Properties and numerical solution of an integral equation system to minimize airplane drag for a multiwing system

Properties and Numerical Solution of an Integral Equation System to Minimize Airplane Drag for a Multiwing System*

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Abstract

We consider an open multiwing system, composed by $N \geq 2$ disjoint open plane curves, not necessarily symmetric, and examine the corresponding (constrained) induced drag minimization problem. To this end, we first derive the associated Euler-Lagrange system of equations, which is then reduced to an equivalent system of Cauchy singular integral equations. By generalizing a previous approach of ours for the case of a single open wing, we obtain existence and uniqueness results for the problem solution in a product of weighted Sobolev type spaces. This system is then solved by applying to it a collocation-quadrature method. For this, we prove stability and derive corresponding error estimates. Finally, to test the efficiency of the proposed numerical method, we apply it to some multiwing systems.

KEYWORDS: Singular integral equations, Discrete collocation method, Constraint minimization

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1 INTRODUCTION

Reduction of induced drag, one of the major airplane drag components, could have a high impact on the reduction of pollution and emissions. This is one of the driving motivations behind the idea of innovative wing configurations. To this end, as a preliminary step, in [3, 6] the authors have considered the case of a single open curve in the y-z Cartesian plane, symmetric with respect to the z-axis. In particular, they have studied the corresponding

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(constrained) induced drag minimization problem. By applying a classical variational approach, they have derived the associated Euler-Lagrange (integral) equation (ELE) for the unknown wing circulation distribution. The ELE is finally rewritten as a Cauchy singular integral equation. In [6], the existence and uniqueness of the solution of this equation has been proved under the assumption that the curve logarithmic capacity is different from 1.

Then, in [6], a biwing systems defined by two disjoint open curves (i.e., lifting wings), in general having unequal shape and wingspans, but both symmetric with respect to the z-axis, was considered. By applying the same classical variational approach, properly generalized, for the corresponding induced drag minimization problem the authors have derived the associated system of Euler-Lagrange (integral) equations (ELE) for the unknown wing circulation distributions. In its final form, these reduce to a system of two Cauchy singular integral equations. Some applications and results have been reported in [7].

Very recently, the same authors of [4] have generalized their results to a system of N disjoint open wings. Also in this case, by applying the same approach mentioned above they have obtained the associated $N \times N$ system of Cauchy singular integral equations. This case is of importance when a complex wing configuration, such as the truss-braced one, is interpreted as the limiting case of a multiwing system, obtained by moving to zero the distances between single (open and disjoint) wings (see [5, 12]).

In all the above three cases, wings are assumed to be symmetric, with respect to the airplane vertical plane of symmetry, and the problem constraint is defined by the prescribed airplane total lift. Most important, existence and uniqueness of the solution of the corresponding ELE, in proper weighted Sobolev type spaces, have been assumed to hold, except in [6] for a single curve. Under these major assumptions, in all three cases the authors have then derived several new results of interest in wing theory. Thus, a mathematical proof for the existence and uniqueness property of the solution of the ELE system, hence of the induced drag minimization problem, is required.

To give numerical evidence that the solution existence and uniqueness assumption holds, the authors of the above papers have performed an intensive numerical testing. To carry out this testing, they have solved the ELE system by applying to it a discrete polynomial collocation method, based on Chebyshev polynomials and a corresponding Gaussian quadrature. Although the convergence of this method has been confirmed by the numerical testing, this property has not been proved, hence no error estimates have been obtained. Thus, these are key results that must be obtained.

In [11], we have considered the first case of a single open wing, not necessarily symmetric. By following an approach different from the more classical one applied in the above papers, without the above mentioned curve restriction we have obtained existence and uniqueness results for the ELE solution in suitable weighted Sobolev type spaces. Then, for the collocation-quadrature method used to solve the singular integral equation, we have proved stability and derived error estimates.

In this new paper, we consider an open multiwing system, composed by $N \geq 2$ disjoint open plane curves, not necessarily symmetric, and more general lift conditions. These are defined by prescribed global lifts on wing subsets. For this more general problem, we examine and numerically solve the associated Euler-Lagrange system of equations. More precisely, for the drag minimization problem defined on a system of (in general non-symmetric) wings, the equivalent system of Cauchy singular integral equations is provided, the unique solvability of this system of equations is proved, and an efficient numerical method for its solution is proposed and tested. An estimate of the convergence behavior of the method has also been obtained.

The main physical quantities and formulas, that are needed to describe the minimization problem, are briefly recalled in Section 2. In Section 3, by generalizing the approach we have used in [11], for the solution of the equivalent Cauchy singular integral equation system we obtain existence and uniqueness results in a product of weighted Sobolev type spaces, without requiring the curve symmetry restriction. Then, in Section 4, for the collocation-quadrature method we use to solve the above system we derive an error estimate. In the case of symmetric lifting lines, this method naturally reduces to that proposed in [4]. To test the efficiency of the proposed method and the error estimate previously obtained for it, in Section 5 we apply the method to two biwing systems. Finally, in the last section we describe a new application of our results and further questions to be answered.

2 THE DRAG MINIMIZATION PROBLEM

Following [4], we consider a system of N wings, each of them defined by a single open lifting line ℓ_k , $k = 1, \ldots, N$, in the Cartesian y-z plane and represented by a curve ℓ_k, having parametric representation $\boldsymbol{\psi}_k(t) = \left[\begin{array}{cc} \psi_{1k}(t) & \psi_{2k}(t) \end{array}\right]^T$, $|\boldsymbol{\psi}'_k(t)| \neq 0$, $t \in [-1, 1]$. It is assumed that $\ell_k \cap \ell_j = \emptyset$ for $j \neq k$. The corresponding arc length abscissa η_k is then defined by

$$\eta_k(t) = \int_0^t |\boldsymbol{\psi}'_k(s)| \, \mathrm{d}s, \tag{1}$$

where, here and in the following, $|\cdot|$ denotes the Euclidean norm. This abscissa will run from $\eta_k(-1) = -a_k$ to $\eta_k(1) = b_k$ for some positive real numbers a_k and b_k. Moreover, $\eta_k(0) = 0$.

For simplicity, it is also assumed that the lifting lines ℓ_k are sufficiently smooth. That is, it is assumed that the $\psi_{ik}(t), i = 1, 2$, are continuous functions together with their first $m \geq 2$ derivatives on the interval $[-1, 1]$ (i.e., $\psi_{ik} \in \mathbf{C}^m[-1, 1]$). A point on the kth lifting line, where the aerodynamic forces are calculated, is denoted by $\mathbf{r}_k = \left[\begin{array}{cc} y_k & z_k \end{array}\right]^T \in \ell_k$, with $\mathbf{r}_k = \mathbf{r}_k(\eta_k) = \left[\begin{array}{cc} y_k(\eta_k) & z_k(\eta_k) \end{array}\right]^T$.

The expressions of the *total lift* L and *induced drag* D_{ind} in terms of the (unknown) *circulations* Γ_k on ℓ_k are given by

$$L = L(\boldsymbol{\Gamma}) = \sum_{k=1}^N L(\Gamma_k) \quad \text{with} \quad L(\Gamma_k) = -\rho_\infty V_\infty \int_{-a_k}^{b_k} \tau_{yk}(\eta_k) \Gamma_k(\eta_k) \, \mathrm{d}\eta_k \tag{2}$$

and

$$D_{\mathrm{ind}} = D_{\mathrm{ind}}(\boldsymbol{\Gamma}) = -\rho_\infty \sum_{j=1}^N \int_{-a_j}^{b_j} v_{nj}(\eta_j) \Gamma_j(\eta_j) \mathrm{d}\eta_j, \tag{3}$$

respectively, where $\boldsymbol{\Gamma} = \left[\begin{array}{c} \Gamma_j \end{array}\right]_{j=1}^N$.

The quantities ρ_∞ and V_∞ are given positive constants which indicate the density and free stream velocity, respectively. Further, $\tau_{yk}(\eta_k) = y_k'(\eta_k)$ is the projection on the y-axis of the unit vector tangent to the lifting line ℓ_k, while v_{nj} is the so-called *normalwash* associated with ℓ_j. This latter has the representation

$$v_{nj}(\eta_j) = \frac{1}{4\pi} \sum_{k=1}^{N} \int_{-a_k}^{b_k} \Gamma_k'(\xi_k)\, Y_{jk}(\eta_j, \xi_k)\, \mathrm{d}\xi_k, \quad -a_j < \eta_j < b_j, \tag{4}$$

where

$$Y_{jk}(\eta_j, \xi_k) = -\frac{\mathrm{d}}{\mathrm{d}\eta_j} \ln |\mathbf{r}_k(\xi_k) - \mathbf{r}_j(\eta_j)| \,. \tag{5}$$

The function $Y_{jk}(\eta_j, \xi_k)$ has a singularity of order 1 when $j = k$ and $\eta_j = \xi_k$, and in that case the integral in (4) is a Cauchy principal value one.

Let m and $n_1, \ldots, n_m$, $1 \le m \le N$, be positive integers, with $0 =: n_0 < n_1 < \ldots < n_m = N$ when $m > 1$ and $n_1 = N$ when $m = 1$. The problem we need to solve is the minimization, in a suitable space, of the functional $D_{\mathrm{ind}}(\Gamma)$, subject to the prescribed lift constraints

$$\sum_{k=n_{j-1}+1}^{n_j} L(\Gamma_k) = L_{\mathrm{pres},j}\,, \quad j = 1, \ldots, m\,. \tag{6}$$

We remark that the multiwing problem examined in [4, 5, 12] corresponds to the case $m = 1$, while the choice $n_j = j$ implies the lift constraint on each wing, an additional case of possible wing theory interest.

In order to get a system of equations, in which every unknown function is defined on a unique interval, we go back to the interval $[-1, 1]$. For this, we use the notations

$$\Gamma_{0k}(t) := \Gamma_k(\eta_k(t)), \quad \mathbf{r}_{0k}(t) := \mathbf{r}_k(\eta_k(t)) = \boldsymbol{\psi}_k(t),$$

and

$$Y_{0jk}(t, s) := -\frac{\mathrm{d}}{\mathrm{d}t} \ln |\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)| \tag{7}$$

for $t, s \in [-1, 1]$, as well as the respective relations

$$\Gamma_{0k}'(t) = \Gamma'(\eta_k(t))\eta_k'(t), \quad \psi_{1k}'(t) = y_k'(\eta_k(t))\eta_k'(t),$$

and

$$Y_{0jk}(t, s) = Y_{jk}(\eta_j(t), \eta_k(s))\eta_j'(t).$$

Conditions (6) then take the new forms

$$\sum_{k=n_{j-1}+1}^{n_j} \int_{-1}^{1} \psi_{1k}'(t)\Gamma_{0k}(t)\, \mathrm{d}t = \gamma_j := -\frac{L_{\mathrm{pres},k}}{\rho_\infty V_\infty}\,, \quad j = 1, \ldots, m\,. \tag{8}$$

Moreover, from (3) and (4) we get

$$D_{\mathrm{ind}} = D_{\mathrm{ind}}(\Gamma_0) = -\frac{\rho_\infty}{4\pi} \sum_{j=1}^{N} \int_{-1}^{1} \sum_{k=1}^{N} \int_{-1}^{1} Y_{0jk}(t, s)\Gamma_{0k}'(s)\, \mathrm{d}s\, \Gamma_{0j}(t)\, \mathrm{d}t, \tag{9}$$

where $\Gamma_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^{N}$.

3 THE EULER-LAGRANGE EQUATION AND ITS PROPERTIES

For a Jacobi weight $\rho(t) := v^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ with $\alpha, \beta > -1$, let us recall the definition of the Sobolev-type space (cf. [1]) $\mathbf{L}^{2,r}_\rho = \mathbf{L}^{2,r}_\rho(-1,1)$, $r \geq 0$. For this, by $\mathbf{L}^2_\rho = \mathbf{L}^{2,0}_\rho$ we denote the real Hilbert space of all (classes of) quadratic summable functions w.r.t. the weight $\rho(t)$ $f : (-1,1) \longrightarrow \mathbb{R}$, equipped with the inner product

$$\langle f, g\rangle_\rho := \int_{-1}^{1} f(t)g(t)\rho(t)\,\mathrm{d}t$$

and the norm $\|f\|_\rho = \sqrt{\langle f, f\rangle_\rho}$. In the case $\alpha = \beta = 0$, i.e., $\rho \equiv 1$, we write $\langle f, g\rangle$ and $\|f\|$ instead of $\langle f, g\rangle_\rho$ and $\|f\|_\rho$, respectively. If $\{p^\rho_n : n \in \mathbb{N}_0\}$ denotes the system of orthonormal w.r.t. $\rho(t)$ polynomials $p^\rho_n(t)$ of degree n with positive leading coefficient, then

$$\mathbf{L}^{2,r}_\rho := \left\{ f \in \mathbf{L}^2_\rho : \sum_{n=0}^{\infty} (1+n)^{2r} \left|\langle f, p^\rho_n\rangle_\rho\right|^2 < \infty \right\}.$$

Equipped with the inner product

$$\langle f, g\rangle_{\rho,r} = \sum_{n=0}^{\infty} (1+n)^{2r} \langle f, p^\rho_n\rangle_\rho \langle g, p^\rho_n\rangle_\rho$$

and the norm $\|f\|_{\rho,r} := \sqrt{\langle f, f\rangle_{\rho,r}}$, the set $\mathbf{L}^{2,r}_\rho$ becomes a Hilbert space. Note that, when $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, the spaces $\mathbf{L}^{2,r}_\rho$ were also introduced in [9, Section 1] with a slightly different notation. Let $\varphi(t) = \sqrt{1-t^2}$ and define

$$\mathbf{V} := \left\{ f = \varphi u : u \in \mathbf{L}^{2,1}_\varphi \right\}$$

together with $\langle f, g\rangle_{\mathbf{V}} := \langle \varphi^{-1} f, \varphi^{-1} g\rangle_{\varphi,1}$ and $\|f\|_{\mathbf{V}} := \|\varphi^{-1} f\|_{\varphi,1}$.

In what follows, we denote by $\mathcal{D}$ the operator of generalized differentiation. An important property of this operator with respect to the $\mathbf{L}^{2,r}_\rho$ spaces is recalled in the next lemma, where we have set $\rho^{(1)}(t) = (1-t)^{1+\alpha}(1+t)^{1+\beta} = \rho(t)(1-t^2)$.

Lemma 3.1 ([2], Lemma 2.7, cf. also [1], Theorem 2.17)**.** *For $r \geq 0$, the operator of generalized differentiation $\mathcal{D} : \mathbf{L}^{2,r+1}_\rho \longrightarrow \mathbf{L}^{2,r}_{\rho^{(1)}}$ is a continuous one.*

Lemma 3.2 ([11], Lemma 2)**.** *For $f \in \mathbf{V}$, we have $f \in \mathbf{C}[-1,1]$ with $f(\pm 1) = 0$.*

Using here and in the following the notation $\left[\ f_k\ \right]_{k=1}^N$ to identify a vector of the form $\left[\ f_1\ \ \ldots\ \ f_N\ \right]^T$, the problem we aim to solve (cf. [3]) can be written as follows.

(P) *Find a function $\mathbf{\Gamma}_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^N \in \mathbf{V}^N$, which minimizes the functional (cf. (9))*

$$F(\mathbf{\Gamma}_0) := -\sum_{j=1}^{N} \int_{-1}^{1} \sum_{k=1}^{N} \int_{-1}^{1} Y_{0jk}(t,s)\Gamma'_{0k}(s)\,\mathrm{d}s\,\Gamma_{0j}(t)\,\mathrm{d}t$$

subject to (cf. (8)) $$\sum_{k=n_{j-1}+1}^{n_j} \langle \psi'_{1k}, \Gamma_{0k}\rangle = \gamma_j\,,\ j = 1,\ldots,m\,.$$

If we define

$$\langle \mathbf{f},\mathbf{g}\rangle_N := \sum_{k=1}^{N} \langle f_k, g_k\rangle \ , \quad \mathbf{f} = \left[\ f_k\ \right]_{k=1}^{N}, \, \mathbf{g} = \left[\ g_k\ \right]_{k=1}^{N} \ ,$$

and the linear operator

$$(\mathcal{A}\mathbf{f})\,(t) := \left[\ -\frac{1}{\pi}\sum_{k=1}^{N}\int_{-1}^{1} Y_{0jk}(t,s) f_k'(s)\,\mathrm{d}s\ \right]_{j=1}^{N}, \quad -1<t<1, \tag{10}$$

then the problem can be reformulated as follows:

(P) *Find a function* $\boldsymbol{\Gamma}_0 = \left[\ \Gamma_{0k}\ \right]_{k=1}^{N} \in \mathbf{V}^N$ *which minimizes the functional* $F(\boldsymbol{\Gamma}_0) := \langle \mathcal{A}\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_0\rangle_N$ *on* $\mathbf{V}^N$ *subject to the conditions* $\displaystyle\sum_{k=n_{j-1}+1}^{n_j} \langle \psi_{1k}', \Gamma_{0k}\rangle = \gamma_j \ , \ j=1,\ldots,m\,.$

The formulation of this problem is correct, which can be seen from the following Lemma 3.4.

Lemma 3.3 ([11], Lemma 3)**.** *If* $\psi_{ik} \in \mathbf{C}^m[-1,1]$ *for all* $i=1,2,\ k=1,\ldots,N,$ *and for some integer* $m\geq 2$ *and* $|\boldsymbol{\psi}_k'(t)| \neq 0$ *for* $t\in[-1,1]$ *and* $k=1,\ldots,N$*, then the functions* $Y_{0jj}(t,s)$ *have the representation*

$$Y_{0jj}(t,s) = \frac{1}{s-t} + K_j(t,s), \tag{11}$$

where the functions $K_j : [-1,1]^2 \longrightarrow \mathbb{R}$ *are continuous together with their partial derivatives* $\dfrac{\partial^{i+\ell}K_j(t,s)}{\partial t^i \partial s^\ell}$*,* $i,\ell\in\mathbb{N}_0,\ i+\ell\leq m-2.$

Lemma 3.4. *The operator* $\mathcal{A} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ *is a linear and bounded one and, consequently,* $\langle \mathcal{A}\mathbf{f},\mathbf{f}\rangle_N$ *is well defined for all* $\mathbf{f}\in\mathbf{V}^N$*.*

Proof. Let $U_n = p_n^\varphi$ and $T_n = p_n^{\varphi^{-1}}$. Then, recalling Lemma 3.2 and the well known property $\mathcal{D}T_n = nU_{n-1}$, for $f\in\mathbf{V}$ we have:

$$\begin{aligned}\|\mathcal{D}f\|_\varphi^2 &= \textstyle\sum_{n=0}^\infty \left|\langle \mathcal{D}f, \varphi^{-1}T_n\rangle_\varphi\right|^2 = \sum_{n=0}^\infty \left|\langle \mathcal{D}f, T_n\rangle\right|^2 = \sum_{n=1}^\infty \left|\langle f, n\,U_{n-1}\rangle\right|^2 \\ &= \textstyle\sum_{n=0}^\infty (1+n)^2 \left|\langle \varphi^{-1}f, U_n\rangle_\varphi\right|^2 = \|\varphi^{-1}f\|_{\varphi,1}^2 = \|f\|_{\mathbf{V}}^2.\end{aligned}$$

Consequently, the operator $\mathcal{D} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ defined by $\mathcal{D}\mathbf{f} := \left[\ \mathcal{D}f_k\ \right]_{k=1}^{N}$ is an isometrical isomorphism, where $\|\mathbf{f}\|_{\mathbf{V}^N} = \left(\displaystyle\sum_{k=1}^{N} \|f_k\|_{\mathbf{V}}^2\right)^{\frac{1}{2}}$, $\|\mathbf{f}\|_{(\mathbf{L}^2_\varphi)^N} = \sqrt{\langle \mathbf{f},\mathbf{f}\rangle_{\varphi,N}}$, and $\langle \mathbf{f},\mathbf{g}\rangle_{\varphi,N} = \displaystyle\sum_{k=1}^{N} \langle f_k, g_k\rangle_\varphi$.

By relation (11), the operator $\mathcal{A}$ defined in (10) can be written in the form $\mathcal{A} = -(\mathcal{S}+\mathcal{K})\mathcal{D}$ with

$$(\mathcal{S}\mathbf{f})(t) := \left[\frac{1}{\pi} \int_{-1}^{1} \frac{f_j(s)\,\mathrm{d}s}{s-t} \right]_{j=1}^{N}, \quad -1 < t < 1$$

and

$$(\mathcal{K}\mathbf{f})(t) := \left[\sum_{k=1}^{N} \frac{1}{\pi} \int_{-1}^{1} K_{jk}(t,s) f_k(s)\,\mathrm{d}s \right]_{j=1}^{N}, \quad -1 < t < 1,$$

where

$$K_{jk}(t,s) := \begin{cases} K_j(t,s) & : \ j = k, \\ Y_{0jk}(t,s) & : \ j \neq k. \end{cases} \tag{12}$$

It is well known that the Cauchy singular integral operator $\mathcal{S} : (\mathbf{L}^2_\varphi)^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is bounded ([8, Theorem 4.1]) and that $\mathcal{K} : (\mathbf{L}^2_\varphi)^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is compact. Consequently, for $\mathbf{f} = \varphi\mathbf{u} \in \mathbf{V}^N$ we have that $\langle \mathcal{A}\mathbf{f}, \mathbf{f} \rangle_N = \langle \mathcal{A}\mathbf{f}, \mathbf{u} \rangle_{\varphi,N}$ is a finite number, since both $\mathcal{A}\mathbf{f}$ and $\mathbf{u}$ belong to $(\mathbf{L}^2_\varphi)^N$. □

In the following lemma we give a representation of the operator $\mathcal{A}$ defined in (10), which is crucial for our further investigations. From this representation, it is seen that the operator $\mathcal{A}$ is an example of an hypersingular integral operator in the sense of Hadamard (cf., for example, the representation of Prandtl's integro-differential operator in [2, Section 1], where $N = 1$, $\mathbf{r}_0(t) = t$, and $\mathcal{B}$ is equal to the Cauchy singular integral operator $\mathcal{S}$).

Lemma 3.5. *For all $\mathbf{f} \in \mathbf{V}^N$, the relation*

$$\mathcal{A}\mathbf{f} = \mathcal{D}\mathcal{B}\mathbf{f} \tag{13}$$

holds true, where

$$(\mathcal{B}\mathbf{f})(t) := \left[\sum_{k=1}^{N} \frac{1}{\pi} \int_{-1}^{1} \ln|\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|\, f_k'(s)\,\mathrm{d}s \right]_{j=1}^{N} \tag{14}$$

and where $\mathcal{D}$ is the operator of generalized differentiation already used in the proof of Lemma 3.2.

Proof. Since $\mathcal{A} = -(\mathcal{S}+\mathcal{K})\mathcal{D}$ and since $\mathcal{D} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is an isometrical mapping (cf. the proof of Lemma 3.4), it suffices to show that $-(\mathcal{S}+\mathcal{K})\mathbf{g} = \mathcal{D}\mathcal{B}_0\mathbf{g}$ is valid for all $\mathbf{g} \in (\mathbf{L}^2_\varphi)^N$, where

$$(\mathcal{B}_0\mathbf{g})(t) = \left[\sum_{k=1}^{N} \frac{1}{\pi} \int_{-1}^{1} \ln|\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|\, g_k(s)\,\mathrm{d}s \right]_{j=1}^{N}.$$

Since

$$Z_{0jj}(t,s) := \ln|\mathbf{r}_{0j}(s) - \mathbf{r}_{0j}(t)| = \ln|s-t| + K_{0j}(t,s) \tag{15}$$

with a function $K_{0j} : [-1,1]^2 \longrightarrow \mathbb{R}$ which is continuous together with

$$\begin{aligned} \tfrac{\partial K_{0j}(t,s)}{\partial t} &= \tfrac{\partial}{\partial t}\big[\ln|\mathbf{r}_{0j}(s) - \mathbf{r}_{0j}(t)| - \ln|s-t|\big] \\ &\overset{(7)}{=} -Y_{0jj}(t,s) + \tfrac{1}{s-t} \overset{(11)}{=} -K_j(t,s) \end{aligned}$$

the operator $\mathcal{B}_0 : (\mathbf{L}^2_\varphi)^N \longrightarrow \left(\mathbf{L}^{2,1}_{\varphi^{-1}}\right)^N$ is bounded (see [10, Section 5] and [1, Lemma 4.2]). Moreover, $\mathcal{D} : \left(\mathbf{L}^{2,1}_{\varphi^{-1}}\right)^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is continuous ([2, Lemma 2.7]), such that on the one hand, the operator $\mathcal{D}\mathcal{B}_0 : (\mathbf{L}^2_\varphi)^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is linear and bounded. On the other hand, the operator $\mathcal{S} + \mathcal{K} : (\mathbf{L}^2_\varphi)^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is also linear and bounded. Thus, it remains to prove that

$$\int_{-1}^1 Y_{0jk}(t,s) g(s)\,\mathrm{d}s = -\frac{d}{dt}\int_{-1}^1 Z_{0jk}(t,s) g(s)\,\mathrm{d}s, \quad -1 < t < 1 \tag{16}$$

for all g from a linear and dense subset of $\mathbf{L}^2_\varphi$, where we have defined the functions $Z_{0jk}(t,s) := \ln|\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|$. In case of $j \neq k$, this is obvious. For the case of $j = k$, we refer to (15) and the proof of [11, Lemma 5]. □

In the following, the symbol Θ will denote the trivial element of the linear space under consideration.

Lemma 3.6. *The operator $\mathcal{A} : \mathbf{V}^N \longrightarrow (\mathbf{L}^2_\varphi)^N$ is symmetric and positive, i.e. $\forall\, \mathbf{f}, \mathbf{g} \in \mathbf{V}^N$, $\langle \mathcal{A}\mathbf{f}, \mathbf{g}\rangle_N = \langle \mathbf{f}, \mathcal{A}\mathbf{g}\rangle_N$ and, $\forall\, \mathbf{f} \in \mathbf{V}^N \setminus \{\Theta\}$, $\langle \mathcal{A}\mathbf{f}, \mathbf{f}\rangle > 0$.*

Proof. Using relation (13), Lemma 3.2, partial integration, and Fubini's theorem, we get, for all $\mathbf{f}, \mathbf{g} \in \mathbf{V}^N$,

$$\langle \mathcal{A}\mathbf{f}, \mathbf{g}\rangle_N = -\frac{1}{\pi}\sum_{j=1}^N\sum_{k=1}^N \int_{-1}^1\int_{-1}^1 f_k'(s) \ln|\mathbf{r}_{0k}(s) - \mathbf{r}_{0j}(t)|\,\mathrm{d}s\, g_j'(t)\,\mathrm{d}t = \langle \mathbf{f}, \mathcal{A}\mathbf{g}\rangle_N. \tag{17}$$

If we set $\ell = \ell_1 \cup \ldots \cup \ell_N$ and, for $z \in \ell$,

$$\mu(z) = f_k(t) \quad \text{if} \quad z = \mathbf{r}_{0k}(t) \in \ell_k,$$

then

$$\begin{aligned} \langle \mathcal{A}\mathbf{f}, \mathbf{f}\rangle_N &= \tfrac{1}{\pi}\textstyle\sum_{j=1}^N\sum_{k=1}^N \int_{-1}^1\int_{-1}^1 \ln\frac{1}{|\mathbf{r}_{0k}(s)-\mathbf{r}_{0j}(t)|}\, f_k'(s) f_j'(t)\,\mathrm{d}s\,\mathrm{d}t \\ &= \tfrac{1}{\pi}\textstyle\int_\ell\int_\ell \frac{1}{\ln|w-z|}\,\mathrm{d}\mu(w)\,\mathrm{d}\mu(z) \end{aligned}$$

corresponds to the logarithmic energy of the signed measure μ on ℓ, where

$$\int_\ell \mathrm{d}\mu(z) = \sum_{k=1}^N \int_{-1}^1 f_k'(t)\,\mathrm{d}t = 0,$$

due to Lemma 3.2. Consequently (see [13], Section I.1, and in particular Lemma 1.8), $\langle \mathcal{A}\mathbf{f}, \mathbf{f}\rangle_N$ is positive if $\mathbf{f}' \neq \Theta$ a.e. Hence, $\langle \mathcal{A}\mathbf{f}, \mathbf{f}\rangle_N = 0$ implies $\mathbf{f}'(t) = \Theta$ for almost all $t \in (-1,1)$ and, due to $\mathbf{f}(\pm 1) = \Theta$, also $\mathbf{f}(t) = \Theta$ for all $t \in [-1,1]$. □

For $\boldsymbol{\gamma} = \left[\ \gamma_j\ \right]_{j=1}^m \in \mathbb{R}^m$, define the corresponding (affine) manifold

$$\mathbf{V}^N_{\boldsymbol{\gamma}} := \left\{ \mathbf{f} = \left[\ f_k\ \right]_{k=1}^N \in \mathbf{V}^N : \sum_{k=n_{j-1}+1}^{n_j} \langle f_k, \psi_{1k}'\rangle = \gamma_j,\ j = 1, \ldots, m \right\}.$$

Moreover, for $\boldsymbol{\beta} = \left[\ \beta_j\ \right]_{j=1}^{m} \in \mathbb{R}^m$ and $\mathbf{f} : [-1,1] \longrightarrow \mathbb{R}^N$, $t \mapsto \left[\ f_k(t)\ \right]_{k=1}^{N}$, we define

$$\boldsymbol{\beta}\mathbf{f} = \left[\ \beta_1 f_1 \quad \ldots \quad \beta_1 f_{n_1} \quad \ldots \quad \beta_m f_{n_{m-1}+1} \quad \ldots \quad \beta_m f_{n_m}\ \right]^T .$$

If we set $\Psi_1 = \left[\ \psi_{1k}\ \right]_{k=1}^{N}$ and $\Psi_1' = \left[\ \psi_{1k}'\ \right]_{k=1}^{N}$, then the following result holds.

Theorem 3.7. *The element $\boldsymbol{\Gamma}_0^* \in \mathbf{V}_{\boldsymbol{\gamma}}^N$ is a solution of Problem* (P) *if and only if there is a $\boldsymbol{\beta} \in \mathbb{R}^m$ such that*

$$\mathcal{A}\boldsymbol{\Gamma}_0^* = \boldsymbol{\beta}\Psi_1' . \tag{18}$$

This solution is unique, if it exists.

Proof. Assume that $\boldsymbol{\Gamma}_0^* \in \mathbf{V}_{\boldsymbol{\gamma}}^N$ and $F(\boldsymbol{\Gamma}_0^*) = \min\left\{F(\boldsymbol{\Gamma}_0) : \boldsymbol{\Gamma}_0 \in \mathbf{V}_{\boldsymbol{\gamma}}^N\right\}$. This implies $G'(0) = 0$ for $G(\alpha) = F(\boldsymbol{\Gamma}_0^* + \alpha\mathbf{f})$ and for all $\mathbf{f} \in \mathbf{V}_{\Theta}^N \setminus \{\Theta\}$. Since

$$G(\alpha) = F(\boldsymbol{\Gamma}_0^*) + 2\alpha\langle \mathcal{A}\boldsymbol{\Gamma}_0^*, \mathbf{f}\rangle_N + \alpha^2\langle \mathbf{f}, \mathbf{f}\rangle_N \tag{19}$$

and

$$G'(\alpha) = 2\langle \mathcal{A}\boldsymbol{\Gamma}_0^*, \mathbf{f}\rangle_N + 2\alpha\langle \mathbf{f}, \mathbf{f}\rangle_N,$$

this condition gives $\langle \mathcal{A}\boldsymbol{\Gamma}_0^*, \mathbf{g}\rangle_{\varphi,N} = 0$ for all $\mathbf{g} = \left[\ g_k\ \right]_{k=1}^{N} \in \left(\mathbf{L}_{\varphi}^{2,1}\right)^N$ satisfying $\sum\limits_{k=n_{j-1}+1}^{n_j} \langle g_k, \psi_{1k}'\rangle_{\varphi} = 0$, $j = 1, \ldots, m$, which is equivalent to (18). On the other hand, if $\boldsymbol{\Gamma}_0^* \in \mathbf{V}_{\boldsymbol{\gamma}}^N$ and $\beta \in \mathbb{R}^m$ fulfil (18) and if $\mathbf{f} \in \mathbf{V}_{\Theta}^N \setminus \{\Theta\}$, then we get from (19) for $\alpha = 1$

$$\begin{aligned} F(\boldsymbol{\Gamma}_0^* + \mathbf{f}) &= F(\boldsymbol{\Gamma}_0^*) + 2\langle \mathcal{A}\boldsymbol{\Gamma}_0^*, \mathbf{f}\rangle_N + \langle \mathbf{f}, \mathbf{f}\rangle_N \\ &= F(\boldsymbol{\Gamma}_0^*) + 2\left\langle \mathcal{A}\boldsymbol{\Gamma}_0^* - \beta\Psi_1', \mathbf{f}\right\rangle_N + \langle \mathbf{f}, \mathbf{f}\rangle_N \\ &= F(\boldsymbol{\Gamma}_0^*) + \langle \mathbf{f}, \mathbf{f}\rangle_N > F(\boldsymbol{\Gamma}_0^*), \end{aligned}$$

which also shows the uniqueness of the solution (if it exists). □

Remark 3.8. *Using relation* (13)*, equation* (18) *can be written equivalently as*

$$\mathcal{B}\boldsymbol{\Gamma}_0^* = \boldsymbol{\beta}\Psi_1 + \boldsymbol{\delta}, \quad \boldsymbol{\Gamma}_0^* \in \mathbf{V}_{\boldsymbol{\gamma}}^N, \ \ (\boldsymbol{\beta} \in \mathbb{R}^m,\, \boldsymbol{\delta} \in \mathbb{R}^N). \tag{20}$$

Moreover, by applying partial integration to the integrals in (14) *and taking into account*

$\mathbf{f}(\pm 1)=\Theta$ *for* $\mathbf{f}\in\mathbf{V}^N$ *(see Lemma 3.2), we get*

$$\begin{aligned}
(\mathcal{B}\mathbf{f})(t) &= \left[\lim_{\varepsilon\to+0}\sum_{k=1}^N \frac{1}{\pi}\left(\int_{-1}^{t-\varepsilon}+\int_{t+\varepsilon}^{1}\right)\ln|\mathbf{r}_{0k}(s)-\mathbf{r}_{0j}(t)|\,f_k'(s)\,\mathrm{d}s\right]_{j=1}^N\\
&= \left[\lim_{\varepsilon\to+0}\sum_{k=1}^N \frac{1}{\pi} f_k(t-\varepsilon)\ln|\mathbf{r}_{0k}(t-\varepsilon)-\mathbf{r}_{0j}(t)|\right]_{j=1}^N\\
&\quad-\left[\lim_{\varepsilon\to+0}\sum_{k=1}^N \frac{1}{\pi} f_k(t+\varepsilon)\ln|r_{0k}(t+\varepsilon)-\mathbf{r}_{0j}(t)|\right]_{j=1}^N\\
&\quad-\left[\lim_{\varepsilon\to+0}\frac{1}{\pi}\sum_{k=1}^N \left(\int_{-1}^{t-\varepsilon}+\int_{t+\varepsilon}^{1}\right) f_k(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\ln|\mathbf{r}_{0k}(s)-\mathbf{r}_{0j}(t)|\,\mathrm{d}s\right]_{j=1}^N\\
&\overset{(7)}{=} \left[\frac{1}{\pi}\sum_{k=1}^N\int_{-1}^{1} Y_{0kj}(s,t)f_k(s)\,\mathrm{d}s\right]_{j=1}^N
\end{aligned}$$

Hence, we obtain the identity

$$\mathcal{B}\mathbf{f}=\mathcal{A}_0\mathbf{f}\quad\forall\,\mathbf{f}\in\mathbf{V}^N, \tag{21}$$

where

$$(\mathcal{A}_0\mathbf{f})(t):=\left[\sum_{k=1}^N\frac{1}{\pi}\int_{-1}^{1}Y_{0kj}(s,t)f_k(s)\,\mathrm{d}s\right]_{j=1}^N=-(\mathcal{S}\mathbf{f})(t)+(\mathcal{K}_0\mathbf{f})(t)$$

with (*cf.* (11) *and* (12))

$$(\mathcal{K}_0\mathbf{f})(t):=\left[\sum_{k=1}^N\frac{1}{\pi}\int_{-1}^{1}K_{kj}(s,t)f_k(s)\,\mathrm{d}s\right]_{j=1}^N. \tag{22}$$

Note that equation (18), hence its equivalent representation one obtains from (20) and equality (21), define the Euler-Lagrange equation for the corresponding drag minimization problem. □

The following Lemma is a consequence of the well-known relation

$$\mathcal{S}\varphi p_n^{\varphi}=-p_{n+1}^{\varphi^{-1}},\quad n\in\mathbb{N}_0. \tag{23}$$

Lemma 3.9. *The operator* $\mathcal{S}:\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N\longrightarrow\left(\mathbf{L}^2_{\varphi^{-1},0}\right)^N$ *and the operator* $\mathcal{S}:\left(\varphi\mathbf{L}^{2,r}_{\varphi}\right)^N\longrightarrow\left(\mathbf{L}^{2,r}_{\varphi^{-1},0}\right)^N$, $r>0$, *with* $\mathbf{L}^2_{\rho,0}=\left\{f\in\mathbf{L}^2_{\rho}:\langle f,1\rangle_\rho=0\right\}$ *and* $\mathbf{L}^{2,r}_{\rho,0}=\mathbf{L}^{2,r}_{\rho}\cap\mathbf{L}^2_{\rho,0}$ *are invertible.*

The next proposition is concerned with the solvability of (20). For this we define the operator

$$\mathcal{G} : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m,$$

$$(\mathbf{f}, \boldsymbol{\beta}, \boldsymbol{\delta}) \mapsto \left(\mathcal{A}_0\mathbf{f} - \boldsymbol{\beta}\boldsymbol{\Psi}_1 - \boldsymbol{\delta}, \left[\sum_{k=n_{j-1}+1}^{n_j} \langle f_k, \psi'_{1k} \rangle \right]_{j=1}^m \right).$$

Theorem 3.10. *Assume that* $\psi_{ik} \in \mathbf{C}^3[-1,1]$, $i = 1,2$, $k = 1, \ldots, N$, *Then,*

(a) *the operator* $\mathcal{A}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ *has a trivial null space, i.e.,*

$$N(\mathcal{A}_0) = \left\{ \mathbf{f} \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N : \mathcal{A}_0\mathbf{f} = \Theta \right\} = \{\Theta\}.$$

If furthermore, the vector valued functions $\left[\ \psi_{1k}(t) \ \right]_{k=n_{j-1}+1}^{n_j}$ *are not constant functions on* $[-1,1]$ *for* $j = 1, \ldots, m$, *then:*

(b) *for every* $(\mathbf{g}, \gamma) \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m$, *the equation*

$$\mathcal{G}(\mathbf{f}, \boldsymbol{\beta}, \boldsymbol{\delta}) = (\mathbf{g}, \boldsymbol{\gamma}) \tag{24}$$

possesses a unique solution;

(c) *equation* (20) *possesses a unique solution* $(\boldsymbol{\Gamma}^*_0, \boldsymbol{\beta}, \boldsymbol{\delta}) \in \mathbf{V}^N_{\boldsymbol{\gamma}} \times \mathbb{R}^m \times \mathbb{R}^N$*;*

(d) *Problem* (P) *is uniquely solvable.*

Proof. Let $\mathbf{f}^0 \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ and $\mathcal{A}_0\mathbf{f}^0 = \Theta$. Hence, $\mathcal{S}\mathbf{f}^0 = \mathcal{K}_0\mathbf{f}^0 \in \left(\mathbf{C}^1[-1,1]\right)^N \subset \left(\mathbf{L}^{2,1}_{\varphi^{-1}}\right)^N$, due to Lemma 3.3. By Lemma 3.9, we get $\mathcal{K}_0\mathbf{f}^0 \in \left(\mathbf{L}^{2,1}_{\varphi^{-1},0}\right)^N$ and, consequently, $\mathbf{f}^0 \in \left(\varphi\mathbf{L}^{2,1}_{\varphi}\right)^N = \mathbf{V}^N$. On the other hand, due to (13) and (21) (cf. also the proof of Lemma 3.6), we have

$$0 < \langle \mathcal{A}\mathbf{f}, \mathbf{f} \rangle_N = -\langle \mathcal{A}_0\mathbf{f}, \mathcal{D}\mathbf{f} \rangle_N \quad \forall\, \mathbf{f} \in \mathbf{V}^N \setminus \{\Theta\}.$$

This implies $\mathbf{f}^0 = \Theta$, and (a) is proved.

Since, by Lemma 3.9, the operator $\mathcal{S} : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ is Fredholm with index $-N$ and since, due to the continuity of the functions $K_{kj}(s,t)$ (cf. Lemma 3.3), the operator $\mathcal{K}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ is compact, also the operator $\mathcal{A}_0 = -\mathcal{S} + \mathcal{K}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ is Fredholm with index $-N$. Hence, we conclude that the codimension of the image

$$R(\mathcal{A}_0) = \left\{ \mathcal{A}_0\mathbf{f} : \mathbf{f} \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \right\}$$

is equal to N. Since

$$\dim \left\{ \boldsymbol{\beta}\Psi_1 + \boldsymbol{\delta} : \boldsymbol{\beta} \in \mathbb{R}^m, \boldsymbol{\delta} \in \mathbb{R}^N \right\} = N + m\,,$$

the intersection $\mathbf{W}_1 := R(\mathcal{A}_0) \cap \left\{\boldsymbol{\beta}\Psi_1 + \boldsymbol{\delta} : \boldsymbol{\beta} \in \mathbb{R}^m, \boldsymbol{\delta} \in \mathbb{R}^N\right\}$ is at least m-dimensional. We show, however, that $\dim \mathbf{W}_1 > m$ is not possible. Indeed, in that case there exist $m+1$ linearly independent $\mathbf{f}^j \in \mathbf{V}^N$ with

$$\mathcal{A}_0\mathbf{f}^j = \mathbf{g}^j \in \left\{\boldsymbol{\beta}\Psi_1 + \boldsymbol{\delta} : \boldsymbol{\beta} \in \mathbb{R}^m, \boldsymbol{\delta} \in \mathbb{R}^N\right\}, \quad j = 1, \ldots, m+1,$$

where $\mathbf{W}_1 = \operatorname{span}\left\{\mathbf{g}^1, \ldots, \mathbf{g}^{m+1}\right\}$. The $m \times (m+1)$ homogeneous system

$$\left[\ \sum_{k=n_{\ell-1}+1}^{n_\ell} \left\langle f_k^j, \psi'_{1k}\right\rangle,\ j = 1 : m+1\ \right]_{\ell=1}^{m} \left[\ \alpha_j\ \right]_{j=1}^{m+1} = \left[\ 0\ \right]_{\ell=1}^{m}$$

has a nontrivial solution. The respective $\mathbf{f} = \left[\ f_k\ \right]_{k=1}^{N} = \sum_{j=1}^{m+1} \alpha_j \mathbf{f}^j$ satisfies

$$\sum_{k=n_{j-1}+1}^{n_j} \langle f_k, \psi'_{1k}\rangle = 0, \quad j = 1, \ldots, m.$$

This means that $\mathbf{f} \in \mathbf{V}_\Theta^N$ solves (20) with $\boldsymbol{\gamma} = \Theta$. By Proposition 3.7 and Remark 3.8, this solution is unique, hence identically zero in contradiction to $\boldsymbol{\alpha} \neq \Theta$. Therefore, $\mathbf{f}^* \neq \Theta$.

Thus $\dim \mathbf{W}_1 = m$. Let $\{\mathbf{h}^j : j = N+1, \ldots, N+m\}$ denote an orthonormal basis of $\mathbf{W}_1$, which can be extended to an orthonormal basis

$$\left\{\mathbf{h}^j = \boldsymbol{\beta}^j \boldsymbol{\Psi}_1 + \boldsymbol{\delta}^j : j = 1, \ldots, N+m\right\}$$

of $\mathbf{M} := \left\{\boldsymbol{\beta}\boldsymbol{\Psi}_1 + \boldsymbol{\delta} : \boldsymbol{\beta} \in \mathbb{R}^m, \boldsymbol{\delta} \in \mathbb{R}^N\right\}$. Set $\mathbf{W}_2 := \operatorname{span}\left\{\mathbf{h}^j : j = 1, \ldots, N\right\}$. We have

$$R(\mathcal{A}_0) \cap \mathbf{W}_2 = \{\Theta\} \quad \text{and} \quad \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N = R(\mathcal{A}_0) + \mathbf{W}_2\,. \tag{25}$$

We show that, for every $(\mathbf{g}, \boldsymbol{\gamma}) \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m$, there is a unique $(\mathbf{f}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*) \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N$ satisfying $\mathcal{G}(\mathbf{f}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*) = (\mathbf{g}, \boldsymbol{\gamma})$. Due to the second relation in (25), there is an $(\mathbf{f}^0, \boldsymbol{\beta}^0, \boldsymbol{\delta}^0) \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N$ such that

$$\widehat{\mathcal{G}}(\mathbf{f}^0, \boldsymbol{\beta}^0, \boldsymbol{\delta}^0) := \mathcal{A}_0\mathbf{f}^0 - \boldsymbol{\beta}^0\boldsymbol{\Psi}_1 - \delta^0 = \mathbf{g}\,. \tag{26}$$

The null space of the operator $\widehat{\mathcal{G}} : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ is equal to

$$N(\widehat{\mathcal{G}}) = \left\{(\mathbf{f}, \boldsymbol{\beta}, \boldsymbol{\delta}) \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N : \mathcal{A}_0\mathbf{f} \in \mathbf{W}_1 \text{ and } \mathcal{A}_0\mathbf{f} = \boldsymbol{\beta}\boldsymbol{\Psi}_1 + \boldsymbol{\delta}\right\}$$

In view of (a), there exist uniquely determined $\mathbf{u}^j \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ such that $\mathcal{A}_0\mathbf{u}^j = \mathbf{h}^j = \boldsymbol{\beta}^j\boldsymbol{\Psi}_1 + \boldsymbol{\delta}^j$ for $j = N+1, \ldots, N+m$. Consequently, the set of all solutions of (26) is given by

$$\left\{(\mathbf{f}, \boldsymbol{\beta}, \boldsymbol{\delta}) = \sum_{j=N+1}^{N+m} \alpha_j(\mathbf{u}^j, \boldsymbol{\beta}^j, \boldsymbol{\delta}^j) + (\mathbf{f}^0, \boldsymbol{\beta}^0, \boldsymbol{\delta}^0) : \alpha_j \in \mathbb{R}\right\}.$$

An element $(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta})$ from this set solves (24) if and only if the respective α_j's fulfil

$$\sum_{\ell=N+1}^{N+m}\sum_{k=n_{j-1}+1}^{n_j}\left\langle u_k^\ell,\psi_{1k}'\right\rangle\alpha_\ell+\sum_{k=n_{j-1}+1}^{n_j}\left\langle f_k^0,\psi_{1k}'\right\rangle=\gamma_j\,,\quad j=1,\dots,m\,. \tag{27}$$

This system has a unique solution $\left[\ \alpha_j\ \right]_{j=N+1}^{N+m}$, since otherwise there is an $\boldsymbol{\alpha}^0=\left[\ \alpha_j^0\ \right]_{j=N+1}^{N+m}\in\mathbb{R}^m\setminus\{\Theta\}$ satisfying

$$\sum_{\ell=N+1}^{N+m}\sum_{k=n_{j-1}+1}^{n_j}\left\langle u_k^\ell,\psi_{1k}'\right\rangle\alpha_\ell=0\,,\quad j=1,\dots,m\,,$$

which implies $\mathbf{u}^0:=\sum_{j=N+1}^{N+m}\alpha_j^0\mathbf{u}^j\neq\Theta$ and $\mathcal{A}_0\mathbf{u}^0=\boldsymbol{\beta}^0\boldsymbol{\Psi}_1+\boldsymbol{\delta}^0$, where

$$\boldsymbol{\beta}^0=\sum_{j=N+1}^{N+m}\alpha_j^0\mathbf{u}^j\quad\text{and}\quad\boldsymbol{\delta}^0=\sum_{j=N+1}^{N+m}\alpha_j^0\boldsymbol{\delta}^j\,.$$

Moreover,

$$\sum_{j=N+1}^{N+m}\left\langle u_k^0,\psi_{1k}'\right\rangle=\sum_{j=N+1}^{N+m}\sum_{\ell=N+1}^{N+m}\alpha_\ell^0\left\langle u_k^\ell,\psi_{1k}'\right\rangle=0\,,\quad j=1,\dots,n\,.$$

Thus, $\mathbf{u}^0\in\mathbf{V}_\Theta$ is a solution of (20). By Proposition 3.7 and Remark 3.8, this solution is unique, hence identically zero in contradiction to $\boldsymbol{\alpha}^0\neq\Theta$. Therefore, if we denote the unique solution of (27) by $\boldsymbol{\alpha}^*=\left[\ \alpha_j^*\ \right]_{j=N+1}^{N+m}$, then

$$(\mathbf{f}^*,\boldsymbol{\beta}^*,\boldsymbol{\delta}^*)=\sum_{j=N+1}^{N+m}\alpha_j^*(\mathbf{u}^j,\boldsymbol{\beta}^j,\boldsymbol{\delta}^j)+(\mathbf{f}^0,\boldsymbol{\beta}^0,\boldsymbol{\delta}^0)$$

is the uniquely defined solution of (24).

Assertion (c) is the special case $(\mathbf{g},\boldsymbol{\gamma})=(\Theta,\boldsymbol{\gamma})$ of (b), and assertion (d) is an immediate consequence of (c), together with Proposition 3.7 and Remark 3.8. □

4 A COLLOCATION-QUADRATURE METHOD

Here, we describe a numerical procedure for the approximate solution of equation (20). For this, we write this equation in the form (cf. (21), (22))

$$\mathcal{A}_0\mathbf{f}=\boldsymbol{\beta}\Psi_1+\boldsymbol{\delta},\quad(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta})\in\mathbf{V}_{\boldsymbol{\gamma}}^N\times\mathbb{R}^m\times\mathbb{R}^N \tag{28}$$

with $\mathcal{A}_0=-\mathcal{S}+\mathcal{K}_0:\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N\longrightarrow\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ and

$$(\mathcal{S}\mathbf{f})(t)=\left[\ \frac{1}{\pi}\int_{-1}^1\frac{f_j(s)\,\mathrm{d}s}{s-t}\ \right]_{j=1}^N,$$

as well as

$$(\mathcal{K}_0 f)(t) = \left[\sum_{k=1}^{N} \frac{1}{\pi} \int_{-1}^{1} K_{kj}(s,t) f_k(s) \, \mathrm{d}s \right]_{j=1}^{N} .$$

For every integer $n \geq 1$, we are looking for an approximate solution $(\mathbf{f}^n, \boldsymbol{\beta}^n, \boldsymbol{\delta}^n)$ in $R(\mathcal{P}_n) \times \mathbb{R}^m \times \mathbb{R}^N$ of (28), where $\mathbf{f}^n = \left[\, f_k^n \, \right]_{k=1}^{N}$, $\boldsymbol{\beta}^n = \left[\, \beta_j^n \, \right]_{j=1}^{m}$ and $\boldsymbol{\delta}^n = \left[\, \delta_k^n \, \right]_{k=1}^{N}$, and where by $R(\mathcal{P}_n)$ we denote the image space of the orthoprojection $\mathcal{P}_n : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ defined by

$$\mathcal{P}_n \mathbf{f} = \left[\sum_{k=0}^{n-1} \langle f_j, U_k \rangle \, \varphi U_k \right]_{j=1}^{N} ,$$

where $U_k = p_k^{\varphi}$ denotes the normalized second kind Chebyshev polynomial of degree k. For that, we solve the collocation equations

$$-(\mathcal{S}\mathbf{f}^n)(t_{\ell n}) + (\mathcal{K}_n^0 \mathbf{f}^n)(t_{\ell n}) = \boldsymbol{\beta}^n \Psi_1(t_{\ell n}) + \boldsymbol{\delta}^n, \quad \ell = 1, \ldots, n+1, \tag{29}$$

together with

$$\frac{\pi}{n+1} \sum_{k=n_{j-1}+1}^{n_j} \sum_{i=1}^{n} \varphi(s_{in}) \psi'_{1k}(s_{in}) f_k^n(s_{in}) = \gamma_j \,, \quad j = 1, \ldots, m \,, \tag{30}$$

where $t_{\ell n} = \cos \dfrac{(2\ell - 1)\pi}{2n+2}$ and $s_{in} = \cos \dfrac{i\pi}{n+1}$ are Chebyshev nodes of first and second kind, respectively, and where

$$(\mathcal{K}_n^0 \mathbf{f}^n)(t) = \left[\frac{1}{n+1} \sum_{k=1}^{N} \sum_{i=1}^{n} \varphi(s_{in}) K_{kj}(s_{in}, t) f_k^n(s_{in}) \right]_{j=1}^{N} . \tag{31}$$

Note that $\mathbf{f}^n(t)$ can be written with the help of the weighted Lagrange interpolation polynomials

$$\widetilde{\ell}_{in}^{\varphi}(t) = \frac{\varphi(t) \ell_{in}^{\varphi}(t)}{\varphi(s_{in})} \quad \text{with} \quad \ell_{in}^{\varphi}(t) = \frac{U_n(t)}{(t - s_{in}) U_n'(s_{in})}, \quad i = 1, \ldots, n.$$

in the form

$$\mathbf{f}^n(t) = \sum_{i=1}^{n} \widetilde{\ell}_{in}^{\varphi}(t) \boldsymbol{\xi}_{in} \,, \quad \boldsymbol{\xi}_{in} = \mathbf{f}^n(s_{in}) \,. \tag{32}$$

Let $\mathcal{L}_n^j$, $j = 1, 2$ denote the interpolation operators, which associate to a function $\mathbf{g} : (-1, 1) \longrightarrow \mathbb{R}^N$ the (vector) polynomials

$$(\mathcal{L}_n^1 \mathbf{g})(t) = \sum_{\ell=1}^{n+1} \frac{T_{n+1}(t)}{(t - t_{\ell n}) T'_{n+1}(t_{\ell n})} \, \mathbf{g}(t_{\ell n}) \,,$$

and

$$(\mathcal{L}_n^2 \mathbf{g})(t) = \sum_{i=1}^{n} \frac{U_n(t)}{(t - s_{in}) U_n'(s_{in})} \, \mathbf{g}(s_{in}) \,,$$

respectively, where $T_n = p_n^{\varphi^{-1}}$. Now, the system (29), (30) can be written as operator equation

$$\mathcal{A}_n \mathbf{f}^n = \boldsymbol{\beta}^n \mathcal{L}_n^1 \boldsymbol{\Psi}_1 + \boldsymbol{\delta}^n, \quad (\mathbf{f}^n, \boldsymbol{\beta}, \boldsymbol{\delta}) \in R(\mathcal{P}_n) \times \mathbb{R}^m \times \mathbb{R}^N \tag{33}$$

together with

$$\sum_{k=n_{j-1}+1}^{n_j} \left\langle \mathcal{L}_n^2 \psi_{1k}', f_k^n \right\rangle = \gamma_j \,, \; j = 1, \ldots, m \,, \tag{34}$$

where $\mathcal{A}_n = -\mathcal{S}_n + \mathcal{K}_n$, $\mathcal{S}_n = \mathcal{L}_n^1 \mathcal{S} \mathcal{P}_n$, and $\mathcal{K}_n = \mathcal{L}_n^1 \mathcal{K}_n^0 \mathcal{P}_n$. The equivalence of (30) and (34) follows from the algebraic accuracy of the Gaussian rule w.r.t. the Chebyshev nodes of second kind. In Lemma 4.1 and Lemma 4.2, we take $N = 1$, the generalization to $N > 1$ is obvious. The assertion of the following lemma is well-known (see [14, Theorem 14.3.1]).

Lemma 4.1. *For all $f \in \mathbf{C}[-1,1]$, we have $\lim\limits_{n\to\infty} \left\| f - \mathcal{L}_n^1 f \right\|_{\varphi^{-1}} = 0$ as well as $\lim\limits_{n\to\infty} \left\| f - \mathcal{L}_n^2 f \right\|_{\varphi} = 0$.*

The next lemma provides convergence rates for the interpolating polynomials and will be used in the proof of Proposition 4.4.

Lemma 4.2 ([1], Theorem 3.4)**.** *If $r > \frac{1}{2}$, then there exists a constant $c > 0$ such that, for any real p, $0 \le p \le r$ and all $n \ge 1$,*

(a) $\|f - \mathcal{L}_n^1 f\|_{\varphi^{-1},p} \le c\, n^{p-r} \|f\|_{\varphi^{-1},r}$ *for all* $f \in \mathbf{L}^{2,r}_{\varphi^{-1}}$,

(b) $\|f - \mathcal{L}_n^2 f\|_{\varphi,p} \le c\, n^{p-r} \|f\|_{\varphi,r}$ *for all* $f \in \mathbf{L}^{2,r}_{\varphi}$.

Lemma 4.3. *The operator $\mathcal{K}_n^0$ can be extended to a linear and bounded operator on $\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ such that, for $\psi_{jk} \in \mathbf{C}^2[-1,1]$, $j = 1, 2$, $k = 1, \ldots, N$, we have*

(a) $\lim\limits_{n\to\infty} \|\mathcal{K}_n - \mathcal{K}_0\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \to \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} = 0.$

(b) *Moreover, a solution $\mathbf{f} \in \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ of $\mathcal{A}_n \mathbf{f} - \boldsymbol{\beta} \mathcal{L}_n^1 \boldsymbol{\Psi}_1 - \boldsymbol{\delta} = \Theta$ automatically belongs to $R(\mathcal{P}_n)$.*

Proof. By definition of $\mathcal{K}_n^0$ and in virtue of the algebraic accuracy of the Gaussian rule, for $\mathbf{f}^n \in R(\mathcal{P}_n)$ we have

$$\begin{aligned}
(\mathcal{K}_n^0 \mathbf{f}^n)(t) &= \left[\sum_{k=1}^N \frac{1}{\pi} \int_{-1}^1 \mathcal{L}_n^2 \left[K_{kj}(.,t) \varphi^{-1} f_k^n \right](s) \varphi(s) \, \mathrm{d}s \right]_{j=1}^N \\
&= \left[\sum_{k=1}^N \frac{1}{\pi} \int_{-1}^1 \mathcal{L}_n^2 \left[K_{kj}(\cdot,t) \right](s) f_k^n(s) \, \mathrm{d}s \right]_{j=1}^N ,
\end{aligned}$$

which implies, that the definition of $\mathcal{K}_n^0$ can be extended to the whole space $\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ by

$$(\mathcal{K}_n^0 \mathbf{f})(t) = \left[\sum_{k=1}^N \frac{1}{\pi} \int_{-1}^1 \mathcal{L}_n^2 \left[K_{kj}(\cdot,t) \right](s) f_k(s) \, \mathrm{d}s \right]_{j=1}^N .$$

Using $\|f\|_\infty = \max\{|f(t)| : -1 \le t \le 1\}$ (the norm in the space $\mathbf{C} = \mathbf{C}[-1,1]$ of continuous functions $f : [-1,1] \longrightarrow \mathbb{R}$) and $\|\mathbf{f}\|_{\infty,N} := \left(\sum_{k=1}^N \|f_k\|_\infty^2 \right)^{\frac{1}{2}}$ (the norm in the space $\mathbf{C}^N$), we get

$$
\begin{aligned}
&\|(\mathcal{K}_n^0 - \mathcal{K}_0)\,\mathbf{f}\|_{\infty,N} \\
&= \frac{1}{\pi}\left(\sum_{j=1}^N \sup_{-1\le t\le 1} \left\{ \left| \sum_{k=1}^N \int_{-1}^1 \Big[\mathcal{L}_n^2\left[K_{kj}(\cdot,t)\right](s) - K_{kj}(s,t) \Big] f_k(s)\,\mathrm{d}s \right|^2 \right\} \right)^{\frac{1}{2}} \\
&\le \frac{1}{\pi}\left(\sum_{j=1}^N \sup_{-1\le t\le 1} \left\{ \sum_{k=1}^N \left\| \mathcal{L}_n^2 K_{kj}(\cdot,t) - K_{kj}(\cdot,t) \right\|_\varphi^2 \sum_{k=1}^N \|f_k\|_{\varphi^{-1}}^2 \right\} \right)^{\frac{1}{2}} \\
&\le \frac{1}{\pi}\left(\sum_{j=1}^N \sup_{-1\le t\le 1} \left\{ \sum_{k=1}^N \left\| \mathcal{L}_n^2 K_{kj}(\cdot,t) - K_{kj}(\cdot,t) \right\|_\varphi^2 \right\} \right)^{\frac{1}{2}} \|\mathbf{f}\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} .
\end{aligned}
\tag{35}
$$

Since, due to Lemma 4.1 and the principle of uniform boundedness, the operator sequence $\mathcal{L}_n^1 : \mathbf{C}^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ is uniformly bounded, the last estimate together with Lemma 4.1 (applied to $\mathcal{L}_n^2$) leads to

$$\lim_{n\to\infty} \left\| \mathcal{K}_n - \mathcal{L}_n^1 \mathcal{K}_0 \mathcal{P}_n \right\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \to \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} = 0 .$$

Again Lemma 4.1, the strong convergence of $\mathcal{P}_n = \mathcal{P}_n^* \longrightarrow \mathcal{I}$ (the identity operator), and the compactness of the operator $\mathcal{K}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \mathbf{C}^N$ give us

$$\lim_{n\to\infty} \left\| \mathcal{L}_n^1 \mathcal{K}_0 \mathcal{P}_n - \mathcal{K}_0 \right\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \to \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} = 0 ,$$

and (a) is proved.

Assertion (b) is a consequence of Lemma 3.9 and relation (23). □

Theorem 4.4. *Assume $\psi_{ik} \in \mathbf{C}^r[-1,1]$, $i = 1,2$, $k = 1,\ldots,N$, for some integer $r > 2$. Let $\gamma_j \ne 0$ for all $j = 1,\ldots,m$, and the vector valued functions $\left[\ \psi_{1k}(t)\ \right]_{k=n_{j-1}+1}^{n_j}$, $j = 1,\ldots,m$, be not constant. Then, for all sufficiently large n (say $n \ge n_0$), there exists a unique solution $(\mathbf{f}^{n*}, \boldsymbol{\beta}^{n*}, \boldsymbol{\delta}^{n*}) \in R(\mathcal{P}_n) \times \mathbb{R}^m \times \mathbb{R}^N$ of (33), (34). Moreover, since for the unique solution $(\mathbf{f}^*, \beta^*, \delta^*)$ of (28) we have $\mathbf{f}^* \in \left(\varphi \mathbf{L}^{2,r-2}_\varphi\right)^N$, the following inequality holds:*

$$\left(\sum_{k=1}^N \|f_k^{n*} - f_k^*\|_{\varphi^{-1}}^2 + \sum_{j=1}^m |\beta_j^{n*} - \beta_j^*|^2 + \sum_{k=1}^N |\delta_k^{n*} - \delta_k^*|^2 \right)^{\frac{1}{2}} \le c\, n^{2-r} \tag{36}$$

with a constant $c > 0$ independent of n.

Proof. Set $\mathbf{X} := \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m \times \mathbb{R}^N$ and $\mathbf{Y} := \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \times \mathbb{R}^m$ as well as

$$\|(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta})\|_{\mathbf{X}} := \left(\|\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + \sum_{j=1}^m \beta_j^2 + \sum_{k=1}^N \delta_j^2\right)^{\frac{1}{2}}$$

and

$$\|(\mathbf{g},\boldsymbol{\gamma})\|_{\mathbf{Y}} := \left(\|\mathbf{g}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + \sum_{j=1}^m \gamma_j^2\right)^{\frac{1}{2}}.$$

In view of Proposition 3.10,(b), the linear and bounded operator $\mathcal{G} : \mathbf{X} \longrightarrow \mathbf{Y}$ is invertible, where, due to Banach's theorem, $\mathcal{G}^{-1} \in \mathcal{L}(\mathbf{Y},\mathbf{X})$. If we denote the map

$$(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta}) \mapsto \left(\mathcal{A}_n\mathbf{f} - \boldsymbol{\beta}\mathcal{L}_n^1\boldsymbol{\Psi}_1 - \boldsymbol{\delta}, \left[\sum_{k=n_{j-1}+1}^{n_j} \left\langle f_k, \mathcal{L}_n^2\psi'_{1k}\right\rangle\right]_{j=1}^m\right)$$

by $\mathcal{G}_n : \mathbf{X} \longrightarrow \mathbf{Y}$, then $(\mathbf{f}^{n*},\boldsymbol{\beta}^{n*},\boldsymbol{\delta}^{n*})$ is a solution of (33), (34) if and only if $\mathcal{G}_n(\mathbf{f}^{n*},\boldsymbol{\beta}^{n*},\boldsymbol{\delta}^{n*}) = (\Theta,\boldsymbol{\gamma})$ and, taking into account the inequality

$$\begin{aligned}\|\boldsymbol{\beta}\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} &= \sum_{j=1}^m \beta_j^2 \sum_{n_{j-1}+1}^{n_j} \|f_k\|^2_{\mathbf{L}^2_{\varphi^{-1}}} \le \left(\max_{j=1,\ldots,m} \beta_j^2\right) \|\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\le \left(\sum_{j=1}^m \beta_j^2\right) \|\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N},\end{aligned}$$

as well as Lemma 4.1 and Lemma 4.3,(a), we get

$$\begin{aligned}&\|(\mathcal{G}_n - \mathcal{G})(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta})\|^2_{\mathbf{Y}} = \|(\mathcal{K}_n - \mathcal{K}_0)\mathbf{f} - \boldsymbol{\beta}\left(\mathcal{L}_n^1\boldsymbol{\Psi}_1 - \boldsymbol{\Psi}_1\right)\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\qquad\qquad + \sum_{j=1}^m \left(\sum_{k=n_{j-1}+1}^{n_j} \left\langle f_k, \mathcal{L}_n^2\psi'_{1k} - \psi'_{1k}\right\rangle\right)^2 \\ &\le 2\,\|(\mathcal{K}_n - \mathcal{K}_0)\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + 2\,\|\boldsymbol{\beta}\left(\mathcal{L}_n^1\boldsymbol{\Psi}_1 - \boldsymbol{\Psi}_1\right)\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\qquad\qquad + \sum_{j=1}^m \sum_{k=n_{j-1}+1}^{n_j} \|f_k\|^2_{\mathbf{L}^2_{\varphi^{-1}}} \left\|\mathcal{L}_n^2\psi'_{1k} - \psi'_{1k}\right\|^2_{\mathbf{L}^2_\varphi} \\ &\le 2\,\|(\mathcal{K}_n - \mathcal{K}_0)\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \to \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \|\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\qquad\qquad + 2\left(\textstyle\sum_{j=1}^m \beta_j^2\right) \|\mathcal{L}_n^1\boldsymbol{\Psi}_1 - \boldsymbol{\Psi}_1\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\qquad\qquad + \left(\max_{k=1,\ldots,N} \|\mathcal{L}_n^2\psi'_{1k} - \psi'_{1k}\|^2_{\mathbf{L}^2_\varphi}\right) \|\mathbf{f}\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\le \alpha_n^2\,\|(\mathbf{f},\boldsymbol{\beta},\boldsymbol{\delta})\|^2_{\mathbf{X}},\end{aligned}$$

with

$$\alpha_n = \Bigg(2\,\|(\mathcal{K}_n - \mathcal{K}_0)\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \to \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + \left\|\mathcal{L}^1_n \mathbf{\Psi}_1 - \mathbf{\Psi}_1\right\|^2_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N}$$

$$+ \max_{k=1,\ldots,N} \left\|\mathcal{L}^2_n \psi'_{1k} - \psi'_{1k}\right\|^2_{\mathbf{L}^2_{\varphi}} \Bigg)^{\frac{1}{2}}$$

tending to zero if n tends to infinity, i.e.,

$$\|\mathcal{G}_n - \mathcal{G}\|_{\mathbf{X}\to\mathbf{Y}} \longrightarrow 0\,. \tag{37}$$

This implies that, for all sufficiently large n, the inverses $\mathcal{G}_n^{-1} : \mathbf{Y} \longrightarrow \mathbf{X}$ exist and are uniformly bounded in norm; moreover,

$$\begin{aligned} \|\mathcal{G}_n^{-1} - \mathcal{G}^{-1}\|_{\mathbf{Y}\to\mathbf{X}} &= \|\mathcal{G}_n^{-1}(\mathcal{G} - \mathcal{G}_n)\mathcal{G}^{-1}\|_{\mathbf{Y}\to\mathbf{X}} \\ \le \|\mathcal{G}_n^{-1}\|_{\mathbf{Y}\to\mathbf{X}}\,\|\mathcal{G} - \mathcal{G}_n\|_{\mathbf{X}\to\mathbf{Y}}\,\|\mathcal{G}^{-1}\|_{\mathbf{Y}\to\mathbf{X}} &\le c\,\alpha_n \longrightarrow 0\,. \end{aligned} \tag{38}$$

Note that this proves that the left hand side of (36) converges to zero, since it is equal to $\|(\mathcal{G}_n^{-1} - \mathcal{G}^{-1})(\Theta, \boldsymbol{\gamma})\|_{\mathbf{X}}$.

To derive the error estimate (36), first we recall that $\psi_{ik} \in \mathbf{C}^r[-1,1]$, $i = 1,2$, $k = 1,\ldots,N$, for some $r > 2$ implies, due to Lemma 3.3, the continuity of the partial derivatives $\dfrac{\partial^\ell K_{kj}(s,t)}{\partial t^\ell}$ and $\dfrac{\partial^\ell K_{kj}(s,t)}{\partial s^\ell}$, $\ell = 1,\ldots,r-2$, $j,k = 1,\ldots,N$, for $(s,t) \in [-1,1]^2$. Consequently, each entry of

$$\mathcal{S}\mathbf{f}^* = -\mathcal{K}_0\mathbf{f}^* - \boldsymbol{\beta}^*\Psi_1 - \boldsymbol{\delta}^*$$

belongs to $\in \mathbf{C}^{r-2}[-1,1] \subset \mathbf{L}^{2,r-2}_{\varphi^{-1}}$, i.e., $\mathbf{f}^* \in \left(\varphi\mathbf{L}^{2,r-2}_{\varphi}\right)^N$ in virtue of Lemma 3.9. Taking into account the uniform boundedness of $\mathcal{L}^1_n : \mathbf{C}^N \longrightarrow \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N$ (see Lemma 4.1) and Lemma 4.2, we get, for all $\mathbf{f}^n \in R(\mathcal{P}_n)$ (cf. (35)),

$$\begin{aligned} &\|(\mathcal{K}_n - \mathcal{K}_0)\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\quad\le \|\mathcal{L}^1_n(\mathcal{K}^0_n - \mathcal{K}_0)\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + \|(\mathcal{L}^1_n\mathcal{K}_0 - \mathcal{K}_0)\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\quad\le c\,\|(\mathcal{K}^0_n - \mathcal{K}_0)\mathbf{f}^n\|_{\infty,N} + \|(\mathcal{L}^1_n - \mathcal{I})\mathcal{K}_0\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} \\ &\quad\le c\,n^{2-r}\left(\|\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N} + \|\mathcal{K}_0\mathbf{f}^n\|_{\left(\mathbf{L}^{2,r}_{\varphi^{-1}}\right)^N}\right) \le cn^{2-r}\,\|\mathbf{f}^n\|_{\left(\mathbf{L}^2_{\varphi^{-1}}\right)^N}, \end{aligned}$$

where we have also used the property that $\mathcal{K}_0 : \left(\mathbf{L}^2_{\varphi^{-1}}\right)^N \longrightarrow \left(\mathbf{C}^{r-2}[-1,1]\right)^N \subset \mathbf{L}^{2,r-2}_{\varphi^{-1}}$ is bounded (cf. [1, Lemma 4.2]). Since $\psi_{1k} \in \mathbf{L}^{2,r}_{\varphi^{-1}}$ and $\psi'_{1k} \in \mathbf{L}^{2,r-1}_{\varphi}$, $k = 1,\ldots,N$, we obtain (cf. Lemma 4.2) $\alpha_n = \mathcal{O}(n^{2-r})$. Bound (36) then easily follows. □

Remark 4.5. *We recall that (see [1], Theorem 2.13) when* $r > 3$ *property* $\mathbf{f}^* \in \left(\varphi \mathbf{L}_{\varphi}^{2,r-2}\right)^N$ *implies* $\mathbf{f}^* \in \left(C^{r-3}[-1+\epsilon, 1-\epsilon]\right)^N$ *with* $0 < \epsilon < 1$ *fixed as small as one likes. Furthermore, we note that if the assumption* $\psi_{ik} \in \mathbf{C}^r[-1,1], r > 2$, *is replaced by* $\psi_{ik} \in \mathbf{C}^2[-1,1]$, *and* $\dim N(\mathcal{A}_0) = 0$ *(cf. Proposition 3.10), from the proof of Proposition 4.4 it is seen that the first assertion and relation* (37) *remain true. Thus, also in this case the left hand side of* (36) *converges to zero, as* $n \to \infty$.

5 NUMERICAL RESULTS

Before applying the proposed numerical method, we rewrite the final algebraic linear system defined by equations (29) and (30) in a more explicit form (see (40)). For this, note that, for $\mathbf{f}^n = \left[\ f_k^n\ \right]_{k=1}^N \in R(\mathcal{P}_n)$,

$$\left(\mathcal{S}\mathbf{f}^n\right)(t_{\ell n}) = \left[\ \sum_{i=1}^n \frac{\varphi(s_{in})}{n+1} \frac{f_j^n(s_{in})}{s_{in} - t_{\ell n}}\ \right]_{j=1}^N$$

(see [11, Section 5]), such that

$$-\left(\mathcal{S}\mathbf{f}^n\right)(t_{jn}) + \left(\mathcal{K}_n^0\mathbf{f}^n\right)(t_{\ell n}) = \left[\ \frac{1}{n+1} \sum_{k=1}^N \sum_{i=1}^n \varphi(s_{in}) Y_{0jk}(s_{in}, t_{\ell n}) f_k^n(t_{\ell n})\ \right]_{j=1}^N .$$

Hence, we have to solve the following system of equations

$$\tfrac{1}{n+1} \textstyle\sum_{k=1}^N \sum_{i=1}^n \varphi(s_{in}) Y_{0kj}(s_{in}, t_{\ell n}) f_k^n(s_{in}) - \beta_{g(j)} \psi_{1j}(t_{\ell n}) - \delta_j^n = 0\,,$$

$$j = 1, \dots, N\,,\ \ell = 1, \dots, n+1\,,$$

$$\tfrac{\pi}{n+1} \textstyle\sum_{k=n_{j-1}+1}^{n_j} \sum_{i=1}^n \varphi(s_{in}) \psi'_{1k}(t_{\ell n}) f_k^n(s_{in}) = \gamma_j\,, \quad j = 1, \dots, m\,,$$

where

$$g(j) = r \ \text{ iff } \ n_{r-1} < j \le n_r. \tag{39}$$

Consequently, we can write (29),(30) as

$$\mathbb{A}_n \boldsymbol{\xi}^n = \boldsymbol{\eta}^n, \tag{40}$$

where $\boldsymbol{\eta}^n = \left[\ \eta_{jn}\ \right]_{j=1}^{N(n+1)+m} = \left[\ 0 \ \dots \ 0 \ \ \gamma_1 \ \dots \ \gamma_m\ \right]^T \in \mathbb{R}^{N(n+1)+m}$ is given and, having set

$$\mathbf{f}_k^n = \left[\ f_k^n(s_{1n}) \ \dots \ f_k^n(s_{nn})\ \right], \quad k = 1, \dots, N,$$

$$\begin{aligned} \boldsymbol{\xi}_n &= \left[\ \xi_{kn}\ \right]_{k=1}^{N(n+1)+m} \\ &= \left[\ \mathbf{f}_1^n \ \dots \ \mathbf{f}_N^n \ \ \beta_1^n \ \dots \ \beta_m^n \ \ \delta_1^n \ \dots \ \delta_N^n\ \right]^T \in \mathbb{R}^{N(n+1)+m} \end{aligned}$$

is the vector we are looking for, i.e., $\xi_{(k-1)n+i} = f_k^n(s_{in})$, $k = 1,\dots,N$, $i = 1,\dots,n$, and $\xi_{Nn+j} = \beta_j^n$, $j = 1,\dots,m$, as well as $\xi_{Nn+m+k} = \delta_k^n$, $k = 1,\dots,N$. The matrix $\mathbb{A}_n = \left[\, a_{jk} \,\right]_{j,k=1}^{N(n+1)+m}$ is defined by

$$a_{(\ell-1)N+j,(k-1)n+i} = \frac{\varphi(s_{in})Y_{0kj}(s_{in},t_{\ell n})}{n+1}\,, \quad j,k = 1,\dots,N,\ i = 1,\dots,n,\ \ell = 1,\dots,n+1\,,$$

$$a_{(\ell-1)N+j,Nn+r} = \left\{ \begin{array}{ccc} -\psi_{1j}(t_{\ell n}) & : & n_{r-1} < j \le n_r\,, \\ 0 & : & \text{otherwise}\,, \end{array} \right\} \quad j,k = 1,\dots,N,\ \ell = 1,\dots,n+1\,,$$

$$a_{(\ell-1)N+j,Nn+m+k} = \left\{ \begin{array}{ccc} 0 & : & k \ne j\,, \\ -1 & : & k = j\,, \end{array} \right\} \quad j,k = 1,\dots,N,\ \ell = 1,\dots,n+1\,,$$

$$a_{N(n+1)+j,(k-1)n+i} = \left\{ \begin{array}{ccc} \dfrac{\pi}{n+1}\varphi(s_{in})\psi_{1k}'(s_{in}) & : & n_{j-1} < k \le n_j\,, \\ 0 & : & \text{otherwise}\,, \end{array} \right\}$$
$$k = 1,\dots,N,\ i = 1,\dots,n,\ j = 1,\dots,m\,,$$

$$a_{N(n+1)+j,Nn+k} = 0\,, \quad k = 1,\dots,m+N\,,\ j = 1,\dots,m\,.$$

Hence, the matrix $\mathbb{A}_n$ has the following block structure

$$\mathbb{A}_n = \left[\begin{array}{cccccc} \mathbb{A}_{11} & \mathbb{A}_{12} & \cdots & \mathbb{A}_{1N} & \mathbb{B}_1 & \mathbb{D}_1 \\ \mathbb{A}_{21} & \mathbb{A}_{22} & \cdots & \mathbb{A}_{2N} & \mathbb{B}_2 & \mathbb{D}_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbb{A}_{N1} & \mathbb{A}_{N2} & \cdots & \mathbb{A}_{NN} & \mathbb{B}_N & \mathbb{D}_N \\ \mathbb{C}_1 & \mathbb{C}_2 & \cdots & \mathbb{C}_N & \Theta & \Theta \end{array} \right] \tag{41}$$

with, using the Kronecker-delta $\delta_{j,k}$,

$$\mathbb{A}_{jk} = \left[\, a_{\ell i}^{(j,k)} \,\right]_{\ell=1,i=1}^{n+1,\,n} = \left[\, \frac{\varphi(s_{in})Y_{0kj}(s_{in},t_{\ell n})}{n+1} \,\right]_{\ell=1,i=1}^{n+1,\,n}\,,$$

$$\mathbb{B}_j = \left[\, b_{\ell s}^{(j)} \,\right]_{\ell=1,s=1}^{n+1,\,m} = \left[\, -\psi_{1j}(t_{\ell n})\delta_{g(j),s} \,\right]_{\ell=1,s=1}^{n+1,\,m}\,,$$

$$\mathbb{C}_k = \left[\, c_{ji}^{(k)} \,\right]_{j=1,i=1}^{m,\ n} = \left[\, \frac{\pi}{n+1}\,\varphi(s_{in})\psi_{1k}'(s_{in})\delta_{g(k),j} \,\right]_{j=1,i=1}^{m,\ n}\,,$$

$$\mathbb{D}_j = \left[\, d_{\ell k}^{(j)} \,\right]_{\ell=1,k=1}^{n+1,\,N} = \left[\, -\delta_{j,k} \,\right]_{\ell=1,k=1}^{n+1,\,N}\,,$$

where $g(j)$ is defined in (39). By $\mathbf{X}_n$ and $\mathbf{Y}_n$ we denote the subspaces of $\mathbf{X}$ and $\mathbf{Y}$, respectively, defined by $\mathbf{X}_n = (\varphi \mathbf{P}_{n-1})^N \times \mathbb{R}^m \times \mathbb{R}^N$ and $\mathbf{Y}_n = \mathbf{P}_n^N \times \mathbb{R}^m$. Set $\omega_n = \sqrt{\frac{\pi}{n+1}}$ and define the operators $\mathcal{E}_n : \mathbf{X}_n \longrightarrow \mathbb{R}^{(n+1)N+m}$ and $\mathcal{F}_n : \mathbf{Y}_n \longrightarrow \mathbb{R}^{(n+1)N+m}$ as the maps

$$\mathcal{E}_n : (\mathbf{f}^n, \boldsymbol{\beta}^n, \boldsymbol{\delta}^n) \mapsto \left[\begin{array}{cccccccccc} \omega_n \mathbf{f}_1^n & \cdots & \omega_n \mathbf{f}_N^n & \beta_1^n & \cdots & \beta_m^n & \delta_1^n & \cdots & \delta_N^n \end{array} \right]^T$$

and

$$\mathcal{F}_n : (\mathbf{p}^n, \boldsymbol{\gamma}^n) \mapsto \left[\begin{array}{cccccc} \omega_n \mathbf{p}_1^n & \cdots & \omega_n \mathbf{p}_N^n & \gamma_1^n & \cdots & \gamma_m^n \end{array} \right]^T ,$$

where, for $\mathbf{p}^n = \left[\, p_k^n \, \right]_{k=1}^N$, we have set $\mathbf{p}_k^n = \left[\begin{array}{ccc} p_k^n(t_{1n}) & \cdots & p_k^n(t_{n+1,n}) \end{array} \right]$ and where the space $\mathbb{R}^{(n+1)N+m}$ is equipped with the usual Euclidean inner product. The operators $\mathcal{E}_n$ and $\mathcal{F}_n$ are unitary operators, since, for $(\mathbf{f}^n, \boldsymbol{\beta}^n, \boldsymbol{\delta}^n) \in \mathbf{X}_n$ and $\boldsymbol{\xi} = \left[\, \xi_i \, \right]_{i=1}^{(n+1)N+m}$, we have

$$\begin{aligned}
&\langle \mathcal{E}_n(\mathbf{f}^n, \boldsymbol{\beta}^n, \boldsymbol{\delta}^n), \boldsymbol{\xi} \rangle_{\mathbb{R}^{(n+1)N+m}} \\
&= \omega_n \sum_{k=1}^N \sum_{i=1}^n f_k^n(s_{in}) \xi_{(k-1)n+i} + \sum_{j=1}^m \eta_j^n \xi_{nN+j} + \sum_{k=1}^N \delta_k^n \xi_{nN+m+k} \\
&= \sum_{k=1}^N \int_{-1}^1 f_k^n(s) \omega_n^{-1} \sum_{i=1}^n \xi_{(k-1)n+i} \widetilde{\ell}_{in}^{\varphi}(s) \mathrm{d}s + \sum_{j=1}^m \eta_j^n \xi_{nN+j} + \sum_{k=1}^N \delta_k^n \xi_{nN+m+k} \\
&= \langle (\mathbf{f}^n, \boldsymbol{\beta}^n, \boldsymbol{\delta}^n), \mathcal{E}_n^{-1} \boldsymbol{\xi} \rangle_{\mathbf{X}_n}
\end{aligned}$$

and analogously, for $(\mathbf{p}^n, \boldsymbol{\gamma}^n) \in \mathbf{Y}_n$ and $\boldsymbol{\eta} \in \mathbb{R}^{(n+1)N+m}$,

$$\langle \mathcal{F}_n(\mathbf{p}^n, \boldsymbol{\gamma}^n), \beta \rangle_{\mathbb{R}^{(n+1)+m}} = \left\langle (\mathbf{p}^n, \boldsymbol{\gamma}^n), \mathcal{F}_n^{-1} \boldsymbol{\eta} \right\rangle_{\mathbf{Y}_n} .$$

Hence, the spectral condition numbers of the matrices $\widetilde{\mathbb{A}}_n = \left[\, b_{jk} \, \right]_{j,k=1}^{N(n+1)+m}$ defined by $\widetilde{\mathbb{A}}_n \boldsymbol{\xi} = \mathcal{F}_n \mathcal{G}_n \mathcal{E}_n^{-1} \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in \mathbb{R}^{(n+1)N+m}$ are equal to $\|\mathcal{G}_n\| \, \|\mathcal{G}_n^{-1}\|$. Moreover, under the assumptions of Proposition 4.4 we have (cf. (37) and (38))

$$\lim_{n\to\infty} \mathrm{cond}(\widetilde{\mathbb{A}}_n) = \|\mathcal{G}\|_{\mathbf{X}\to\mathbf{Y}} \left\| \mathcal{G}^{-1} \right\|_{\mathbf{Y}\to\mathbf{X}} .$$

Since, by setting $\mathbf{f} = \left[\, f_k \, \right]_{k=1}^N := \left[\sum_{i=1}^n \xi_{(k-1)n+i} \widetilde{\ell}_{in}^{\varphi} \right]_{k=1}^N$, i.e., $f_k(s_{in}) = \xi_{(k-1)n+i}$, we have

$$\begin{aligned}
&\mathcal{F}_n \mathcal{G}_n \mathcal{E}_n^{-1} \boldsymbol{\xi} \\
&= \mathcal{F}_n \mathcal{G}_n \left(\left[\omega_n^{-1} \sum_{i=1}^n \xi_{(k-1)n+i} \widetilde{\ell}_{in}^{\varphi} \right]_{k=1}^N, \xi_{nN+1}, \ldots, \xi_{(n+1)N+m} \right) \\
&= \mathcal{F}_n \Bigg(\omega_n^{-1} \mathcal{A}_n \mathbf{f} - \left[\, \xi_{nN+j} \, \right]_{j=1}^m \mathcal{L}_n^1 \boldsymbol{\Psi}_1 - \left[\, \xi_{nN+m+k} \, \right]_{k=1}^N , \\
&\qquad\qquad \left[\sum_{k=n_{j-1}+1}^{n_j} \left\langle f_k, \mathcal{L}_n^2 \psi_{1k}' \right\rangle \right]_{j=1}^m \Bigg)
\end{aligned}$$

$$= \Bigg(\Big[\Big[\frac{1}{n+1} \sum_{k=1}^{N} \varphi(s_{in}) Y_{0kj}(s_{in}, t_{\ell n}) f_k(s_{in}) - \omega_n \psi_{1j}(t_{\ell n}) \xi_{nN+g(j)}$$

$$-\omega_n \xi_{nN+m+j} \Big]_{\ell=1}^{n} \Big]_{j=1}^{N}, \Big[\sum_{k=n_{j-1}+1}^{n_j} \omega_n \sum_{i=1}^{n} \varphi(s_{in}) \psi'_{1k}(s_{in}) \xi_{(k-1)n+i} \Big]_{j=1}^{m} \Bigg)$$

$$= \Bigg(\Big[\Big[\sum_{k=1}^{N} a_{(\ell-1)N+j,(k-1)n+i} \xi_{(k-1)n+i} + \omega_n a_{(\ell-1)N+j,Nn+g(j)} \xi_{Nn+g(j)}$$

$$+\omega_n a_{(\ell-1)N+j,Nn+m+j} \xi_{Nn+m+j} \Big]_{\ell=1}^{n} \Big]_{j=1}^{N},$$

$$\Big[\sum_{k=1}^{N} \sum_{i=1}^{n} \omega_n^{-1} a_{N(n+1)+j,(k-1)n+i} \xi_{(k-1)n+i} \Big]_{j=1}^{m} \Bigg) \,,$$

for the entries of the matrix $\widetilde{\mathbb{A}}_n = \left[\ \widetilde{a}_{jk} \ \right]_{j,k=1}^{N(n+1)+m}$ we get the formulas (having in mind the block structure (41) of $\mathbb{A}_n$)

$$\widetilde{\mathbb{A}}_{jk} = \mathbb{A}_{jk} \,, \qquad j,k = 1,\dots,N \,,$$

$$\widetilde{\mathbb{B}}_j = \omega_n \mathbb{B}_j \,, \qquad j = 1,\dots,N \,,$$

$$\widetilde{\mathbb{C}}_k = \omega_n^{-1} \mathbb{C}_k \,, \qquad k = 1,\dots,N \,,$$

$$\widetilde{\mathbb{D}}_j = \omega_n \mathbb{D}_j \,, \qquad j = 1,\dots,N \,.$$

This is equivalent to

$$\widetilde{\mathbb{A}}_n = \mathbb{F}_n \mathbb{A}_n \mathbb{E}_n^{-1} \tag{42}$$

with the diagonal matrices

$$\mathbb{E}_n = \mathrm{diag}\Big[\underbrace{1 \ \cdots \ 1}_{Nn} \ \underbrace{\omega_n^{-1} \ \cdots \ \omega_n^{-1}}_{m+N} \Big] \quad \text{and} \quad \mathbb{F}_n = \mathrm{diag}\Big[\underbrace{1 \ \cdots \ 1}_{N(n+1)} \ \underbrace{\omega_n^{-1} \ \cdots \ \omega_n^{-1}}_{m} \Big] \,.$$

Thus, we solve the system $\widetilde{\mathbb{A}}_n \widetilde{\boldsymbol{\xi}}^n = \widetilde{\boldsymbol{\eta}}^n$ instead of (40), where $\widetilde{\boldsymbol{\eta}}^n = \mathbb{F}_n \boldsymbol{\eta}^n$ and $\widetilde{\boldsymbol{\xi}}^n = \mathbb{E}_n \boldsymbol{\xi}^n$. A simple algorithm for the construction of the matrix $\widetilde{\mathbb{A}}_n$ is given here:

$\nu = 0$

for $\ell = 1 : n+1$

for $j = 1 : N$

$\widetilde{a}_{\nu+j,(k-1)n+i} = \frac{\varphi(s_{in}) Y_{0kj}(s_{in}, t_{\ell n})}{n+1}, \quad i = 1 : n;\ k = 1 : N$

for $k = 1 : m$

if $n_{k-1} < j \leq n_k$ **then** $\widetilde{a}_{\nu+j,Nn+k} = -\omega_n \psi_{1j}(t_{\ell n})$

else $\widetilde{a}_{\nu+j,Nn+k} = 0$

end

for $k = 1 : N$

if $j = k$ **then** $\widetilde{a}_{\nu+j,Nn+m+k} = -\omega_n$

else $\widetilde{a}_{\nu+j,Nn+m+k} = 0$

end

end

$\nu = \nu + N$

end

for $j = 1, \ldots, m$

$$\widetilde{a}_{N(n+1)+j,(k-1)n+i} = \left\{ \begin{array}{ccc} \dfrac{\pi \varphi(s_{in}) \psi'_{1k}(s_{in})}{\omega_n (n+1)} & : & n_{j-1} < k \leq n_j\,, \\ 0 & : & \text{otherwise}\,, \end{array} \right\} \quad i = 1 : n,\ k = 1 : N\,,$$

$\widetilde{a}_{N(n+1)+j,Nn+i} = 0\,, \quad i = 1 : m + N$

end

Note that, assuming that the above system has a unique solution, in the case of symmetric wings all unknown functions $f_k^n(s)$ are also symmetric. Thus, we can take advantage of this property to halve the size of the system.

The numerical method we have examined in this paper has been extensively applied to several multiwing (symmetric) configurations, including the truss-braced wing one; see [6, 7, 4, 5, 12]. Thus, to verify the error estimates we have derived in the previous section, here we consider only symmetric and non symmetric biwing systems. For simplicity, in both examples the first wing is a symmetric (bounded) interval, while the second wing is symmetric and arbitrarily smooth in Example 1, and non symmetric with C^3 degree of smoothness in Example 2. Since for a biwing we have $N = 2$ and $1 \le m \le N$, we will consider first the case $m = 1$ and then $m = 2$. We recall that only in the first case the error estimate proof (see Proposition 4.4) has been completed. When $m = 1$ we take $\gamma_1 = 1$, while when $m = 2$ we set $\gamma_1 = \gamma_2 = 0.5$ in Example 1 and $\gamma_1 = 0.3, \gamma_2 = 0.7$ in Example 2.

To have a sequence of nested mesh points $\{s_{in}\}$, we have chosen

$$n = 5, 11, 23, 47, 95, 191, 383.$$

This choice allows us to check also the (pointwise) convergence rate at these points.

In the tables below, the error estimates reported in the columns labeled (36) are obtained by approximating the left hand side of bound (36) by the following discretization of it:

$$\text{err} = \sqrt{\text{err}_1^2 + \text{err}_2^2}, \tag{43}$$

where

$$\text{err}_1^2 = \frac{\pi}{M+1} \sum_{k=1}^{N} \sum_{i=1}^{M} |f_k^{n*}(s_{iM}) - f_k^{M*}(s_{iM})|^2$$

and

$$\text{err}_2^2 = \sum_{j=1}^{m} \left|\beta_j^{n*} - \beta_j^{M*}\right|^2 + \sum_{k=1}^{N} \left|\delta_k^{n*} - \delta_k^{M*}\right|^2,$$

with $M >> n$ and $s_{iM} = \cos \frac{i\pi}{M+1}$. The reference values are those obtained by applying the numerical method with $n = M = 383$ or 767 or 1535, as indicated in the table headings.

In the below graphs, the two unknown functions f_1^{M*}, f_2^{M*} are drawn with the blue and red colors, respectively.

Example 1. The parametric representations in $[-1, 1]$ of the two wings are:

$$\psi_{11}(t) = t, \quad \psi_{21}(t) = a, \qquad (wing\ 1)$$

$$\left.\begin{aligned} \psi_{12}(t) = 0.75\cos(\theta_t),\ \psi_{22}(t) = 0.2\sin(\theta_t), \\ \theta_t = \left(3\tfrac{\pi}{8} + 0.01\right) t + 3\tfrac{\pi}{2} \end{aligned}\right\} \qquad (wing\ 2)$$

Figure 1: Example 1, $a = 1, m = 1, \gamma = 1$ (left); $m = 2, \gamma = [0.5, 0.5]$ (right)

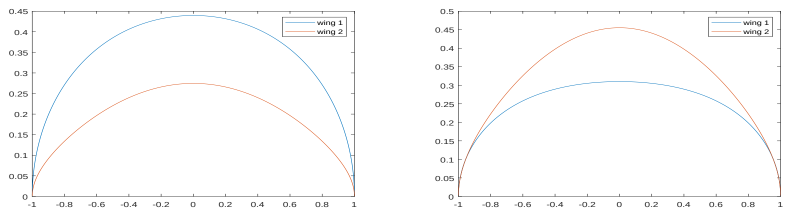
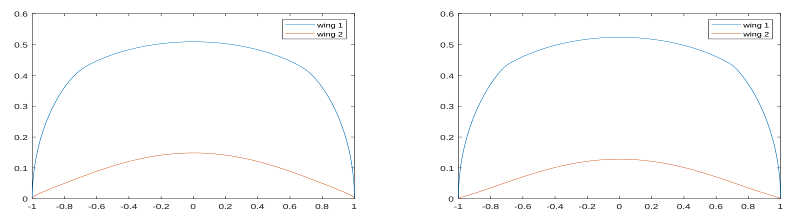


Figure 2: Example 1, $m = 1, \gamma = 1$, $a = 0$ (left); $a = -0.05$ (right)



In the tables below, in each column estimates are stopped before reaching the maximum value of n, whenever the maximum accuracy is achieved.

Table 1: Example 1, $a = 1, M = 383, N = 2, m = 1, \gamma = 1$

n	(36)	β_1^{n*}	δ_1^{n*}	δ_2^{n*}	$\lvert\beta_1^{n*} - \beta_1^*\rvert$	$\Vert\gamma^{n*} - \gamma^*\Vert$	cond($\widetilde{\mathbb{A}}_n$)
5	6.35e-05	4.8878226e-01	2.01e-17	1.37e-16	1.12e-06	1.32e-15	2.55
11	3.51e-08	4.8878338e-01			4.38e-12		2.55
23	9.28e-14	4.8878338e-01			7.22e-16		2.55
47	5.21e-14				1.11e-16		2.55
95							2.55
191							2.55

Without preconditioning, in Table 1 we would have had

$$\text{cond}(\mathbb{A}_n) = [2.9,\ 4.8,\ 9.1,\ 17.8,\ 35.2,\ 70.0].$$

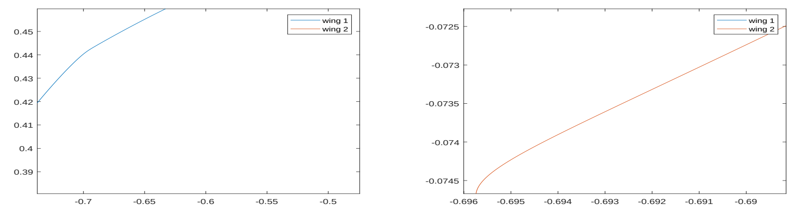
Table 2: Example 1, $a = 1, M = 383, N = 2, m = 2, \gamma_i = 0.5$

n	(36)	β_1^{n*}	β_2^{n*}	δ_1^{n*}	δ_2^{n*}	$\lvert\beta_1^{n*} - \beta_1^*\rvert$	$\Vert\gamma^{n*} - \gamma^*\Vert$
5	6.5e-05	3.9148840e-01	7.1414539e-01	1.3e-17	2.6e-16	1.4e-06	4.2e-06
11	5.8e-08	3.9148699e-01	7.1414975e-01			6.1e-14	1.3e-11
23	1.5e-13	3.9148699e-01	7.1414975e-01			2.8e-16	5.6e-16
47	7.2e-14						

Table 2': Example 1, $a = 1, M = 383, N = 2, m = 2, \gamma_i = 0.5$

n	5	11	23	47
cond($\widetilde{\mathbb{A}}_n$)	3.67	3.67	3.67	3.67
cond($\mathbb{A}_n$)	5.0	8.6	15.8	30.1

Figure 3: Example 1, $a = -0.07, m = 1, \gamma = 1$; f_1^{M*} (left) and f_2^{M*} (right)



To see the (convergence) behavior of the problem solution as wing 1 moves, by vertical translation, towards wing 2, we have considered the case $m = 1$ and let the parameter a in $\psi_{21}(t)$ of wing 1 moving towards $\psi_{22}(\pm 1) = -0.074685$. Figures 1,2 and Tables 1,3,4 show what happens when $a = 1, 0, -0.05$; since the graph for $a = -0.07$ coincides with that of $a = -0.05$, we have not reported it. However, in these latter two cases, to show the behavior of f_1^{M*} near the quasi-singular points $\psi_{12}(\pm 1) = \pm 0.695745$, and of f_2^{M*} near its endpoints, in Figure 3 we have plotted a zoom of these two behaviors.

Table 3: Example 1, $a = 0, M = 767, N = 2, m = 1, \gamma = 1$

n	(36)	β_1^{n*}	δ_1^{n*}	δ_2^{n*}	$\lvert\beta_1^{n*} - \beta_1^*\rvert$	$\lVert\gamma^{n*} - \gamma^*\rVert$	cond$(\widetilde{\mathbb{A}}_n)$
5	4.6e-01	6.5491499e-01	-8.6e-18	-2.3e-16	3.2e-02	1.3e-15	14.2
11	7.3e-02	6.1982953e-01			2.7e-03		10.2
23	5.6e-03	6.2242333e-01			1.6e-04		10.4
47	2.1e-05	6.2257853e-01			3.3e-07		10.4
95	1.2e-08	6.2257885e-01			2.6e-11		10.4
191	2.1e-13	6.2257885e-01			1.4e-15		10.4

Table 4: Example 1, $a = -0.05, M = 1535, N = 2, m = 1, \gamma = 1$

n	(36)	β_1^{n*}	δ_1^{n*}	δ_2^{n*}	$\lvert\beta_1^{n*} - \beta_1^*\rvert$	$\lVert\gamma^{n*} - \gamma^*\rVert$	cond$(\widetilde{\mathbb{A}}_n)$
5	2.9e-00	7.9746486e-01	-9.1e-17	-1.2e-15	1.7e-01	2.3 e-15	62.5
11	4.7e-01	6.1610383e-01			1.3e-02		16.6
23	2.1e-01	6.2667585e-01			2.5e-03		18.5
47	2.8e-02	6.2901791e-01			2.0e-04		19.1
95	4.7e-04	6.2922391e-01			4.1e-06		18.9
191	5.1e-07	6.2921982e-01			3.7e-09		18.9
383	3.7e-11	6.2921982e-01			4.2e-15		18.9

Without preconditioning, in the the last two tables we would have had, respectively,

$$\text{cond}(\mathbb{A}_n) = [15.4,\ 13.5,\ 19.4, 28.1,\ 44.3,\ 80.0]$$

and

$$\text{cond}(\mathbb{A}_\text{n}) = [67.0,\ 22.1,\ 33.0,\ 48.0,\ 68.6,\ 102.7,\ 171.8].$$

Example 2. The parametric representations of the two wings are:

$$\psi_{11}(t) = t, \quad \psi_{21}(t) = 1, \quad -1 \le t \le 1 \qquad \text{(wing 1)}$$

and (wing 2)

$$\psi_{12}(t) = t, \qquad -1 \le t \le 1,$$
$$\psi_{22}(t) = \begin{cases} \dfrac{t^4}{4}, & -1 \le t \le 0 \\ \dfrac{t^4}{2}, & 0 < t \le 1. \end{cases}$$

Figure 4: Example 2, $a = 1$; $m = 1, \gamma = 1$ (left); $m = 2, \gamma = [0.3, 0.7]$ (right)

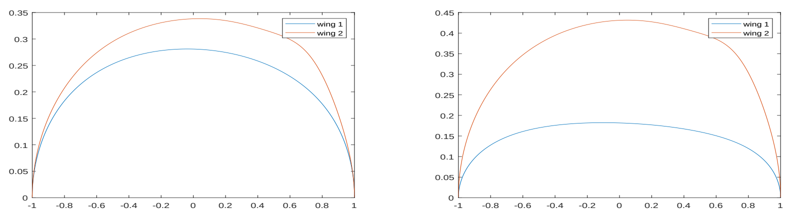


Table 5: Example 2, $a = 1$; $M = 1535, N = 2, m = 1, \gamma = 1$

n	(36)	β_1^{n*}	δ_1^{n*}	δ_2^{n*}	$\lvert\beta_1^{n*} - \beta_1^*\rvert$	$\lVert\gamma^{n*} - \gamma^*\rVert$	cond($\widetilde{\mathbb{A}}_n$)
5	8.2e-03	3.9086573e-01	4.9915303e-03	-1.0269478e-02	2.8e-04	5.1e-05	4.07
11	4.5e-04	3.9058419e-01	4.9579663e-03	-1.0310741e-02	6.5e-07	2.9e-06	4.07
23	2.2e-06	3.9058482e-01	4.9606810e-03	-1.0310595e-02	1.7e-08	1.9e-07	4.07
47	1.4e-08	3.9058487e-01	4.9608572e-03	-1.0310553e-02	9.2e-10	1.2e-08	4.07
95	8.8e-10	3.9058484e-01	4.9608681e-03	-1.0310550e-02	5.5e-11	7.5e-10	4.07
191	5.5e-11	3.9058484e-01	4.9608688e-03	-1.0310550e-02	3.4e-12	4.7e-11	4.07
383	3.4e-12		4.9608689e-03		2.1e-13	2.9e-12	4.07

In Table 5, without preconditioning we would have had

$$\text{cond}(\mathbb{A}_n) = [4.6,\ 6.0,\ 8.6,\ 15.9,\ 31.3,\ 62.0,\ 123.4].$$

Table 6: Example 2, $a = 1, M = 1535, N = 2, m = 2, \gamma_1 = 0.3, \gamma_2 = 0.7$

n	(36)	β_1^{n*}	β_2^{n*}	δ_1^{n*}	δ_2^{n*}	$\lVert\beta^{n*} - \beta^*\rVert$	$\lVert\gamma^{n*} - \gamma^*\rVert$
5	1.0e-02	3.2236367e-01	4.4587044e-01	6.6386284e-03	-1.6329797e-02	1.4e-04	5.3e-04
11	5.7e-04	3.2222830e-01	4.4534082e-01	6.5899128e-03	-1.6369827e-02	3.8e-07	4.0e-06
23	2.8e-06	3.2222863e-01	4.4534207e-01	6.5934033e-03	-1.6369647e-02	1.8e-09	2.4e-07
47	1.8e-08	3.2222864e-01	4.4534210e-01	6.5936284e-03	-1.6369596e-02	6.9e-11	1.5e-08
95	1.1e-09	3.2222864e-01	4.4534210e-01	6.5936423e-03	-1.6369593e-02	7.1e-12	9.4e-10
191	6.8e-11			6.5936432e-03	-1.6369592e-02	4.9e-13	5.8e-11
383	4.2e-12			6.5936432e-03	-1.6369592e-02	3.1e-14	3.6e-12

Table 6': Example 2, $a = 1, M = 1535, N = 2, m = 2, \gamma_1 = 0.3, \gamma_2 = 0.7$

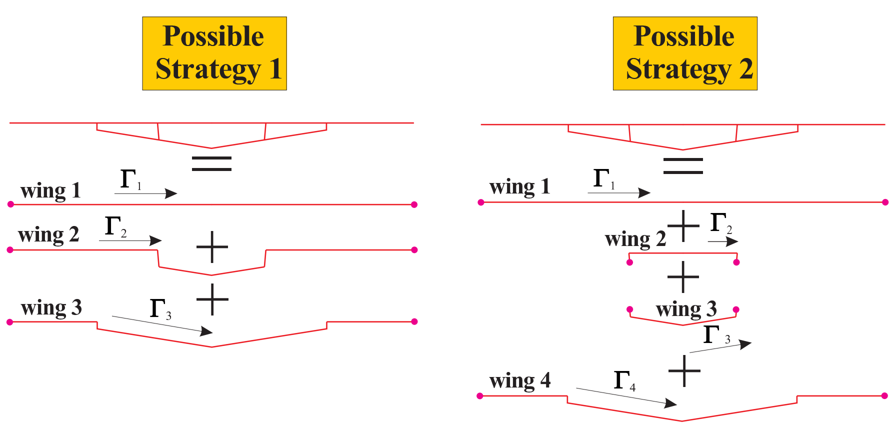
n	5	11	23	47	95	191	383
cond($\widetilde{\mathbb{A}}_n$)	2.60	2.60	2.60	2.60	2.60	2.60	2.60
cond($\mathbb{A}_n$)	3.4	6.1	11.5	22.3	43.9	87.2	173.7

Remark 5.1. *In the numerical examples reported above, where, for simplicity, only the biwing case has been considered, preconditioning does not have a significant effect on the solution accuracy. We note however that in the last table, for the reference solution we have cond($\mathbb{A}_n$)= 693.5. Furthermore, when we take the wings very close to each other, to simulate a TBW configuration, the required value of n can be significantly higher. In such situation, also the number of wing elements N is generally higher than 2. Thus, in that case preconditioning could be mandatory.*

6 A NEW APPLICATION AND A NEW TOPIC OF INVESTIGATION

Let us consider a symmetric TBW configuration; for example that of Figure 5 in [12] and partially replicated in Figure 5 for convenience. For such type of multiple wings the corresponding constrained minimization problem is not well defined. First of all it is not clear which functional spaces one should choose or define, to look for a problem solution. Therefore, it is not clear how to derive the problem ELE. Based on the physical properties of

Figure 5: Example of possible conceptual strategies adopted to study TBW



the circulation distributions, in [12] the authors have proposed a possible definition of this problem, whose validity seems to be confirmed by the intensive numerical testing they have performed.

To describe this approach, we first recall that the TBW elements never intersect, but are locally smoothly joined. In the case of Figure 5 this is accomplished by using a local Hermite polynomial interpolation of degree 5 (see Figure 4 in [5] or Figure 3 in [12]). Then, the TBW configuration is decomposed into N open curves, each one being at least C^3 continuous. Of course, there are several ways of performing this decomposition. For example, in the case of the two strategies adopted in Figure 5 we have $N = 3, 4$, respectively. In addition, every single curve is conceptually imagined to be made of several basic open and smooth elements. Let M be the total number of elements constituting the entire wing system. Each element is identified by its parametrization interval $\{(\alpha_j, \beta_j), j = 1 : M\}$, these being not necessarily all distinct, and by its parametrization function. We denote the associated (unknown) circulations by $\Gamma_j^M, j = 1 : M$, and the overall circulation array by $\mathbf{\Gamma}^M$. Then we have:

$$D_{\text{ind}}(\mathbf{\Gamma}^M) = \sum_{j=1}^{M} D_{\text{ind}}(\Gamma_j^M, \mathbf{\Gamma}^M) \qquad \text{and} \qquad L(\mathbf{\Gamma}^M) = \sum_{j=1}^{M} L(\Gamma_j^M),$$

where we have set (see (2.3),(2.4))

$$D_{\text{ind}}(\Gamma_j^M, \mathbf{\Gamma}^M) := -\rho_\infty \sum_{j=1}^{M} \int_{\alpha_j}^{\beta_j} v_{nj}^M(\eta_j)\Gamma_j^M(\eta_j)\mathrm{d}\eta_j,$$

$$v_{nj}^M(\eta_j) := \frac{1}{4\pi} \sum_{k=1}^{M} \int_{\alpha_k}^{\beta_k} \Gamma_k'^M(\xi_k)\, Y_{jk}(\eta_j, \xi_k)\, \mathrm{d}\xi_k, \quad \alpha_j < \eta_j < \beta_j.$$

Note that the last integral is defined in the Hadamard finite part sense whenever $\mathbf{r}_j(\alpha_j)$ or $\mathbf{r}_j(\beta_j)$ coincides with $\mathbf{r}_k(\alpha_k)$ or $\mathbf{r}_k(\beta_k)$.

Now some of the above chosen wing elements can be further split into n_e coincident elements (obtaining an element of multiplicity n_e), each one associated with a new (unknown) circulation. This is the case, for example, of all strategies of Figure 5. The sum of the individual circulations gives the circulation of the original element of multiplicity 1. Next we assembly the new basic elements to obtain N smooth curves, each one defining an imaginary open wing and having a corresponding circulation. As said before, there are several possibilities to obtain a TBW decomposition of this type, as shown in the above Figure 5.

As in Section 2, we denote these wings by ℓ_k, and the associated parametrization intervals by $(-a_k, b_k), k = 1 : N$, $a_k, b_k > 0$; we have:

$$D_{\text{ind}}(\mathbf{\Gamma}) = \sum_{k=1}^{N} D_{\text{ind}}(\Gamma_k, \mathbf{\Gamma}) \qquad \text{and} \qquad L(\mathbf{\Gamma}) = \sum_{k=1}^{N} L(\Gamma_k).$$

At this stage it is still impossible to define a proper TBW constrained minimization formulation, that can then be solved. To this end, taking into account the theoretical results we have obtained in Sections 3,4 for multiwing systems, we separate the N wings ℓ_k by translating them away, one from each other, along the z-axis by a quantity $\epsilon > 0$. Having now a multisystem of disjoint wings, the minimization problem is well-posed and we can apply the results given in the above sections. Of course, each TBW decomposition will produce its solution, and in general, different decompositions will give rise to different solutions, hence to different $D_{\text{ind}}^{\text{opt}}$. Let us denote these optimal terms by $\{\Gamma_{k\epsilon}^{\text{opt}}, k = 1 : N\}$, $\boldsymbol{\Gamma}_{\epsilon}^{\text{opt}}$ and $D_{\epsilon}^{\text{opt}}$. The final crucial property one should now prove is the existence of the following limits:

$$\lim_{\epsilon \to 0} \Gamma_{k\epsilon}^{\text{opt}}, k = 1 : N, \quad \lim_{\epsilon \to 0} \boldsymbol{\Gamma}_{\epsilon}^{\text{opt}}, \quad \lim_{\epsilon \to 0} D_{\epsilon}^{\text{opt}}.$$

If this is so, we define these limits to be a solution of our original minimization problem.

The intensive numerical testing performed in [12] seem to confirm that indeed the above limits exist. Most important, although we have a different solution $\{\Gamma_k^{\text{opt}}, k = 1 : N\}$, Γ^{opt} for each TBW decomposition, the values of their derivatives $\Gamma_j'^M, j = 1 : M$, on each TBW element, and that of $D_{\text{ind}}^{\text{opt}}$, are unique. Note that the latter property follows from the first one, since from the expression $D_{\text{ind}}(\boldsymbol{\Gamma})$ given in Section 2, by performing integration by parts we obtain the alternative expression

$$D_{\text{ind}}(\boldsymbol{\Gamma}) = \frac{\rho_\infty}{4\pi} \sum_{j=1}^{N} \int_{-a_j}^{b_j} \left[\sum_{k=1}^{N} \int_{-a_k}^{b_k} \ln |\mathbf{r}_k(\xi_k) - \mathbf{r}_j(\eta_j)| \Gamma_k'(\xi_k) d\xi_k \right] \Gamma_j'(\eta_j) d\eta_j.$$

Although the above derivatives appear to be independent from the chosen TBW decomposition, the overall circulation $\boldsymbol{\Gamma}^{\text{opt}}$ depends on it. This because, being $\Gamma_j'^M$ uniquely defined on each TBW element represented by the interval (α_j, β_j), the function Γ_k^{opt} is uniquely defined up to an arbitrary constant. However, the endpoint vanishing property of each Γ_k^{opt} and the continuity of the latter on ℓ_k uniquely determine the associated constants. Note that the values of the latter depend on the TBW elements we have chosen to define ℓ_k.

A further topic of investigation is the interesting case of a TBW mixed decomposition into open and closed wing elements.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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