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Rapid Communications

# Engineered swift equilibration of a Brownian gyrator

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(Received 29 May 2020; accepted 8 September 2020; published 30 September 2020)

In the context of stochastic thermodynamics, a minimal model for nonequilibrium steady states has been recently proposed: the Brownian gyrator (BG). It describes the stochastic overdamped motion of a particle in a two-dimensional harmonic potential, as in the classic Ornstein-Uhlenbeck process, but considering the simultaneous presence of two independent thermal baths. When the two baths have different temperatures, the steady BG exhibits a rotating current, a clear signature of nonequilibrium dynamics. Here, we consider a time-dependent potential, and we apply a reverse-engineering approach to derive exactly the required protocol to switch from an initial steady state to a final steady state in a finite time $\tau$. The protocol can be built by first choosing an arbitrary quasistatic counterpart, with few constraints, and then adding a finite-time contribution which only depends upon the chosen quasistatic form and which is of order $1/\tau$. We also get a condition for transformations which, in finite time, conserve internal energy, useful for applications such as the design of microscopic thermal engines. Our study extends finite-time stochastic thermodynamics to transformations connecting nonequilibrium steady states.

DOI: 10.1103/PhysRevE.102.030105

*Introduction*. Fast switching through two or more modes of operation in microscopic experiments, where fluctuations dominate, is a goal for several applications: cyclical mesoscopic thermal machines such as colloids in time-dependent optical traps [1–6], thermal engines realized in bacterial baths [7,8], realization of bit operation under noisy environment with connection to information theory [9], and much more. Experiments and theory have recently demonstrated the existence of special protocols that in finite time realize conditions which are usually realized in infinite time: these protocols can be deduced by reverse-engineering the desired, fast path of evolution of given observables, including the probability distribution in phase space [10,11].

A paradigmatic example has been given in one effective dimension with a harmonic trap, realized by optical radiation confining a colloidal particle [12]. The colloidal particle has reached a steady state in the trap with a stiffness $k^i$. Then the trap is modulated from the initial stiffness $k^i$ to a new stiffness $k^f$ in some finite time $\tau$. If the change $k^i \to k^f$ is realized in too short a time, e.g., taking ideally $\tau = 0$ (what is called "STEP" protocol), then the colloidal particle will take some uncontrolled additional time to reach the steady state compatible with the final stiffness. Such a "natural" time is related to the typical relaxation times of the system and can be very long, depending upon the situation. Interestingly, it is possible to design one or more "swift equilibration" (SE) protocols $k(t)$, with $k(0) = k^i$ and $k(\tau) = k^f$, such that at time $\tau$ the final steady state is reached and no additional relaxation time is needed. The shape of $k(t)$ can be nonintuitive: when $\tau$ is smaller than the typical relaxation times, such protocols may exhibit large excursions well outside of the range $[k^i, k^f]$. In fact, there are cases where $k(t)$ can even become negative, posing problems to its experimental realization. Additional constraints may be introduced into the mathematical design problem, in order to limit the protocol excursion [13]. Other possibilities have been suggested, where the trap position is also modulated by additional noise [14]. The SE protocol has been demonstrated also in atomic force microscopy experiments [15].

Here, we discuss the problem of swift equilibration in a two-dimensional harmonic trap. The generalization may seem a pure increase of dimensionality, but in fact it allows us to step outside of the realm of pure equilibrium steady states. A two-dimensional harmonic trap may be coupled to different thermostats and, in general, may exhibit rotating currents which break the time-reversal symmetry even in the steady state [16–23]. It is therefore a sound test ground for the study of SE protocols for switching between two different nonequilibrium steady states in a finite time.

*Swift equilibration protocol*. Firstly, let us introduce the general strategy for the SE. We adopted a different and, in a sense, more general approach with respect to [12]. Consider an experimental system that can be leveraged controlling some forcing parameters, which will be noted in a vector $\mathbf{\Pi}$. The instantaneous statistical state of the system can be described by a set of parameters, which will be denoted by a vector $\boldsymbol{\gamma}(t)$: for instance, in a Gaussian process, as in our case below, these can be the parameters of a multivariate Gaussian. The value of the parameters $\boldsymbol{\gamma}(t)$ depends, through a dynamical equation, on the history of the applied forcing $\mathbf{\Pi}$ up to time $t$. We prepare the system in a stationary condition, given a value for the forcing $\mathbf{\Pi}^i$. This means that we observe

 

a time constant value of the system parameters $\boldsymbol{\gamma}$ that depends on the forcing $\boldsymbol{\Pi}^i$: we note this value as $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\Pi}^i]$. Our goal is to lead the system into a new stationary state with a final set of parameters $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\Pi}^f]$, *in a finite time* $\tau$.

Of course, if $\tau$ is very large, any transformation behaves as a SE protocol. More precisely, let us consider an arbitrary function $\mathcal{P}(s)$, with $s \in [0,1]$, such that $\mathcal{P}(0) = \boldsymbol{\Pi}^i$ and $\mathcal{P}(1) = \boldsymbol{\Pi}^f$. Then the evolution of parameters $\boldsymbol{\Pi}(t) = \mathcal{P}(t/\tau)$ is a SE protocol in the limit $\tau$ much larger than the largest characteristic time of the system dynamics. In fact, in this case the change in the forcing parameters is so slow that the system is always in its stationary state. In this quasistatic forcing, $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}^{\rm st}[\mathcal{P}(t/\tau)]$ at any time, including the final one.

On the contrary, when $\tau$ is finite (smaller than the largest characteristic time of the system), $\mathcal{P}(t/\tau)$ is not a SE and must be *modified* with appropriate finite-time corrections, i.e., a finite-$\tau$ SE protocol reads $\boldsymbol{\Pi} = \mathcal{P} + \frac{1}{\tau}\delta\mathcal{P}$: here we denote the finite-time corrections with $\delta\mathcal{P}$. The quantity $\delta\mathcal{P}$, hereafter named the finite-time correction to the quasistatic protocol, depends upon the choice of the quasistatic protocol $\mathcal{P}$. This is the relevant quantity one has to know to experimentally perform the desired SE. The exact, explicit and general formula for $\delta\mathcal{P}[\mathcal{P}(s)]$ in the case of the Brownian gyrator constitutes the main result of this Rapid Communication, and it allows a number of interesting theoretical considerations.

*The Brownian gyrator*. The system we consider has been introduced in [17] and then studied in [18] and [19], with an experimental realization obtained recently in [21–23]. It is widely known as Brownian gyrator (BG). Its stochastic differential equation takes the form

$$\begin{aligned} dx &= -(k_x\, x + u\, y)dt + \sqrt{2T_x}dW_x, \\ dy &= -(k_y\, y + u\, x)dt + \sqrt{2T_y}dW_y, \end{aligned} \tag{1}$$

which fairly describes an overdamped particle subject to a potential $V(x,y) = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + uxy$ in contact with two thermal baths at temperatures $T_x$ and $T_y$. Note that the condition of confining potential, required for the steady states, is $k_x k_y - u^2 > 0$ [24]. In this Rapid Communication we consider the case where $k_x, k_y, u$ may depend upon time (on the contrary, we keep the temperatures constant). For compactness we denote the set of parameters by the vector $\boldsymbol{\Pi}$, i.e., $\Pi_1 = k_x$, $\Pi_2 = k_y$, and $\Pi_3 = u$. The associated Fokker-Planck equation reads

$$\partial_t p = \partial_x(p\partial_x V) + \partial_y(p\partial_y V) + T_x\partial_x^2 p + T_y\partial_y^2 p, \tag{2}$$

where $p(x,y;t)$ is the one time distribution of the stochastic process. The process is Gaussian and for Gaussian initial condition keeps the Gaussian form at all times:

$$p(x,y;t) = \frac{\exp\left(-\frac{1}{2}\gamma_1 x^2 - \frac{1}{2}\gamma_2 y^2 - \gamma_3\, x\, y\right)}{2\pi\left(\gamma_1\gamma_2 - \gamma_3^2\right)^{-1/2}}, \tag{3}$$

where $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ depends on time. The introduction of form (3) in Eq. (2) leads to the equations governing the time evolution of $\boldsymbol{\gamma}(t)$, since

$$\begin{aligned} \dot\gamma_1 &= 2\big(k_x\gamma_1 - T_x\gamma_1^2 + u\gamma_3 - T_y\gamma_3^2\big), \\ \dot\gamma_2 &= 2\big(k_y\gamma_2 - T_y\gamma_2^2 + u\gamma_3 - T_x\gamma_3^2\big), \\ \dot\gamma_3 &= \gamma_3(k_x + k_y) + u(\gamma_1 + \gamma_2) + \\ &\quad - 2\gamma_3(T_x\gamma_1 + T_y\gamma_2). \end{aligned} \tag{4}$$

If the parameter vector $\boldsymbol{\Pi}$ of Eq. (1) does not depend on time, then the time dependence of $\boldsymbol{\gamma}(t)$ is only due to the relaxation from initial conditions. In that case, assuming that the potential is confining, a steady state is reached asymptotically, and—for ergodicity—coincides with the solution $\partial_t p_{\rm st} = 0$, uniquely determined by the values $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\Pi}]$ that obey Eqs. (4) with all left-hand sides set to zero (see Supplemental Material [25]). When $T_x = T_y = T$ ("thermodynamic equilibrium"), the Boltzmann distribution $p_{\rm st} \propto e^{-V/T}$ is recovered, i.e., $\gamma_1^{\rm st} = \frac{k_x}{T}$, $\gamma_2^{\rm st} = \frac{k_y}{T}$, $\gamma_3^{\rm st} = \frac{u}{T}$. On the contrary, when $T_x \neq T_y$, the steady state is not of the Boltzmann form and, most importantly, contains a current: $\mathbf{J}(x,y) = (-p_{\rm st}\partial_x V - T_x\partial_x p_{\rm st}, -p_{\rm st}\partial_y V - T_y\partial_y p_{\rm st}) \neq (0,0)$ which is rotational, with null divergence. The steady current breaks time-reversal invariance (detailed balance) and for this reason the BG has been proposed as a minimal model for nonequilibrium steady states [17].

*SE for the Brownian gyrator*. We look for the forcing protocol $\boldsymbol{\Pi}$ that in a finite time $\tau$ leads the system from the stationary state $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\Pi}^i]$ to the stationary state $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\Pi}^f]$. We require that the vector $\boldsymbol{\Pi}(t) \equiv \{k_x, k_y, u\}$ has the form $\boldsymbol{\Pi}(t) = \mathcal{P}(t/\tau) + \frac{1}{\tau}\delta\mathcal{P}(t/\tau)$, where $\mathcal{P}(s) \equiv \{\mathcal{K}_x, \mathcal{K}_y, \mathcal{U}\}$ is a given quasistatic protocol, and $\delta\mathcal{P}(s) \equiv \{\delta k_x, \delta k_y, \delta u\}$ is its finite-time correction.

In order to accomplish our task, first we invert the dynamical equations (4) in order to get rid of explicit time and obtain a set of expressions for $k_x, k_y, u$ as functions of $\gamma_i$ and $\dot\gamma_i$ only ($i \in [1,3]$). For the full formula see Supplemental Material [25]: the important fact is that such expressions can be written in the form $\boldsymbol{\Pi} = \boldsymbol{A}[\boldsymbol{\gamma}] + \boldsymbol{B}[\boldsymbol{\gamma}]\cdot\dot{\boldsymbol{\gamma}}$ with $\boldsymbol{A}[\boldsymbol{\gamma}]$ a vector and $\boldsymbol{B}[\boldsymbol{\gamma}]$ a matrix. If we require that $\boldsymbol{\gamma}(t)$ is a function of $s = \frac{t}{\tau}$, then $\dot{\boldsymbol{\gamma}} = \frac{1}{\tau}\frac{d}{ds}\boldsymbol{\gamma}$ and the second term vanishes in the $\tau \to \infty$ limit. Then it is natural to identify $\mathcal{P} = \boldsymbol{A}[\boldsymbol{\gamma}]$, and $\delta\mathcal{P} = \boldsymbol{B}[\boldsymbol{\gamma}]\cdot\frac{d}{ds}\boldsymbol{\gamma}$ (see Supplemental Material [25] for the full formula of both terms).

In order to close our loop, now we need to express everything as a function of the quasistatic protocol. This is done in two steps. The first step is to invert the relation $\mathcal{P} = A(\boldsymbol{\gamma})$. Since this relation is valid even in the $\tau \to \infty$ limit, the result is nothing but the expression of $\boldsymbol{\gamma}^{\rm st}[\mathcal{P}]$ that solve Eqs. (4) in the stationary condition and parameters set to $\mathcal{P}$. Finally, we have to express $\frac{d}{ds}\boldsymbol{\gamma}$ as a function of the quasistationary protocols. This is done considering that $\frac{d}{ds} = \mathcal{K}'_x\partial_{\mathcal{K}_x} + \mathcal{K}'_y\partial_{\mathcal{K}_y} + \mathcal{U}'\partial_{\mathcal{U}}$ and applying this operator to $\boldsymbol{\gamma}^{\rm st}[\mathcal{P}]$ (here and in the following $f'$ stands for $\frac{d}{ds}f$). Putting back $\gamma_i$ and $\frac{d}{ds}\gamma_i$ in the definition of the forcing protocol, we obtain the final expression:

$$\begin{aligned} \boldsymbol{\Pi}(t) &= \mathcal{P}(t/\tau) + \frac{1}{\tau}\delta\mathcal{P}(t/\tau), \\ \delta\mathcal{P}(s) &= \boldsymbol{\mathcal{B}}[\mathcal{P}(s)]\cdot\mathcal{P}'(s), \end{aligned} \tag{5}$$

with a matrix $\boldsymbol{\mathcal{B}}$ which is fully defined in the Supplemental Material [25]. We recall the operative meaning of this formula: one chooses an arbitrary [26] quasistatic protocol $\mathcal{P}(s) = \{\mathcal{K}_x(s), \mathcal{K}_y(s), \mathcal{U}(s)\}$ and this corresponds to a particular form of $\delta\mathcal{P}(s) = \{\delta k_x(s), \delta k_y(s), \delta u(s)\}$ for finite-time

corrections. Before giving a handier expression of $\delta\mathcal{P}$, we discuss some special cases.

Firstly, we consider the case where there is no interaction among $x$ and $y$, i.e., when $u = 0$ both at the beginning and at the end. Then it does not make sense to switch on $u$ during the protocol, so that the choice $\mathcal{U}(s) \equiv 0$ is quite general (hence $\mathcal{U}' \equiv 0$). The two degrees of freedom are independent; each one follows a separate equation and the finite-time corrections take the characteristic log derivative of the quasistatic protocol: $\delta k_x = \frac{1}{2}\frac{\mathcal{K}'_x}{\mathcal{K}_x}$, $\delta k_y = \frac{1}{2}\frac{\mathcal{K}'_y}{\mathcal{K}_y}$, $\delta u = 0$. This result coincides with that in [12].

As a second step, we consider the case of two interacting degrees of freedom $\mathcal{U} \neq 0$ in contact with the same thermal bath at temperature $T_x = T_y = T$. In this case the finite-time corrections to the quasistatic forcing read:

$$\delta k_x^{\rm eq} = \frac{(\Delta + \mathcal{K}_y^2)\mathcal{K}'_x + \mathcal{U}^2\mathcal{K}'_y - 2\mathcal{K}_y\mathcal{U}\mathcal{U}'}{2\mathcal{K}_s\Delta},$$
$$\delta u^{\rm eq} = -\frac{\mathcal{U}(\mathcal{K}_y\mathcal{K}'_x + \mathcal{K}_x\mathcal{K}'_y) - 2\mathcal{K}_x\mathcal{K}_y\mathcal{U}'}{2\mathcal{K}_s\Delta},$$

where $\mathcal{K}_s = \mathcal{K}_x + \mathcal{K}_y$ and $\Delta = \mathcal{K}_x\mathcal{K}_x - \mathcal{U}^2$. The value for $\delta k_y^{\rm eq}$ is obtained swapping the subscripts $x$ and $y$ in the expression for $\delta k_x^{\rm eq}$. Note that the result does not depend on the temperature $T$. This result generalizes [12] in two dimensions.

A richer phenomenology is obtained in the realm of nonequilibrium, when $T_x \neq T_y$ and a nonzero current appears in the stationary state (a brief study of the current during the SE can be found in the Supplemental Material [25]).

For instance, for $T_y = T_x + dT$, we observe interesting deviations

$$\delta k_x = \delta k_x^{\rm eq} - \frac{\mathcal{K}_s\mathcal{U}' - \mathcal{U}\mathcal{K}'_s}{T_x\mathcal{K}_s^3}dT + O(dT^2),$$
$$\delta k_y = \delta k_y^{\rm eq} + \frac{\mathcal{K}_s\mathcal{U}' - \mathcal{U}\mathcal{K}'_s}{T_x\mathcal{K}_s^3}dT + O(dT^2),$$
$$\delta u = \delta u^{\rm eq} - \frac{\mathcal{U}(\mathcal{K}'_y\mathcal{K}_x - \mathcal{K}'_x\mathcal{K}_y)}{T_x\mathcal{K}_s^3}dT + O(dT^2).$$

Note that for a symmetric protocol $\mathcal{K}_x = \mathcal{K}_y = \mathcal{K}$ the finite-time correction to $u$ becomes of the second order in $dT$. For such a symmetric protocol one has that $\delta k_x^{\rm eq} = \delta k_y^{\rm eq}$ and both are proportional to a logarithmic derivative, as happens to the noninteracting case: $\delta k_x^{\rm eq} = \delta k_y^{\rm eq} = \frac{1}{2}\frac{d}{ds}\ln(\mathcal{K}^2 - \mathcal{U}^2)$ (which is minus the "free energy" of the system divided by $2T$; see Supplemental Material [25]). In the same symmetric case, one can consider a quasistatic protocol involving only weak interactions $\mathcal{U} \ll 1$. In this case the general nonequilibrium case reads:

$$\delta k_x = \frac{1}{2}\frac{\mathcal{K}'}{\mathcal{K}} - \frac{\mathcal{U}}{T_x}\frac{(T_x + T_y)\mathcal{U}'}{4\mathcal{K}^2} + O(\mathcal{U}^2),$$
$$\delta k_y = \frac{1}{2}\frac{\mathcal{K}'}{\mathcal{K}} - \frac{\mathcal{U}}{T_y}\frac{(T_x + T_y)\mathcal{U}'}{4\mathcal{K}^2} + O(\mathcal{U}^2),$$
$$\delta u = \frac{1}{2}\frac{\mathcal{U}'}{\mathcal{K}} - \frac{\mathcal{U}}{2}\frac{\mathcal{K}'}{\mathcal{K}^2} + O(\mathcal{U}^2).$$

Note that the corrections to $\delta k_x$ and $\delta k_y$ at first order in $\mathcal{U}$ are different: the finite-time correction to a symmetric quasistatic protocol should not be the same, since the symmetry is broken by the nonequilibrium condition $T_y \neq T_x$. Nevertheless we note that they differ only by a factor $T_x/T_y$. It turns out that this is a general mathematical feature of the solution for the general (nonsymmetric) quasistatic protocol. In fact, in the general case, the finite-time corrections $\delta k_{xy}$ and $\delta u$ have a striking mathematical structure:

$$\begin{aligned}\delta k_x &= \delta k_x^{\rm eq} + \frac{1}{T_x}\frac{d}{ds}\mathcal{F},\\ \delta k_x &= \delta k_y^{\rm eq} - \frac{1}{T_y}\frac{d}{ds}\mathcal{F},\\ \delta u &= \delta u^{\rm eq} + (T_y - T_x)\mathcal{J} + \frac{d}{ds}\mathcal{G},\end{aligned} \tag{6}$$

where $\mathcal{F}$ and $\mathcal{G}$ are functions of $\mathcal{K}_s$ and $\mathcal{U}$. The function $\mathcal{J}$ remains finite in the equilibrium limit $T_y \to T_x$, while $\mathcal{F}$ and $\mathcal{G}$ vanish. The explicit expressions for $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{J}$ are quite simple and are given in the Supplemental Material [25]. Equations (6) are the main result of this work.

*Energetics*. SE protocols represent an interesting theoretical framework to study energetics and thermodynamics. For instance, we consider internal energy for the model in this study:

$$E \quad = \quad \langle V\rangle = \tfrac{1}{2}k_x\langle x^2\rangle + \tfrac{1}{2}k_y\langle y^2\rangle + u\langle xy\rangle,$$

where, calling $\det\Gamma = \gamma_1\gamma_2 - \gamma_3^2$, we have $\langle x^2\rangle = \gamma_2/\det\Gamma$, $\langle y^2\rangle = \gamma_1/\det\Gamma$, and $\langle xy\rangle = -\gamma_3/\det\Gamma$. Since during the SE, the distribution parameters $\boldsymbol{\gamma}(t)$ depend on the quasistatic protocol as $\boldsymbol{\gamma}^{\rm st}[\boldsymbol{\mathcal{P}}(t/\tau)]$, the expressions for $\langle x^2\rangle$, $\langle y^2\rangle$, and $\langle xy\rangle$ can be written in terms of the quasistatic protocol (see Supplemental Material [25]). Hence, using the expression for the forcing protocols $\boldsymbol{\Pi} = \boldsymbol{\mathcal{P}} + \frac{1}{\tau}\delta\boldsymbol{\mathcal{P}}$, one can compute the explicit expression of the internal energy $E$. Remarkably, it

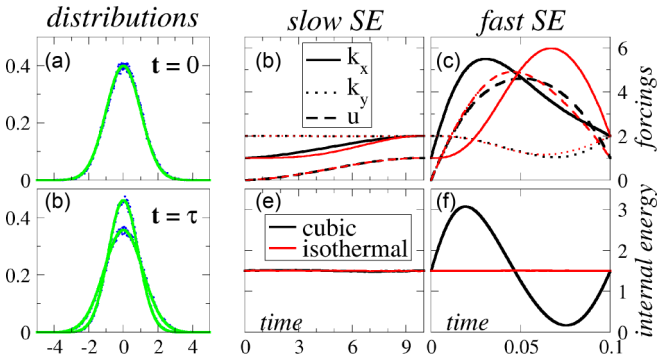


FIG. 1. SE in action (details in the Supplemental Material [25]). On the left: (a) initial marginal distribution $p(x, t = 0)$ and $p(y, t = 0)$ (points for simulations) compared with the theoretical stationary distributions (line); (b) same quantities at $t = \tau$, the end of the SE transformation. Middle and right plots show the use of two quasistatic protocols, with the same initial and final parameters $\boldsymbol{\Pi}^i$ and $\boldsymbol{\Pi}^f$: a simple *cubic* protocol (black) and an *isothermal* protocol (red). Upper plots: (c) forcings for a large $\tau = 10$, which are very close to the quasistatic protocol chosen; (d) for a much faster SE, $\tau = 0.1$, the forcings have large corrections given by Eq. (6). Lower plots (e) and (f): the values of the internal energy $E$ in the four different cases (*cubic* and *isothermal* during slow and fast SE).

turns out, after careful algebra, that this expression is simple:

$$E = \frac{T_x + T_y}{2} - \frac{1}{\tau}\frac{d}{ds}\left[\frac{\mathcal{K}_x T_y + \mathcal{K}_y T_x}{4(\mathcal{K}_x\mathcal{K}_y - \mathcal{U}^2)}\right]_{s=t/\tau}. \quad (7)$$

Several comments about this equation are in order. Firstly we note that during a quasistatic protocol ($\tau \to \infty$) the internal energy is constant, as one could expect: this constant value does not depend on the forcing parameters, neither on $k_{x,y}$ nor on $u$. More interestingly, the finite-time correction to the internal energy can be written with a single differential form $\frac{1}{\tau}\Theta'$, where $\Theta = \frac{\mathcal{K}_x T_y + \mathcal{K}_y T_x}{4(\mathcal{K}_x\mathcal{K}_y - \mathcal{U}^2)}$. An immediate reward of this result is that it allows one to identify a specific class of quasistatic protocols that we call the *finite-time isothermal protocols*, defined as the SE protocols that keep $\Theta$ constant. Using such a quasistatic protocol one can perform a SE procedure with constant internal energy in any finite time $\tau$, provided to force the system with the appropriate finite-time corrections (6). In Fig. 1 we show simple numerical simulations giving a demonstration of our results.

*Conclusions*. Here we have proposed a general framework for studying finite-time transformations in stochastic processes under the important request of connecting two steady states without the need of further relaxation time ("swift equilibration"). Our general framework is based upon the idea of fixing an arbitrary quasistatic protocol and then computing the finite-time corrections to it. We have applied our idea to a model ("Brownian gyrator") with a harmonic potential in contact with two different thermal baths, a minimal nonequilibrium generalization of the celebrated Ornstein-Uhlenbeck process. In this sense, the model can be considered as the harmonic oscillator or the "perfect gas" for nonequilibrium steady states. Despite the linearity of the model, the problem of SE discloses a rich and promising terrain for theoretical explorations. We give the exact explicit expression for the general SE, and also a simple condition to obtain finite-time transformations that conserve internal energy. The existence of experimental realizations of the steady Brownian gyrator [16,21–23] let us foresee interesting experimental investigations of our procedures in the near future. An important theoretical perspective concerns the research of optimal protocols with respect to work or other thermodynamic relevant quantities; for instance, a suitable definition of finite-time adiabatic transformations [27] to the case where two thermal baths are present.

*Acknowledgments.* A.B. and A.P. acknowledge the financial support of Regione Lazio through the Grant "Progetti Gruppi di Ricerca" No. 85-2017-15257 and from the MIUR PRIN 2017 project 201798CZLJ. L.S. acknowledges interesting discussions with Andrea Pagnani and Antonio Prados.

[1] V. Blickle and C. Bechinger, Nat. Phys. **8**, 143 (2012).
[2] I. A. Martinez, É. Roldán, L. Dinis, D. Petrov, and R. A. Rica, Phys. Rev. Lett. **114**, 120601 (2015).
[3] T. Schmiedl and U. Seifert, Europhys. Lett. **81**, 20003 (2007).
[4] S. Bo and A. Celani, Phys. Rev. E **87**, 050102(R) (2013).
[5] I. A. Martinez, É. Roldán, L. Dinis, D. Petrov, J. M. Parrondo, and R. A. Rica, Nat. Phys. **12**, 67 (2016).
[6] P. A. Quinto-Su, Nat. Commun. **5**, 5889 (2014).
[7] S. Krishnamurthy, S. Ghosh, D. Chatterji, R. Ganapathy, and A. Sood, Nat. Phys. **12**, 1134 (2016).
[8] R. Di Leonardo, L. Angelani, D. Dell'Arciprete, G. Ruocco, V. Iebba, S. Schippa, M. P. Conte, F. Mecarini, F. De Angelis, and E. Di Fabrizio, Proc. Natl. Acad. Sci. USA **107**, 9541 (2010).
[9] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, Nature (London) **483**, 187 (2012).
[10] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, Rev. Mod. Phys. **91**, 045001 (2019).
[11] D. Guéry-Odelin, J. G. Muga, M. J. Ruiz-Montero, and E. Trizac, Phys. Rev. Lett. **112**, 180602 (2014).
[12] I. A. Martinez, A. Petrosyan, D. Guéry-Odelin, E. Trizac, and S. Ciliberto, Nat. Phys. **12**, 843 (2016).
[13] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados, Phys. Rev. E **99**, 012140 (2019).
[14] M. Chupeau, B. Besga, D. Guéry-Odelin, E. Trizac, A. Petrosyan, and S. Ciliberto, Phys. Rev. E **98**, 010104(R) (2018).
[15] A. Le Cunuder, I. A. Martínez, A. Petrosyan, D. Guéry-Odelin, E. Trizac, and S. Ciliberto, Appl. Phys. Lett. **109**, 113502 (2016).
[16] S. Ciliberto, A. Imparato, A. Naert, and M. Tanase, Phys. Rev. Lett. **110**, 180601 (2013).
[17] R. Filliger and P. Reimann, Phys. Rev. Lett. **99**, 230602 (2007).
[18] V. Dotsenko, A. Maciołek, O. Vasilyev, and G. Oshanin, Phys. Rev. E **87**, 062130 (2013).
[19] S. Cerasoli, V. Dotsenko, G. Oshanin, and L. Rondoni, Phys. Rev. E **98**, 042149 (2018).
[20] D. Villamaina, A. Baldassarri, A. Puglisi, and A. Vulpiani, J. Stat. Mech. (2009) P07024.
[21] A. Argun, J. Soni, L. Dabelow, S. Bo, G. Pesce, R. Eichhorn, and G. Volpe, Phys. Rev. E **96**, 052106 (2017).
[22] K.-H. Chiang, C.-L. Lee, P.-Y. Lai, and Y.-F. Chen, Phys. Rev. E **96**, 032123 (2017).
[23] J. Soni, A. Argun, L. Dabelow, S. Bo, R. Eichhorn, G. Pesce, and G. Volpe, in *Optical Trapping Applications* (Optical Society of America, Washington, D.C., 2017), p. OtM2E–1.
[24] The concavity condition is necessary to reach a steady state but it can be relaxed in transient states. However, it can become necessary in experimental realizations as discussed, for instance, in [13,14,28] .
[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.102.030105, which includes Refs. [13,29,30], for details on analytical and numerical results.
[26] In the following we always consider continuous functions, which attain the values $\mathbf{\Pi}^i$ and $\mathbf{\Pi}^f$ at the border with zero

derivatives. However, some of these requirements can be relaxed, if one admits jumps in the finite-time forcing $\mathbf{\Pi}(t)$, as explained in [13] .

[27] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados, Phys. Rev. E **101**, 032129 (2020).

[28] M. Chupeau, S. Ciliberto, D. Guéry-Odelin, and E. Trizac, New J. Phys. **20**, 075003 (2018).

[29] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).

[30] C. Jarzynski, C. R. Phys. **8**, 495 (2007).