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EXAMPLES OF SURFACES WHICH ARE ULRICH–WILD

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ABSTRACT. We give examples of surfaces which are Ulrich–wild, i.e. that support families of dimension p of pairwise non–isomorphic, indecomposable, Ulrich bundles for arbitrary large p.

1. Introduction and Notation

Throughout the whole paper k will denote an algebraically closed field and $\mathbb{P}^N$ the projective space over k of dimension N.

If $X \subseteq \mathbb{P}^N$ is a *variety*, i.e. an integral closed subscheme, then the study of coherent sheaves on X is an important tool for understanding its geometric properties. From a cohomological viewpoint, the simplest sheaves on such an X are the *Ulrich* ones with respect to the very ample line bundle $\mathcal{O}_X(h_X) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$. There are several equivalent characterizations for such sheaves (e.g. see Proposition 2.1 of [22]). In this paper we call the sheaf $\mathcal{F}$ Ulrich if

$$h^i\big(X, \mathcal{F}(-ih_X)\big) = h^j\big(X, \mathcal{F}(-(j+1)h_X)\big) = 0$$

for each $i > 0$ and $j < \dim(X)$. Ulrich sheaves are aCM, i.e. $h^i\big(X, \mathcal{F}(th_X)\big) = 0$ for $0 < i < \dim(X)$ and each $t \in \mathbb{Z}$: thus they are vector bundles when restricted to the smooth part of X.

In [22], the authors asked the following questions.

Question 1.1. Is every variety (or even scheme) $X \subseteq \mathbb{P}^N$ the support of an Ulrich sheaf? If so, what is the smallest possible rank for such a sheaf?

At present, answers to the questions above are known in a number of particular cases: e.g., see [2], [4], [6], [7], [12], [13], [14], [18], [19], [20], [35], [36], [37], [41].

The existence of many Ulrich sheaves on a variety X can be viewed as a sign of the complexity of the variety itself. For example one could ask if $X \subseteq \mathbb{P}^N$ is of *Ulrich–wild representation type*, i.e. if it supports families of dimension p of pairwise non–isomorphic, indecomposable, Ulrich sheaves for arbitrary large p.

Ulrich–wildness is known for several classes of varieties. The case of *surfaces*, i.e. smooth integral projective varieties of dimension 2 is of particular interest. In this paper we prove the following result (K_S denotes the canonical class of S).

Theorem 1.2. *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h_S)$ and let $d := h_S^2$.*

If S supports an Ulrich bundle with respect to $\mathcal{O}_S(h_S)$ and

$$d^2 + 4(\chi(\mathcal{O}_S) - 2)d - (h_S K_S)^2 > 0, \tag{1.1}$$

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then S is Ulrich–wild with respect to $\mathcal{O}_S(h_S)$.

Unfortunately, the above result is not sharp. E.g. if S is a del Pezzo surface, $h_SK_S = -d$ and $\chi(\mathcal{O}_S) = 1$, thus the first member of Inequality (1.1) is strictly negative. Nevertheless del Pezzo surfaces are Ulrich–wild as shown in [41] and [36]: see also Section 5.3 of [24] where the authors prove the Ulrich–wildness of each 2–dimensional maximally del Pezzo variety (see [11] and the references therein for details about such varieties).

In particular, Theorem 1.2 yields some interesting examples. The first one are complete intersection surfaces of degree $4 \le d \le 9$ in $\mathbb{P}^N$ (see Example 3.1). As a consequence of results by A. Beauville and D. Faenzi, we show in Example 3.2 that *abelian surfaces* (i.e. surfaces S with $K_S = 0$ and $q(S) = 2$) and $K3$ *surfaces* (i.e., surfaces S with $K_S = 0$ and $q(S) = 0$) are Ulrich–wild. A similar result has been recently proved in [23] for $K3$ *surfaces* (i.e. surfaces S with $K_S = 0$ and $q(S) = 0$) extending the previous works [19] and [4]. Thus each minimal surface S with Kodaira dimension $\kappa(S) = 0$ is Ulrich–wild (see Proposition 6 of [8]).

We also deal with *surfaces of general type* (i.e. minimal surfaces S with $\kappa(S) = 2$: see Corollary 3.3) such that $\mathcal{O}_S(h_S) \cong \mathcal{O}_S(\lambda K_S)$ for some positive $\lambda \in \mathbb{Z}$.

In order to find other examples, we prove the following helpful theorem: it makes slightly more precise the results from [27].

Theorem 1.3. *Let $X, Y \subseteq \mathbb{P}^N$ be varieties endowed with Ulrich sheaves $\mathcal{A}$ and $\mathcal{B}$ respectively.*

If X and Y intersect properly, $\mathcal{A}$ and $\mathcal{B}$ are locally free of respective ranks a and b along $X \cap Y$ and Y is locally complete intersection at the points of X, then $\mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}$ is an Ulrich bundle of rank ab on $X \cap Y$.

Since $\mathcal{O}_{\mathbb{P}^n}$ is the unique Ulrich bundle on $\mathbb{P}^n$ with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that the above Theorem generalizes the well–known obvious fact that the restriction of an Ulrich sheaf to a general linear section of a variety is still Ulrich. One can construct some other applications besides hyperplane sections.

E.g. in [10] the authors considered the intersection of two varieties $G_1, G_2 \subseteq \mathbb{P}^9$ projectively isomorphic to the Plücker model of the grassmannian $G(2,5)$ of lines in $\mathbb{P}^4$. If G_1 and G_2 are general, then $F := G_1 \cap G_2$ is a smooth threefold which is *Calabi–Yau*, i.e. $\mathcal{O}_F(K_F) \cong \mathcal{O}_F$ and $h^1\big(F, \mathcal{O}_F\big) = 0$: F is called a *GPK^3 threefold*. In Example 4.1 we show that F supports Ulrich bundles by applying Theorem 1.3 above, though F is not a hypersurface section.

A second application is the construction of Ulrich bundles on hypersurface sections of del Pezzo threefolds (see Example 4.2).

We end the paper by collecting what we know about surfaces of degree up to 8. We call *special* each Ulrich bundle $\mathcal{E}$ on S of even rank such that

$$c_1(\mathcal{E}) = \frac{\mathrm{rk}(\mathcal{E})}{2}(3h + K_S).$$

If $\mathcal{E}$ is any Ulrich bundle on S, then $\mathcal{E} \oplus \mathcal{E}^\vee(3h_S + K_S)$ is special: thus S supports Ulrich bundles if and only if it supports special ones. The existence of a special Ulrich bundle $\mathcal{E}$ on S has some interesting consequences: e.g. if $\mathrm{rk}(\mathcal{E}) = 2$, then the Chow form of S is pfaffian.

We then prove the following results over the complex field $\mathbb{C}$ (if $S \subseteq \mathbb{P}^N$, then $\pi(S)$ is the genus of a general hyperplane section of S).

Theorem 1.4. *Let $S \subseteq \mathbb{P}^N$ be a surface of degree $d \le 8$ and $\kappa(S) \neq 1$.*

Then S supports special Ulrich bundles of rank r^{sp}_{Ulrich} as indicated in the Table A.

Theorem 1.5. *Let $S \subseteq \mathbb{P}^N$ be a surface of degree $d \le 8$ and $\kappa(S) \ne 1$.*
Then S is Ulrich–wild if and only if either $d \ge 5$, or $d \le 4$ and $\pi(S) \ge 1$.

When $\kappa(S) = 1$ we are not able to prove or disprove the existence of Ulrich bundles, though [37] provides some evidences in this direction.

In Section 2 we list some results about Ulrich bundles. In Section 3 we prove Theorem 1.2. In Section 4 we prove Theorem 1.3. In Section 5 we prove Theorems 1.4 and 1.5.

For reader's benefit we incorporate in Table A below all the important informations about surfaces in $\mathbb{P}^N$ of degree up to 8. Their classification can be essentially found in [31] and [32]: see also [39] and the references therein. In [32] and [39] the authors ruled out the case of irregular surfaces of degree 8 in $\mathbb{P}^4$ with different incomplete arguments. Such a case is completely described in the proof of Proposition 2.1 of [1]: see also [42], Section 1.3 and Lemma 1.4 for some further details.

Remark 1.6. In Table A we use the following notation.

- $X_{d_1,\dots,d_{N-2}}$ denotes any complete intersection of hypersurfaces of degrees $d_1, \dots, d_{N-2}$ in $\mathbb{P}^N$. In particular there is a natural rational map from a non–empty open subset of a Segre product $\prod_{i=1}^{N-2} \mathbb{P}\left(H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i)\big)\right)$ to the Hilbert scheme whose image corresponds to the locus of points representing such surfaces.
- Consider a ruled surface $X \cong \mathbb{P}(\mathcal{H})$, where $\mathcal{H}$ is a rank 2 vector bundle on a curve C which is normalized in the sense of Section V.2 of [26]. Then $p\colon X \to C$ denotes the canonical map, $e := -\deg(\det(\mathcal{H}))$ the invariant of X, ξ the class of the divisor $\mathcal{O}_X(1)$ and $\mathfrak{a}f$ the class of the pull back of the divisor $\mathfrak{a}$ on the curve C via p: if $a := \deg(\mathfrak{a})$ we will also write af for its numerical class in $\mathrm{Num}(X)$. If $C \cong \mathbb{P}^1$ we set $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, for each $e \ge 0$.
- $\mathrm{Bl}_{P_1,\dots,P_t} X$ denotes the blow up of X at the points $P_1, \dots, P_t$: in this case σ denotes the blow up map, $e_i := \sigma^{-1}(P_i)$.
- The number r^{sp}_{Ulrich}, if any, denotes the minimal rank of a special (not necessarily indecomposable) Ulrich bundle on S: in the case S is also known to support or not Ulrich line bundles '$\exists$ line bundles' and 'no line bundles' are added here. For the details see the proof of Theorem 1.4.

  In two cases, denoted by the sentence 'no results' in the last column in the table, we are unable to prove or disprove the existence of Ulrich bundles on S: these are exactly the cases when $\kappa(S) = 1$ (see Remark 5.1).

Table A: Surfaces of degree $d \le 8$ in $\mathbb{P}^N$

Class	Abstract model of S, $\mathcal{O}_S(h_S)$	d	κ	p_g	q	K_S^2	h_SK_S	N	r^{sp}_{Ulrich}
(I)	$X_1 \cong \mathbb{P}^2,\ \mathcal{O}_{\mathbb{P}^2}(1)$	1	$-\infty$	0	0	9	-3	2	2, $\exists$ line bundles
(II)	$X_2 \cong \mathbb{P}^1\times\mathbb{P}^1,\ \mathcal{O}_{\mathbb{P}^1}(1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(1)$	2	$-\infty$	0	0	8	-4	3	2, $\exists$ line bundles
(III)	$X_3 \cong \mathrm{Bl}_{P_1,\dots,P_6}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{i=1}^6 e_i\right)$	3	$-\infty$	0	0	3	-3	3	2, $\exists$ line bundles
(IV)	$\mathbb{F}_1,\ \mathcal{O}_S(\xi+2f)$	3	$-\infty$	0	0	8	-5	4	2, $\exists$ line bundles
(V)	$X_{2,2} \cong \mathrm{Bl}_{P_1,\dots,P_5}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{i=1}^5 e_i\right)$	4	$-\infty$	0	0	4	-4	4	2, $\exists$ line bundles
(VI)	$\mathbb{F}_e,\ e=0,2,\ \mathcal{O}_S\left(\xi+\frac{e+4}{2}f\right)$	4	$-\infty$	0	0	8	-6	5	2, $\exists$ line bundles
(VII)	$\mathbb{P}^2,\ \mathcal{O}_{\mathbb{P}^2}(2)$	4	$-\infty$	0	0	9	-6	5	2, no line bundles
(VIII)	$X_4,\ \mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathcal{O}_S$	4	0	1	0	0	0	3	2
(IX)	$\mathrm{Bl}_{P_1,\dots,P_8}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-2e_1-\sum_{i=2}^8 e_i\right)$	5	$-\infty$	0	0	2	-3	4	2, $\exists$ line bundles
(X)	$\mathrm{Bl}_{P_1,\dots,P_4}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{i=1}^4 e_i\right)$	5	$-\infty$	0	0	5	-5	5	2, $\exists$ line bundles
(XI)	$\mathbb{F}_e,\ e=1,3,\ \mathcal{O}_S\left(\xi+\frac{e+5}{2}f\right)$	5	$-\infty$	0	0	8	-7	6	2, $\exists$ line bundles
(XII)	elliptic ruled surface with $e=-1,\ \mathcal{O}_S(\xi+2f)$	5	$-\infty$	0	1	0	-5	6	2, $\exists$ line bundles
(XIII)	$X_5,\ \mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathcal{O}_S$	5	2	4	0	5	5	3	$\gg 0$, generically ≤ 2
(XIV)	$\mathrm{Bl}_{P_1,\dots,P_{10}}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-\sum_{i=1}^{10} e_i\right)$	6	$-\infty$	0	0	-1	-2	4	2, $\exists$ line bundles
(XV)	$\mathrm{Bl}_{P_1,\dots,P_3}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{i=1}^3 e_i\right)$	6	$-\infty$	0	0	6	-6	6	2, $\exists$ line bundles
(XVI)	$\mathbb{F}_e,\ e=0,2,4,\ \mathcal{O}_S\left(\xi+\frac{e+6}{2}f\right)$	6	$-\infty$	0	0	8	-8	7	2, $\exists$ line bundles
(XVII)	elliptic ruled surface with $e=0,\ \mathcal{O}_S(\xi+3f)$	6	$-\infty$	0	1	0	-6	5	2, $\exists$ line bundles
(XVIII)	$X_{2,3},\ \mathcal{O}_{\mathbb{P}^4}(1)\otimes\mathcal{O}_S$	6	0	1	0	0	0	4	2
(XIX)	$X_6,\ \mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathcal{O}_S$	6	2	10	0	24	12	3	$\gg 0$, generically ≤ 2
(XX)	$\mathrm{Bl}_{P_1,\dots,P_{11}}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(6)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^6 e_i-\sum_{j=7}^{11} e_j\right)$	7	$-\infty$	0	0	-2	-1	4	2
(XXI)	$\mathrm{Bl}_{P_1,\dots,P_8}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(6)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^7 e_i-e_8\right)$	7	$-\infty$	0	0	1	-3	5	2
(XXII)	$\mathrm{Bl}_{P_1,\dots,P_9}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-\sum_{i=1}^9 e_i\right)$	7	$-\infty$	0	0	0	-3	5	2
(XXIII)	$\mathrm{Bl}_{P_1,\dots,P_9}\mathbb{F}_e,\ e=0,\dots,3,$ $\sigma^*\mathcal{O}_{\mathbb{F}_e}(2\xi+(4+e)f)\otimes\mathcal{O}_S\left(-\sum_{i=1}^9 e_i\right)$	7	$-\infty$	0	0	-1	-3	5	2
(XXIV)	$\mathrm{Bl}_{P_1,\dots,P_6}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-2e_1-\sum_{i=2}^6 e_i\right)$	7	$-\infty$	0	0	3	-5	6	2
(XXV)	$\mathrm{Bl}_{P_1,P_2}\mathbb{P}^2,\ \sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{i=1}^2 e_i\right)$	7	$-\infty$	0	0	7	-7	7	2, $\exists$ line bundles
(XXVI)	$\mathbb{F}_e,\ e=1,3,5,\ \mathcal{O}_S\left(\xi+\frac{e+7}{2}f\right)$	7	$-\infty$	0	0	8	-9	8	2, $\exists$ line bundles

Table A: Surfaces of degree $d \leq 8$ in $\mathbb{P}^N$

Class	Abstract model of S, $\mathcal{O}_S(h_S)$	d	κ	p_g	q	K_S^2	h_SK_S	N	r_{Ulrich}
(XXVII)	elliptic ruled surface with $e=-1,1$, $\mathcal{O}_S\left(\xi+(4+\lceil\frac{e}{2}\rceil)f\right)$	7	$-\infty$	0	1	0	-7	8	2, $\exists$ line bundles
(XXVIII)	$\mathrm{Bl}_{P_1} X_{2,2,2}$, $\sigma^*\mathcal{O}_X(h_X)\otimes\mathcal{O}_S(-e_1)$	7	0	1	0	-1	1	4	2
(XXIX)	Proper elliptic, $\mathcal{O}_{\mathbb{P}^4}(1)\otimes\mathcal{O}_S$	7	1	2	0	0	3	4	no results
(XXX)	X_7	7	2	20	0	63	21	3	$\gg 0$, generically 2
(XXXI)	$\mathrm{Bl}_{P_1,\dots,P_{16}}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(6)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^4 e_i-\sum_{j=5}^{16}e_j\right)$	8	$-\infty$	0	0	-7	2	4	2
(XXXII)	$\mathrm{Bl}_{P_1,\dots,P_{12}}\mathbb{F}_e$, $e=0,\dots,4$, $\sigma^*\mathcal{O}_{\mathbb{F}_e}(2\xi+(5+e)f)\otimes\mathcal{O}_S\left(-\sum_{i=1}^{12}e_i\right)$	8	$-\infty$	0	0	-4	-2	5	2
(XXXIII)	$\mathrm{Bl}_{P_1,\dots,P_{10}}\mathbb{P}^1\times\mathbb{P}^1$, $\sigma^*(\mathcal{O}_{\mathbb{P}^1}(3)\boxtimes\mathcal{O}_{\mathbb{P}^1}(3))\otimes\mathcal{O}_S\left(-\sum_{i=1}^{10}e_i\right)$	8	$-\infty$	0	0	-2	-2	5	2
(XXXIV)	$\mathrm{Bl}_{P_1,\dots,P_{11}}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(7)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^{10}e_i-e_{11}\right)$	8	$-\infty$	0	0	-2	0	4	2
(XXXV)	$\mathrm{Bl}_{P_1,\dots,P_{10}}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(6)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^6 e_i-\sum_{j=7}^{10}e_j\right)$	8	$-\infty$	0	0	-1	-2	5	2
(XXXVI)	$\mathrm{Bl}_{P_1,\dots,P_8}\mathbb{F}_e$, $e=0,\dots,3$, $\sigma^*\mathcal{O}_{\mathbb{F}_e}(2\xi+(4+e)f)\otimes\mathcal{O}_S\left(-\sum_{i=1}^{8}e_i\right)$	8	$-\infty$	0	0	0	-4	6	2
(XXXVII)	$\mathrm{Bl}_{P_1,\dots,P_8}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-\sum_{i=1}^8 e_i\right)$	8	$-\infty$	0	0	1	-4	6	2
(XXXVIII)	$\mathrm{Bl}_{P_1,\dots,P_7}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(6)\otimes\mathcal{O}_S\left(-2\sum_{i=1}^7 e_i\right)$	8	$-\infty$	0	0	2	-4	6	2
(XXXIX)	$\mathrm{Bl}_{P_1,\dots,P_5}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(4)\otimes\mathcal{O}_S\left(-2e_1-\sum_{i=2}^5 e_i\right)$	8	$-\infty$	0	0	4	-6	7	2
(XL)	$\mathbb{F}_e$, $e=0,2,4,6$, $\mathcal{O}_S\left(\xi+\frac{e+8}{2}f\right)$	8	$-\infty$	0	0	8	-10	9	2, $\exists$ line bundles
(XLI)	$\mathrm{Bl}_{P_1}\mathbb{P}^2$, $\sigma^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S(-e_1)$	8	$-\infty$	0	0	8	-8	8	2, no line bundles
(XLII)	$\mathbb{P}^1\times\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{O}_{\mathbb{P}^1}(2)$	8	$-\infty$	0	0	8	-8	8	2, $\exists$ line bundles
(XLIII)	elliptic ruled surface with $e=0,2$, $\mathcal{O}_S\left(\xi+(4+\frac{e}{2})f\right)$	8	$-\infty$	0	1	0	-8	7	2, $\exists$ line bundles
(XLIV)	$\mathrm{Bl}_{P_1,\dots,P_8}X$, X elliptic ruled surface with $e=-1,1$, $\sigma^*\mathcal{O}_X(2\xi+(4+e)f)\otimes\mathcal{O}_S\left(-\sum_{i=1}^8 e_i\right)$	8	$-\infty$	0	1	-8	0	4	2
(XLV)	elliptic ruled surface with $e=-1$, $\mathcal{O}_S(2\xi+f)$	8	$-\infty$	0	1	0	4	5	2, no line bundles
(XLVI)	scroll with $e=-2$ over a curve C of genus 2, $\mathcal{O}_S(\xi+3f)$	8	$-\infty$	0	2	-8	-6	5	2, $\exists$ line bundles
(XLVII)	$C\times\mathbb{P}^1$, $C\subseteq\mathbb{P}^2$ smooth with $\deg(C)=4$, $\mathcal{O}_C(1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(1)$	8	$-\infty$	0	3	-16	-4	5	2, $\exists$ line bundles
(XLVIII)	$\mathrm{Bl}_{P_1}X$, $X\subseteq\mathbb{P}^7$ $K3$ surface, $\sigma^*\mathcal{O}_X(h_X)\otimes\mathcal{O}_S(-2e_1)$	8	0	1	0	-1	2	4	2
(IL)	$K3$, $\mathcal{O}_{\mathbb{P}^5}(1)\otimes\mathcal{O}_S$	8	0	1	0	0	0	5	2
(L)	Proper elliptic, $\mathcal{O}_{\mathbb{P}^4}(1)\otimes\mathcal{O}_S$	8	1	2	0	0	4	4	no results
(LI)	$X_{2,4}$, $\mathcal{O}_{\mathbb{P}^4}(1)\otimes\mathcal{O}_S$	8	2	5	0	8	8	4	$\gg 0$, generically ≤ 4
(LII)	X_8, $\mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathcal{O}_S$	8	2	35	0	128	32	3	$\gg 0$, generically 2

2. General results

Let $\mathcal{E}$ be an Ulrich sheaf on a variety $X \subseteq \mathbb{P}^N$ with $\dim(X) = n$. As pointed out in Section 2 of [22] (see in particular Proposition 2.1 therein) we know that $\mathcal{E}$ is aCM and $E := \bigoplus_{t\in\mathbb{Z}} H^0\big(X, \mathcal{E}(th_X)\big)$ has a minimal free resolution over $P := k[x_0, \dots, x_N]$ of the form

$$\begin{gathered}0 \longrightarrow P(n-N)^{\oplus\alpha_{N-n}} \longrightarrow P(n-N+1)^{\oplus\alpha_{N-n-1}} \longrightarrow \dots \\ \longrightarrow P(-1)^{\oplus\alpha_1} \longrightarrow P^{\oplus\alpha_0} \longrightarrow E \longrightarrow 0.\end{gathered} \tag{2.1}$$

Ulrich bundles also behave well with respect to the notions of (semi)stability and μ–(semi)stability. For each coherent sheaf $\mathcal{F}$ on X, the slope $\mu(\mathcal{F})$ and the reduced Hilbert polynomial $p_{\mathcal{F}}(t)$ (with respect to $\mathcal{O}_X(h_X)$) are defined as follows:

$$\mu(\mathcal{F}) = c_1(\mathcal{F})h_X^{\dim(X)-1}/\mathrm{rk}(\mathcal{F}), \qquad p_{\mathcal{F}}(t) = \chi(\mathcal{F}(th_X))/\mathrm{rk}(\mathcal{F}).$$

The coherent sheaf $\mathcal{F}$ is *μ–semistable* (resp. *μ–stable*) if for all subsheaves $\mathcal{G}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$ we have $\mu(\mathcal{G}) \le \mu(\mathcal{F})$ (resp. $\mu(\mathcal{G}) < \mu(\mathcal{F})$).

The coherent sheaf $\mathcal{F}$ is called *semistable* (resp. *stable*) if for all proper non–zero subsheaves $\mathcal{G}$, $p_{\mathcal{G}}(t) \le p_{\mathcal{F}}(t)$ (resp. $p_{\mathcal{G}}(t) < p_{\mathcal{F}}(t)$) for $t \gg 0$. We recall that in order to check the semistability and stability of $\mathcal{F}$ one can restrict the attention to subsheaves with torsion–free quotient. The following chain of implications holds for $\mathcal{F}$:

$$\mathcal{F} \text{ is } \mu\text{–stable} \ \Rightarrow\ \mathcal{F} \text{ is stable} \ \Rightarrow\ \mathcal{F} \text{ is semistable} \ \Rightarrow\ \mathcal{F} \text{ is } \mu\text{–semistable}.$$

We revert below some of the above implications.

Theorem 2.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety. If $\mathcal{E}$ is a Ulrich bundle on X the following assertions hold.*

(1) $\mathcal{E}$ is semistable and μ–semistable.
(2) $\mathcal{E}$ is stable if and only if it is μ–stable.
(3) If
$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{H} \longrightarrow 0$$
is an exact sequence of coherent sheaves with $\mathcal{H}$ torsion free and $\mu(\mathcal{G}) = \mu(\mathcal{E})$, then both $\mathcal{G}$ and $\mathcal{H}$ are Ulrich bundles.

Proof. See Theorem 2.9 of [12]. □

Thus, Ulrich bundles on X of minimal rank (if any) are both stable and μ–stable. It is interesting to estimate the size of the families of Ulrich bundles.

Theorem 2.2. *Let X be a smooth variety endowed with a very ample line bundle $\mathcal{O}_X(h_X)$. If $\mathcal{A}$ and $\mathcal{B}$ are simple Ulrich bundles on X such that $h^1\big(X, \mathcal{A}\otimes\mathcal{B}^\vee\big) \ge 3$ and every non–zero morphism $\mathcal{A} \to \mathcal{B}$ is an isomorphism, then X is Ulrich–wild.*

Proof. See [24, Theorem A, Corollary 2.1 and Remark 1.6 iii)]. Indeed $\mathcal{A}$ and $\mathcal{B}$, being Ulrich, satisfy $p_{\mathcal{A}}(t) = p_{\mathcal{B}}(t)$ by [12, Lemma 2.6] and are semistable by [12, Theorem 2.9]. □

From now on we restrict our attention to Ulrich bundles $\mathcal{E}$ on a surface S. The Serre duality for $\mathcal{E}$ is $h^i\big(S, \mathcal{E}\big) = h^{2-i}\big(S, \mathcal{E}^\vee(K_S)\big)$, for $i = 0, 1, 2$. The Riemann–Roch theorem for $\mathcal{E}$ is

$$h^0\big(S,\mathcal{E}\big) + h^2\big(S,\mathcal{E}\big) = \mathrm{rk}(\mathcal{E})\chi(\mathcal{O}_S) + \frac{c_1(\mathcal{E})(c_1(\mathcal{E}) - K_S)}{2} - c_2(\mathcal{E}) + h^1\big(S,\mathcal{E}\big), \tag{2.2}$$

where $\chi(\mathcal{O}_S) := 1 - q(S) + p_g(S)$.

Proposition 2.3. *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h_S)$ and let $d := h_S^2$. If $\mathcal{E}$ is a vector bundle on S, then the following assertions are equivalent:*

(1) $\mathcal{E}$ is an Ulrich bundle;
(2) $\mathcal{E}^\vee(3h_S + K_S)$ is an Ulrich bundle;
(3) $\mathcal{E}$ is an aCM bundle and

$$\begin{aligned} c_1(\mathcal{E})h_S &= \frac{\mathrm{rk}(\mathcal{E})}{2}(3d + h_S K_S), \\ c_2(\mathcal{E}) &= \frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_S) - \mathrm{rk}(\mathcal{E})(d - \chi(\mathcal{O}_S)); \end{aligned} \tag{2.3}$$

(4) $h^0\big(S, \mathcal{E}(-h_S)\big) = h^0\big(S, \mathcal{E}^\vee(2h_S + K_S)\big) = 0$ and Equalities (2.3) hold.

Proof. See [13, Proposition 2.1]. □

If $\mathcal{E}$ is an Ulrich bundle on S, then the first Equality (2.3) and the Hodge theorem applied to h_S and $c_1(\mathcal{E})$ yields

$$c_1(\mathcal{E})^2 \le \frac{\mathrm{rk}(\mathcal{E})^2}{4d}(3d + h_S K_S)^2. \tag{2.4}$$

If $\mathcal{F}$ is another Ulrich bundle on S

$$\chi(\mathcal{E}^\vee \otimes \mathcal{F}) = \mathrm{rk}(\mathcal{F})c_1(\mathcal{E})K_S - c_1(\mathcal{E})c_1(\mathcal{F}) + \mathrm{rk}(\mathcal{E})\mathrm{rk}(\mathcal{F})(2d - \chi(\mathcal{O}_S)) \tag{2.5}$$

(see Proposition 2.12 of [12] for the details).

Finally, the hypothesis $\mathrm{rk}(\mathcal{E}) = 1$ yields $c_2(\mathcal{E}) = 0$. Thus, if $\mathcal{E} \cong \mathcal{O}_S(D)$, then Equalities (2.3) become

$$Dh_S = \frac{1}{2}(3d + h_S K_S), \qquad D^2 = 2(d - \chi(\mathcal{O}_S)) + DK_S. \tag{2.6}$$

3. Ulrich–wildness of surfaces

In this section we first prove Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. Let $\mathcal{E}$ be an Ulrich bundle of minimal rank r on S: Theorem 2.1 implies that $\mathcal{E}$ is stable, hence simple (see [30], Corollary 1.2.8).

The bundle $\mathcal{F} := \mathcal{E}^\vee(3h_S + K_S)$ is stable and simple too. In particular, every non–zero morphism $\mathcal{E} \to \mathcal{F}$ is an isomorphism, thanks to [30, Proposition 1.2.7]. Moreover, $\mathcal{F}$ is Ulrich by Proposition 2.3.

Since $c_1(\mathcal{F}) = r(3h_S + K_S) - c_1(\mathcal{E})$, it follows from Equality (2.5) that

$$\begin{aligned} h^1\big(S, \mathcal{E}^\vee \otimes \mathcal{F}\big) &= -\chi(\mathcal{E}^\vee \otimes \mathcal{F}) + h^2\big(S, \mathcal{E}^\vee \otimes \mathcal{F}\big) \ge -\chi(\mathcal{E}^\vee \otimes \mathcal{F}) = \\ &= -rc_1(\mathcal{E})K_S + rc_1(\mathcal{E})(3h_S + K_S) - c_1(\mathcal{E})^2 - r^2(2d - \chi(\mathcal{O}_S)). \end{aligned}$$

The first Equality (2.3) combined with Inequalities (2.4) and (1.1) yield

$$h^1\big(S, \mathcal{E}^\vee \otimes \mathcal{F}\big) \ge \frac{r^2}{4d}(d^2 + 4\chi(\mathcal{O}_S)d - (h_S K_S)^2) > 2.$$

Thus the statement follows from Theorem 2.2. □

Example 3.1. Thanks to [9] and [27] each complete intersection surface is the support of an Ulrich bundle as well.

Let $S \subseteq \mathbb{P}^N$ be a non–degenerate complete intersection surface of degree d. Then Theorem 1.2 implies that S is certainly Ulrich–wild if either $5 \le d \le 9$, or $N = 3$ and $d = 4$. When $d = N = 3$ (resp. $d = N = 4$) the Ulrich–wildness of S was proved in

[12] (resp [41]). If $d \le 2$, then S supports a finite number of indecomposable Ulrich (and aCM) bundles. If $d \ge 10$ and $N = 3$, S does not satisfy Inequality (1.1), thus we cannot say anything about its Ulrich–wildness.

Example 3.2. Let S be a minimal surface with $\kappa(S) = 0$ over $\mathbb{C}$. The Enriques–Kodaira classification implies that S is either abelian, or a $K3$ surface, or an Enriques surface, or a bielliptic surface. The Riemann–Roch and the Kodaira vanishing theorems on S imply that

$$d = 2h^0\big(S, \mathcal{O}_S(h_S)\big) - 2\chi(\mathcal{O}_S) > 4(2 - \chi(\mathcal{O}_S)).$$

Since trivially $h_S K_S = 0$ in all these cases, we immediately deduce from Theorem 1.2, [8, Proposition 6] and [23, Theorem 1] that every such S is Ulrich–wild.

The Ulrich–wildness of Enriques, $K3$ and bielliptic surfaces were already shown in [13], [23] and [14].

We close this section with the following corollary of Theorem 1.2, where $k = \mathbb{C}$.

Corollary 3.3. *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h_S)$ such that $\mathcal{O}_S(h_S) \cong \mathcal{O}_S(\lambda K_S)$ for some positive integer λ. If S supports a bundle which is Ulrich with respect to $\mathcal{O}_S(h_S)$, then S is Ulrich–wild.*

Proof. If $\mathcal{O}_S(h_S) := \mathcal{O}_S(\lambda K_S)$ is very ample, then $\mathcal{O}_S(K_S)$ is ample. The ampleness of $\mathcal{O}_S(K_S)$ and the Nakai criterion yield that S is minimal. Thus both K_S^2 and $\chi(\mathcal{O}_S)$ are positive (see Theorem VII.1.1 of [5]).

Since the degree of S is $d = \lambda^2 K_S^2$ and $h_S K_S = \lambda K_S^2$, it follows that Inequality (1.1) is equivalent to

$$(\lambda^2 - 1)K_S^2 + 4\chi(\mathcal{O}_S) > 8, \tag{3.1}$$

which is immediately satisfied if $\lambda \ge 3$.

If $\lambda = 2$, then $\mathcal{O}_S(2K_S)$ is very ample, hence $h^0\big(S, \mathcal{O}_S(2K_S)\big) \ge 4$. Proposition VII.5.3 of [5] implies $K_S^2 + \chi(\mathcal{O}_S) \ge 4$, whence we deduce Inequality (3.1).

If $\lambda = 1$, then $\mathcal{O}_S(K_S)$ is very ample, hence $p_g(S) \ge 4$. In this case Inequality (3.1) is equivalent to $\chi(\mathcal{O}_S) > 2$. If $\chi(\mathcal{O}_S) \le 2$, then either $q(S) = p_g(S)$, or $q(S) + 1 = p_g(S)$.

The Bogomolov–Miyaoka–Yau inequality (see Theorem VII.4.1 of [5]) and the equality $12\chi(\mathcal{O}_S) = K_S^2 + c_2(\Omega_S^1)$ imply $K_S^2 \le 9\chi(\mathcal{O}_S)$. By combining the latter inequality with Theorem 3.2 of [21] we obtain $q(S) = p_g(S) = 4$ if $\chi(\mathcal{O}_S) = 1$, and $6 \ge q(S)+1 = p_g(S) \ge 4$ if $\chi(\mathcal{O}_S) = 2$. Since every surface in $\mathbb{P}^3$ is regular, it follows that we can restrict to the case $6 \ge q(S) + 1 = p_g(S) \ge 5$.

If $p_g(S) = 5$, then the double point formula for S (see [26, Example A.4.1.3]) implies that $d = h_S^2 = h_S K_S = K_S^2$ satisfies $d^2 - 17d + 24 = 0$ which has no integral solutions. We deduce that $q(S) + 1 = p_g(S) = 6$, hence S should be the product of a curve of genus 2 and a curve of genus 3 (see the Theorem in the Appendix of [21]): since the canonical map for such a surface has degree at least 2, it follows a contradiction.

Thus, if S supports an Ulrich bundle, then it is Ulrich–wild by Theorem 1.2. □

4. Ulrich bundles on intersections

In this section we prove the following generalization of the main result of [27], which is usually stated only for complete intersections.

Proof of Theorem 1.3. Let $n := \dim(X)$, $m := \dim(Y)$: thanks to the hypothesis we have that $\dim(X \cap Y) = m + n - N$. If $m + n \le N$, then the statement is trivial, thus we will assume $m + n > N$ from now on.

There is an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(n-N)^{\oplus\alpha_{N-n}} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(n-N+1)^{\oplus\alpha_{N-n-1}} \longrightarrow \dots \\ \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus\alpha_0} \longrightarrow \mathcal{A} \longrightarrow 0$$

obtained by sheafifying Sequence (2.1) with $E := \bigoplus_{t\in\mathbb{Z}} H^0\big(X, \mathcal{A}(th_X)\big)$.

The variety Y intersects properly X and it is locally complete intersection at each point $x \in X \cap Y$. Thus its local equations in $\mathcal{O}_{\mathbb{P}^N,x}$ are regular elements for $\mathcal{O}_{X,x}$, hence also in $\mathcal{A}_x \cong \mathcal{O}_{X,x}^{\oplus a}$. When $x \in Y \setminus X$ then $\mathcal{A}_x$ is zero. Thus the above sequence tensored by $\mathcal{O}_Y$ is everywhere exact along Y, thanks to Theorem 7 of [38].

Taking into account that $\mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{O}_Y \otimes_Y \mathcal{B} \cong \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}$, by tensoring such a restricted sequence with $\mathcal{B}$, we obtain the complex on Y

$$0 \longrightarrow \mathcal{B}((n-N)h_Y)^{\oplus\alpha_{N-n}} \xrightarrow{\varphi_{N-n}} \mathcal{B}((n-N+1)h_Y)^{\oplus\alpha_{N-n-1}} \xrightarrow{\varphi_{N-n-1}} \dots \\ \xrightarrow{\varphi_2} \mathcal{B}(-h_Y)^{\oplus\alpha_1} \xrightarrow{\varphi_1} \mathcal{B}^{\oplus\alpha_0} \longrightarrow \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B} \longrightarrow 0.$$

Such a complex is actually exact on $X \cap Y$, because $\mathcal{B}_x \cong \mathcal{O}_{Y,x}^{\oplus b}$. When $x \in Y \setminus X$ it is still exact because $\mathcal{A}_x$ is zero.

Since $\mathcal{B}$ is Ulrich on Y, it follows that

$$h^i\big(Y, \mathcal{B}(-ih_Y)\big) = h^j\big(Y, \mathcal{B}(-(j+1)h_Y)\big) = 0, \tag{4.1}$$

for $i > 0$ and $j < m$ by definition (see the introduction).

Let $\mathcal{C}_0 := \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}$ and $\mathcal{C}_\lambda := \operatorname{im}(\varphi_\lambda)$ for $1 \le \lambda \le N-n$: notice that $\mathcal{C}_{N-n} = \mathcal{B}((n-N)h_Y)^{\oplus\alpha_{N-n}}$. We have the exact sequences

$$0 \longrightarrow \mathcal{C}_{\lambda+1} \longrightarrow \mathcal{B}(-\lambda h_Y)^{\oplus\alpha_\lambda} \longrightarrow \mathcal{C}_\lambda \longrightarrow 0.$$

Tensoring the above exact sequences by $\mathcal{O}_Y((\lambda-i)h_Y)$ and $\mathcal{O}_Y((\lambda-j-1)h_Y)$ and taking their cohomologies, Equalities (4.1) yield respectively

$$h^i\big(Y, \mathcal{C}_\lambda((\lambda-i)h_Y)\big) \le h^{i+1}\big(Y, \mathcal{C}_{\lambda+1}((\lambda+1-i-1)h_Y)\big), \\ h^j\big(Y, \mathcal{C}_\lambda((\lambda-j-1)h_Y)\big) \le h^{j+1}\big(Y, \mathcal{C}_{\lambda+1}((\lambda+1-j-2)h_Y)\big),$$

for each $i > 0$, $j < m$ and $0 \le \lambda \le N-n-1$. Taking into account that $\mathcal{C}_0 = \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}$ and $\mathcal{C}_{N-n} = \mathcal{B}((n-N)h_Y)^{\oplus\alpha_{N-n}}$, the above inequalities return

$$0 \le h^i\big(X \cap Y, \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}(-ih_Y)\big) \le h^{i+N-n}\big(Y, \mathcal{B}((n-N-i)h_Y)^{\oplus\alpha_{N-n}}\big), \\ 0 \le h^j\big(X \cap Y, \mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}(-(j+1)h_Y)\big) \le h^{j+N-n}\big(Y, \mathcal{B}((n-N-j-1)h_Y)^{\oplus\alpha_{N-n}}\big),$$

for $i > 0$ and $j+N-n < m$, i.e. $j < n+m-N$. Equalities (4.1) then imply that $\mathcal{A} \otimes_{\mathbb{P}^N} \mathcal{B}$ is also Ulrich by definition.

Finally, $\mathcal{A}$ and $\mathcal{B}$ are locally free along $X \cap Y$, hence the same is true for $\mathcal{A} \otimes \mathcal{B}$. □

Example 4.1. Let $F := G_1 \cap G_2$ be a GPK^3 threefold over $\mathbb{C}$ (see the introduction).

Theorem 3.5 and Corollary 4.6 of [20] guarantee the existence of an Ulrich bundle $\mathcal{E}_i$ on G_i of rank 3. Since $\operatorname{Pic}(G_i)$ is generated by $\mathcal{O}_{G_i}(h_{G_i})$, there is $\alpha_i \in \mathbb{Z}$ such that $c_1(\mathcal{E}_i) = \alpha_i h_{G_i}$. The rational number $\mu(\mathcal{E}_i)$ does not depend on the choice of the Ulrich bundle on G_i (this is a standard property of Ulrich bundles: e.g. see [20], Proposition 2.5), hence we finally obtain $\alpha_i = 3$ by combining the previous discussion with Corollary 3.7 in [20].

Theorem 1.3 implies that F supports the Ulrich bundle $\mathcal{E} := \mathcal{E}_1 \otimes_{\mathbb{P}^9} \mathcal{E}_2$ which has rank 9. Moreover, $c_1(\mathcal{E}) = \operatorname{rk}(\mathcal{E}_1)c_1(\mathcal{E}_2)F + \operatorname{rk}(\mathcal{E}_2)c_1(\mathcal{E}_1)F = 18h_F$.

Each hyperplane section S of F supports Ulrich bundles as well. The adjunction formula implies that S is canonically embedded, hence it is Ulrich–wild thanks to Corollary 3.3. It is not difficult to check that $q(S) = 0$, $p_g(S) = 9$, and $K_S^2 = 25$. In particular S is not complete intersection.

Example 4.2. If F is a del Pezzo threefold over $\mathbb{C}$, $F \subseteq \mathbb{P}^{a+1}$, $3 \le a \le 8$, then it supports an Ulrich bundle of rank 2 with $c_1(\mathcal{F}) = 2h_F$ (see [8, Proposition 8]: see also [3], [15], [16], [17] and the methods described therein). By combining this fact with the existence of Ulrich bundles on each hypersurface in $\Delta \subseteq \mathbb{P}^N$ of degree δ (see [9]), we deduce that each smooth hypersurface section $S = F \cap \Delta$ certainly supports Ulrich bundles of high rank.

The surface S has degree δa and $K_S = (\delta - 2)h_S$ by adjunction. In particular $h_S K_S = (\delta - 2)\delta a$. Moreover the cohomology of

$$0 \longrightarrow \mathcal{O}_F(-\delta h_F) \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_S \longrightarrow 0$$

yields $\chi(\mathcal{O}_S) = 1 - \chi(\mathcal{O}_F(-\delta h_F))$. The Riemann–Roch theorem on F and the equality $h_F c_2(\Omega^1_{F|k}) = 12$ finally return

$$\chi(\mathcal{O}_S) = -\frac{1}{6}\delta(\delta - 1)(\delta - 2)a + \delta.$$

Simple computations show that Inequality (1.1) is then equivalent to

$$-\delta(\delta - 1)(\delta - 5)a > 24 - 12\delta.$$

When $a \ge 3$, the above inequality is satisfied if and only if $2 \le \delta \le 5$. Thus S is Ulrich–wild in this range, thanks to Theorem 1.2. Notice that S is a surface of general type when $\delta \ge 3$ and it is a $K3$ surface when $\delta = 2$: the existence of special Ulrich bundles of rank 2 on each $K3$ surface has been recently proved in [23].

We spend some further words in the latter case. If the rank of $\Delta \subseteq \mathbb{P}^{a+1}$ is 4, then Δ is endowed with two pencils of linear spaces of dimension $a - 1$. They induce two fibrations on S with elliptic normal curves of degree a as fibres.

If we choose projective coordinates in $\mathbb{P}^{a+1}$ such that $\Delta = \{\ x_0x_2 - x_1x_3 = 0\ \}$, then the matrix

$$\begin{pmatrix} x_0 & x_3 \\ x_1 & x_2 \end{pmatrix},$$

defines a monomorphism $\varphi\colon \mathcal{O}_{\mathbb{P}^{a+1}}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^{a+1}}^{\oplus 2}$, whose cokernel $\mathcal{S}$ is an Ulrich sheaf of rank 1 on Δ. Theorem 1.3 yields that $\mathcal{E} := \mathcal{F} \otimes \mathcal{S}$ is an Ulrich bundle of rank 2 on S. Each non–zero section of $\mathcal{S}$ vanishes on a linear space of one of the aforementioned pencils, thus $\mathcal{S} \otimes \mathcal{O}_S$ is the line bundle $\mathcal{O}_S(A)$ associated to the corresponding elliptic fibration on S. It follows that $c_1(\mathcal{E}) = 2h_S + 2A$.

Notice that $A^2 = 0$ and $h_S A = a$, thus $c_1(\mathcal{E})^2 = 16a \ne 18a = (3h_S)^2$: in particular $\mathcal{E}$ is not special. For further examples in this direction see [18].

Remark 4.3. Even if $\mathcal{A}$ and $\mathcal{B}$ are Ulrich vector bundles of minimal rank on X and Y, the bundle $\mathcal{A} \otimes \mathcal{B}$ need not be of minimal rank on $X \cap Y$.

For example, let $F \subseteq \mathbb{P}^4$ be a smooth cubic threefold and $\mathcal{L} \in \operatorname{Pic}(F)$. If $\mathcal{L}$ were Ulrich, then $h^0\big(F, \mathcal{L}(-h_F)\big) = 0$ and $h^0\big(F, \mathcal{L}\big) \ne 0$. Since the $\operatorname{Pic}(F)$ is generated by $\mathcal{O}_F(h_F)$, it follows that $\mathcal{L} \cong \mathcal{O}_F$, which is not Ulrich because $h^3\big(F, \mathcal{O}_F(-3h_F)\big) \ne 0$.

Nevertheless, for each general hyperplane $H \subseteq \mathbb{P}^4$, the surface $S := F \cap H$ supports Ulrich line bundles (see [6], Corollaries 6.4 and 1.12).

5. Ulrich wildness of some surfaces

In this section we will prove Theorems 1.4 and 1.5 stated in the introduction.

Proof of Theorem 1.4. If S is in the classes (VIII), (XVIII), (IL) the existence of special Ulrich bundles of rank 2 have been proved in [19, Proposition 7.6] and [23, Theorem 1]. Every surface in class (XXVIII) is the blow up of a polarized surface in class (IL) at a point. Thus [33, Theorem 0.1] implies that S supports a special Ulrich bundle of rank 2.

In the cases (XIII), (XIX), (XXX), (LI) and (LII), then S is either a surface in $\mathbb{P}^3$ of degree $5 \le d \le 8$, or quadro–quartic complete intersection in $\mathbb{P}^4$. For the existence of Ulrich bundles (possibly of very high rank) in these cases see [27] and the references therein.

In [7, Proposition 7.6] the author proves the existence of special Ulrich bundles of rank 2 on the general surface in $\mathbb{P}^3$ of degree $d \le 15$. This proves the statement in cases (XIII), (XIX), (XXX) and (LII).

We now examine the case (LI). We know the existence of a quadric Q and a quartic F in $\mathbb{P}^4$ such that $S = Q \cap F$. If F is general, then Proposition 8.9 of [7] guarantees the existence of a special Ulrich bundle $\mathcal{F}$ of rank 2 on F: we have $c_1(\mathcal{F}) = 3h_F$. If Q is smooth, then it supports a unique spinor bundle $\mathcal{S}$. Theorem 2.3 and Remark 2.9 of [40] imply that $\mathcal{S}(h_Q)$ is aCM, initialized and the Serre duality implies $h^3\big(Q, \mathcal{S}(-2h_Q)\big) = h^0\big(Q, \mathcal{S}\big)$. Thus $\mathcal{S}(h_Q)$ is Ulrich: we have $c_1(\mathcal{S}(h_Q)) = h_Q$ thanks to Remark 2.9 of [40]. Theorem 1.3 implies that $\mathcal{E} := \mathcal{F} \otimes \mathcal{S}(h_Q)$ is Ulrich on S of rank 4 with $c_1(\mathcal{E}) = 2c_1(\mathcal{F})S + 2c_1(\mathcal{S}(h_Q))S = 8h_S$. Since $K_S = h_S$, it follows that $\mathcal{E}$ is special.

Surfaces S in classes (I), (II), (IV), (VI), (XI), (XII), (XVI), (XVII), (XXVI), (XXVII), (XL), (XLIII), (XLVI), (XLVII) support Ulrich line bundles, because they are embedded as scroll in these cases (see [8, Proposition 5]). The Bordiga (that is the surfaces in class (IX)) and Castelnuovo surfaces (surfaces in class (XIV)) also carry Ulrich line bundles (see [35]). The same is true for del Pezzo surfaces of degree $d = 3, \dots, 7$ (surfaces in classes (III), (V), (X), (XV), (XXV): see [41]).

Surfaces in classes (VII), (XLI), (XLV) do not support Ulrich line bundles (see [13], Example 2.1, [41], [13] respectively).

Surfaces in classes (VII), (XX), (XXI), (XXII), (XXIII), (XXIV), (XXXII), (XXXIII), (XXXIV), (XXXV), (XXXVI), (XXXVII), (XXXVIII), (XXXIX), (XLI), (XLII) have $p_g(S) = q(S) = 0$. Moreover, the line bundle $\mathcal{O}_S(h_S)$ is *non–special* (i.e. $h^1\big(S, \mathcal{O}_S(h_S)\big) = 0$), as one can check by confronting the value of N listed in the penultimate column in Table A with the expected dimension of the linear system. Thus the existence of a special Ulrich bundle of rank 2 on them follows from Theorem 1.1 of [13].

Surfaces S in class (XLV) have $p_g(S) = 0$ and $q(S) = 1$. Moreover the linear system $\mathcal{O}_S(h_S)$ is non–special thanks to [25], Proposition 3.1. The existence of a special Ulrich bundle of rank 2 on such an S follows from Theorem 1.1 of [14] (the case $e = 1$ is also covered by [2]).

In the case (XXXI) the linear system $\mathcal{O}_S(h_S)$ is special with $h^1\big(S, \mathcal{O}_S(h_S)\big) = 1$. The surface S can be obtained in two steps as follows.

In the first step we blow up $\mathbb{P}^2$ at some points $P_1, \dots, P_4$, obtaining a surface S_1: if $\sigma_1 \colon S_1 \to \mathbb{P}^2$ is the blow up morphism, then we embed S_1 via $\mathcal{O}_{S_1}(h_{S_1}) := \sigma_1^*\mathcal{O}_{\mathbb{P}^2}(6) \otimes \mathcal{O}_{S_1}(-2\sum_{i=1}^4 e_i)$. The points $P_1, \dots, P_4$ are in general linear position in $\mathbb{P}^2$, i.e. any three of them are not collinear: otherwise, there would exist an effective divisor D on S_1 (the proper transform of the line through the three collinear points) such that $h_{S_1}D = 0$, contradicting the ampleness of $\mathcal{O}_S(h_{S_1})$. With this in mind it is immediate to check that

$\mathcal{O}_{S_1}(h_{S_1})$ is non–special. Thus S_1 supports a special Ulrich bundle of rank 2 with respect to $\mathcal{O}_{S_1}(h_{S_1})$ also in this case.

In the second step we blow up S_1 at some points $P_5, \dots, P_{16}$: if $\sigma_2\colon S \to S_1$ is the blow up morphism, then we embed S via $\mathcal{O}_S(h_S) := \sigma_2^*\mathcal{O}_{S_1}(h_{S_1}) \otimes \mathcal{O}_S(-\sum_{j=5}^{16} e_j)$. Thus Theorem 0.1 of [33] implies that S supports a special Ulrich bundle of rank 2 also in this case.

Consider the case (XLIV). On the one hand $h^0\big(S, \mathcal{O}_S(h_S)\big) = 5$. On the other hand the Riemann–Roch theorem yields $\chi(\mathcal{O}_S(h_S)) = 4$, hence $h^1\big(S, \mathcal{O}_S(h_S)\big) = 1$. Thus the surface is specially polarized in this case too.

Let $\mathfrak{a}$ be an arbitrary divisor on C of degree $4+e$. Then $\mathcal{O}_X(h_X) := \mathcal{O}_X(2\xi + \mathfrak{a}f)$ is very ample on X, thanks to [28, Theorem 3.3] when $e = 1$ and [29, Theorem 3.4] when $e = -1$. Moreover, it is also non–special, thanks to [25, Proposition 3.1]. Thus X supports a special Ulrich bundle of rank 2 with respect to $\mathcal{O}_X(h_X)$, thanks to Corollary 3.5 of [14] (the case $e = 1$ being also covered by [2]). The existence on S of a special Ulrich bundle of rank 2 follows again from Theorem 0.1 of [33].

We now construct an Ulrich bundle on a surface S in class (XLVIII) (see [39] for details). Let $Z \subseteq S$ be a set of 9 general points. Since $\mathcal{O}_S(h_S + 4e_1) \cong \sigma^*\mathcal{O}_X(h_X) \otimes \mathcal{O}_S(2e_1)$, it follows that each divisor in $|\mathcal{O}_S(h_S + 4e_1)|$ is the sum of a divisor in $|\sigma^* h_X|$ plus $2e_1$, hence

$$h^0\big(S, \mathcal{O}_S(h_S + 4e_1)\big) = h^0\big(S, \sigma^*\mathcal{O}_X(h_X)\big).$$

Moreover, $R^i\sigma_*\sigma^*\mathcal{O}_X(h_X) \cong \mathcal{O}_X(h_X) \otimes R^i\sigma_*\mathcal{O}_S \cong 0$ if $i \geq 1$ (see [26], Proposition V.3.4), hence

$$h^0\big(S, \sigma^*\mathcal{O}_X(h_X)\big) = h^0\big(X, \mathcal{O}_X(h_X)\big) = 8. \tag{5.1}$$

We deduce from Theorem 5.1.1 in [30] that Z has the Cayley–Bacharach property with respect to $\mathcal{O}_S(h_S + 4e_1)$, hence there is an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{Z|S}(h_S - K_S + 4e_1) \longrightarrow 0$$

where $\mathcal{F}$ is a vector bundle of rank 2 with $c_1(\mathcal{F}) = h_S - K_S + 4e_1$ and $c_2(\mathcal{F}) = 9$.

The bundle $\mathcal{E} := \mathcal{F}(h_S + K_S - 2e_1)$ fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_S(h_S + K_S - 2e_1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|S}(2h_S + 2e_1) \longrightarrow 0 : \tag{5.2}$$

we have $c_1(\mathcal{E}) = 3h_S + K_S$, whence $h^0\big(S, \mathcal{E}(-h_S)\big) = h^0\big(S, \mathcal{E}^\vee(2h_S + K_S)\big)$ and $c_2(\mathcal{E}) = 27$. Moreover, computing the cohomology of Sequence (5.2) tensored by $\mathcal{O}_S(-h_S)$, taking into account the isomorphisms $\mathcal{O}_S(K_S) = \mathcal{O}_S(e_1)$, $\sigma^*\mathcal{O}_X(h_X) \cong \mathcal{O}_S(h_S + 2e_1)$ and Equality (5.1), we finally obtain

$$h^0\big(S, \mathcal{E}(-h_S)\big) \leq h^0\big(S, \mathcal{O}_S(-e_1)\big) + h^0\big(S, \mathcal{I}_{Z|S} \otimes \mathcal{O}_X(h_X)\big) = 0,$$

because $\deg(Z) = 9$ and $h^0\big(S, \sigma^*\mathcal{O}_X(h_X)\big) = 8$. Proposition 2.3 yields that $\mathcal{E}$ is a special Ulrich bundle of rank 2 on S. □

Remark 5.1. If S is in classes (XXIX) and (L), then its canonical map is an elliptic fibration $\epsilon\colon S \to \mathbb{P}^1$, fibres being elliptic normal curves of degrees 3 and 4.

In [34, Theorem III.4.2 and Observation III.3.5], the author proves that $\operatorname{Pic}(S)$ is generated by h_S and K_S for the very general surface S as above. The map ϵ has no sections, otherwise there would exist integers x, y such that $(xK_S + yh_S)K_S = 1$, contradicting the conditions $K_S^2 = 0$ and $3 \leq h_S K_S \leq 4$. Thus, we cannot use results from [37] for proving the existence of Ulrich bundles on such surfaces.

We are now ready to prove Theorem 1.5 as an easy corollary of Theorem 1.2.

Proof of Theorem 1.5. Surfaces S in classes (III), (V), (IX), (X), (XI), (XIV), (XV), (XVI), (XX), (XXI), (XXII), (XXIII), (XXIV), (XXV), (XXVI), (XXXII), (XXXIII), (XXXIV), (XXXV), (XXXVI), (XXXVII), (XXXVIII), (XXXIX), (XL), (XLI), (XLII) are non–special and have $p_g(S) = q(S) = 0$. Thus their Ulrich–wildness follows from Theorem 1.3 of [13]. By the same theorem, if $d \le 4$ and $\pi(S)$ vanishes, then S is not Ulrich–wild: these cases are (I), (II), (IV), (VI), (VII).

Similarly surfaces S in classes (XII), (XVII), (XXVII), (XLIII), (XLV) are Ulrich–wild, thanks to [14, Theorem 1.3] and [25, Proposition 3.1].

In cases (VIII), (XIII), (XVIII), (XIX), (XXVIII), (XXX), (XXXI), (XLIV), (XLVIII), (IL), (LI), (LII) the statement follows easily from Theorem 1.2, thanks to the invariants listed in the table.

In cases (XLV), (XLVI) the surface S is geometrically ruled on a curve C and it is embedded in $\mathbb{P}^N$ as a scroll. Following the notation in Remark 1.6, we know that the invariant e is -2 and 0 in cases (XLV) and (XLVI) respectively. Thus, we have $\mathcal{O}_S(h_S) \cong \mathcal{O}_S(\xi + p^*\mathfrak{b})$, hence $2\deg(\mathfrak{b}) - e = \deg(S) = 8$.

Assertion 2) of Proposition 5 in [8] implies that for each general $\mathfrak{u} \in \mathrm{Pic}^{g-1}(C)$ then $\mathcal{L} := \mathcal{O}_S(h_S + p^*\mathfrak{u}) \cong \mathcal{O}_S(\xi + p^*\mathfrak{b} + p^*\mathfrak{u})$ is Ulrich. It follows from Proposition 2.3 that $\mathcal{M} := \mathcal{O}_S(2h_S + K_S - p^*\mathfrak{u}) \cong p^*\mathcal{O}_E(2\mathfrak{b} + \mathfrak{h} + \mathfrak{k} - \mathfrak{u})$ is Ulrich too.

Such bundles are trivially simple and $h^0\big(S, \mathcal{L} \otimes \mathcal{M}^\vee\big) = h^0\big(S, \mathcal{M} \otimes \mathcal{L}^\vee\big) = 0$ because $\mathcal{L} \not\cong \mathcal{M}$. Since $\mathcal{L} \otimes \mathcal{M}^\vee \cong \mathcal{O}_S(\xi - p^*\mathfrak{b} - p^*\mathfrak{h} - p^*\mathfrak{k} + 2\mathfrak{u})$, it follows from Equality (2.2) that

$$h^1\big(S, \mathcal{L} \otimes \mathcal{M}^\vee\big) \ge -\chi(\mathcal{L} \otimes \mathcal{M}^\vee) = 2\deg(\mathfrak{b}) - e = 8.$$

The statement thus follows from Theorem 2.2. □

Remark 5.2. The proofs of Theorems 1.3 of [13] and [14] used above contain a gap which can be overcome by assuming that k is uncountable (see the erratum). Thus, such theorems certainly hold when $k = \mathbb{C}$ as we assume in Theorem 1.5.

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