Positive solutions for autonomous and non-autonomous nonlinear critical elliptic problems in unbounded domains

Positive solutions for autonomous and non-autonomous nonlinear critical elliptic problems in unbounded domains

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Abstract. The paper concerns with positive solutions of problems of the type $-\Delta u+a(x)\,u = u^{p-1}+\varepsilon u^{2^*-1}$ in $\Omega \subseteq \mathbb{R}^N$, $N \ge 3$, $2^* = \frac{2N}{N-2}$, $2 < p < 2^*$. Here Ω can be an exterior domain, i.e. $\mathbb{R}^N \setminus \Omega$ is bounded, or the whole of $\mathbb{R}^N$. The potential $a \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$ is assumed to be strictly positive and such that there exists $\lim_{|x|\to\infty} a(x) := a_\infty > 0$. First, some existence results of ground state solutions are proved. Then the case $a(x) \ge a_\infty$ is considered, with $a(x) \equiv a_\infty$ or $\Omega = \mathbb{R}^N$. In such a case, no ground state solution exists and the existence of a bound state solution is proved, for small ε.

Keywords. Schrödinger equations, Unbounded domains, Critical nonlinearity.

1. Introduction and main results

This paper deals with a class of problems of the type

$$(P_\varepsilon)\qquad \begin{cases} -\Delta u + a(x)\,u = u^{p-1} + \varepsilon u^{2^*-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u \in H^1_0(\Omega) \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \ge 3$, and we consider both the case $\Omega = \mathbb{R}^N$ and $\mathbb{R}^N \setminus \Omega$ bounded with smooth boundary; $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\varepsilon > 0$, $2 < p < 2^*$ and on the potential we assume

$$a \in L^{N/2}_{\rm loc}(\mathbb{R}^N), \qquad \lim_{|x|\to\infty} a(x) = a_\infty, \qquad a(x) \ge a_0 > 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{1.1}$$

Problem (P_ε) has a variational structure: its solutions correspond to the nonnegative functions that are critical points of the functional $E_\varepsilon : H^1_0(\Omega) \to \mathbb{R}$

defined by

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)\, u^2)\, dx - \frac{1}{p}\int_\Omega |u|^p\, dx - \frac{\varepsilon}{2^*}\int_\Omega |u|^{2^*}\, dx.$$

Problems of the type (P_ε) have been widely studied: it is well known that they come from problems in Physics and Mathematical Physics like Schrödinger equations and Klein-Gordon equations, and from other applied and theoretical sciences. From a mathematical point of view, problems like (P_ε) present a number of difficulties related to the lack of compactness due both to the critical exponent and to the unboundedness of the domain. If $\mathbb{R}^N \setminus \Omega$ is a ball and a is radially symmetric, then a classical feature is to employ the compactness of the embedding of $H^1_{rad}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, that allows to recover existence results and qualitative properties of solutions for equations of the type $-\Delta u + a(|x|)u = f(|x|, u)$ ([12, 30]).

For exterior domains and potentials without any symmetry, several papers treat the subcritical case, i.e. $\varepsilon = 0$ in (P_ε), starting from the seminal papers [7], concerning the autonomous case, and [5, 6], concerning also the nonautonomous case; in those papers the authors analyze how the lack of compactness works. Then, many papers deal with the non autonomous case, in the subcritical setting (see [15, 17, 25, 26] and references therein). We refer the reader to [3, 4, 9–11, 21, 28] and references therein for related problems in unbounded domains, with non homogeneous nonlinearities having different asymptotic behaviour at zero and at infinity.

When $\varepsilon > 0$ it is interesting to study problem (P_ε) because there is an overlapping between the effects of the subcritical and the critical growth in the nonlinearity. Actually, if $\varepsilon > 0$, the analysis of the Palais-Smale sequences done in the subcritical case does not work, so that it is not possible to apply in a straight way the methods developed in the cited papers. Indeed, some concentration phenomena can appear, related to the critical nonlinearity. Of course, if ε is very large the effect of the critical nonlinearity is relevant, as one can see, for example, in [27]. In [27] the authors prove the existence of solutions of problems similar to (P_ε), in bounded domains, and point out some concentration effects as $\varepsilon \to \infty$.

Here we want to analyze the problem for small ε, so that we have a critical perturbation of the subcritical case. Then, besides the analysis of the lack of compactness as in [7], we make a further study of the Palais-Smale sequences, that takes into account the concentration phenomena in the spirit of [8, 14, 24, 29]. We emphasize that in the problems considered in this paper, in order to study the compactness, it is not possible to use only the classical analysis of the compactness developed in the subcritical case, nor the classical analysis developed in the critical case, but some delicate estimates involving both cases need (see Proposition 3.2). The aim of this analysis will be not only to show that compactness is restored below a "bad energy level", but also that it is restored in a suitable range above this "bad level". This done, we can recover a result similar to the well known result stated in [6] about the existence of a bound state solution in the subcritical case.

The first results we prove concern ground state solutions.

Theorem 1.1. *If* $\Omega = \mathbb{R}^N$, $a(x)$ *verifies* (1.1) *and*

$$a(x) \le a_\infty \quad a.e.\ in\ \mathbb{R}^N, \tag{1.2}$$

then there exists $\varepsilon_0 > 0$ *such that problem* (P_ε) *has a ground state solution for every* $\varepsilon \in (0, \varepsilon_0)$.

In Theorem 1.1 assumption (1.2) allows to apply in a straight way concentration-compactness arguments. Let us consider now the case in which at least one of the assumptions of Theorem 1.1 is not true, that is either $a(x) > a_\infty$ in a positive measure subset of $\mathbb{R}^N$, or $\Omega \neq \mathbb{R}^N$. Then, the existence of a ground state solution is not guarantee. To check the existence of such a solution, the potential $a(x)$ has to be below a_∞ in a suitable large region of Ω, to balance the effect of the boundary of Ω or of the part of $\mathbb{R}^N$ in which $a(x)$ has higher values than a_∞. In order to state a quantitative assumption, we introduce the problem

$$(P_\infty) \qquad \begin{cases} -\Delta u + a_\infty u = |u|^{p-2} u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Then, we denote by w the ground state, positive, radial solution of (P_∞) and we call

$$w_z(x) := \vartheta(x)\, w(x - z), \qquad z \in \mathbb{R}^N, \tag{1.3}$$

where $\vartheta \equiv 1$ if $\Omega = \mathbb{R}^N$, otherwise ϑ is a cut-off function verifying

$$\vartheta \in C^\infty(\mathbb{R}^N, [0, 1]), \qquad \begin{cases} \vartheta(x) = 1 \text{ if } \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega) \ge 1 \\ \vartheta(x) = 0 \text{ if } x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

Theorem 1.2. *Assume that* $a(x)$ *verifies* (1.1). *If there exists* $z \in \mathbb{R}^N$ *such that*

$$\int_\Omega \left| \nabla \frac{w_z}{|w_z|_{L^p(\Omega)}} \right|^2 dx + \int_\Omega a(x) \left(\frac{w_z}{|w_z|_{L^p(\Omega)}} \right)^2 dx < \frac{\|w\|^2_{H^1(\mathbb{R}^N)}}{|w|^2_{L^p(\mathbb{R}^N)}}, \tag{1.5}$$

then problem (P_ε) *has a ground state solution for small* ε.

We point out that the r.h.s. in (1.5) is a constant independent of the domain and the potential $a(x)$, and we observe that if $a(x) \equiv a_\infty$ and $\Omega = \mathbb{R}^N$ then in (1.5) the equality holds for every z in $\mathbb{R}^N$.

Consider now $a(x) \ge a_\infty$. In Proposition 4.1 we state that if $\Omega = \mathbb{R}^N$ or $a(x) = a_\infty$, then a ground state solution for (P_ε) does not exist. In this setting, to find a solution one has to look at higher energy critical levels and this is more difficult than the minimizing problem. A first difficulty to be faced concerns compactness above the ground state of some related limit problems. By the concentration phenomenon due to the critical nonlinearity, the problem to be considered is

$$(CP_\varepsilon) \qquad \begin{cases} -\Delta u = \varepsilon |u|^{2^*-2} u & \text{in } \mathbb{R}^N, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

while the natural limit problem related to the translations is

$$(P_{\varepsilon,\infty}) \qquad \begin{cases} -\Delta u + a_\infty u = |u|^{p-2}u + \varepsilon |u|^{2^*-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

A proof of the existence of a ground state solution of $(P_{\varepsilon,\infty})$, for small ε, is proved in [1]. Looking for least energy solutions of $(P_{\varepsilon,\infty})$, the minimization problem to deal with is

$$m_\varepsilon := \inf_{\mathcal{N}_{\varepsilon,\infty}} E_{\varepsilon,\infty}, \tag{1.6}$$

where

$$E_{\varepsilon,\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a_\infty u^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx$$

and

$$\mathcal{N}_{\varepsilon,\infty} = \left\{ u \in H^1(\mathbb{R}^N)\setminus\{0\} : E'_{\varepsilon,\infty}(u)[u] = 0 \right\}.$$

Testing the functional $E_{\varepsilon,\infty}$ on a concentrating sequence of least energy solutions of (CP_ε), in Proposition 2.3 we show that for all $\varepsilon > 0$

$$m_\varepsilon \le \frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}, \tag{1.7}$$

where S is the best Sobolev constant. We observe that the value $\frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}$ in (1.7) is the ground state level of the solutions of problem (CP_ε), that is

$$\frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}} = \min \left\{ \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx : \ u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |\nabla u|^2 dx = \varepsilon \int_{\mathbb{R}^N} |u|^{2^*} dx \right\} \tag{1.8}$$

(see Proposition 2.3). So, the compactness cannot hold at the level $\frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}$ neither for problem (P_ε) nor for problem $(P_{\varepsilon,\infty})$, by the concentration phenomenon described.Here we give an alternative proof of the existence of solutions of (1.6), that will be useful in the paper. As a consequence of that proof, we get that actually $m_\varepsilon < \frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}$, for small ε (see Theorem 2.2 and Corollary 2.4). Nevertheless, neither unicity nor nondegeneracy of the positive solution of $(P_{\varepsilon,\infty})$ are known. Hence, it is not possible to obtain a complete picture of the lack of compactness,as in the purely subcritical or critical case. Anyway, a local Palais-Smale condition can be restored for small ε by using the solutions of (P_∞). This done, we can prove the existence of a solution both for the autonomous and for the non autonomous problem, in $\mathbb{R}^N$ or in exterior domains.

In [1] the authors consider problem (P_ε) in the autonomous case $a(x) \equiv a_\infty$ and they found a solution assuming that $\mathbb{R}^N \setminus \Omega$ is contained in a small ball. That result here is improved, because we have no assumption on the size of $\mathbb{R}^N \setminus \Omega$. In order to find a solution for every exterior domain, a fundamental tool is a fine estimate of the interactions of "almost minimizing" functions.

Indeed, this estimate allows us to work in a suitable compactness range (see Lemma 4.4).

Our result is the following

Theorem 1.3 *Assume that $a(x)$ verifies (1.1) and*

$$a(x) \geq a_\infty, \qquad \int_{\mathbb{R}^N} (a(x) - a_\infty)|x|^{N-1} e^{2\sqrt{a_\infty}|x|}\, dx < \infty, \tag{1.9}$$

then there exists $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ problem (P_ε) has at least one positive solution, that is a bound state solution when $\Omega \neq \mathbb{R}^N$ or $a(x) \not\equiv a_\infty$

Remark 1.4 If both $\Omega = \mathbb{R}^N$ and $a \equiv a_\infty$ hold, problem (P_ε) is nothing but $(P_{\varepsilon,\infty})$ and Theorem 1.3 coincides with Theorem 2.2.

The paper is organized as follows: in Sect. 2 we introduce some notations and recall some known facts we use; Sect. 3 deals with ground state solutions; in Sect. 4 the proof of Theorem 1.3 is developed, moreover we report some remarks that describe the asymptotic shape of the solution given by Theorem 1.3 and some ways to use it to get multiplicity results (see Remarks 4.12 and 4.13).

2. Notations and preliminary results

Without any loss of generality we may assume $a_\infty = 1$, up to a rescaling, and $0 \in \mathbb{R}^N \setminus \Omega$ if $\Omega \neq \mathbb{R}^N$. Throughout the paper we make use of the following notation:

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) := \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)dx; \qquad \|u\|^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx. \tag{2.10}$$

  We shall use also the equivalent norm

$$\|u\|_a^2 := \int_\Omega \left(|\nabla u|^2 + a(x)u^2\right) dx.$$

- H^{-1} denotes the dual space of $H^1(\mathbb{R}^N)$.
- $D^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_D := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}$.
- $L^q(\mathcal{O})$, $1 \leq q \leq \infty$, $\mathcal{O} \subseteq \mathbb{R}^N$ a measurable set, denotes the Lebesgue space, the norm in $L^q(\mathcal{O})$ is denoted by $|\cdot|_{L^q(\mathcal{O})}$ when $\mathcal{O}$ is a proper measurable subset of $\mathbb{R}^N$ and by $|\cdot|_q$ when $\mathcal{O} = \mathbb{R}^N$.
- For $u \in H_0^1(\Omega)$ we denote by u also the function in $H^1(\mathbb{R}^N)$ obtained setting $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.
- S denotes the best Sobolev constant, namely

$$S = \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{|u|_{2^*}^2}.$$

- For any $\rho > 0$ and for any $z \in \mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centered at z, and for any measurable set $O \subset \mathbb{R}^N$, $|O|$ denotes its Lebesgue measure.
- $c, c', C, C', C_i, \ldots$ denote various positive constants.

When $\varepsilon = 0$, (P_ε) becomes

$$(P)\qquad \begin{cases} -\Delta u + a(x)\, u = u^{p-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u \in H^1_0(\Omega) \end{cases}$$

and the related action functional is $E : H^1_0(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)\, u^2)\, dx - \frac{1}{p}\int_\Omega |u|^p\, dx.$$

Furthermore, we denote by E_∞, $E_{\varepsilon,\infty} : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functionals related to (P_∞) and $(P_{\varepsilon,\infty})$ respectively, defined by

$$E_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\, dx,$$

$$E_{\varepsilon,\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\, dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\, dx.$$

In a standard way, we consider the following Nehari manifolds:

$$\mathcal{N} = \left\{ u \in H^1_0(\Omega)\setminus\{0\} : E'(u)[u] = 0 \right\},$$

$$\mathcal{N}_\varepsilon = \left\{ u \in H^1_0(\Omega)\setminus\{0\} : E'_\varepsilon(u)[u] = 0 \right\},$$

$$\mathcal{N}_\infty = \left\{ u \in H^1(\mathbb{R}^N)\setminus\{0\} : E'_\infty(u)[u] = 0 \right\},$$

$$\mathcal{N}_{\varepsilon,\infty} = \left\{ u \in H^1(\mathbb{R}^N)\setminus\{0\} : E'_{\varepsilon,\infty}(u)[u] = 0 \right\}.$$

Remark that there exists $c > 0$ independent of small ε such that

$$\|u\| \geq c \quad \forall u \in \mathcal{N}_{\varepsilon,\infty}, \qquad \|u\|_a \geq c \quad \forall u \in \mathcal{N}_\varepsilon, \tag{2.11}$$

indeed

$$0 = \|u\|^2 - |u|_p^p - \varepsilon|u|_{2^*}^{2^*} \geq \|u\|^2 - c_1\|u\|^p - c_1\varepsilon\|u\|^{2^*}, \quad \forall u \in \mathcal{N}_{\varepsilon,\infty},$$

$$0 = \|u\|_a^2 - |u|_p^p - \varepsilon|u|_{2^*}^{2^*} \geq \|u\|_a^2 - c_2\|u\|_a^p - c_2\varepsilon\|u\|_a^{2^*}, \quad \forall u \in \mathcal{N}_\varepsilon.$$

Straight computations allow to state the following

Lemma 2.1. *Let $u \in H^1_0(\Omega)\setminus\{0\}$ and $v \in H^1(\mathbb{R}^N)\setminus\{0\}$, then:*

- *$tu \in \mathcal{N}$ if and only if* $t = \left(\frac{\|u\|_a^2}{|u|_p^p}\right)^{\frac{1}{p-2}}$;
- *$tv \in \mathcal{N}_\infty$ if and only if* $t = \left(\frac{\|v\|^2}{|v|_p^p}\right)^{\frac{1}{p-2}}$;

- $tu \in \mathcal{N}_\varepsilon$ if and only if $\|u\|_a^2 = t^{p-2}|u|_p^p + \varepsilon t^{2^*-2}|u|_{2^*}^{2^*}$;
- $tv \in \mathcal{N}_{\varepsilon,\infty}$ if and only if $\|v\|^2 = t^{p-2}|v|_p^p + \varepsilon t^{2^*-2}|v|_{2^*}^{2^*}$.

Moreover, $t_u > 0$ *such that* $t_u u \in \mathcal{N}$ *is characterized as the unique real value such that*

$$E(t_u u) = \max_{t>0} E(tu)$$

and $u \to t_u$ *is a continuous map from* $H_0^1(\Omega)\setminus\{0\}$ *in* $\mathbb{R}^+$. *Analogous results hold if we consider* E_∞, E_ε *and* $E_{\varepsilon,\infty}$ *respectively on* $\mathcal{N}_\infty$, $\mathcal{N}_\varepsilon$ *and* $\mathcal{N}_{\varepsilon,\infty}$.

Let us define:

$$m = \inf_{\mathcal{N}_\infty} E_\infty, \qquad m_\varepsilon = \inf_{\mathcal{N}_{\varepsilon,\infty}} E_{\varepsilon,\infty}. \tag{2.12}$$

We denote by w the unique positive solution, up to translations, of the problem (P_∞); it is well known that $w \in C^\infty(\mathbb{R}^N)$, w is radially symmetric about the origin, and

$$w(|x|)\, e^{|x|}\, |x|^{(N-1)/2} \to c \quad \text{as } |x| \to +\infty, \tag{2.13}$$

$$w'(|x|)\, e^{|x|}\, |x|^{(N-1)/2} \to -c \quad \text{as } |x| \to +\infty, \tag{2.14}$$

with $c > 0$; moreover $w \in \mathcal{N}_\infty$ and $E_\infty(w) = m$, namely w is the ground state solution of (P_∞) (see [12, 22, 23] and also (2.19), (2.20) in [6] for a precis estimate of c in (2.13) and (2.14)).

For the limit problem $(P_{\varepsilon,\infty})$ the following existence result holds.

Theorem 2.2. *There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *problem* $(P_{\varepsilon,\infty})$ *has a positive radially symmetric ground state solution* w_ε.

Proof. We first observe that $m_\varepsilon \le m$, $\forall \varepsilon > 0$. Indeed let $\tau_\varepsilon > 0$ be such that $\tau_\varepsilon w \in \mathcal{N}_{\varepsilon,\infty}$, then

$$m_\varepsilon \le E_{\varepsilon,\infty}(\tau_\varepsilon w) \le E_\infty(\tau_\varepsilon w) \le E_\infty(w) = m. \tag{2.15}$$

As shown in [30], by Schwartz symmetrization, in order to solve the minimization problem for m_ε we can restrict our considerations to

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ radially symmetric}\}, \qquad \mathcal{N}_r = \mathcal{N}_{\varepsilon,\infty} \cap H_r^1(\mathbb{R}^N).$$

Let $\{u_n\}_n$ in $\mathcal{N}_r$ be a minimizing sequence, that is

$$\|u_n\|^2 = |u_n|_p^p + \varepsilon |u_n|_{2^*}^{2^*}, \tag{2.16}$$

$$E_{\varepsilon,\infty}(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right)\varepsilon|u_n|_{2^*}^{2^*} = m_\varepsilon + o(1). \tag{2.17}$$

Inequalities (2.15) and (2.17) imply that

$$\|u_n\|^2 \le \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} m_\varepsilon + o(1) \le \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} m + o(1). \tag{2.18}$$

Observe that from (2.16), (2.11), (2.18) and the Sobolev embedding Theorem it follows the existence of $\varepsilon_0 > 0$ such that, for all $n \in \mathbb{N}$,

$$|u_n|_p^p \ge \|u_n\|^2 - c\,\varepsilon \|u_n\|^{2^*} \ge \text{const} > 0 \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{2.19}$$

Now, since $H^1_r(\mathbb{R}^N)$ embeds compactly in $L^p(\mathbb{R}^N)$ (see [30]) we deduce the existence of $w_\varepsilon \in H^1_r(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \xrightarrow{n\to\infty} w_\varepsilon \qquad \begin{array}{l}\text{strongly in } L^p(\mathbb{R}^N)\\ \text{weakly in } H^1(\mathbb{R}^N) \text{ and in } L^{2^*}(\mathbb{R}^N),\end{array} \tag{2.20}$$

moreover by (2.19) $w_\varepsilon \neq 0$. By Ekeland's variational principle the minimizing sequence $\{u_n\}_n$ in $\mathcal{N}_r$ can be chosen such that

$$E'_{\varepsilon,\infty}(u_n)[v] = \lambda_n G'(u_n)[v] + o(1)\|v\| \qquad \forall v \in H^1_r(\mathbb{R}^N) \tag{2.21}$$

where, for all $n \in \mathbb{N}$, $\lambda_n \in \mathbb{R}$ is the Lagrange multiplier and $G(u) = E'_{\varepsilon,\infty}(u)[u]$. By definition of $\mathcal{N}_{\varepsilon,\infty}$, $G(u_n) = 0$ for all $n \in \mathbb{N}$, so using (2.21), we deduce

$$0 = G(u_n) = E'_{\varepsilon,\infty}(u_n)[u_n] = \lambda_n G'(u_n)[u_n] + o(1)\|u_n\|. \tag{2.22}$$

Hence, taking into account that $\|u_n\|$ is bounded and that $G'(u_n)[u_n] \leq c < 0$ on $\mathcal{N}_r$, we get $\lambda_n = o(1)$. Choosing $v = w_\varepsilon$ in (2.21), by (2.20) and standard arguments it follows that $w_\varepsilon \in \mathcal{N}_{\varepsilon,\infty}$.

Using again (2.20), we get

$$m_\varepsilon \leq E_{\varepsilon,\infty}(w_\varepsilon) \leq \liminf_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{2^*}\right)\|u_n\|^2 - \left(\frac{1}{p}-\frac{1}{2^*}\right)|u_n|_p^p\right] = m_\varepsilon,$$

that is w_ε is the minimizing function we are looking for. Thus, w_ε solves

$$-\Delta u + u = |u|^{p-2}u + \varepsilon|u|^{2^*-2}u \qquad \text{in } \mathbb{R}^N. \tag{2.23}$$

In order to verify that w_ε is strictly positive we just observe that $|w_\varepsilon|$ too is a minimizer of $E_{\varepsilon,\infty}$ constrained on $\mathcal{N}_{\varepsilon,\infty}$, so we can assume $w_\varepsilon \geq 0$. Furthermore, since w_ε solves (2.23), $w_\varepsilon > 0$ as a consequence of the maximum principle.

Proposition 2.3. The following estimate holds:

$$m_\varepsilon \leq \frac{1}{N} S^{N/2}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}} \qquad \forall \varepsilon > 0. \tag{2.24}$$

Sketch of the proof. Since the computations that prove (2.24) are classical, we only sketch them. First we verify (1.8). Observe that for every $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}$ the function tu verifies $\int_{\mathbb{R}^N}|\nabla(tu)|^2dx = \varepsilon\int_{\mathbb{R}^N}|tu|^{2^*}dx$ if and only if $t = \left(\frac{1}{\varepsilon}\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{\int_{\mathbb{R}^N}|u|^{2^*}dx}\right)^{\frac{1}{2^*-2}}$, and

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N}|\nabla(tu)|^2dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N}|tu|^{2^*}dx &= \left(\frac{1}{2}-\frac{1}{2^*}\right)t^2\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&= \frac{1}{N}\left(\frac{1}{\varepsilon}\right)^{\frac{2}{2^*-2}}\frac{\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{2}{2^*-2}+1}}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*-2}}}\\
&= \frac{1}{N}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}\left(\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{|u|^2_{2^*}}\right)^{\frac{N}{2}}\\
&\geq \frac{1}{N}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}S^{N/2}.
\end{aligned} \tag{2.25}$$

So we obtain (1.8) by (2.25), taking into account that the equality in (2.25) is attained by choosing u as a minimizing function for the Sobolev constant (see [2, 20, 31]).

Now, let $\bar{U} \in D^{1,2}(\mathbb{R}^N)$ be a fixed radial function that realizes the minimum in (1.8), for example consider $\bar{U}(x) = \frac{C}{(1+|x|^2)^{\frac{N-2}{2}}}$, where C is a normalizing constant. In order to prove (2.24), we consider the concentrating sequence of functions $v_n(x) = \zeta(|x|)\, n^{\frac{N-2}{2}} \bar{U}(nx)$, $n \in \mathbb{N}$, where $\zeta \in C_0^\infty(\mathbb{R}^+, [0,1])$ is a cut-off function such that $\zeta(s) = 1$, for $s \in [0,1]$. Then we test the functional $E_{\varepsilon,\infty}$ on the sequence of functions $u_n := t_n v_n$, $n \in \mathbb{N}$, where t_n is such that $u_n \in N_{\varepsilon,\infty}$, that is

$$|\nabla v_n|_2^2 + |v_n|_2^2 = t_n^{p-2}\, |v_n|_p^p + \varepsilon t_n^{2^*-2}\, |v_n|_{2^*}^{2^*}. \tag{2.26}$$

Well known estimates provided in [13] ensure that

$$|v_n - n^{\frac{N-2}{2}} \bar{U}(nx)|_{2^*} \xrightarrow{n\to\infty} 0, \tag{2.27}$$

$$|\nabla v_n - \nabla(n^{\frac{N-2}{2}} \bar{U}(nx))|_2 \xrightarrow{n\to\infty} 0, \tag{2.28}$$

$$v_n \xrightarrow{n\to\infty} 0, \text{ in } L^2(\mathbb{R}^N). \tag{2.29}$$

From (2.29) and the boundedness of $\{v_n\}_n$ in $L^{2^*}(\mathbb{R}^N)$ we obtain also $v_n \to 0$ in $L^p(\mathbb{R}^N)$, by interpolation. Hence, $t_n \to 1$ follows from (2.26) – (2.29) and so

$$E_{\varepsilon,\infty}(u_n) - \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla(n^{\frac{N-2}{2}} \bar{U}(nx))|^2 dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |n^{\frac{N-2}{2}} \bar{U}(nx)|^{2^*} dx\right] \xrightarrow{n\to\infty} 0.$$

Since

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla(n^{\frac{N-2}{2}} \bar{U}(nx))|^2 dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |n^{\frac{N-2}{2}} \bar{U}(nx)|^{2^*} dx = \frac{1}{N}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}} S^{N/2}, \qquad \forall n \in \mathbb{N},$$

then (2.24) is proved.

Corollary 2.4. For ε small the following estimate holds:

$$m_\varepsilon < \frac{1}{N} S^{N/2} \left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}. \tag{2.30}$$

Indeed, in the proof of Proposition 2.3 we have exhibited a sequence $\{v_n\}_n$ of radial functions in $N_{\varepsilon,\infty}$ that converges weakly to 0 in $L^{2^*}(\mathbb{R}^N)$ and such that $E_{\varepsilon,\infty}(v_n) \to \frac{1}{N}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}} S^{N/2}$. But in the proof of Theorem 2.2 we have proved that for small ε *every* minimizing sequence of radial functions converges weakl to a nonzero minimizing function of $E_{\varepsilon,\infty}$ on $N_{\varepsilon,\infty}$, up to a subsequence. Hence (2.30) must hold, for small ε.

Let us give another estimate of m_ε, more precise for small ε, and analyze its asymptotic behaviour.

Lemma 2.5. *For all $\varepsilon > 0$ the relation $m_\varepsilon \le m$ holds and*

$$\lim_{\varepsilon\to 0} m_\varepsilon = m.$$

Proof. Inequality $m_\varepsilon \le m$ has been shown in (2.15).

Now, for $\varepsilon \in (0, \varepsilon_0)$ let w_ε be the minimizing function whose existence is stated in Theorem 2.2 and $t_\varepsilon > 0$ be such that $t_\varepsilon w_\varepsilon \in N_\infty$, namely

$$t_\varepsilon = \left(\frac{\|w_\varepsilon\|^2}{|w_\varepsilon|_p^p}\right)^{\frac{1}{p-2}}. \tag{2.31}$$

Observe that w_ε is bounded, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, because

$$E_{\varepsilon,\infty}(w_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p}\right)\|w_\varepsilon\|^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right)\varepsilon|w_\varepsilon|_{2^*}^{2^*} = m_\varepsilon \le m.$$

Moreover, $|w_\varepsilon|_p^p \ge c > 0$ follows from (2.19). As a consequence, t_ε is bounded by (2.31) and

$$m \le E_\infty(t_\varepsilon w_\varepsilon) = E_{\varepsilon,\infty}(t_\varepsilon w_\varepsilon) + \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N}(t_\varepsilon w_\varepsilon)^{2^*}dx \tag{2.32}$$

$$\le E_{\varepsilon,\infty}(w_\varepsilon) + \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N}(t_\varepsilon w_\varepsilon)^{2^*}dx$$

$$= m_\varepsilon + o(1). \tag{2.33}$$

Inequality (2.32) completes the proof.

3. Existence of a ground state solution

In this section we prove Theorems 1.1 and 1.2, which provide some cases in which a least energy solution $\bar{u}$ of (P_ε) exists, that is $\bar{u} \in N_\varepsilon$ verifies

$$E_\varepsilon(\bar{u}) = \min_{N_\varepsilon} E_\varepsilon.$$

A basic tool to prove the existence of a ground state is the analysis of the Palais-Smale sequences at a level c ($(PS)_c$*-sequences for short*) below the minimum of the limit problem $(P_{\varepsilon,\infty})$. We start with the following lemma.

Lemma 3.1. *Let $c \in \mathbb{R}$ and let $\{u_n\}_n$ be a $(PS)_c$-sequence for E_ε, then $\{u_n\}_n$ is bounded and $c \ge 0$.*

Proof. From

$$E'_\varepsilon(u_n)[u_n] = \|u_n\|_a^2 - |u_n|_p^p - \varepsilon|u_n|_{2^*}^{2^*} = o(1)\|u_n\|$$

we infer

$$E_\varepsilon(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_a^2 + \varepsilon\left(\frac{1}{p} - \frac{1}{2^*}\right)|u_n|_{2^*}^{2^*} + o(1)\|u_n\| = c + o(1),$$

that implies our claims.

Proposition 3.2. Assume that $a(x)$ verifies (1.1). Let $\varepsilon > 0$ and $\{u_n\}_n$ be a $(PS)_c$-sequence for E_ε constrained on N_ε. If $c < m_\varepsilon$ then $\{u_n\}_n$ is relatively compact.

Proof. First, let us observe that the sequence $\{\|u_n\|_a\}_n$ is bounded away from 0, by (2.11), and that it is bounded above, by the argument developed in Lemma 3.1.

Then, arguing exactly as in (2.21) and (2.22), we get that $\{u_n\}_n$ is a $(PS)_c$-sequence also for the free functional E_ε, namely $\forall v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_n \cdot \nabla v\, dx + \int_\Omega a u_n v\, dx - \int_\Omega |u_n|^{p-2} u_n v\, dx - \varepsilon \int_\Omega |u_n|^{2^*-2} u_n v\, dx = o(1)\|v\|. \tag{3.34}$$

From now on, we denote by $\{u_n\}_n$ not only the sequence $\{u_n\}_n$ but also its subsequences.

Since $\{u_n\}_n$ is bounded in $H_0^1(\Omega)$, there exists a function $\bar{u} \in H_0^1(\Omega)$ such that

$$u_n \xrightarrow{n\to\infty} \bar{u} \quad \begin{cases} \text{weakly in } H_0^1(\Omega) \text{ and in } L^{2^*}(\Omega) \\ \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ and in } L^2_{\text{loc}}(\mathbb{R}^N) \\ \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.35}$$

By (3.35) and (3.34), $\bar{u}$ is a weak solution of (P_ε), hence

$$\|\bar{u}\|_a^2 = |\bar{u}|_p^p + \varepsilon|\bar{u}|_{2^*}^{2^*}. \tag{3.36}$$

We have to prove that $u_n \to \bar{u}$ in $H^1(\Omega)$. Assume by contradiction that $u_n \not\to \bar{u}$ in $H^1(\Omega)$, so the sequence $v_n := u_n - \bar{u}$ verifies $\|v_n\| \geq \hat{c} > 0$, $\forall n \in \mathbb{N}$. By (3.35) and the Brezis-Lieb Lemma,

$$E_\varepsilon(u_n) = E_\varepsilon(\bar{u}) + E_\varepsilon(v_n) + o(1) \tag{3.37}$$

and, since $\bar{u}$ is a solution of (P_ε), $\{v_n\}_n$ turns out to be a (PS)-sequence for E_ε. We claim that

$$|v_n|_{2^*}^{2^*} \geq \tilde{c} > 0. \tag{3.38}$$

If this is not the case, $u_n \to \bar{u}$ in $L^{2^*}(\Omega)$ and by interpolation in $L^p(\Omega)$, because $\{u_n\}_n$ is bounded in $L^2(\Omega)$. So, from $\|u_n\|_a^2 = |u_n|_p^p + \varepsilon|u_n|_{2^*}^{2^*}$ and (3.36) we get

$$\lim_{n\to\infty} \|u_n\|_a^2 = |\bar{u}|_p^p + \varepsilon|\bar{u}|_{2^*}^{2^*} = \|\bar{u}\|_a^2$$

which implies $u_n \to \bar{u}$ in $H^1(\Omega)$, contradicting our assumption.

Let $\{y_i\}_i = \mathbb{Z}^N$ and let us decompose $\mathbb{R}^N$ in the N-dimensional hypercubes Q_i with unitary sides and vertices in y_i. Since $v_n \in L^{2^*}(\mathbb{R}^N)$, we can define

$$d_n = \max_{i\in\mathbb{N}} |v_n|_{L^{2^*}(Q_i)} \qquad \forall n \in \mathbb{N}.$$

By (3.38) and the boundedness of $\{u_n\}_n$ in $H^1(\Omega)$

$$\begin{aligned} 0 < \tilde{c} \leq |v_n|_{2^*}^{2^*} &= \sum_{i=1}^{\infty} |v_n|_{L^{2^*}(Q_i)}^{2^*} \\ &\leq d_n^{2^*-2} \sum_{i=1}^{\infty} |v_n|_{L^{2^*}(Q_i)}^2 \leq c\, d_n^{2^*-2} \sum_{i=1}^{\infty} \|v_n\|_{H^1(Q_i)}^2 \\ &\leq \bar{c}\, d_n^{2^*-2}, \end{aligned} \tag{3.39}$$

and, then, $d_n \geq \gamma > 0$ $\forall n \in \mathbb{N}$, where $\gamma > 0$.

Now, let us call z_n the center of a hypercube Q_{i_n} such that

$$|v_n|_{L^{2^*}(Q_{i_n})} = d_n$$

and put

$$w_n(x) = v_n(x + z_n).$$

Since $\{v_n\}_n$ is a (PS)-sequence, $\{w_n\}_n$ is a (PS)-sequence, too.

Setting $Q_0 = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$, one of the following two cases occurs:

$$\begin{aligned} &(a) \quad \int_{Q_0} |w_n(x)|^p dx \geq c > 0 \\ &(b) \quad \int_{Q_0} |w_n(x)|^p dx \xrightarrow{n\to\infty} 0. \end{aligned} \tag{3.40}$$

Assume first that (3.40)(a) holds. Then $|z_n| \to \infty$ because $v_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, so, since $\{u_n\}_n$ is a (PS)-sequence and $\bar{u}$ is a solution of (P_ε), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla w_n \cdot \nabla\varphi\, dx + \int_{\mathbb{R}^N} w_n\varphi\, dx - \int_{\mathbb{R}^N} |w_n|^{p-2} w_n\varphi\, dx - \varepsilon\int_{\mathbb{R}^N} |w_n|^{2^*-2} w_n\varphi\, dx \\ &= \int_{\mathbb{R}^N} [1 - a(\cdot + z_n)]w_n\varphi\, dx + o(1)\varphi = o(1)\varphi, \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{3.41}$$

The sequence $\{w_n\}_n$ is bounded in $H^1(\mathbb{R}^N)$, so $\bar{w} \in H^1(\mathbb{R}^N)$ exists such that

$$w_n \xrightarrow{n\to\infty} \bar{w} \quad \begin{cases} \text{weakly in } H^1(\mathbb{R}^N) \text{ and in } L^{2^*}(\mathbb{R}^N) \\ \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \text{ and in } L^2_{\mathrm{loc}}(\mathbb{R}^N) \\ \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.42}$$

Now, from (3.40)(a), (3.41), (3.42) we deduce that $\bar{w}$ is a nonzero solution of $(P_{\varepsilon,\infty})$. Then $\{w_n - \bar{w}\}_n$ is a (PS)-sequence for $E_{\varepsilon,\infty}$ and $E_{\varepsilon,\infty}(w_n - \bar{w}) \geq o(1)$ can be deduced arguing as in Lemma 3.1. Hence, applying the Brezis-Lieb Lemma, we get

$$\begin{aligned} c = E_\varepsilon(u_n) + o(1) &= E_\varepsilon(\bar{u}) + E_{\varepsilon,\infty}(\bar{w}) \\ &+ E_{\varepsilon,\infty}(w_n - \bar{w}) + o(1) \geq E_{\varepsilon,\infty}(\bar{w}) + o(1) \geq m_\varepsilon + o(1) \end{aligned}$$

contrary to the assumption $c < m_\varepsilon$ and proving that (3.40) (a) can not be true.

To conclude the argument, we assume that (3.40) (b) holds and show tha a contradiction arises again. Remark that in this case we can also assume that

$$\tilde{d}_n = \max_{i\in\mathbb{N}} |v_n|_{L^p(Q_i)} = \max_{i\in\mathbb{N}} |w_n|_{L^p(Q_i)} \xrightarrow{n\to\infty} 0. \tag{3.43}$$

Indeed, if it is not true, we can argue by substituting Q_{i_n} with a cube $Q_{\tilde{i}_n}$ such that $|v_n|_{L^p(Q_{\tilde{i}_n})} \geq c_1 > 0$ and then proceed as in case (3.40)(a). So, let us assume (3.43). Then, rewriting the inequalities in (3.39) with the L^p-norm in place of the L^{2^*}-norm and $\tilde{d}_n$ in place of d_n, we obtain

$$|v_n|_p = |w_n|_p \xrightarrow{n\to\infty} 0. \tag{3.44}$$

Notice that (3.44) implies

$$w_n \xrightarrow{n\to\infty} 0 \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N). \tag{3.45}$$

Now, assume that $\{z_n\}_n$ is bounded, so that in our argument we can consider $z_n = 0$, $\forall n \in \mathbb{N}$. Let $R > 0$ be such that $|a(x) - 1| < \eta$ $\forall x \in \mathbb{R}^N \setminus B_R(0)$, where η is a suitable small constant to be fixed later. Consider the functionals $\hat{f}, \hat{f}_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\hat{f}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{B_R(0)} (a(x) - 1)u^2 dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx,$$

$$\hat{f}_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx.$$

Then, (3.34), (3.44), and (3.45) imply that $\{w_n\}_n$ is a (PS)-sequence also for $\hat{f}$. So, Theorem 2.5 of [8] applies: there exist a number $k \in \mathbb{N}\setminus\{0\}$, k sequences of points $\{y_n^j\}_n$, $1 \le j \le k$, k sequences of positive numbers $\{\sigma_n^j\}_n$, $1 \le j \le k$, with $\sigma_n^j \to 0$ because of (3.45), such that

$$w_n(x) = \sum_{j=1}^{k} (\sigma_n^j)^{-\frac{N-2}{2}} U_j\left(\frac{x - y_n^j}{\sigma_n^j}\right) + \phi_n(x),$$

with $\phi_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and U_j nontrivial solutions of

$$-\Delta U(x) = \varepsilon|U(x)|^{2^*-2}U(x) \qquad x \in \mathbb{R}^N; \tag{3.46}$$

moreover,

$$\hat{f}(w_n) = \sum_{j=1}^{k} \hat{f}_\infty(U_j) + o(1). \tag{3.47}$$

By the estimate of the ground state level of the solutions of (3.46) given in (1.8), we get

$$\hat{f}_\infty(U_j) \ge \frac{1}{N} S^{N/2}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}}. \tag{3.48}$$

Finally, by (3.37), (3.47), (3.44), (3.48) and Proposition 2.3 we have

$$\begin{aligned} E_\varepsilon(u_n) &= E_\varepsilon(\bar{u}) + E_\varepsilon(v_n) + o(1) \\ &\ge E_\varepsilon(\bar{u}) + \hat{f}(w_n) - \frac{\eta}{2}|w_n|_2^2 - \frac{1}{p}|w_n|_p^p + o(1) \\ &\ge E_\varepsilon(\bar{u}) + \sum_{j=1}^{k} \hat{f}_\infty(U_j) - \hat{c}\eta + o(1) \\ &\ge \frac{1}{N} S^{N/2}\left(\frac{1}{\varepsilon}\right)^{\frac{N-2}{2}} - \hat{c}\eta + o(1) \\ &\ge m_\varepsilon - \hat{c}\eta + o(1) \\ &> c \end{aligned} \tag{3.49}$$

for η small and large n. So a contradiction arises because of the assumption $E_\varepsilon(u_n) \to c$.

Finally, let us consider $|z_n| \to \infty$. In such a case, the argument developed in the case $\{z_n\}_n$ bounded can be repeated in an easier way. Indeed by (3.35), (3.44) and (3.45) we can simply consider the functional $\hat{f}_\infty$ in place of $\hat{f}$ and get a contradiction with $E_\varepsilon(u_n) \to c < m_\varepsilon$ as in (3.49). So the proof is completed.

Proof of Theorem 1.1. By Remark 1.4 we may assume that $a(x) \equiv 1$. We claim that

$$\inf_{\mathcal{N}_\varepsilon} E_\varepsilon < m_\varepsilon. \tag{3.50}$$

Since $\Omega = \mathbb{R}^N$, we can consider the minimizing function w_ε introduced in Theorem 2.2 and $t > 0$ such that $tw_\varepsilon \in \mathcal{N}_\varepsilon$. Then

$$\inf_{\mathcal{N}_\varepsilon} E_\varepsilon \le E_\varepsilon(tw_\varepsilon) < E_{\varepsilon,\infty}(tw_\varepsilon) \le E_{\varepsilon,\infty}(w_\varepsilon) = m_\varepsilon.$$

By (3.50) and Proposition 3.2 the existence of a minimizing function $\bar{u}$ for the functional E_ε constrained on $\mathcal{N}_\varepsilon$ follows. Arguing as in the proof of Theorem 2.2 one can verify that $\bar{u}$ is a constant sign function, which can be chosen strictly positive.

Proof of Theorem 1.2. Let $z \in \mathbb{R}^N$ be such that (1.5) holds and $t_\varepsilon > 0$ be such that $t_\varepsilon w_z \in \mathcal{N}_\varepsilon$. In order to obtain the statement, it is enough to prove that for small ε

$$E_\varepsilon(t_\varepsilon w_z) < m_\varepsilon. \tag{3.51}$$

Indeed, once (3.51) is proved, $\inf_{\mathcal{N}_\varepsilon} E_\varepsilon < m_\varepsilon$ follows and we can argue as in the proof of Theorem 1.1.

Let $s > 0$ be such that $sw_z \in \mathcal{N}$, namely $s = \left(\frac{\|w_z\|_a^2}{|w_z|_p^p} \right)^{\frac{1}{p-2}}$. We claim that (1.5) implies

$$E(sw_z) < m. \tag{3.52}$$

Let us evaluate

$$E(sw_z) = \left(\frac{1}{2} - \frac{1}{p}\right) \|sw_z\|_a^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|w_z\|_a^2}{|w_z|_p^p} \right)^{\frac{2}{p-2}} \|w_z\|_a^2$$
$$= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\left\| \frac{w_z}{|w_z|_p} \right\|_a^2 \right)^{\frac{p}{p-2}}. \tag{3.53}$$

Observe that, by (1.5),

$$\left\| \frac{w_z}{|w_z|_p} \right\|_a^2 < \frac{\|w\|^2}{|w|_p^2} \tag{3.54}$$

and that, since w is the ground state of (P_∞),

$$\|w\|^2 = |w|_p^p \quad \text{and} \quad E_\infty(w) = \left(\frac{1}{2} - \frac{1}{p}\right) \|w\|^2 = m. \tag{3.55}$$

Then, putting (3.54) in (3.53) and using (3.55), we get (3.52).

Finally, remark that $t_\varepsilon \to s$, as $\varepsilon \to 0$, because $\|w_z\|_a^2 = t_\varepsilon^{p-2} |w_z|_p^p + \varepsilon t_\varepsilon^{2^*-2} |w_z|_{2^*}^{2^*-2}$, and that $m_\varepsilon \to m$ as $\varepsilon \to 0$, by Lemma 2.5, so for small ε (3.51) follows from (3.52).

4. Existence of a bound state solution

In this section we construct the tools for the proof of Theorem 1.3 and prove it. We assume $\Omega = \mathbb{R}^N$ or $a(x) \equiv 1$. First we prove that no ground state solution can exist, then we show that, in spite of the difficulties due to the few information about the solutions of $(P_{\varepsilon,\infty})$, a local compactness can be recovered in some interval of the functional values.

Proposition 4.1. Assume $\varepsilon \in [0, \varepsilon_0)$. Let $a(x) \geq 1$ and suppose that at least one assumption between $\Omega = \mathbb{R}^N$ and $a(x) \equiv 1$ holds true, then

$$\inf_{\mathcal{N}_\varepsilon} E_\varepsilon = m_\varepsilon \tag{4.56}$$

and the minimization problem (4.56) has no solution (here we mean $E_0 = E$, $\mathcal{N}_0 = \mathcal{N}$, $m_0 = m, \ldots$).

Proof. Let $u \in \mathcal{N}_\varepsilon$ and $t_u \in \mathbb{R}$ be such that $t_u u \in \mathcal{N}_{\varepsilon,\infty}$. Since $a(x) \geq 1$ a.e. in $\mathbb{R}^N$, we have

$$m_\varepsilon \leq E_{\varepsilon,\infty}(t_u u) \leq E_\varepsilon(t_u u) \leq E_\varepsilon(u).$$

Hence $\inf_{\mathcal{N}_\varepsilon} E_\varepsilon \geq m_\varepsilon$. Let us prove that $\inf_{\mathcal{N}_\varepsilon} E_\varepsilon \leq m_\varepsilon$.

First, assume $\Omega = \mathbb{R}^N$. In order to exhibit a sequence $\{u_n\}_n$ in $\mathcal{N}_\varepsilon$ such that $E_\varepsilon(u_n) \to m_\varepsilon$, we define $u_n = t_n[\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1)]$, where w_ε is the minimizing function introduced in Theorem 2.2, e_1 is the first element of the canonical basis of $\mathbb{R}^N$, ϑ is the cut-off function introduced in (1.4) and $t_n > 0$ is such that $u_n = t_n[\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1)] \in \mathcal{N}_\varepsilon$.

Let us fix $r > 1$ such that $\mathbb{R}^N \setminus \Omega \subset B_{r-1}(0)$, then

$$|\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1) - w_\varepsilon(\cdot - ne_1)|_p^p = \int_{B_r(0)} |(\vartheta(x) - 1)\, w_\varepsilon(x - ne_1)|^p\, dx$$

$$\leq \int_{B_r(0)} |w_\varepsilon(x - ne_1)|^p\, dx = \int_{B_r(-ne_1)} |w_\varepsilon(z)|^p\, dz = o(1).$$

Hence $|\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1)|_p^p \to |w_\varepsilon|_p^p$. Analogously we have $|\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1)|_{2^*}^{2^*} \to |w_\varepsilon|_{2^*}^{2^*}$ and $\|\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)\|_a^2 \to \|w_\varepsilon\|^2$.

Taking into account $u_n = t_n[\vartheta(\cdot)\, w_\varepsilon(\cdot - ne_1)] \in \mathcal{N}_\varepsilon$ and Lemma 2.1, we have

$$\|\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)\|_a^2 - t_n^{p-2}\, |\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)|_p^p - \varepsilon\, t_n^{2^*-2}\, |\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)|_{2^*}^{2^*} = 0, \tag{4.57}$$

so that

$$\begin{aligned} t_n^{p-2}\, |\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)|_p^p + \varepsilon\, t_n^{2^*-2}\, |\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)|_{2^*}^{2^*} \\ = \|\vartheta(\cdot) w_\varepsilon(\cdot - ne_1)\|_a^2 = \|w_\varepsilon\|^2 + o(1). \end{aligned}$$

Hence $\{t_n\}_n$ is bounded and, up to a subsequence, $t_n \to t$. Getting $n \to \infty$ in (4.57) we obtain

$$\|w_\varepsilon\|^2 - t^{p-2}\, |w_\varepsilon|_p^p - \varepsilon\, t^{2^*-2}\, |w_\varepsilon|_{2^*}^{2^*} = 0,$$

namely $tw_\varepsilon \in \mathcal{N}_{\varepsilon,\infty}$. Since $w_\varepsilon \in \mathcal{N}_{\varepsilon,\infty}$, we deduce that $t = 1$. It follows that $\|u_n\|^2 \to \|w_\varepsilon\|^2$, $|u_n|_p^p \to |w_\varepsilon|_p^p$ and $|u_n|_{2^*}^{2^*} \to |w_\varepsilon|_{2^*}^{2^*}$. Then $E_\varepsilon(u_n) \to E_{\varepsilon,\infty}(w_\varepsilon) = m_\varepsilon$ and we can conclude $\inf_{\mathcal{N}_\varepsilon} E_\varepsilon \le m_\varepsilon$.

If $\Omega = \mathbb{R}^N$, then we set $u_n = t_n w_\varepsilon(\cdot - ne_1)$ and the same argument developed in the case $\Omega \neq \mathbb{R}^N$ shows that $E(u_n) \to m_\varepsilon$, so that $\inf_{\mathcal{N}_\varepsilon} E_\varepsilon \le m_\varepsilon$ holds again.

Now, let us prove that m_ε is not attained in $\mathcal{N}_\varepsilon$. By contradiction, assume that $u \in \mathcal{N}_\varepsilon$ verifies $E_\varepsilon(u) = m_\varepsilon$.

First, assume $\Omega \neq \mathbb{R}^N$. Let $t > 0$ be such that $tu \in \mathcal{N}_{\varepsilon,\infty}$, then

$$m_\varepsilon \le E_{\varepsilon,\infty}(tu) \le E_\varepsilon(tu) \le E_\varepsilon(u) = m_\varepsilon,$$

i.e. tu is a minimizing function for $E_{\varepsilon,\infty}$ on $\mathcal{N}_{\varepsilon,\infty}$. But then the arguments developed in the proof of Theorem 2.2 show that $|tu| > 0$, contrary to $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

Now, assume $\Omega = \mathbb{R}^N$ and $a \not\equiv 1$. Again, let $t > 0$ be such that $tu \in \mathcal{N}_{\varepsilon,\infty}$, then

$$m_\varepsilon = E_\varepsilon(u) \ge E_\varepsilon(tu) > E_{\varepsilon,\infty}(tu) \ge m_\varepsilon,$$

that is a contradiction, and the proof is complete.

About the compactness, in the subcritical case we remind an almost classical result (see f.i. [7]).

Proposition 4.2. Let $\{v_n\}_n$ be a $(PS)_c$-sequence of E, let c belong to the interval $(m, 2m)$, then $\{v_n\}_n$ is relatively compact and, up to a subsequence, converges to a nonzero function $\bar v \in H_0^1(\Omega)$ such that $E(\bar v) \in (m, 2m)$.

Here we prove:

Proposition 4.3. To every $\delta \in (0, m/2)$ there corresponds $\varepsilon_\delta > 0$ having the following property: $\forall \varepsilon \in (0, \varepsilon_\delta)$, $\forall c \in (m + \delta, 2m - \delta)$, if $\{u_n\}_n$ is a $(PS)_c$-sequence of E_ε constrained on $\mathcal{N}_\varepsilon$, then $u_n \rightharpoonup \bar u \not\equiv 0$ weakly in $H^1(\Omega)$. Moreover $\bar u$ is a critical point of E_ε on $\mathcal{N}_\varepsilon$ and $E_\varepsilon(\bar u) \le c$.

Proof. As in the proof of Proposition 3.2, we deduce that every $(PS)_c$-sequence for the constrained functional is also a $(PS)_c$-sequence for the free functional, and its weak limit is a critical point. Moreover, by Lemma 3.1 every (PS)-sequence is bounded in $H^1(\mathbb{R}^N)$, so it has a weak limit in $H^1(\mathbb{R}^N)$. Arguing by contradiction, then we can assume that there exist $\delta \in (0, m/2)$, a sequence $\{c_n\}_n$ in $(m + \delta, 2m - \delta)$, a sequence $\{\varepsilon_n\}_n$ in $(0, +\infty)$, with $\varepsilon_n \to 0$, and, for every $n \in \mathbb{N}$, a sequence $\{u_k^n\}_k$ in $H_0^1(\Omega)$ such that

$$E_{\varepsilon_n}(u_k^n) \xrightarrow{k\to\infty} c_n, \qquad E'_{\varepsilon_n}(u_k^n) \xrightarrow{k\to\infty} 0,$$

$$u_k^n \xrightarrow{k\to\infty} 0 \quad \text{weakly in } H^1(\Omega).$$

Since p is subcritical, we can also assume

$$u_k^n \xrightarrow{k\to\infty} 0 \quad \text{in } L^p_{loc}(\mathbb{R}^N).$$

Now, up to a subsequence, $c_n \to \bar{c} \in [m + \bar{\delta}, 2m - \bar{\delta}]$ and by a diagonal argument we build a sequence $\{v_n\}_n := \{u^n_{k_n}\}_n$ such that

$$E_{\varepsilon_n}(v_n) \xrightarrow{n\to\infty} \bar{c}, \qquad E'_{\varepsilon_n}(v_n) \xrightarrow{n\to\infty} 0, \qquad v_n \xrightarrow{n\to\infty} 0 \ \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N). \tag{4.58}$$

Furthermore

$$E_{\varepsilon_n}(v_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|v_n\|^2_a + \left(\frac{1}{p} - \frac{1}{2^*}\right)\varepsilon_n |v_n|^{2^*}_{2^*} = \bar{c} + o(1) \tag{4.59}$$

implies $\{\|v_n\|_a\}_n$ bounded. Hence we obtain

$$E(v_n) = E_{\varepsilon_n}(v_n) + \frac{\varepsilon_n}{2^*}\int_\Omega |v_n|^{2^*} dx \xrightarrow{n\to\infty} \bar{c}$$

$$\|E'(v_n)\|_{H^{-1}} \le \|E'_{\varepsilon_n}(v_n)\|_{H^{-1}} + C\,\varepsilon_n \|v_n\|^{2^*-1} \xrightarrow{n\to\infty} 0,$$

so that $\{v_n\}_n$ is a $(PS)_{\bar{c}}$-sequence of E, with $\bar{c} \in (m, 2m)$. Then, by Proposition 4.2, $\bar{v} \in H^1(\Omega)$, $\bar{v} \neq 0$, exists such that $v_n \to \bar{v}$, contrary to (4.58).

Finally, if $\{u_n\}_n$ is a $(PS)_c$-sequence for E_ε, constrained on N_ε, and $u_n \rightharpoonup \bar{u}$, then $E_\varepsilon(\bar{u}) \le c$ by (4.59) with $\varepsilon_n \equiv \varepsilon$ and c in place of $\bar{c}$.

4.1. Energy estimates

Here we first construct some test functions to explore some sublevels of the functional E_ε and we prove some basic estimates on the action of these test functions. Later, we introduce a barycenter map to analyse some features of the sublevels.

Let us set $\Sigma = \partial B_2(e_1)$, where e_1 is the first element of the canonical basis of $\mathbb{R}^N$, and for any $\rho > 0$ define the map $\psi_\rho : [0, 1] \times \Sigma \longrightarrow H^1_0(\Omega)$ by

$$\psi_\rho[s, y](x) = \vartheta(x)\,[(1 - s)w(x - \rho e_1) + sw(x - \rho y)],$$

where w is the ground state solution of (P_∞) and ϑ is the cut-off function defined in (1.4). Let us denote by $t_{\rho,s,y}$ and $\tau_{\rho,s,y}$ the positive real numbers such that $t_{\rho,s,y}\,\psi_\rho[s, y] \in N_\varepsilon$ and $\tau_{\rho,s,y}\,\psi_\rho[s, y] \in N$.

Lemma 4.4. *There exists $\bar{\rho} > 0$ and $A \in (m, 2m)$ such that for any $\rho > \bar{\rho}$ and for any $\varepsilon > 0$*

$$A_{\varepsilon,\rho} = \max\{E_\varepsilon(t_{\rho,s,y}\,\psi_\rho[s, y]) : s \in [0, 1], y \in \Sigma\} < A < 2m.$$

Before proving Lemma 4.4, let us recall two technical lemmas. We refer the readers to [18] for the proof of Lemma 4.5 while the proof of Lemma 4.6 is in [5] (see also Lemma 2.9 in [16]).

Lemma 4.5. *For all $a, b \ge 0$, for all $p \ge 2$, the following relation holds true*

$$(a + b)^p \ge a^p + b^p + (p - 1)(a^{p-1} b + ab^{p-1}).$$

Lemma 4.6. *If $g \in L^\infty(\mathbb{R}^N)$ and $h \in L^1(\mathbb{R}^N)$ are such that, for some $\alpha \ge 0$, $b \ge 0$, $\gamma \in \mathbb{R}$*

$$\lim_{|x|\to\infty} g(x)e^{\alpha|x|}\,|x|^b = \gamma \tag{4.60}$$

and

$$\int_{\mathbb{R}^N} |h(x)| e^{\alpha|x|} |x|^b dx < \infty, \tag{4.61}$$

then, for every $z \in \mathbb{R}^N \setminus \{0\}$,

$$\lim_{\rho\to\infty} \left(\int_{\mathbb{R}^N} g(x+\rho z)h(x)dx\right) e^{\alpha|\rho z|} |\rho z|^b = \gamma \int_{\mathbb{R}^N} h(x) e^{-\alpha \frac{x\cdot z}{|z|}} dx.$$

Proof of Lemma 4.4. In this proof we shall consider $r > 1$ fixed such that $\mathbb{R}^N \setminus \Omega \subset B_{r-1}(0)$, if $\Omega \neq \mathbb{R}^N$, and any fixed $r > 1$ if $\Omega = \mathbb{R}^N$.

Let us set $\delta_\rho = \left(\rho^{(N-1)/2} e^{2\rho}\right)^{-1}$ and, in order to simplify the notations, we omit s, y and write $t_\rho = t_{\rho,s,y}$, $\tau_\rho = \tau_{\rho,s,y}$ and $\psi_\rho = \psi_\rho[s,y]$. Being $\tau_\rho \psi_\rho \in \mathcal{N}$,

$$\|\tau_\rho \psi_\rho\|_a^2 = |\tau_\rho \psi_\rho|_p^p, \qquad \tau_\rho = \left(\frac{\|\psi_\rho\|_a^2}{|\psi_\rho|_p^p}\right)^{1/p-2}$$

hold true, so, for every $\varepsilon > 0$, we have

$$\begin{aligned} E_\varepsilon(t_\rho \psi_\rho) &\le E(t_\rho \psi_\rho) \le E(\tau_\rho \psi_\rho) \\ &= \frac{1}{2}\|\tau_\rho \psi_\rho\|_a^2 - \frac{1}{p}|\tau_\rho \psi_\rho|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \tau_\rho^2 \|\psi_\rho\|_a^2 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\psi_\rho\|_a^2}{|\psi_\rho|_p^2}\right)^{\frac{p}{p-2}}. \end{aligned} \tag{4.62}$$

So, to get the statement of the Lemma, we need to estimate the ratio in the last line of (4.62).

Estimate of $\|\psi_\rho\|_a^2$: we have

$$\begin{aligned} \|\psi_\rho\|_a^2 &= \|\vartheta(\cdot)\,[(1-s)w(\cdot-\rho e_1) + sw(\cdot-\rho y)]\|_a^2 \\ &\le \int_{\mathbb{R}^N} |\nabla\vartheta(x)|^2 \left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^2 dx \\ &\quad + 2\int_{\mathbb{R}^N} \Big(\vartheta(x)\nabla\vartheta(x) \cdot [(1-s)w(x-\rho e_1) + sw(x-\rho y)] \cdot \\ &\quad \cdot \nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]\Big) dx \\ &\quad + \int_{\mathbb{R}^N} \Big(|\nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]|^2 \\ &\quad + a(x)\left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^2\Big) dx. \end{aligned} \tag{4.63}$$

Let us evaluate the addends in (4.63). By direct computation and since w is a solution of (P_∞), we obtain

$$\int_{\mathbb{R}^N} \Big(|\nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]|^2 + a(x)[(1-s)w(x-\rho e_1) + sw(x-\rho y)]^2\Big) dx$$

$$= \left[(1-s)^2 + s^2\right]\|w\|^2 + 2s(1-s)\int_{\mathbb{R}^N} w^{p-1}(x-\rho e_1)w(x-\rho y)\,dx$$
$$+ \int_{\mathbb{R}^N}(a(x)-1)\left[(1-s)w(x-\rho e_1) + s\,w(x-\rho y)\right]^2 dx. \tag{4.64}$$

By Lemma 4.6 there exists $c_1 > 0$ such that

$$\lim_{\rho\to\infty} \delta_\rho^{-1}\int_{\mathbb{R}^N} w^{p-1}(x-\rho e_1)\,w(x-\rho y)\,dx = \tag{4.65}$$

$$= \lim_{\rho\to\infty} \delta_\rho^{-1}\int_{\mathbb{R}^N} w(x-\rho e_1)\,w^{p-1}(x-\rho y)\,dx = c_1.$$

Taking into account assumption (1.9) and (2.13), by Lemma 4.6 we have

$$\int_{\mathbb{R}^N}(a(x)-1)\left[(1-s)w(x-\rho e_1) + s\,w(x-\rho y)\right]^2 dx$$
$$\le 2\int_{\mathbb{R}^N}(a(x)-1)\left[w^2(x-\rho e_1) + w^2(x-\rho y)\right] dx = o(\delta_\rho).$$

Hence (4.64) becomes

$$\int_{\mathbb{R}^N}\Big(|\nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]|^2$$
$$+a(x)[(1-s)w(x-\rho e_1) + sw(x-\rho y)]^2\Big)\,dx$$
$$\le \left[(1-s)^2 + s^2\right]\|w\|^2 + 2s(1-s)c_1\delta_\rho + o(\delta_\rho). \tag{4.66}$$

Since $\nabla\vartheta$ has support in $B_r(0)$ and $|y| \ge 1$ $\forall y \in \Sigma$, from (2.13) it follows

$$\int_{\mathbb{R}^N}|\nabla\vartheta(x)|^2\left[(1-s)w(x-\rho e_1) + sw(x-\rho y)\right]^2 dx$$
$$\le 2|\nabla\vartheta|_\infty \int_{B_r(0)}\left[w^2(x-\rho e_1) + w^2(x-\rho y)\right] dx = o(\delta_\rho). \tag{4.67}$$

Taking into account (2.14) and arguing as above we obtain

$$2\int_{\mathbb{R}^N}\vartheta(x)\,\nabla\vartheta(x)\cdot[(1-s)w(x-\rho e_1) + sw(x-\rho y)]\cdot$$
$$\cdot\,\nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]\,dx \tag{4.68}$$

$$= \frac{1}{2}\int_{B_r(0)}\nabla(\vartheta(x))^2\cdot\nabla[(1-s)w(x-\rho e_1) + sw(x-\rho y)]^2\,dx = o(\delta_\rho).$$

By (4.63), (4.66), (4.67) and (4.68) we deduce

$$\|\Psi_\rho\|_a^2 \le \left[(1-s)^2 + s^2\right]\|w\|^2 + 2s(1-s)c_1\delta_\rho + o(\delta_\rho). \tag{4.69}$$

Estimate of $|\psi_\rho|_p^p$: since $0 \le \vartheta(x) \le 1$ in $\mathbb{R}^N$ and $\vartheta \equiv 1$ in $\mathbb{R}^N \setminus B_r(0)$, we get

$$
\begin{aligned}
|\psi_\rho|_p^p &= \int_{\mathbb{R}^N} \vartheta(x)\,[(1-s)w(x-\rho e_1) + sw(x-\rho y)]^p\,dx \\
&\ge \int_{\mathbb{R}^N} \left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^p dx \\
&\quad - \int_{B_r(0)} \left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^p dx.
\end{aligned}
$$

By the asymptotic behaviour of w,

$$
\begin{aligned}
&\int_{B_r(0)} \left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^p dx \\
&\le 2^{p-1} \int_{B_r(0)} [w^p(x-\rho e_1) + w^p(x-\rho y)]\,dx = o(\delta_\rho).
\end{aligned}
$$

Therefore, from Lemma 4.5 and (4.65) it follows

$$
\begin{aligned}
|\psi_\rho|_p^p &\ge \int_{\mathbb{R}^N} \left((1-s)w(x-\rho e_1) + sw(x-\rho y)\right)^p dx + o(\delta_\rho) \\
&\ge [(1-s)^p + s^p]\,|w|_p^p + (p-1)\left((1-s)^{p-1} s + (1-s)s^{p-1}\right) c_1 \delta_\rho + o(\delta_\rho).
\end{aligned} \tag{4.70}
$$

Estimate of (4.62): combining estimates (4.69) and (4.70) and taking advantage of a Taylor expansion, we obtain for any $s \in [0, 1]$ and $y \in \Sigma$

$$
\frac{\|\psi_\rho\|_a^2}{|\psi_\rho|_p^2} \le \frac{[(1-s)^2 + s^2]\,\|w\|^2 + 2s(1-s)c_1\delta_\rho + o(\delta_\rho)}{\left([(1-s)^p + s^p]\,|w|_p^p + (p-1)\,[(1-s)^{p-1} s + (1-s)s^{p-1}]\,c_1\delta_\rho + o(\delta_\rho)\right)^{2/p}}
$$

$$
= \frac{(1-s)^2 + s^2}{[(1-s)^p + s^p]^{2/p}}\,\frac{\|w\|^2}{|w|_p^2} + \gamma(s)\delta_\rho + o(\delta_\rho),
$$

where

$$
\gamma(s) = \frac{2s(1-s)c_1}{[(1-s)^p + s^p]^{2/p}\,|w|_p^2}\left(1 - \frac{p-1}{p}\,\frac{(1-s)^2 + s^2}{(1-s)^p + s^p}\left((1-s)^{p-2} + s^{p-2}\right)\right).
$$

Since $p > 2$ we have that $\gamma(1/2) < 0$, so there exist $\bar{c} > 0$ and a neighbourhood $I(1/2)$ of $1/2$ such that for any $s \in I(1/2)$ and any $y \in \Sigma$

$$
\begin{aligned}
E_\varepsilon(t_\rho\psi_\rho) &\le \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\|\psi_\rho\|_a^2}{|\psi_\rho|_p^2}\right)^{\frac{p}{p-2}} \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{(1-s)^2 + s^2}{[(1-s)^p + s^p]^{2/p}}\,\frac{\|w\|^2}{|w|_p^2} + \gamma(s)\delta_\rho + o(\delta_\rho)\right)^{\frac{p}{p-2}} \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right)\left(2^{\frac{p-2}{p}}\,\frac{\|w\|^2}{|w|_p^2}\right)^{\frac{p}{p-2}} - \bar{c}\delta_\rho + o(\delta_\rho) \\
&= 2\left(\frac{1}{2} - \frac{1}{p}\right)|w|_p^p - \bar{c}\delta_\rho + o(\delta_\rho) \\
&= 2m - \bar{c}\delta_\rho + o(\delta_\rho),
\end{aligned}
$$

where we have used $\|w\|^2 = |w|_p^p$ and $E_\infty(w) = \left(\frac{1}{2} - \frac{1}{p}\right)|w|_p^p = m$.

Similar computations show that for any $s \in [0,1]\backslash I(1/2)$ and $y \in \Sigma$ we have

$$
\begin{aligned}
&\lim_{\rho\to\infty} \max\{E_\varepsilon(t_\rho\psi_\rho) : \; s \in [0,1]\backslash I(1/2), \; y \in \Sigma\} \\
&\le \max\left\{\left(\frac{(1-s)^2 + s^2}{[(1-s)^p + s^p]^{2/p}}\right)^{\frac{p}{p-2}} m : \; s \in [0,1]\backslash I(1/2)\right\} < 2m.
\end{aligned}
$$

Finally, we may conclude that the relation

$$A_{\varepsilon,\rho} = \max\{E_\varepsilon(t_{\rho,s,y}\,\psi_\rho[s,y]) : \; s \in [0,1], y \in \Sigma\} < 2m$$

holds true for ρ large enough, independent of $\varepsilon > 0$.

Corollary 4.7. There exist $\bar{\rho}, \bar{\varepsilon} > 0$ such that for any $\rho > \bar{\rho}$ and for any $\varepsilon \in (0, \bar{\varepsilon})$

$$A_{\varepsilon,\rho} = \max\{E_\varepsilon(t_{\rho,s,y}\,\psi_\rho[s,y]) : \; s \in [0,1], y \in \Sigma\} < 2m_\varepsilon.$$

Proof. It is a direct consequence of Lemmas 2.5 and 4.4.

The following *definition of barycenter* of a function $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, has been introduced in [19]. We set

$$\mu(u)(x) = \frac{1}{|B_1(0)|}\int_{B_1(x)} |u(y)|\,dy \qquad x \in \mathbb{R}^N \tag{4.71}$$

and we remark that $\mu(u)$ is bounded and continuous, so we can introduce the function

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2}\max\mu(u)\right]^+ \qquad x \in \mathbb{R}^N, \tag{4.72}$$

that is continuous and has compact support. Thus, we can set $\beta : H^1(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}^N$ as

$$\beta(u) = \frac{1}{|\hat{u}|_1}\int_{\mathbb{R}^N} \hat{u}(x)\,x\,dx.$$

The map β has the following properties:

$$\beta \text{ is continuous in } H^1(\mathbb{R}^N)\setminus\{0\}; \tag{4.73}$$

$$\text{if } u \text{ is a radial function, then } \beta(u) = 0; \tag{4.74}$$

$$\beta(tu) = \beta(u) \qquad \forall t \in \mathbb{R}\setminus\{0\}, \quad \forall u \in H^1(\mathbb{R}^N)\setminus\{0\}; \tag{4.75}$$

$$\beta(u(x-z)) = \beta(u) + z \qquad \forall z \in \mathbb{R}^N \quad \forall u \in H^1(\mathbb{R}^N)\setminus\{0\}. \tag{4.76}$$

Let us set

$$C_0 = \inf\{E(u) : \ u \in \mathcal{N},\ \beta(u) = 0\},\ C_{0,\varepsilon} = \inf\{E_\varepsilon(u) : \ u \in \mathcal{N}_\varepsilon,\ \beta(u) = 0\}.$$

Lemma 4.8. *The following facts hold:*

(a) $C_0 > m$;
(b) $\lim_{\varepsilon\to 0} C_{0,\varepsilon} = C_0$.

Proof. Let us prove inequality *a*). By Proposition 4.1, $C_0 \geq m$. Assume by contradiction that $C_0 = m$. Let $\{u_n\}_n$ be a sequence in $\mathcal{N}$ with $\beta(u_n) = 0$ such that $E(u_n) \to m$ and $t_n > 0$ be such that $t_n u_n \in \mathcal{N}_\infty$, $\forall n \in \mathbb{N}$. Since $a(x) \geq 1$ a.e. in $\mathbb{R}^N$ we have

$$m \leq E_\infty(t_n u_n) \leq E(t_n u_n) \leq E(u_n) = m + o(1), \tag{4.77}$$

that implies that $\{t_n u_n\}_n$ is a minimizing sequence for E_∞ on $\mathcal{N}_\infty$. Hence there exists a sequence $\{y_n\}_n$ in $\mathbb{R}^N$ such that

$$t_n u_n(x) = w(x - y_n) + \varphi_n(x), \qquad \varphi_n \to 0 \text{ strongly in } H^1(\mathbb{R}^N)$$

(see [7, Lemma 3.1]). By (4.75), (4.76) we have

$$0 = \beta(u_n) = \beta(t_n u_n) = \beta(w(\cdot - y_n) + \varphi_n) = \beta(w + \varphi(\cdot + y_n)) + y_n.$$

From $\varphi_n \to 0$ strongly in $H^1(\mathbb{R}^N)$ and (4.73), (4.74), it follows that $\beta(w + \varphi(\cdot + y_n)) \to \beta(w) = 0$, because w is radially symmetric. Hence $y_n \to 0$ and $t_n u_n \to w$ strongly in $H^1(\mathbb{R}^N)$. We shall prove that this is not possible. If $\Omega \neq \mathbb{R}^N$ then $t_n u_n \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ would imply $w \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, contrary to $w > 0$ in $\mathbb{R}^N$. If $\Omega = \mathbb{R}^N$ and $a(x) \not\equiv 1$, then, taking into account (4.77), we have

$$m = E_\infty(w) < E(w) = \lim_{n\to\infty} E(t_n u_n) \leq \lim_{n\to\infty} E(u_n) = m,$$

a contradiction. So *a*) is proved.

Let us prove *b*). Let $\varepsilon > 0$ be fixed and for every $\eta > 0$ let $u_\eta \in \mathcal{N}$ be such that $\beta(u_\eta) = 0$ and $E(u_\eta) \leq C_0 + \eta$, moreover let $s_\eta > 0$ be such that $s_\eta u_\eta \in \mathcal{N}_\varepsilon$. Then

$$C_{0,\varepsilon} \leq E_\varepsilon(s_\eta u_\eta) \leq E(s_\eta u_\eta) \leq E(u_\eta) \leq C_0 + \eta,$$

so, by the arbitrary choice of η, we get

$$C_{0,\varepsilon} \leq C_0 \qquad \forall \varepsilon > 0. \tag{4.78}$$

Let $v_\varepsilon \in N_\varepsilon$ so that $\beta(v_\varepsilon) = 0$ and $E_\varepsilon(v_\varepsilon) \leq C_{0,\varepsilon} + \varepsilon$, and let $t_\varepsilon > 0$ such that $t_\varepsilon u_\varepsilon \in N$. Then

$$\begin{aligned} C_0 &\leq E(t_\varepsilon v_\varepsilon) = E_\varepsilon(t_\varepsilon v_\varepsilon) + \frac{\varepsilon}{2^*}|t_\varepsilon v_\varepsilon|_{2^*}^{2^*} \\ &\leq E_\varepsilon(v_\varepsilon) + \frac{\varepsilon}{2^*}|t_\varepsilon v_\varepsilon|_{2^*}^{2^*} \\ &\leq C_{0,\varepsilon} + \varepsilon + \frac{\varepsilon}{2^*} t_\varepsilon^{2^*}|v_\varepsilon|_{2^*}^{2^*}. \end{aligned} \tag{4.79}$$

Now, observe that by (4.78)

$$E_\varepsilon(v_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p}\right) v_\varepsilon {}_a^2 + \varepsilon \left(\frac{1}{p} - \frac{1}{2^*}\right) |v_\varepsilon|_{2^*}^{2^*} \leq C_{0,\varepsilon} + \varepsilon \leq C_0 + \varepsilon,$$

that implies that $|v_\varepsilon|_{2^*}^{2^*}$ is bounded. Moreover, taking into account $a(x) \geq 1$ a.e. in $\mathbb{R}^N$ and $v_\varepsilon \in N_\varepsilon$, and arguing as in the proof of (2.11), we deduce that v_ε $_a$ → 0. Hence, as in (2.19), we conclude that $|\mu_{\varepsilon}| \to 0$ too. Finally, since $t_\varepsilon v_\varepsilon \in N$, by Lemma 2.1 we have that $\{t_\varepsilon\}$ is bounded. So, from (4.79) we infer $\liminf_{\varepsilon \to 0} C_{0,\varepsilon} \geq C_0$ that, combined with (4.78), gives *b*).

Lemma 4.9. *There exists* $\varepsilon > 0$ *such that for any* $\varepsilon \in (0, \varepsilon)$ *the inequality* $C_{0,\varepsilon} > \frac{C_0 + m}{2}$ *holds.*

Proof. The assertion follows combining (a) and (b) of Lemma 4.8.

Lemma 4.10. *Let* $A_{\varepsilon,\rho}$ *be as in Lemma 4.4. Then* $\rho > 0$ *exists such that* $C_{0,\varepsilon} \leq A_{\varepsilon,\rho}$ $\forall \rho > \rho$, $\forall \varepsilon > 0$.

Proof. We claim that, for ρ large, $\beta\left(\vartheta(\cdot)w(\cdot - \rho y)\right) \cdot y > 0$ $\forall y \in \Sigma$. Indeed, by (4.73)–(4.76) we have

$$\beta\left(\vartheta(\cdot)w(\cdot - \rho y)\right) - \rho y = \beta\left(\vartheta(\cdot + \rho y)\, w\right) \xrightarrow{\rho \to \infty} 0,$$

because $\vartheta(\cdot + \rho y)w \to w$ in $H^1(\mathbb{R}^N)$ as $\rho \to \infty$. Hence

$$\beta(\vartheta(\cdot)w(\cdot - \rho y)) = \rho y + o(1),$$

that implies the claim. So, for ρ large, the deformation $G : [0, 1]\times\Sigma \to \mathbb{R}^N \setminus \{0\}$ given by

$$G(s, y) = s\beta(\psi_\rho[1, y]) + (1 - s)\, y \tag{4.80}$$

is well defined. Then, the existence of $(s_\rho, y_\rho) \in [0, 1]\times\Sigma$ such that $\beta(\psi_\rho[s_\rho, y_\rho]) = 0$ follows, because by the continuity of the maps β and ψ and the invariance of the topological degree by homotopy we have shown that $0 = d(G, \Sigma \times [0, 1), 0) = d(\beta \circ \psi_\rho, \Sigma \times [0, 1), 0)$.

By (4.75) we also have $\beta(t_{\rho,s_\rho,y_\rho}\, \psi_\rho[s_\rho, y_\rho]) = 0$. Since $t_{\rho,s_\rho,y_\rho}\, \psi_\rho[s_\rho, y_\rho] \in N_\varepsilon$, the assertion follows.

Lemma 4.11. *Let* $\tilde{\varepsilon}$ *as in Lemma 4.9 and* $\varepsilon \in (0, \tilde{\varepsilon})$. *There exists* $\rho > 0$ *such that for any* $\rho > \rho$

$$B_{\varepsilon,\rho} := \max\{E_\varepsilon(t_{\rho,1,y}\, \psi_\rho[1, y]) : y \in \Sigma\} < C_{0,\varepsilon}.$$

Proof. Let us set $t_\rho = t_{\rho,1,y}$ and $\psi_\rho = \psi_\rho[1, y]$. By contradiction, assume that there exist $\rho_n \to \infty$ and $y_n \in \Sigma$ such that $E_\varepsilon(t_{\rho_n}\psi_{\rho_n}) \geq C_{0,\varepsilon}$ for every $n \in \mathbb{N}$.

Since $t_{\rho_n}\psi_{\rho_n} \in \mathcal{N}_\varepsilon$ we can write

$$\begin{aligned} E_\varepsilon(t_{\rho_n}\psi_{\rho_n}) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|t_{\rho_n}\psi_{\rho_n}\|_a^2 + \varepsilon\left(\frac{1}{p}-\frac{1}{2^*}\right)|t_{\rho_n}\psi_{\rho_n}|_{2^*}^{2^*} \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)t_{\rho_n}^2\|\vartheta w(\cdot-\rho_n y_n)\|_a^2 + \varepsilon\left(\frac{1}{p}-\frac{1}{2^*}\right)t_{\rho_n}^{2^*}|\vartheta w(\cdot-\rho_n y_n)|_{2^*}^{2^*}. \end{aligned} \tag{4.81}$$

Observe that in our setting $0 < m \leq C_{0,\varepsilon} \leq E_\varepsilon(t_{\rho_n}\psi_{\rho_n}) \leq A_{\varepsilon,\rho} < 2m$ and that $0 < c \leq \|\vartheta w(\cdot-\rho_n y_n)\|_a \leq C < \infty$, $\forall n \in \mathbb{N}$. Hence from (4.81) it follows that $0 < c_1 \leq t_{\rho_n} \leq C_1 < \infty$. So, up to a subsequence, we can assume $t_{\rho_n} \to t > 0$.

Since $\rho_n \to \infty$, the same estimates provided in the proof of Lemma 4.4 prove $E_\varepsilon(t_{\rho_n}\psi_{\rho_n}) \to E_{\varepsilon,\infty}(tw)$, and we get

$$\begin{aligned} C_{0,\varepsilon} \leq E_{\varepsilon,\infty}(tw) &= E_\infty(tw) - \frac{\varepsilon}{2^*}|tw|_{2^*}^{2^*} \\ &\leq E_\infty(w) - \frac{\varepsilon}{2^*}|tw|_{2^*}^{2^*} \\ &= m - \frac{\varepsilon}{2^*}|tw|_{2^*}^{2^*} < m, \end{aligned}$$

contrary to Lemmas 4.9 and 4.8 (a).

4.2. Proof of theorem 1.3

Let us recall the values

$$\begin{aligned} A_{\varepsilon,\rho} &= \max\{E_\varepsilon(t_{\rho,s,y}\psi_\rho[s,y]) : s \in [0,1], y \in \Sigma\}, \\ B_{\varepsilon,\rho} &= \max\{E_\varepsilon(t_{\rho,1,y}\psi_\rho[1,y]) : y \in \Sigma\}, \\ C_{0,\varepsilon} &= \inf\{E_\varepsilon(u) : u \in \mathcal{N}_\varepsilon, \beta(u) = 0\}. \end{aligned} \tag{4.82}$$

By Corollary 4.7 and Lemmas 4.4, 4.8, 4.9, 4.10 and 4.11, the inequalities

$$\begin{cases} (a) & B_{\varepsilon,\rho} < C_{0,\varepsilon} \leq A_{\varepsilon,\rho} \\ (b) & m < \frac{C_0+m}{2} < C_{0,\varepsilon} \leq A_{\varepsilon,\rho} \leq A < 2m \\ (c) & A_{\varepsilon,\rho} < 2\, m_\varepsilon \end{cases} \tag{4.83}$$

hold true for every $\rho > \max\{\bar\rho, \rho, \rho\}$ and for every $0 < \varepsilon < \min\{\bar\varepsilon, \varepsilon\}$. Let $0 < \delta < \min\left\{\frac{m}{2}, 2m - A, \frac{C_0-m}{2}\right\}$ and let us consider ε_δ according to Proposition 4.3.

We claim that E_ε constrained on $\mathcal{N}_\varepsilon$ has a (PS)-sequence in $[C_{0,\varepsilon}, A_{\varepsilon,\rho}]$ for every $0 < \varepsilon < \varepsilon := \min\{\varepsilon_\delta, \bar\varepsilon, \varepsilon\}$. This done, the existence of a non-zero critical point $\bar u$ with $E_\varepsilon(\bar u) \leq A_{\varepsilon,\rho}$ follows from Proposition 4.3.

Assume, by contradiction, that no (PS)-sequence exists in $[C_{0,\varepsilon}, A_{\varepsilon,\rho}]$. Then, usual deformation arguments imply the existence of $\eta > 0$ such that the sublevel $E_\varepsilon^{C_{0,\varepsilon}-\eta} := \{u \in \mathcal{N}_\varepsilon : E_\varepsilon(u) \leq C_{0,\varepsilon} - \eta\}$ is a deformation retract of the sublevel $E_\varepsilon^{A_{\varepsilon,\rho}} := \{u \in \mathcal{N}_\varepsilon : E_\varepsilon(u) \leq A_{\varepsilon,\rho}\}$, namely there exists a continuous function $\sigma : E_\varepsilon^{A_{\varepsilon,\rho}} \to E_\varepsilon^{C_{0,\varepsilon}-\eta}$ such that

$$\sigma(u) = u \qquad \text{for any } u \in E_\varepsilon^{C_{0,\varepsilon}-\eta}. \tag{4.84}$$

Furthermore, by (4.83) (*a*) we can also assume η so small that

$$C_{0,\varepsilon} - \eta > B_{\varepsilon,\rho}. \tag{4.85}$$

Let us define the map $H : [0, 1] \times \Sigma \to \mathbb{R}^N$ by

$$H(s, y) = \beta\left(\sigma\left(t_{\rho,s,y}\, \psi_\rho[s, y]\right)\right).$$

By (4.85), (4.84) and by using the map G introduced in (4.80), we deduce that H maps $\{1\} \times \Sigma$ in a set homotopically equivalent to $\rho\Sigma$ (and then to Σ) in $\mathbb{R}^N \setminus \{0\}$. Moreover, taking also into account Lemma 2.1, we see that H is a continuous map. Hence, by the argument developed in the proof of Lemma 4.10, a point $(\tilde{s}, \tilde{y}) \in [0, 1] \times \Sigma$ must exist, for which

$$0 = H(\tilde{s}, \tilde{y}) = \beta(\sigma(t_{\rho,\tilde{s},\tilde{y}}\, \psi_\rho[\tilde{s}, \tilde{y}])).$$

Then, $E_\varepsilon(\sigma(t_{\rho,\tilde{s},\tilde{y}}\, \psi_\rho[\tilde{s}, \tilde{y}])) \geq C_{0,\varepsilon}$, contrary to $\sigma\left(t_{\rho,s,y}\, \psi_\rho[s, y]\right) \in E_\varepsilon^{C_{0,\varepsilon} - \eta}$ for every $(s, y) \in [0, 1] \times \Sigma$, so the claim must be true.

Let $\bar{u} \in E_\varepsilon^{A_{\varepsilon,\rho}}$ be the critical point we have found. To show that $\bar{u}$ is a constant sign function, assume, by contradiction, that $\bar{u} = \bar{u}^+ - \bar{u}^-$, with $\bar{u}^\pm \neq 0$. Multiplying the equation in (P_ε) by $\bar{u}^\pm$ we deduce that $\bar{u}^\pm \in N_\varepsilon$, so

$$E_\varepsilon(\bar{u}) = E_\varepsilon(\bar{u}^+) + E_\varepsilon(\bar{u}^-) \geq 2m_\varepsilon,$$

contrary to (4.83) (*c*).

Remark 4.12 Let us set

$$R(\Omega) = \max\{r > 0 : \quad \exists x_r \in \mathbb{R}^N \ \text{ such that } B_r(x_r) \subset \mathbb{R}^N \setminus \Omega\}.$$

Assume $B_{R(\Omega)}(0) \subset \mathbb{R}^N \setminus \Omega$ and call $u_{a,\Omega}$ the solution provided by Theorem 1.3. Arguing as in [26], the following asymptotic behaviour of $u_{a,\Omega}$ can be described, as $R(\Omega) \to \infty$, up to some sequence:

$$u_{a,\Omega}(x) = w_{1,\varepsilon}(x - x_{1,\Omega}) + w_{2,\varepsilon}(x - x_{2,\Omega}) + O(\Omega),$$

where $O(\Omega) \longrightarrow 0$ in $H^1(\mathbb{R}^N)$, as $R(\Omega) \to \infty$, $x_{1,\Omega}, x_{2,\Omega} \in \mathbb{R}^N$ verify

$$|x_{1,\Omega} - x_{2,\Omega}| \longrightarrow \infty \quad \text{and} \quad \frac{x_{1,\Omega} + x_{2,\Omega}}{2} \longrightarrow 0, \qquad \text{as } R(\Omega) \to \infty,$$

and $w_{1,\varepsilon}$, $w_{2,\varepsilon}$ are solutions of $(P_{\varepsilon,\infty})$. The same behaviour of $u_{a,\Omega}$ can be obtained considering a sequence of potentials $a_n(x)$ verifying (1.1) and (1.9) and such that

$$\lim_{n\to\infty} a_n(x) = \infty \qquad \text{a.e. in } \mathbb{R}^N.$$

On the contrary, if the capacity of $\mathbb{R}^N \setminus \Omega$ goes to zero and $|a_n - a_\infty|_{N/2} \to 0$, then $u_{a_n,\Omega}$ converges to a solution of the limit problem $(P_{\varepsilon,\infty})$.

Remark 4.13 The behaviour of the solution $u_{a,\Omega}$ described in Remark 4.12 can be employed to obtain multiplicity of solutions of (P_ε) when $\Omega = \mathbb{R}^N \setminus \bigcup_{i=1}^h \omega_i$ and $a(x) = a_\infty + \sum_{j=1}^k \alpha_j(x)$, with suitable $\omega_i \subset\subset \mathbb{R}^N$, $i = 1, \ldots, h$, and $\alpha_j \in L^{N/2}(\mathbb{R}^N)$, $j = 1, \ldots, k$. See [25] for a description of the method.

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