

An effective-gravity perspective on the Sun–Jupiter–comet three-body system

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An effective-gravity perspective on the Sun-Jupiter-comet three-body system

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Abstract

Within the solar system, approximate realizations of the three-body problem occur when a comet approaches a planet while being affected mainly by such a planet and the Sun, and this configuration was investigated by Tisserand within the framework of Newtonian gravity. The exact relativistic treatment of the problem is not an easy task, but the present paper develops an approximate calculational scheme which computes for the first time the tiny effective-gravity correction to the equation of the surface for all points of which it

is equally advantageous to regard the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. This analysis completes the previous theoretical investigations of effective-gravity corrections to the Newtonian analysis of three-body systems, and represents an intermediate step towards relativistic effects on cometary motions.

1 Introduction

A complete understanding of potentialities, applications and limits of Newton's and Einstein's theories of gravity has required dedicated efforts along more than three centuries, by now. For example, at the end of nineteenth century the monumental treatise by Tisserand on celestial mechanics [1] presented in great detail the work of d'Alembert and Laplace on the motion of comets when they are approaching a planet. This analysis stimulated Fermi himself, when he wrote his Scuola Normale Superiore dissertation [2], devoted to an investigation of cometary orbits with the help of probability theory and of the classical theory of restricted three-body problems.

What has motivated our research has been therefore, on the one hand, the many (recent) investigations of three-body problems in general relativity [3, 4, 5, 6, 7, 8] and effective-field-theory models of gravity [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] (see also [20] for a new original approach), and on the other hand the consideration that the passage of comets provides in the solar system some very interesting realizations of three-body systems in celestial mechanics. Short-period comets [21] are thought to generate in the Kuiper belt and have predictable orbits with short periods, i.e., up to 200 years. Two major families of short-period comets are the *Jupiter family* with periods of less than 20 years and the Halley family with periods in between 20 and 200 years. Interestingly, even though their orbits can be predicted with some accuracy, some of these short-period comets might be gravitationally perturbed and become long-period objects. More precisely, gravitational effects of the outer planets can cause these bodies to alter their paths into highly elliptical orbits that take them close to the Sun. Long-period comets are instead thought to generate in the Oort cloud and have unpredictable orbits, with periods much longer than 200 years. Their detection is extremely difficult for mankind because they can return on their steps after thousands or even millions of year (or not at all).

Several processes deserve careful consideration, e.g., the capture of comets with parabolic orbit by Jupiter [22]. Another intriguing difficulty is the choice of the appropriate formalism for our analysis. In the last decade

of the twentieth century, the outstanding work by Damour, Soffel and Xu [23, 24, 25, 26, 27] led indeed to a complete prescription for studying equations of motion of N bodies in celestial mechanics, which involve a repeated application of the inverse of the wave operator and are therefore nonlocal. Since, for a three-body problem, we are still far from the level of accuracy reached for relativistic binary systems [28, 29, 30], we here resort first to a shortcut, i.e., an approximation method, a hybrid scheme, which is nevertheless of physical interest, to the same extent that ordinary quantum mechanics, despite not being a relativistic quantum theory, is of much help in evaluating bound states and transition probabilities. The milestones we rely upon are as follows.

(i) In the effective-field-theory approach to quantum gravity, one discovers that the Newtonian potential between two bodies of masses M_A and M_B should be replaced by the expression [14, 16, 18, 12]

$$V_E(r) = -\frac{GM_A M_B}{r_{AB}} \left[1 + \kappa_1 \frac{(L_A + L_B)}{r_{AB}} + \kappa_2 \left(\frac{L_P}{r_{AB}} \right)^2 + \mathcal{O}(G^2) \right], \quad (1.1)$$

where L_A and L_B are their gravitational radii

$$L_A \equiv \frac{GM_A}{c^2}, \quad L_B \equiv \frac{GM_B}{c^2}, \quad (1.2)$$

L_P is the Planck length

$$L_P \equiv \sqrt{\frac{G\hbar}{c^3}}, \quad (1.3)$$

the length r_{AB} is the mutual distance between the bodies, while κ_1 and κ_2 are dimensionless parameters obtained either from a detailed application of Feynman diagrams' technique [14, 15, 16] or the modern on-shell unitary-based methods [31, 32]. It should be stressed that, although the numerical effect of L_P is immaterial in the calculations devoted to Lagrangian points and their stability [9, 10, 11, 12], the dependence of κ_1 on κ_2 implies that *the weight factor κ_1 for the classical gravitational radii depends actually on the underlying quantum world.*

(ii) The formula (1.1) tells us that quantum corrections map the dimensionless ratio

$$U_A \equiv \frac{L_A}{r_{AB}}, \quad (1.4)$$

into [12]

$$V_A \sim \left[1 + \kappa_2 \left(\frac{L_P}{r_{AB}} \right)^2 \right] U_A + \kappa_1 (U_A)^2. \quad (1.5)$$

Thus, if (1.5) is applied to replace U_A and U_B in the Newtonian formula of effective potential W_{eff} for circular restricted three-body problem, one finds in $c = 1$ units [12], denoting by ω the angular frequency for rotation of the ξ, η axes about the ζ axis [4],

$$\begin{aligned} W_{\text{eff}} &= \frac{\omega^2}{2}(\xi^2 + \eta^2) + [U_A + U_B + \kappa_1 ((U_A)^2 + (U_B)^2)] \\ &+ O(G^2). \end{aligned} \tag{1.6}$$

Remarkably, the general relativity formula in $c = 1$ units is instead [4, 12]

$$\begin{aligned} W_{\text{eff}} &\sim \frac{\omega^2}{2}(\xi^2 + \eta^2) + \left[U_A + U_B - \frac{1}{2} ((U_A)^2 + (U_B)^2) \right] \\ &+ \text{remainder}, \end{aligned} \tag{1.7}$$

and the first lines of (1.6) and (1.7) agree if $\kappa_1 = -\frac{1}{2}$. The value $\kappa_1 = -\frac{1}{2}$ is indeed allowed by the effective-field-theory approach to quantum gravity, and it corresponds to the so-called bound-state option for the underlying Feynman diagrammatics [14, 16, 18, 12]. This means that, upon application of (1.1) or (1.5) to the Newtonian formulas for three-body problems, one can obtain many of the relevant terms of the associated effective potential, so that a valuable recipe is available. According to it, one can insert (1.1) (or (1.5)) in all Newtonian formulas, and obtain valuable information on the otherwise (too) lengthy fully relativistic calculations. Bearing in mind also this feature, which was first pointed out in Ref. [12], the plan of our paper is as follows.

Section 2 describes the Newtonian formulation of perturbations of cometary motions when comets come very close to planets. Section 3 finds the leading relativistic corrections of this treatment with the help of the map (1.1), by focusing on the equation of the surface for all points of which it is equally advantageous to view the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. Concluding remarks are made in Sec. 4.

2 Perturbations of cometary motions in Newtonian gravity

Following the monograph of Tisserand [1], we denote by (x, y, z) the heliocentric rectangular coordinates of the comet, by (x', y', z') the coordinates of the planet, here taken to be Jupiter, and by (ξ, η, ζ) the jovicentric coordinates of the comet, with respect to axes that are parallel to the fixed axes.

Moreover, $m_{\odot} = 2 \times 10^{30}$ Kg is the mass of the Sun, m' is the mass of Jupiter:

$$\frac{m'}{m_{\odot}} = \frac{1}{1047}, \quad (2.1)$$

ρ is the Euclidean distance comet-Jupiter, r , r' the Euclidean distances comet-Sun and Jupiter-Sun, respectively. The Newtonian equations of motion turn out to be [1]

$$\left[\frac{d^2}{dt^2} + \frac{Gm_{\odot}}{r^3} \right] x = Gm' \left(\frac{(x' - x)}{\rho^3} - \frac{x'}{r'^3} \right), \quad (2.2)$$

$$\left[\frac{d^2}{dt^2} + \frac{Gm_{\odot}}{r^3} \right] y = Gm' \left(\frac{(y' - y)}{\rho^3} - \frac{y'}{r'^3} \right), \quad (2.3)$$

$$\left[\frac{d^2}{dt^2} + \frac{Gm_{\odot}}{r^3} \right] z = Gm' \left(\frac{(z' - z)}{\rho^3} - \frac{z'}{r'^3} \right), \quad (2.4)$$

$$\left(\frac{d^2}{dt^2} + \Omega^2 \right) x' = 0, \quad (2.5)$$

$$\left(\frac{d^2}{dt^2} + \Omega^2 \right) y' = 0, \quad (2.6)$$

$$\left(\frac{d^2}{dt^2} + \Omega^2 \right) z' = 0, \quad (2.7)$$

where

$$\Omega^2 = \frac{G(m_{\odot} + m')}{r'^3}. \quad (2.8)$$

The unprimed, primed and Greek lower case coordinates are related by linear equations, i.e.

$$x = x' + \xi, \quad y = y' + \eta, \quad z = z' + \zeta, \quad (2.9)$$

and by virtue of (2.2)-(2.9) one obtains the system of second-order equations

$$\left[\frac{d^2}{dt^2} + \frac{Gm'}{\rho^3} \right] \xi = Gm_{\odot} \left(\frac{x'}{r'^3} - \frac{x}{r^3} \right), \quad (2.10)$$

$$\left[\frac{d^2}{dt^2} + \frac{Gm'}{\rho^3} \right] \eta = Gm_{\odot} \left(\frac{y'}{r'^3} - \frac{y}{r^3} \right), \quad (2.11)$$

$$\left[\frac{d^2}{dt^2} + \frac{Gm'}{\rho^3} \right] \zeta = Gm_{\odot} \left(\frac{z'}{r'^3} - \frac{z}{r^3} \right). \quad (2.12)$$

Equations (2.2)-(2.4) pertain to the heliocentric motion of the comet, whereas (2.5)-(2.8) refer to the elliptical motion of Jupiter about the barycentre of

the system, which for simplicity coincides exactly with the position of the Sun; let now R be the modulus of the force per unit mass resulting from the attraction of the Sun, and let F be the modulus of the perturbing force per unit mass. They are given by [1]

$$R = \frac{Gm_{\odot}}{r^2}, \quad (2.13)$$

$$F = Gm' \sqrt{\left(\frac{(x' - x)}{\rho^3} - \frac{x'}{r'^3}\right)^2 + \left(\frac{(y' - y)}{\rho^3} - \frac{y'}{r'^3}\right)^2 + \left(\frac{(z' - z)}{\rho^3} - \frac{z'}{r'^3}\right)^2}. \quad (2.14)$$

Equations (2.10)-(2.12) pertain instead to the jovicentric motion produced by the attraction R' of Jupiter, and the perturbing force exerted by the Sun. The modulus of such forces per unit mass reads as

$$R' = \frac{Gm'}{\rho^2}, \quad (2.15)$$

and

$$F' = Gm_{\odot} \sqrt{\left(\frac{x'}{r'^3} - \frac{x}{r^3}\right)^2 + \left(\frac{y'}{r'^3} - \frac{y}{r^3}\right)^2 + \left(\frac{z'}{r'^3} - \frac{z}{r^3}\right)^2}, \quad (2.16)$$

respectively.

2.1 Heliocentric vs. jovicentric motion

A concept of crucial importance is expressed by the condition [1]

$$\frac{F}{R} = \frac{F'}{R'}, \quad (2.17)$$

which defines implicitly a surface for all points of which it is equally advantageous to regard the heliocentric motion as being perturbed by the attraction of Jupiter, or the jovicentric motion as being perturbed by the attraction of the Sun. On denoting simply by m' the ratio in (2.1), condition (2.17) reads eventually as [1]

$$m'^2 r^2 \sqrt{\frac{1}{\rho^4} + \frac{1}{r'^4} + 2 \frac{(x'\xi + y'\eta + z'\zeta)}{\rho^3 r'^3}} = \rho^2 \sqrt{\frac{1}{r^4} + \frac{1}{r'^4} - 2 \frac{(xx' + yy' + zz')}{r^3 r'^3}}, \quad (2.18)$$

where we have exploited the identities

$$r^2 = x^2 + y^2 + z^2, \quad r'^2 = x'^2 + y'^2 + z'^2, \quad \rho^2 = \xi^2 + \eta^2 + \zeta^2. \quad (2.19)$$

At this stage, it is convenient to introduce the variables θ and u by means of the definitions [1]

$$\cos(\theta) \equiv \frac{(x'\xi + y'\eta + z'\zeta)}{\rho r'}, \quad (2.20)$$

$$u \equiv \frac{\rho}{r'}. \quad (2.21)$$

The dimensionless parameter u approaches 0, since it is only at short distance from Jupiter that the transformation considered can be convenient. Moreover, by virtue of the linear relations (2.9), a term on the right-hand side of (2.18) reads as

$$xx' + yy' + zz' = r'^2(1 + u \cos(\theta)), \quad (2.22)$$

while the squared distance r^2 can be expressed in the form

$$r^2 = (x' + \xi)^2 + (y' + \eta)^2 + (z' + \zeta)^2 = r'^2(1 + 2u \cos(\theta) + u^2). \quad (2.23)$$

In light of (2.20)-(2.23), we can express the squared Jupiter mass in (2.18) by means of u and θ only, after writing $\rho = u r'$ and then expressing $\frac{r'}{r}$ from (2.23). Hence one finds [1]

$$m'^2 = \frac{u^4}{(1 + 2u \cos(\theta) + u^2)^2 \sqrt{1 + 2u^2 \cos(\theta) + u^4}} \sqrt{P(u, \theta)}. \quad (2.24)$$

The exact form of the function P is [1]

$$P(u, \theta) \equiv 1 + (1 + 2u \cos(\theta) + u^2)^2 - 2\sqrt{1 + 2u \cos(\theta) + u^2} (1 + u \cos(\theta)). \quad (2.25)$$

Since u approaches 0 in our physical model, we only need the small- u expansion of $P(u, \theta)$. For this purpose, we first notice that, at fixed θ ,

$$\begin{aligned} f(u) &\equiv \sqrt{1 + 2u \cos(\theta) + u^2} \\ &= 1 + u \cos(\theta) + \frac{1}{2} \sin^2(\theta) u^2 - \frac{1}{2} \cos(\theta) \sin^2(\theta) u^3 + O(u^4), \end{aligned} \quad (2.26)$$

and hence we find, after following patiently a number of exact or partial cancellations,

$$P(u, \theta) = u^2[1 + 3 \cos^2(\theta)] + 4u^3 \cos(\theta) + O(u^4). \quad (2.27)$$

By virtue of (2.24) and (2.27), we obtain first the approximate formula [1]

$$\begin{aligned} m'^2 &= \frac{u^4 \sqrt{u^2[1 + 3 \cos^2(\theta)] + 4u^3 \cos(\theta) + O(u^4)}}{(1 + 4u \cos(\theta) + O(u^2))(1 + O(u^2))} \\ &= u^5(1 - 4u \cos(\theta) + O(u^2)) \sqrt{1 + 3 \cos^2(\theta) + 4u \cos(\theta)} \\ &= u^5 \sqrt{1 + 3 \cos^2(\theta)} \left(1 - 2u \cos(\theta) \frac{[1 + 6 \cos^2(\theta)]}{[1 + 3 \cos^2(\theta)]} + O(u^2) \right) \end{aligned} \quad (2.28)$$

and eventually we solve approximately for the dimensionless parameter u in the form [1]

$$\begin{aligned}
u &= \left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{\frac{1}{5}} \left(1 + 2u \cos(\theta) \frac{[1+6\cos^2(\theta)]}{[1+3\cos^2(\theta)]} \right)^{\frac{1}{5}} \\
&\approx \left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{\frac{1}{5}} \\
&\quad + \frac{2}{5} \cos(\theta) \left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{\frac{2}{5}} \frac{[1+6\cos^2(\theta)]}{[1+3\cos^2(\theta)]}. \tag{2.29}
\end{aligned}$$

By virtue of (2.1), one has in all cases

$$\left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{\frac{1}{5}} < m'^{\frac{2}{5}} = 0.062, \tag{2.30}$$

and hence one can safely use the approximate formula

$$\rho = r' \left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{\frac{1}{5}}. \tag{2.31}$$

This is the approximate equation of the desired surface in polar coordinates ρ and θ , the polar axis being obtained by extending the line SP joining the Sun with the planet, having set the origin at the centre of the planet. This is a surface of revolution about the SP line, and does not differ much from a sphere, since ρ varies in between the two limits

$$m'^{\frac{2}{5}} r' \quad \text{and} \quad \frac{m'^{\frac{2}{5}}}{2^{\frac{1}{5}}} r',$$

which correspond to $\theta = \frac{\pi}{2}$ and $\theta = 0$, respectively, and whose ratio equals 1.15. One can therefore say, with little error, that the surface defined by the condition (2.17) is a sphere of radius $m'^{\frac{2}{5}} r'$, called by Laplace the *sphere of influence* of the planet.¹ Outside of such a sphere, one has $\frac{F}{R} < \frac{F'}{R'}$, and it is therefore advantageous to start from the heliocentric motion of the comet, and to evaluate the perturbations caused by the planet. Within the sphere of influence, one has instead $\frac{F'}{R'} < \frac{F}{R}$, and it is more advantageous to consider the jovicentric motion, and to evaluate as a next step the perturbations resulting from the Sun.

¹Strictly, Laplace considered the fifth root of half the square of m' , as pointed out by Tisserand [1].

3 Perturbations of cometary motions in effective field theories of gravity

The quantum effects considered in (1.1) affect the potential, whereas the Newtonian model outlined in Sec. 2 relies upon the evaluation of forces. Thus, we need to propose first a modified force formula in order to write down the effective-gravity counterpart of Sec. 2. For this purpose, we assume that we can still express the force as minus the gradient of the effective potential, i.e.,

$$F_k = (-\text{grad } V_E)_k, \quad k = 1, 2, 3, \quad x_1 = x, x_2 = y, x_3 = z, \quad (3.1)$$

which implies that (since $\frac{\partial r}{\partial x_k} = \frac{x_k}{r}$)

$$F_k = -\frac{GM_A M_B}{r^3} x_k \left[1 + 2\kappa_1 \frac{(L_A + L_B)}{r} + 3\kappa_2 \left(\frac{L_P}{r} \right)^2 + \text{O}(G^2) \right]. \quad (3.2)$$

The Newtonian formulas (2.13) and (2.14) for the modulus of perturbing forces receive therefore an additional contribution from κ_1 according to the prescriptions (we use the subscript E to denote the influence of effective-gravity calculations, and we neglect the gravitational radius L_J of Jupiter with respect to the gravitational radius L_S of the Sun)

$$R_E \sim \frac{Gm_\odot}{r^2} \left[1 + 2\kappa_1 \frac{L_S}{r} + \text{O}(L_P^2) \right], \quad (3.3)$$

$$F_E \sim Gm' \sqrt{\sum_{k=1}^3 \left[\frac{(x'_k - x_k)}{\rho^3} \left(1 + 2\kappa_1 \frac{L_J}{\rho} \right) - \frac{x'_k}{r'^3} \left(1 + 2\kappa_1 \frac{L_S}{r'} \right) + \text{O}(L_P^2) \right]^2}, \quad (3.4)$$

where $x'_1 = x', x'_2 = y', x'_3 = z'$. Now we point out that, up to $\text{O}(L_P^2)$ terms,

$$\begin{aligned} & \left[\frac{(x'_k - x_k)}{\rho^3} \left(1 + 2\kappa_1 \frac{L_J}{\rho} \right) - \frac{x'_k}{r'^3} \left(1 + 2\kappa_1 \frac{L_S}{r'} \right) \right]^2 \\ &= \left(\frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r'^3} \right)^2 + 4\kappa_1 \left(\frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r'^3} \right) \left(\frac{L_J}{\rho} \frac{(x'_k - x_k)}{\rho^3} - \frac{L_S}{r'} \frac{x'_k}{r'^3} \right) \\ &+ \text{O}(G^2), \end{aligned} \quad (3.5)$$

and hence we find

$$F_E \sim Gm' \sqrt{P_1}, \quad (3.6)$$

where

$$\begin{aligned}
P_1(u, \theta) &= \frac{1}{\rho^4} + \frac{1}{r'^4} + 2 \frac{(x'\xi + y'\eta + z'\zeta)}{\rho^3 r'^3} \\
&+ 4\kappa_1 \sum_{k=1}^3 \left[\left(\frac{(x'_k - x_k)}{\rho^3} - \frac{x'_k}{r'^3} \right) \left(\frac{L_J (x'_k - x_k)}{\rho} - \frac{L_S x'_k}{r' r'^3} \right) \right] + \mathcal{O}(G^2) \\
&= \frac{1}{\rho^4} + \frac{1}{r'^4} + \frac{2}{(\rho r')^2} \cos(\theta) + 4\kappa_1 f_1 + \mathcal{O}(G^2), \tag{3.7}
\end{aligned}$$

having set (see (2.9))

$$f_1 \equiv \sum_{k=1}^3 \left[\left(\frac{\xi_k}{\rho^3} + \frac{x'_k}{r'^3} \right) \left(\frac{L_J \xi_k}{\rho} + \frac{L_S x'_k}{r' r'^3} \right) \right], \tag{3.8}$$

with $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \zeta$.

With analogous procedure, we propose to replace the Newtonian formulas (2.15) and (2.16) with their effective-gravity counterparts

$$R'_E \sim \frac{Gm'}{\rho^2} \left(1 + 2\kappa_1 \frac{L_J}{\rho} + \mathcal{O}(L_P^2) \right), \tag{3.9}$$

$$F'_E \sim Gm_\odot \sqrt{\sum_{k=1}^3 \left[\frac{x'_k}{r'^3} - \frac{x_k}{r^3} + 2\kappa_1 \left(\frac{L_S x'_k}{r' r'^3} - \frac{L_S x_k}{r r^3} \right) + \mathcal{O}(L_P^2) \right]^2} \tag{3.10}$$

Therefore, we can write

$$F'_E \sim Gm_\odot \sqrt{P_2}, \tag{3.11}$$

where up to $\mathcal{O}(L_P^2)$ (see (2.22))

$$P_2(u, \theta) = \frac{1}{r'^4} + \frac{1}{r^4} - 2 \frac{r'^2 (1 + u \cos(\theta))}{r^3 r'^3} + 4\kappa_1 L_S f_2 + \mathcal{O}(G^2), \tag{3.12}$$

having set

$$f_2 \equiv \sum_{k=1}^3 \left[\left(\frac{x'_k}{r'^3} - \frac{x_k}{r^3} \right) \left(\frac{x'_k}{r'^4} - \frac{x_k}{r^4} \right) \right]. \tag{3.13}$$

The effective-gravity counterpart of condition (2.17), i.e.,

$$\frac{F_E}{R_E} = \frac{F'_E}{R'_E}, \tag{3.14}$$

leads therefore to the equation (cf. (2.18) and (2.24))

$$m'^2 = \left(\frac{\rho}{r}\right)^2 \left(1 + 2\kappa_1 \frac{L_S}{r} + \mathcal{O}(L_P^2)\right) \left(1 + 2\kappa_1 \frac{L_J}{\rho} + \mathcal{O}(L_P^2)\right)^{-1} \sqrt{\frac{P_2(u, \theta)}{P_1(u, \theta)}}. \quad (3.15)$$

At this stage, by exploiting Eqs. (2.9) and (2.19)-(2.23) we can rewrite Eqs. (3.8) and (3.13) as follows (the calculations below are both new and very instructive, hence they deserve our special care):

$$\begin{aligned} f_1 &= \frac{1}{r'^7} \sum_{k=1}^3 \left[\left(x'_k + \frac{\xi_k}{u^3}\right) \left(L_S x'_k + \frac{L_J \xi_k}{u^4}\right) \right] \\ &= \frac{1}{\rho^7} \left[L_J \rho^2 + (uL_S + L_J) \rho r' \cos(\theta) u^3 + L_S r'^2 u^7 \right] \\ &= \frac{1}{\rho^5} \left[L_J + (uL_S + L_J) \cos(\theta) u^2 + L_S u^5 \right], \end{aligned} \quad (3.16)$$

$$\begin{aligned} f_2 &= \frac{1}{r'^7} \sum_{k=1}^3 \left\{ \left[x'_k - \gamma^{-\frac{3}{2}} (x'_k + \xi_k) \right] \left[x'_k - \gamma^{-2} (x'_k + \xi_k) \right] \right\} \\ &= \frac{1}{r'^7} \left\{ r'^2 + \gamma^{-\frac{7}{2}} \left[r^2 - (r'^2 + \rho r' \cos(\theta)) \left(\gamma^{\frac{3}{2}} + \gamma^2 \right) \right] \right\} \\ &= \frac{u^5}{\rho^5} \left\{ 1 + \gamma^{-\frac{5}{2}} \left[1 - \left(\gamma^{\frac{1}{2}} + \gamma \right) (1 + u \cos(\theta)) \right] \right\}, \end{aligned} \quad (3.17)$$

where

$$\gamma \equiv 1 + 2u \cos(\theta) + u^2. \quad (3.18)$$

In view of the forthcoming calculations, it will be useful to write (3.17) as

$$f_2 = \left(\frac{u}{\rho}\right)^5 B(\gamma), \quad (3.19)$$

with

$$B(\gamma) \equiv 1 + \gamma^{-\frac{5}{2}} \left[1 - \left(\gamma^{\frac{1}{2}} + \gamma \right) (1 + u \cos(\theta)) \right], \quad (3.20)$$

and to take into account that

$$\frac{\rho}{r} = u \frac{r'}{r} = u \gamma^{-\frac{1}{2}}. \quad (3.21)$$

Bearing in mind the above calculations, we can write Eq. (3.15) as

$$m'^2 = \frac{u^2}{\gamma} \frac{\left(1 + 2\kappa_1 \frac{L_S}{r} + \mathcal{O}(L_P^2)\right)}{\left(1 + 2\kappa_1 \frac{L_J}{\rho} + \mathcal{O}(L_P^2)\right)} \sqrt{\frac{P_2(u, \theta)}{P_1(u, \theta)}}. \quad (3.22)$$

Inspired by the classical calculations of Sec. 2, we extract the factor $1/r^4$ and $1/\rho^4$ from $P_2(u, \theta)$ and $P_1(u, \theta)$, respectively, and hence on exploiting (3.21) the previous equation can be expressed in the form

$$m'^2 = \frac{u^4 \left(1 + 2\kappa_1 \frac{L_S}{r} + O(L_P^2)\right)}{\gamma^2 \left(1 + 2\kappa_1 \frac{L_J}{\rho} + O(L_P^2)\right)} \sqrt{\frac{N(u, \theta)}{D(u, \theta)}}. \quad (3.23)$$

where (cf. Eqs. (2.24) and (2.25))

$$N(u, \theta) = 1 + \gamma^2 - 2\sqrt{\gamma} (1 + u \cos(\theta)) + 4\kappa_1 L_S (r^4 f_2) + O(G^2) = r^4 P_2(u, \theta), \quad (3.24)$$

$$D(u, \theta) = 1 + 2u^2 \cos(\theta) + u^4 + 4\kappa_1 (\rho^4 f_1) + O(G^2) = \rho^4 P_1(u, \theta). \quad (3.25)$$

In order to consider the effective gravity version of the sphere of influence of the planet, we need to evaluate the small- u behaviour of Eq. (3.23). We begin with Eq. (3.24). The last term of this equation represents a pure effective gravity effect and its expansion reads as (see Eqs. (3.19)-(3.21))

$$4\kappa_1 L_S r^4 f_2 = 4\kappa_1 \left(\frac{L_S}{\rho}\right) [u\gamma^2 B(\gamma)] = 4\kappa_1 \left(\frac{L_S}{\rho}\right) \left[(1 + 5 \cos^2(\theta)) u^3 + O(u^4)\right]. \quad (3.26)$$

Therefore, the expansion of Eq. (3.24) leads to (cf. Eq. (2.27))

$$\begin{aligned} N(u, \theta) &= u^2 (1 + 3 \cos^2(\theta)) + 4u^3 \cos(\theta) + u^3 \left(4\kappa_1 \frac{L_S}{\rho}\right) (1 + 5 \cos^2(\theta)) \\ &+ O(u^4) + O(G^2). \end{aligned} \quad (3.27)$$

In order to obtain the expansion of Eq. (3.25), we first need to take into account that in our model we can safely write

$$\frac{L_S}{r}, \frac{L_S}{\rho}, \frac{L_J}{\rho} \ll 1. \quad (3.28)$$

This leads to (see Eq. (3.16))

$$\begin{aligned} D(u, \theta) &= 1 + 4\kappa_1 \left(\frac{L_J}{\rho}\right) + 2u^2 \cos(\theta) + 4\kappa_1 u^2 \left(\frac{L_J}{\rho}\right) \cos(\theta) \\ &+ 4\kappa_1 u^3 \left(\frac{L_S}{\rho}\right) \cos(\theta) + O(u^4) + O(G^2). \end{aligned} \quad (3.29)$$

As a result of the last calculations, we can now evaluate the small- u behaviour

of Eq. (3.23), that is ruled by the following computation:

$$\begin{aligned}
& \frac{u^4 \sqrt{N(u, \theta)}}{\gamma^2 \sqrt{D(u, \theta)}} \\
&= u^4 \frac{\sqrt{N(u, \theta)}}{(1 + 4u \cos(\theta) + O(u^2)) \left(\sqrt{1 + \frac{4\kappa_1 L_J}{\rho}} + O(u^2) + O(G^2) \right)} \\
&= \frac{u^4}{\sqrt{1 + \frac{4\kappa_1 L_J}{\rho}}} (1 - 4u \cos(\theta) + O(u^2) + O(G^2)) \sqrt{N(u, \theta)} \\
&= \frac{u^5}{\sqrt{1 + \frac{4\kappa_1 L_J}{\rho}}} (1 - 4u \cos(\theta) + O(u^2) + O(G^2)) \sqrt{1 + 3 \cos^2(\theta)} \\
&\times \left[1 + \frac{2u}{(1 + 3 \cos^2(\theta))} \left(\cos(\theta) + k_1 \frac{L_S}{\rho} + 5\kappa_1 \frac{L_S}{\rho} \cos^2(\theta) \right) \right. \\
&+ \left. O(u^2) + O(G^2) \right] \\
&= \frac{u^5 \sqrt{1 + 3 \cos^2(\theta)}}{\sqrt{1 + 4\kappa_1(L_J/\rho)}} \\
&\times \left[1 + \frac{-2u \cos(\theta) (1 + 6 \cos^2(\theta) - 5\kappa_1(L_S/\rho) \cos(\theta)) + 2u\kappa_1(L_S/\rho)}{(1 + 3 \cos^2(\theta))} \right. \\
&+ \left. O(u^2) + O(G^2) \right]. \tag{3.30}
\end{aligned}$$

Therefore, up to $O(u^2)$ and $O(G^2)$ terms in the square bracket, we have (cf. Eq. (2.28))

$$\begin{aligned}
m'^2 &= u^5 g(\kappa_1) \sqrt{1 + 3 \cos^2(\theta)} \\
&\times \left[1 + \frac{-2u \cos(\theta) (1 + 6 \cos^2(\theta) - 5\kappa_1(L_S/\rho) \cos(\theta))}{(1 + 3 \cos^2(\theta))} \right. \\
&+ \left. \frac{2u\kappa_1(L_S/\rho)}{(1 + 3 \cos^2(\theta))} \right], \tag{3.31}
\end{aligned}$$

where we have defined, up to $O(L_P^2)$,

$$g(\kappa_1) \equiv \frac{1 + 2\kappa_1(L_S/r)}{(1 + 2\kappa_1(L_J/\rho)) \sqrt{1 + 4\kappa_1(L_J/\rho)}}. \tag{3.32}$$

At this stage, we can invert Eq. (3.31), finding

$$\begin{aligned}
u^5 &= \frac{m'^2}{g(\kappa_1)\sqrt{1+3\cos^2(\theta)}} \\
&\times \left[1 + \frac{2u\cos(\theta)(1+6\cos^2(\theta)-5\kappa_1(L_S/\rho)\cos(\theta))}{(1+3\cos^2(\theta))} \right. \\
&\quad \left. - \frac{2u\kappa_1(L_S/\rho)}{(1+3\cos^2(\theta))} \right], \tag{3.33}
\end{aligned}$$

whose lowest order solution reads as

$$u = \left(\frac{m'^2}{g(\kappa_1)\sqrt{1+3\cos^2(\theta)}} \right)^{1/5} \equiv u_0^{1/5}. \tag{3.34}$$

Therefore, by solving approximately Eq. (3.33) we obtain

$$\begin{aligned}
u &\approx u_0^{1/5} + \frac{2}{5}\cos(\theta)u_0^{2/5} \left[\frac{1+6\cos^2(\theta)-5\kappa_1(L_S/\rho)\cos(\theta)}{(1+3\cos^2(\theta))} \right] \\
&\quad - \frac{2}{5}u_0^{2/5} \frac{\kappa_1(L_S/\rho)}{(1+3\cos^2(\theta))}. \tag{3.35}
\end{aligned}$$

Since Eq. (3.28) implies that $g(\kappa_1) \approx 1$, we obtain a result similar to the Newtonian case

$$u_0^{1/5} \approx \left(\frac{m'^2}{\sqrt{1+3\cos^2(\theta)}} \right)^{1/5} < m'^{2/5}, \tag{3.36}$$

and hence from (3.35) we conclude that the (approximate) equation defining the sphere of influence of the planet within the effective field theories of gravity picture reads as

$$\rho = r' \left(\frac{m'^2}{g(\kappa_1)\sqrt{1+3\cos^2(\theta)}} \right)^{1/5}. \tag{3.37}$$

From the above equation it is clear that the effective field theory framework produces a tiny variation of the radius of the sphere of influence by means of the factor $1/g(\kappa_1)$.

3.1 Comet trajectories in effective and Newtonian gravity

In this section we show the trajectories followed by the comet both in Newtonian and in effective field theory of gravity. As we will see, the corrections introduced in the comet motion by the effective picture turn out to be too tiny to be testable.

According to the effective field theory prescriptions (3.1)-(3.4) the heliocentric motion is described by the set of differential equations (hereafter we neglect terms $O(L_P^2)$)

$$\begin{aligned} & \left[\frac{d^2}{dt^2} + \frac{Gm_\odot}{r^3} \left(1 + 2\kappa_1 \frac{L_S}{r} \right) \right] x_k \\ = & Gm' \left[\frac{(x'_k - x_k)}{\rho^3} \left(1 + 2\kappa_1 \frac{L_J}{\rho} \right) - \frac{x'_k}{r'^3} \left(1 + \frac{2\kappa_1 L_S}{r'} \right) \right], \\ & (k = 1, 2, 3), \end{aligned} \quad (3.38)$$

whereas the motion of Jupiter is given in terms of Eqs. (2.5)-(2.7) provided that Eq. (2.8) is subjected to the change

$$\Omega^2 \rightarrow \tilde{\Omega}^2 = \frac{G(m_\odot + m')}{r'^3} \left(1 + 2\kappa_1 \frac{L_S}{r'} \right). \quad (3.39)$$

Therefore, Eqs. (2.5)-(2.7) are replaced by

$$\left(\frac{d^2}{dt^2} + \tilde{\Omega}^2 \right) x'_k = 0, \quad (k = 1, 2, 3). \quad (3.40)$$

The plots of the heliocentric trajectory of the comet in Newtonian gravity (see Eqs. (2.2)-(2.7)) and in effective field theory of gravity are displayed in Figs. 1 and 2, respectively. In obtaining these figures, Eqs. (2.2)-(2.7) and their effective gravity counterpart (3.38)-(3.40) have been integrated by choosing the following initial conditions at the time $t_0 = 0$: $\vec{r}(t_0) = (7.2 \times 10^{11} \text{ m}, 3 \times 10^{10} \text{ m}, 3 \times 10^{10} \text{ m})$, $\frac{d}{dt}\vec{r}(t_0) = (80 \text{ m/s}, 0, 0)$, $\vec{r}'(t_0) = (7.78 \times 10^{11} \text{ m}, 0, 0)$, $\frac{d}{dt}\vec{r}'(t_0) = (0, 1.3 \times 10^4 \text{ m/s}, 0)$. The solution $\vec{r}(t) = (x(t), y(t), z(t))$ of Eqs. (3.38)-(3.40) is drawn in Figs. 3-5.

Bearing in mind Eqs. (3.1), (3.2), (3.9), and (3.10), within the effective

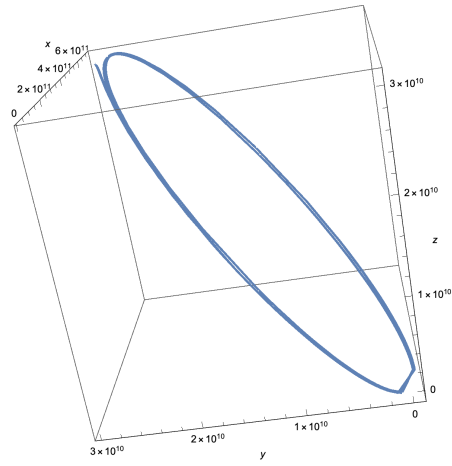


Figure 1: Heliocentric comet motion in Newtonian gravity.

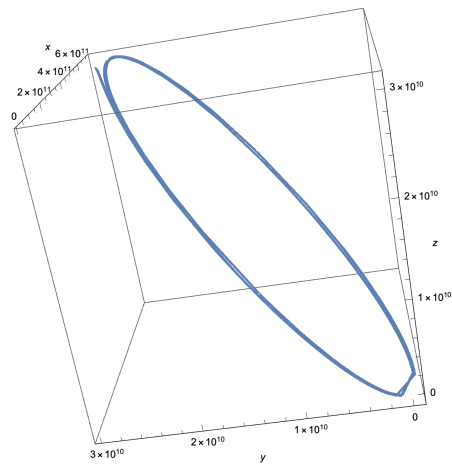


Figure 2: Heliocentric comet motion in effective field theory of gravity.

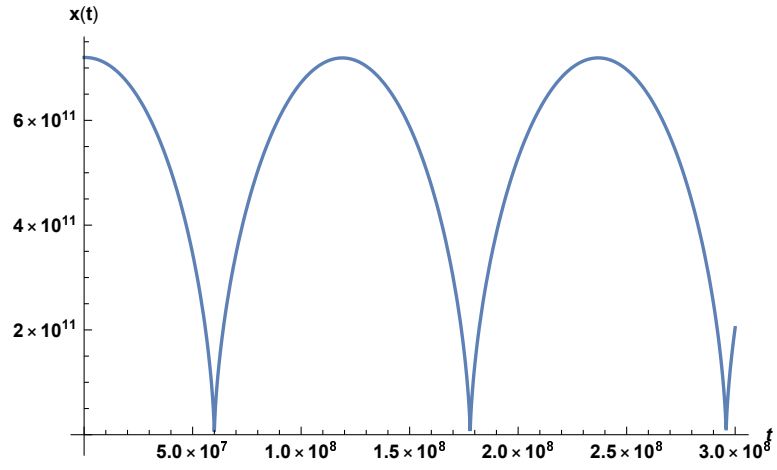


Figure 3: The component $x(t)$ of the solution $\vec{r}(t)$ of the effective field theory equations (3.38)-(3.40).

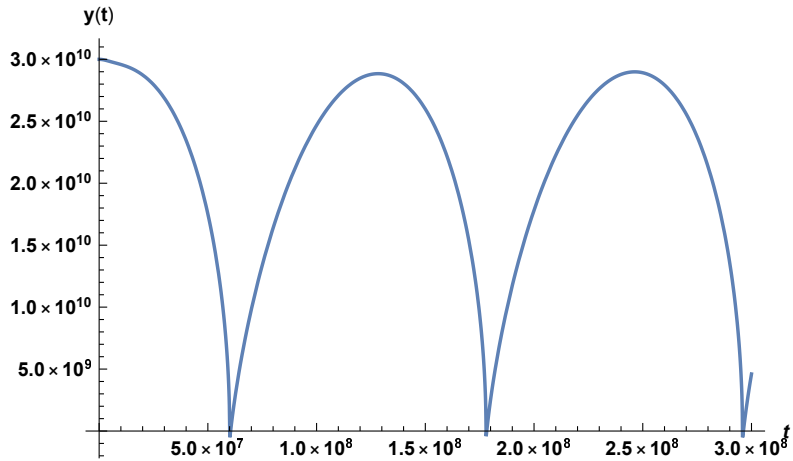


Figure 4: The component $y(t)$ of the solution $\vec{r}(t)$ of the effective field theory equations (3.38)-(3.40).

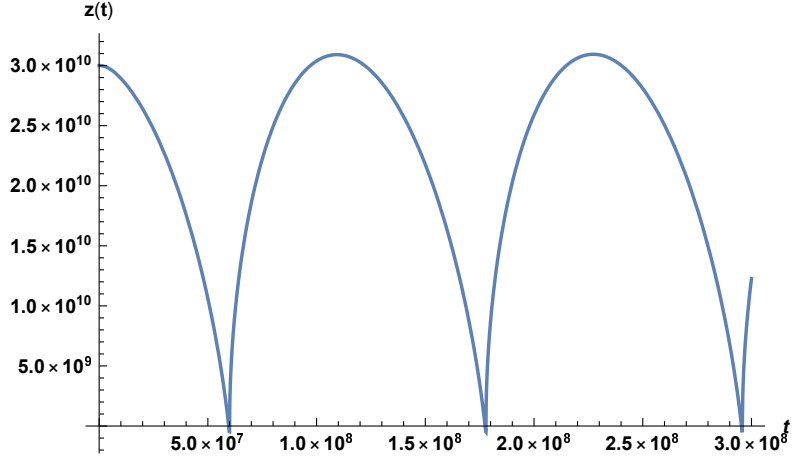


Figure 5: The component $z(t)$ of the solution $\vec{r}(t)$ of the effective field theory equations (3.38)-(3.40).

field theory domain the joventric motion of the comet is given by

$$\begin{aligned}
 & \left[\frac{d^2}{dt^2} + \frac{Gm'}{\rho^3} \left(1 + 2\kappa_1 \frac{L_J}{\rho} \right) \right] \xi_k \\
 &= Gm_\odot \left[\frac{x'_k}{r'^3} - \frac{x_k}{r^3} + 2\kappa_1 \left(\frac{L_S x'_k}{r' r'^3} - \frac{L_S x_k}{r r^3} \right) \right], \\
 & (k = 1, 2, 3)
 \end{aligned} \tag{3.41}$$

along with Eqs. (3.39) and (3.40).

The joventric motion of the comet both in Newtonian (cf. Eqs. (2.5)-(2.7) and (2.10)-(2.12)) and effective gravity is depicted in Figs. 6 and 7. Equations (2.5)-(2.7), (2.10)-(2.12) and their effective gravity counterpart (3.39) -(3.41) have been integrated by employing the following initial conditions at the time $t_0 = 0$: $\vec{\rho}(t_0) = (1.0 \times 10^7 \text{ m}, 1.0 \times 10^7 \text{ m}, 1.0 \times 10^8 \text{ m})$, $\frac{d}{dt}\vec{\rho}(t_0) = (80 \text{ m/s}, 0, 0)$, $\vec{r}'(t_0) = (7.78 \times 10^{11} \text{ m}, 0, 0)$, $\frac{d}{dt}\vec{r}'(t_0) = (0, 1.3 \times 10^4 \text{ m/s}, 0)$ The solution $\vec{\rho}(t) = (\xi(t), \eta(t), \zeta(t))$ can be read from Figs. 8-10.

In obtaining Figs. 1-10 the initial conditions regarding the motion of Jupiter reflect its average orbital speed and its average distance from the Sun. With these choices, the radius ρ_0 evaluated at the initial time $t_0 = 0$ of the (classical) sphere of influence of Jupiter reads as (restoring the solar mass m_\odot)

$$\rho_0 = \left(\frac{m'}{m_\odot} \right)^{2/5} r'(t_0) = 4.82 \times 10^{10} \text{ m}. \tag{3.42}$$

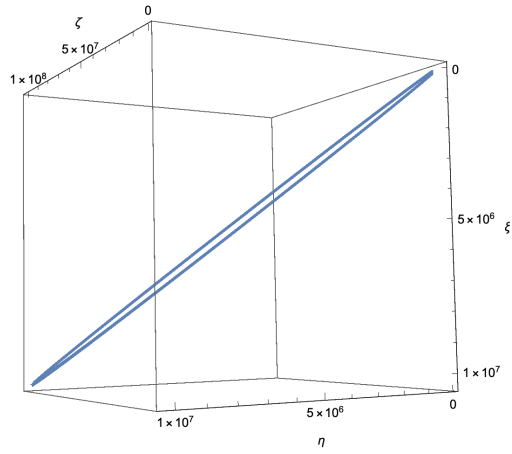


Figure 6: Jovicentric comet motion in Newtonian gravity.

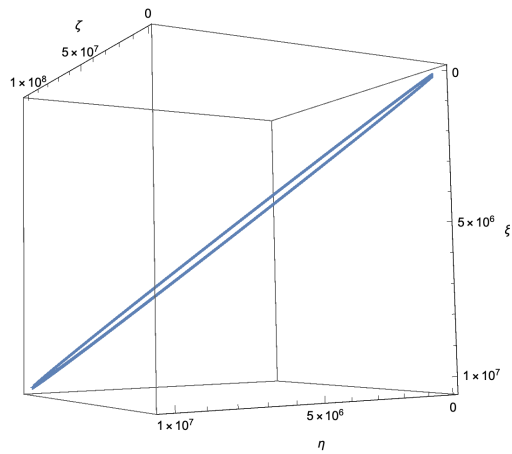


Figure 7: Jovicentric comet motion in effective field theory of gravity.

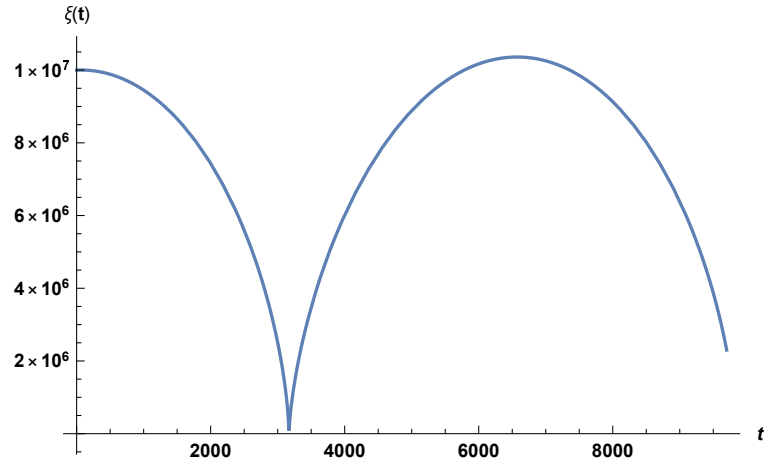


Figure 8: The component $\xi(t)$ of the solution $\vec{\rho}(t)$ of the effective field theory equations (3.39)-(3.41).

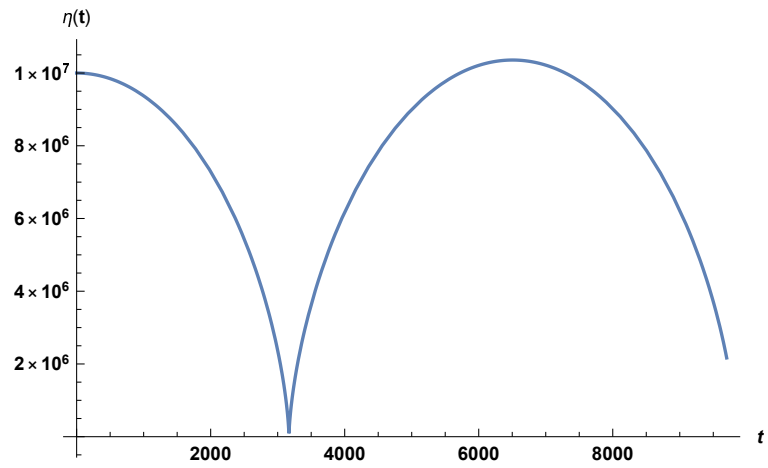


Figure 9: The component $\eta(t)$ of the solution $\vec{\rho}(t)$ of the effective field theory equations (3.39)-(3.41).

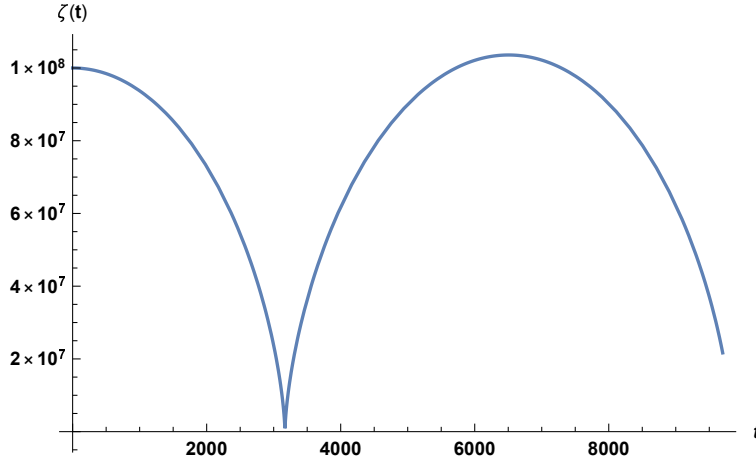


Figure 10: The component $\zeta(t)$ of the solution $\vec{\rho}(t)$ of the effective field theory equations (3.39)-(3.41).

We have analysed several alternatives for the initial conditions of the comet motion. In all cases, the differences between the classical and the effective regime are unperceivable. The initial conditions featuring the heliocentric motion sketched in Figs. 1-5 are such that the initial distances comet-Sun $r_h(t_0)$ and comet-Jupiter $\rho_h(t_0)$ are

$$r_h(t_0) = 7.21 \times 10^{11} \text{ m}, \quad \rho_h(t_0) = 7.19 \times 10^{10} \text{ m}, \quad (3.43)$$

respectively, with the parameter $u_h(t_0)$ given by (cf. Eq. (2.21))

$$u_h(t_0) = \frac{\rho_h(t_0)}{r'(t_0)} = 0.092. \quad (3.44)$$

On the other hand, the joventric motion (Figs. 6-10) is characterized by (the reader should be aware that, in Figs. 6 and 7, the closed orbit results from a particular choice of initial conditions, while for other choices an open orbit is instead obtained)

$$r_j(t_0) = 7.78 \times 10^{11} \text{ m}, \quad \rho_j(t_0) = 1.01 \times 10^8 \text{ m}, \quad (3.45)$$

so that

$$u_j(t_0) = \frac{\rho_j(t_0)}{r'(t_0)} = 1.30 \times 10^{-4}. \quad (3.46)$$

The analysis performed in this section clearly shows that the corrections to the comet motion resulting from the effective field theory approach are negligible. This agrees with the outcome of the previous section, where Eq. (3.37) describing the sphere of influence of the planet indicates a tiny departure from the classical case.

4 Concluding remarks

For the first time in the literature, the dynamics of a comet has been investigated within a framework where the dimensionless weight factors for gravitational radii are obtained from quantum field theory. We have proposed that the effective-gravity regime should rely on the set of original prescriptions (3.1)-(3.4) and we have found that our model predicts very tiny departures from the Newtonian picture. This is witnessed both by Eq. (3.37) defining the quantum corrected sphere of influence of Jupiter and by the comet trajectories of Figs. 1, 2, 6 and 7.

The next task is of course the analysis of cometary motions in general relativity. Within this framework, a full classification of the orbits can be found in Ref. [33], devoted to the orbital dynamics in the post-Newtonian planar circular restricted Sun-Jupiter system. Moreover, one should exploit the Hamiltonian for the three-body problem at second post-Newtonian order in the Arnowitt-Deser-Misner gauge obtained in Ref. [34], and also the important work on the three-body equations of motion in Ref. [35], and the results on post-Newtonian dynamics for time-dependent metrics [36].

Third, one should bear in mind that three-body systems display chaotic behaviour over sufficiently long times [37]. This implies that there might exist critical combinations of some parameters, and in the neighborhood of such critical values, even a very small perturbation could give rise to orbits that differ a lot from each other. From this point of view, even the detailed calculations of Sec. 3 are not just of academical interest. Furthermore, a numerical or analytic investigation of non-gravitational effects such as evaporation of the comet's head, and radiation pressure, might prove useful.

We are aware of the fact that the description of cometary dynamics does not need in principle a quantum description of the low-energy regime of the gravitational field. In spite of that, we strongly believe that this represents one of the reasons for which our paper can be of some interest. Indeed, we have shown that quantum gravity leads to some low-energy modifications even in a purely classical and observationally well established setting like cometary dynamics. This circumstance can give rise to a valuable resource since we can devise a favourable scenario where Eqs. (3.32) and (3.37) can be tested and hence used in order to constrain the factors κ_1 and κ_2 occurring in Eq. (1.1)². This is not a pointless issue since in the literature there is not a general agreement on the value assumed by the term κ_2 occurring in Eq. (1.1) (see *e.g.* Tab. I in Ref. [38] where our κ_2 is indicated with the

²We recall that in effective field theory of gravity the corrections κ_1 and κ_2 occurring in Eq. (1.1) are intimately intertwined and the output of one factor affects the value of the other.

Greek letter γ). This means that the research of observables of low-energy quantum gravity constitutes a context where further efforts must be made. It is in this spirit that the content of our paper, representing the natural continuation of a research project undertaken by two of us some years ago, should be considered.

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