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.20 & 0.80 & 0 & 0 \\ 0 & 0 & 0.45 & 0.55 & 0 & 0 \\ 0.10 & 0.30 & 0.10 & 0.20 & 0.10 & 0.20 \\ 0.30 & 0.25 & 0.15 & 0.10 & 0.20 & 0 \end{bmatrix} \Rightarrow \mathbf{V} = \left[\begin{array}{cccc|cc} 0.4643 & 0.5357 & 0 & 0 & 0 & 0 \\ 0.4643 & 0.5357 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3600 & 0.6400 & 0 & 0 \\ 0 & 0 & 0.3600 & 0.6400 & 0 & 0 \\ \hline 0.2753 & 0.3177 & 0.1465 & 0.2605 & 0 & 0 \\ 0.3104 & 0.3582 & 0.1193 & 0.2121 & 0 & 0 \end{array}\right].$$

A trivial (single-node) sink component is constituted by a so-called *absorbing* node with $w_{ii} = 1$ and $w_{ij} = 0\,\forall j \neq i$. Such a node accumulates quantity transferred from other nodes but does not distribute quantity. A Markov chain model is called

absorbing if there are no other sink components, in other words, every node is connected by a path to one of the absorbing nodes. Absorbing Markov chains arise in abundance in economics and ecology (Suh 2005); in particular, they are used to model pathways in resource flow networks (Duchin and Levine 2010). Renumbering, if necessary, the strong components, matrix $\mathbf{V}$ can be transformed into a standard form. Let $s \geq 2$ be the number of the network's trivial or nontrivial sink components. Without loss of generality, we may assume that the node(s) of the first sink component $\mathcal{C}_1$ are indexed 1 through $m_1 \geq 1$, the node(s) of the second sink component $\mathcal{C}_2$ are indexed $m_1 + 1$ through $m_1 + m_2$, and so on, to the node(s) of last sink component $\mathcal{C}_s$ with indices $m_1 + \cdots + m_{s-1} + 1, \ldots, M := m_1 + \cdots + m_{s-1} + m_s$. Then it can be shown that the matrix $\mathbf{V}$ has the following standard structure:

$$\mathbf{V} = \left(\begin{array}{cccc|c} \mathbf{V}_{11} & \mathbf{0} & \cdots & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{V}_{22} & & \vdots & 0 \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & & & \mathbf{V}_{ss} & \\ \hline * & * & * & * & \mathbf{0} \end{array}\right), \qquad \mathbf{V}_{ii} = \mathbf{1}_{m_i}(\hat{\mathbf{v}}^i)^\top, \mathbf{1}_{m_i}^\top \hat{\mathbf{v}}^i = 1.$$

The left-top part of this matrix (dimensioned $M \times M$) is block-diagonal and is constituted by s different blocks $\mathbf{V}_{ii}$ of dimension m_i, $i = 1, \ldots, s$. If a sink is a trivial strong component, its dimension is $m_i = 1$ and its $v_{ii} = 1$. If a sink is a nontrivial strong component, its dimension is $m_i > 1$ and $\mathbf{V}_{ii}$ has identical rows: its rows $\hat{v}_i^\top$ are nonnegative v_{ij} values that sum to 1. The right-top part of $\mathbf{V}$ is an $M \times (n - M)$ zero matrix. The bottom $n - M$ rows of matrix $\mathbf{V}$ (denoted as * * * *) are *convex combinations* of the top M rows. The coefficients can be computed explicitly and depend only on matrix $\mathbf{W}$.

Consider a node $i \in \{1, \ldots, m_1\}$ from a nontrivial sink component $\mathcal{C}_1$. Recalling that $\mathbf{x}(\infty)^\top = \mathbf{x}(0)^\top \mathbf{V}$, it can shown that node i accumulates the following amount of quantity:

$$x_i(\infty) = \sum_{j=1}^{m_1} \hat{v}_i^1 x_j(0) + \sum_{j=M+1}^{n} v_{ji} x_j(0).$$

Hence, each node i accumulates an *equal* proportion v_j^1 of quantities initially stored in its sink component $\mathcal{C}_1$ and some (in general, unequal) proportions of the quantities stored at nodes $M + 1, \ldots, n$ that do not belong to sink components. Similarly, for $\mathcal{C}_2$,

$$x_i(\infty) = \sum_{j=m_1+1}^{m_2} \hat{v}_i^2 x_j(0) + \sum_{j=M+1}^{n} v_{ji} x_j(0) \quad \forall i = m_1 + 1, \ldots, m_2,$$

and each node i from sink component $\mathcal{C}_2$ accumulates an *equal* proportion v_j^1 of quantities initially stored in its sink component $\mathcal{C}_2$, and so on. Nodes $M + 1, \ldots, n$ are emptied of quantity and accumulate zero percentage of quantity from any other node.

Notice that there is another implicit constraint on the set of possible final distributions $\mathbf{x}(\infty)$. Without loss of generality, assume that the total amount of quantity is $c = 1$, so that $\sum_i x_i(0) = 1$. Then, $\mathbf{x}(\infty)^\top$ is a *convex combination* of the rows of matrix $\mathbf{V}$. Recalling that the bottom $(n - M)$ rows are convex combinations of the top M rows, the vector $\mathbf{x}(\infty)^\top$ belongs to the convex hull, spanned by the top M rows, that is, the set of vectors

$$[\alpha_1 \mathbf{v}^1 \quad \alpha_2 \mathbf{v}^2 \dots \quad \alpha_s \mathbf{v}^s \quad \mathbf{0}_{1\times n-M}], \quad \alpha_i \geq 0, \sum_i \alpha_i = 1.$$

This set is a convex polytope with s vertices, which has affine dimension $s - 1$. In the above example, this set is a line segment in six-dimensional space. Because usually $s \ll n$, the Markov dynamics invisibly restricts the final distribution to a very "thin" set in the space of all possible distributions. The generalization, which we introduce below, relaxes these properties.

Generalization

Here we describe an alternative quantity flow model that relaxes the key restrictions of the classic Equation (1) Markov model. Along with the matrix $\mathbf{W}$, this model introduces a diagonal $n \times n$ matrix $\mathbf{A}$, where $0 < a_{ii} \leq 1$ and $a_{ij} = 0 \, \forall i \neq j$. Obviously, the matrices $\mathbf{AW}$ and $\mathbf{W}$ determine the same topology of the network. The case $\mathbf{A} = \mathbf{I}_n$ will correspond to the classic model.

The dynamical system that we propose is a mixed dissipative-aggregative state transition process as follows:

$$\mathbf{y}(k+1)^\top = \mathbf{z}(k)^\top(\mathbf{I} - \mathbf{A}) + \mathbf{y}(k)^\top, \;\; \mathbf{z}(0) = \mathbf{x}(0), \;\; \mathbf{y}(0) = \mathbf{0}, \tag{4}$$

$$\mathbf{z}(k+1)^\top = \mathbf{z}(k)^\top \mathbf{AW} = \cdots = \mathbf{z}(0)^\top (\mathbf{AW})^{k+1}, \tag{5}$$

$$\mathbf{x}(k+1)^\top = \mathbf{y}(k+1)^\top + \mathbf{z}(k+1)^\top \tag{6}$$

$$= \mathbf{x}(0)^\top \left[\left(\sum_{i=0}^{k} (\mathbf{AW})^i \right) (\mathbf{I} - \mathbf{A}) + (\mathbf{AW})^{k+1} \right] \quad \forall k \geq 0 \tag{7}$$

$$= \mathbf{x}(0)^\top \mathbf{V}(k+1).$$

The rationale of these equations is as follows. Along with distributing quantity, the nodes of a network can also accumulate it. Hence, the total amount of quantity at node i is a sum of two components: $x_i(k) = y_i(k) + z_i(k)$, where y_i is the aggregative component and z_i is the dissipative component. At each step, the amount of $a_{ii} z_i(k)$ is distributed by node i to self and other nodes; similar to the usual Markov model, w_{ij} is the proportion of this quantity transferred to node j. After this redistribution, the dissipative component of node i becomes

$$z_i(k+1) = \sum_{\ell=1}^{n} a_{\ell\ell} w_{\ell i} z_\ell(k) = [\mathbf{z}(k)\mathbf{AW}]_i.$$

The remaining amount of $(1 - a_{ii})z_i(k)$ is withdrawn from circulation and remains at the node forever, being added to the aggregative component: $y_i(k+1) = y_i(k) + (1 - a_{ii})z_i(k)$.

Notice that a node with $a_{ii} = 0$ behaves as an *absorbing* node in the classic Markov chain and has no dissipative component: such a node only accumulates quantity. On the other hand, nodes with $a_{ii} = 1$ do not have aggregative component and distribute all quantity stored at them at each period. Our model thus naturally extends Markovian models with absorbing states (Duchin and Levine 2010), allowing the nodes to both distribute and absorb the quantity.

It can be easily shown that the model preserves total quantity $\sum_i x_i(k) = \sum_i cx_i(0) = c$ for all i and k. The matrix $\mathbf{V}(k)$ evolves as follows:

$$\mathbf{V}(k) = \mathbf{AWV}(k-1) + \mathbf{I} - \mathbf{A},\ k = 1, 2, \ldots,\ \ \mathbf{V}(0) = \mathbf{I}. \tag{8}$$

All $\mathbf{V}(k)$ satisfy $0 \le v_{ij}(k) \le 1\ \forall ij$ and $\sum_{j=1}^{n} v_{ij}(k) = 1\ \forall i$.

Most typically, the matrix $\mathbf{AW}$ is Schur stable (all eigenvalues are less than 1 in modulus), and

$$\lim_{k\to\infty} (\mathbf{AW})^k = \mathbf{0},\ \text{ and} \tag{9}$$

$$\lim_{k\to\infty} \left[\sum_{i=0}^{k-1} (\mathbf{AW})^i\right](\mathbf{I} - \mathbf{A}) = (\mathbf{I} - \mathbf{AW})^{-1}(\mathbf{I} - \mathbf{A}) := \mathbf{V}. \tag{10}$$

With some abuse of notation, we use the same symbol $\mathbf{V}$ that denoted the matrix $\lim_{k\to\infty} \mathbf{W}^k$ in the classic Markov model. This redefinition appears to be natural, because, similar to Equation (3), $\mathbf{x}(k)$ converges to the equilibrium distribution

$$\begin{aligned} \mathbf{x}(\infty)^\top &= \mathbf{x}(0)^\top (\mathbf{I} - \mathbf{AW})^{-1}(\mathbf{I} - \mathbf{A}) \\ &= \mathbf{x}(0)^\top \mathbf{V}. \end{aligned} \tag{11}$$

Equations (9) to (11) obviously hold if $\mathbf{A} < \mathbf{I}_n$. In this situation, the matrix $\mathbf{V}$ has positive diagonal entries because $\mathbf{V} \ge \mathbf{I} - \mathbf{A}$. More generally, the stability can be guaranteed (Parsegov et al. 2017) if each node i either aggregates quantity ($a_{ii} < 1$) or is connected by a path to some aggregating node j with $a_{jj} < 1$. This always holds if $\mathbf{A} \ne \mathbf{I}_n$ and the network $\mathcal{D}$ is strongly connected. The resulting matrix $\mathbf{V}$ may have heterogeneous rows, as well as its submatrices corresponding to sink components. Furthermore, if a node has $a_{ii} < 1$, it accumulates some amount of quantity. Hence, the submatrices corresponding to nonsink strong components may be nonzero.

The new model thus relaxes the restrictions of the classic Markov model. (1) In a strongly connected aperiodic $\mathcal{D}$, the rows of $\mathbf{V}$ may be heterogeneous. (2) In a unilaterally connected $\mathcal{D}$, all strong components may retain positive quantity, and, if the unique sink component is a nontrivial component, then the corresponding submatrix of $\mathbf{V}$ can have heterogeneous rows. (3) In general, for the network $\mathcal{D}$, any nontrivial sink component may correspond to a submatrix with heterogeneous rows, and all other components of $\mathcal{D}$ may retain positive quantity.

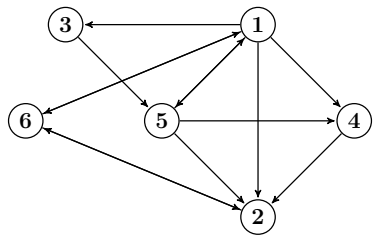

Figure 4: Strongly connected $\mathcal{D}$ ($a_{ii}w_{ii} > 0$ loops are not displayed).

Notice also that if $a_{ii} < 1 \forall i$, then one has a one-to-one correspondence between the final and the initial distribution $\mathbf{x}(0) = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{AW})\mathbf{x}(\infty)$. Hence, unlike the classic Markov model, all distributions $\mathbf{x}(\infty)$ such that $x_i(\infty) \geq 0$ and $\sum_i x_i(\infty) = c$ can emerge as a result of the quantity distribution process.

Social Science Applications

The social science applications of the generalized Markov process include systems of monetized quantity flows, pay-it-forward altruism flows, resource flows in organizations, and refugee flows among nations. In this section, we illustrate the different steady state implications of the classic and generalized Markovian flow models. Each illustration is based on a particular $\mathbf{W}$ and $\mathbf{x}(0)$. Because the normal Markov model is a special case of the generalized model, we use the generalized model with $\mathbf{A} = \mathbf{I}$ for the classic model results and a random $\mathbf{0} < \mathbf{A} < \mathbf{I}$ in which all $0 < a_{ii} < 1$ for the contrasting results. We display the values of the matrix constructs in $\mathbf{x}(\infty)^T = \mathbf{x}(0)^T\mathbf{V}$ for the $\mathbf{V}$ of the classic and generalized model. Each row of the displayed $\mathbf{V}$ sums to 1, and the maximum rounding error of any row is $\sum_j v_{ij} = 1 \pm 0.0001$.

Monetized Quantity Flows and Income Distribution Inequalities

Consider a system of monetized quantity flows in which the n nodes are agents that are transferring money to another agent $i \xrightarrow{a_{ii}w_{ij}>0} j$ in exchange for goods or services, and the income distribution inequalities that result from the network of such exchanges (Kuznets 1955; Amiel and Cowell 1999; Esping-Andersen 2007). Figure 4 illustrates a strongly connected network of such flows.

Let $\mathbf{x}(0)^T = [422\ 359\ 558\ 742\ 424\ 429]$, $\sum_i x_i(0) = 2934$, and

$$\mathbf{W} = \begin{bmatrix} 0 & 0.22 & 0.15 & 0.22 & 0.10 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.49 & 0 & 0.51 & 0 \\ 0 & 0.69 & 0 & 0.31 & 0 & 0 \\ 0.01 & 0.46 & 0 & 0.23 & 0.30 & 0 \\ 0.62 & 0.38 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The normal Markov process gives

$$\begin{bmatrix} 588 \\ 762 \\ 173 \\ 257 \\ 210 \\ 944 \end{bmatrix}^T = \begin{bmatrix} 422 \\ 359 \\ 558 \\ 742 \\ 424 \\ 429 \end{bmatrix}^T \begin{bmatrix} 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \\ 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \\ 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \\ 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \\ 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \\ 0.2003 & 0.2598 & 0.0589 & 0.0877 & 0.0715 & 0.3219 \end{bmatrix},$$

in which each i's $x_i(\infty)$ is based on the same fraction of every node's initial quantity, for example, 0.2003 in the case of node 1. Let the $\mathbf{A}$ of the generalized Markov flow process be

$$\mathbf{A} = \begin{bmatrix} 0.5472 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1386 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1493 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2575 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8407 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2543 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 240 \\ 797 \\ 552 \\ 755 \\ 107 \\ 483 \end{bmatrix}^T = \begin{bmatrix} 422 \\ 359 \\ 558 \\ 742 \\ 424 \\ 429 \end{bmatrix}^T \begin{bmatrix} 0.4676 & 0.1757 & 0.0778 & 0.1136 & 0.0135 & 0.1517 \\ 0.0104 & 0.8770 & 0.0017 & 0.0025 & 0.0003 & 0.1081 \\ 0.0009 & 0.0410 & 0.9180 & 0.0174 & 0.0175 & 0.0052 \\ 0.0020 & 0.1693 & 0.0003 & 0.8074 & 0.0001 & 0.0209 \\ 0.0111 & 0.4993 & 0.0019 & 0.2114 & 0.2133 & 0.0630 \\ 0.0747 & 0.1124 & 0.0124 & 0.0181 & 0.0022 & 0.7801 \end{bmatrix},$$

in which each i's $x_i(\infty)$ is based on a different fraction of every node's initial quantity.

Tournaments of Pay-It-Forward Altruism

A network $\mathcal{D}$ is a tournament if and only if it is a complete asymmetric structure. Such structures may be either strongly or unilaterally connected. Figure 5 is an example of an unilateral tournament. Let $\mathbf{x}(0)$ be the distribution of n individuals' capacities for altruistic actions (also referred as generalized or unilateral exchange) that transfer gifts to other individuals without any expectation of immediate recompense (Malinowski 1920; Yamagishi and Cook 1993; Bearman 1997; Michalski 2003; Molm, Collett, and Schaefer 2007). The $\mathbf{W}$ that is associated with $\mathcal{D}$ governs the distribution of gifts. The system's total supply of gifts, $\sum_{i=1}^{n} x_i(k) = \sum_{i=1}^{n} x_i(0)$ $\forall k$, remains constant over time periods. Because this tournament is an unilaterally connected structure, it has only one sink component (trivial or nontrivial), and the steady state of the normal Markov process will concentrate the total supply of gifts in it. In contrast, the generalized model allows all individuals to withhold gifts, and each individual in a nontrivial component will have a heterogeneous accumulation of gifts.

Let $\mathbf{x}(0)^T = [3\ 10\ 7\ 3\ 6\ 9]$, $\sum_i x_i(0) = 38$, and

$$\mathbf{W} = \begin{bmatrix} 0.13 & 0.17 & 0.13 & 0.10 & 0.47 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.26 & 0.10 & 0.10 & 0.14 & 0.40 \\ 0 & 0.06 & 0 & 0.34 & 0.60 & 0 \\ 0 & 0.55 & 0 & 0 & 0.45 & 0 \\ 0.27 & 0.24 & 0 & 0.17 & 0.18 & 0.14 \end{bmatrix}.$$

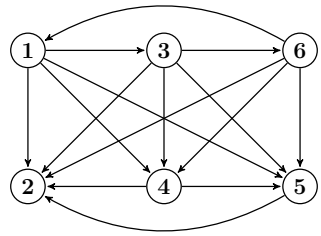

Figure 5: Unilateral tournament ($a_{ii}w_{ii} > 0$ loops are not displayed).

The normal Markov process gives

$$\begin{bmatrix} 0 \\ 38 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 3 \\ 10 \\ 7 \\ 3 \\ 6 \\ 9 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

in which each i's $x_i(\infty)$ is based on the same fraction of every node's initial supply of gifts. Let the $\mathbf{A}$ of the generalized Markov flow process be

$$\mathbf{A} = \begin{bmatrix} 0.7094 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7547 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2760 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6797 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6551 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1626 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 \\ 16 \\ 5 \\ 2 \\ 5 \\ 9 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 17 \\ 5 \\ 2 \\ 5 \\ 8 \end{bmatrix}^T \begin{bmatrix} 0.3203 & 0.3573 & 0.0757 & 0.0339 & 0.2030 & 0.0099 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.0016 & 0.1124 & 0.7449 & 0.0133 & 0.0303 & 0.0973 \\ 0 & 0.3240 & 0 & 0.4166 & 0.2594 & 0 \\ 0 & 0.5109 & 0 & 0 & 0.4891 & 0 \\ 0.0144 & 0.0804 & 0.0034 & 0.0133 & 0.0311 & 0.8574 \end{bmatrix},$$

in which each i may withhold gifts, and each i's $x_i(\infty)$ is based on a different fraction of every node's initial quantity.

Resource Flows in Organizations' Tree Structures

A weakly connected $\mathcal{D}$ is a tree from unique node if and only if exactly one node has indegree 0 and every other node has indegree 1. Figure 6 is an example of such structures. Let a countable quantity (funds or materials) be distributed from the indegree 0 node to the other nodes of the tree (Pfeffer and Salancik 1974; Winkofsky, Baker, and Sweeney 1981; Garcia, Calantone, and Levine 2003). The $\mathbf{W}$ associated with the structure admits a family of possible steady state distributions of quantity under the rule that each $i \xrightarrow{w_{ij}} j$, $i \neq j$, arc of the tree has a $w_{ij} > 0$. The inevitable steady state quantity distribution locates all quantity in the four sink nodes. In contrast, the generalized model allows all nodes with $0 \leq w_{ii} < 1$ to retain a positive fraction of the quantity supply.

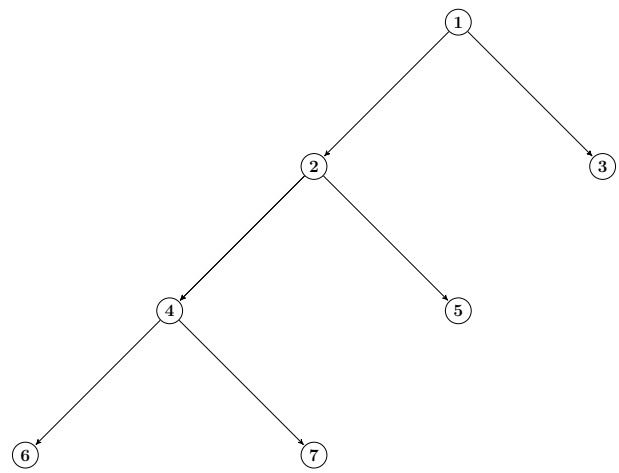

Figure 6: Tree structure ($a_{ii}w_{ii} > 0$ loops are not displayed).

Let $\mathbf{x}(0)^T = [1000\ 0\ 0\ 0\ 0\ 0]$ and

$$\mathbf{W} = \begin{bmatrix} 0.10 & 0.70 & 0.20 & 0 & 0 & 0 & 0 \\ 0 & 0.20 & 0 & 0.60 & 0.20 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 & 0 & 0.40 & 0.45 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The normal Markov process gives

$$\begin{bmatrix} 0 \\ 0 \\ 222 \\ 0 \\ 194 \\ 275 \\ 309 \end{bmatrix}^T = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0.2222 & 0 & 0.1944 & 0.2745 & 0.3088 \\ 0 & 0 & 0 & 0 & 0.2500 & 0.3529 & 0.3971 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4706 & 0.5294 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

in which all quantity is distributed to the four sink nodes. Let the $\mathbf{A}$ of the generalized Markov flow process be

$$\mathbf{A} = \begin{bmatrix} 0.6221 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3510 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5132 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4018 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0760 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2399 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1233 \end{bmatrix}.$$

Then the result,

$$\begin{bmatrix} 403 \\ 324 \\ 133 \\ 67 \\ 35 \\ 18 \\ 20 \end{bmatrix}^T = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0.4030 & 0.3241 & 0.1327 & 0.0669 & 0.0351 & 0.0180 & 0.0202 \\ 0 & 0.6980 & 0 & 0.1442 & 0.0755 & 0.0387 & 0.0436 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6366 & 0 & 0.1710 & 0.1924 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

is that each i on the path leading to a sink node may store a positive fraction of the distributed quantity.

Refugee Flows

Our last illustration considers refugee flows (Pan and Nagurney 1994; Keely 1996; Moorthy and Brathwaite 2019) and returns to the Figure 1 weakly connected network $\mathcal{D}$ of refugees flows in which each node is a nation state. Let $\mathbf{x}(0)^T = [1000, 0 \ \ldots \ 0]$. The w_{ii} of the two sink nations (14 and 17) must be $w_{ii} = 1$, the two w_{ij} arcs of all other 15 nations must be $0 < w_{ij} < 1$, and the w_{ii} of the 15 nations must be $0 \leq w_{ii} < 1$. Then, in the steady state of the classic Markov process, the 1000 refugees must be completely distributed into the two sink nations. The only variable is the proportion of refugees distributed in the two sinks. The steady state of the generalized Markov flow process with $a_{11} = 1$, and $0 < a_{ii} < 1$ for all other nations, allows a settlement of a positive fraction of the refugees in every nation 2 to 17. Let $\mathbf{x}(0)^T = [1000\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ and

$$\mathbf{W} = \begin{bmatrix}
0.34 & 0.34 & 0.32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.30 & 0 & 0.03 & 0.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.40 & 0 & 0.23 & 0.36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.39 & 0 & 0 & 0.43 & 0.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.37 & 0 & 0 & 0.14 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.29 & 0 & 0 & 0.51 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.32 & 0 & 0 & 0 & 0.23 & 0 & 0 & 0.45 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.43 & 0 & 0 & 0.22 & 0.35 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0 & 0 & 0.59 & 0.39 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0.17 & 0 & 0 & 0 & 0.75 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.39 & 0 & 0 & 0.37 & 0.25 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.41 & 0 & 0 & 0.23 & 0.37 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.13 & 0 & 0 & 0.37 & 0.50 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.47 & 0 & 0.47 & 0.06 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0 & 0.20 & 0.72 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

The normal Markov process, $\mathbf{x}(k+1)^T = \mathbf{x}(k)^T\mathbf{W}$, $k = 0, 1, 2, \ldots$, concentrates the refugees in the two sink nations (14 and 17):

$$\mathbf{x}(\infty)^T = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 209\ 0\ 0\ 791]$$

Let the a_{ii} values of the generalized Markov flow process be $a_{11} = 1$ for nation 1 with the following a_{ii}, $i \neq 1$, for nations 2 to 17: 0.7730, 0.7715, 0.8546, 0.5427, 0.5943, 0.2631, 0.4957, 0.9914, 0.1346, 0.3307, 0.0151, 0.8104, 0.7276, and 0.8010, respectively.

Then the equilibrium distribution of the 1000 refugees,

$$\mathbf{x}(\infty)^T = [0\ 329\ 304\ 5\ 148\ 32\ 2\ 16\ 28\ 12\ 3\ 47\ 11\ 4\ 4\ 7\ 48],$$

is a distribution in which each nation on the paths leading to the two sink nations has a positive settled fraction of the 1000 refugees. The assumption of $0 < a_{ii} < 1$ for all nations $i \neq 1$ may be relaxed to allow other instances of $a_{ii} = 1$, in which case the steady state fraction of refugee settlement would be zero in those nations. For example, if nations 2 and 3 have $a_{ii} = 1$, then all refugees from nation 1 to them will flow to nations 4 to 6. Thus, for a given $\mathbf{W}$, the generalization allows a dramatically larger domain of refugee distributions.

Discussion

We have endeavored to be careful in emphasizing the social science motivations for introducing our generalization of the classic Markov model. Given the vast literature on Markov processes, we also have endeavored to find instances of prior work that have addressed the concerns that prompted the present formulation. We currently believe that the formulation is novel and that it is a potentially useful movement toward greater realism, especially in regard to social science applications. It is interesting to note that the fixed point of the iteration

$$\mathbf{V}(k) = \mathbf{AWV}(k-1) + \mathbf{I} - \mathbf{A},\ k = 1, 2, \ldots,\ \ \mathbf{V}(0) = \mathbf{I},$$

is found from the equation $\mathbf{V} = \mathbf{AWV} + \mathbf{I} - \mathbf{A}$ and, in turn,

$$\mathbf{x}(\infty)^\top = \mathbf{x}(0)^\top \mathbf{AWV} + \mathbf{x}(0)^\top (\mathbf{I} - \mathbf{A}).$$

It also is interesting to note the duality of the classic Markov model and the DeGroot (1974) influence system model. A similar duality occurs with the generalized model and the Friedkin and Johnsen (1999) influence system model. The existence of mathematical dualities of systems that apply to substantively different phenomena is always intriguing when they point to an open question of why it is so that these different phenomena are mathematically related.

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