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# Leonov's method of nonlocal reduction and its further development

Vera B. Smirnova<sup>1</sup> and Anton V. Proskurnikov<sup>2</sup>

**Abstract**—The method of nonlocal reduction has been proposed by G.A. Leonov in the 1980s for stability analysis of nonlinear feedback systems. The method combines the comparison principle with Lyapunov techniques. A feedback system is investigated via its reduction to a simpler “comparison” system, whose dynamics can be studied efficiently. The trajectories of the comparison system are explicitly used in the design of Lyapunov functions. Leonov's method proves to be an efficient tool for analysis of Lur'e-type systems with periodic nonlinearities and infinite sets of equilibria. In this paper, we further refine the nonlocal reduction method for periodic systems and obtain new sufficient frequency–algebraic conditions ensuring the convergence of every solution to some equilibrium point (gradient-like behavior).

**Index Terms**—Nonlinear system, periodic nonlinearity, Lagrange stability, Lyapunov function

## I. INTRODUCTION

The term “comparison principle” in differential equations theory stands for a broad class of methods, reducing analysis of a general system (which can have a high order, be nonlinear and partially uncertain) to investigation of a simpler system whose properties can be studied efficiently. The first comparison principle established in stability theory was the Lyapunov criterion of local stability, comparing the system to its linearization. Many comparison principles are based on the differential inequalities [1]–[3], which e.g. arise in generalizations of the Lyapunov direct method [4]–[7].

A novel elegant comparison principle has been developed by G.A. Leonov [8]–[13] under the name of *nonlocal reduction* technique (NRT). A low-order comparison system employed in the NRT is supposed to have the same structure of nonlinearities; its trajectories are explicitly used to construct Lyapunov-type functions for the original system. The NRT proves to be particularly efficient for a class of Lur'e systems with periodic nonlinearities and infinite sets of equilibria (such systems are sometimes called pendulum-like or synchronization systems [14], [15]), which involves models of damped pendulums, electric motors, power generators, vibrational units, and various synchronization circuits

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<sup>1</sup> Saint-Petersburg State University of Architecture and Civil Engineering (Department of Mathematics) and Saint-Petersburg State University (Faculty of Mathematics and Mechanics), St.Petersburg, Russia; smirnova.Vera\_b@mail.ru

<sup>2</sup> Department of Electronics and Telecommunications (DET), Politecnico di Torino, Turin, Italy and the Institute for Problems of Mechanical Engineering of the Russian Academy of Sciences (IPME RAS), St.Petersburg, Russia anton.p.1982@ieee.org

such as phase and frequency locked loops (PLL/FLL) [16]–[20].

In this paper we further extend and refine the NRT to study stability of periodic systems. For this purpose, we combine the NRT with the method of periodic Lyapunov functions [21] and the Kalman–Yakubovich–Popov (KYP) lemma [11]. The stability theorems have the form of “frequency–algebraic” criteria. The efficiency of new stability criteria is illustrated by the example of a PLL with a proportional integrating low-pass filter.

## II. NONLOCAL REDUCTION TECHNIQUE FOR SYNCHRONIZATION SYSTEMS

In this section we show how by means of NRT together with the KYP–lemma it is possible to generate frequency–domain criteria of Lagrange stability (every solution is bounded) and of gradient–like behavior (every solution converges).

Consider the system

$$\begin{aligned} \frac{dz(t)}{dt} &= Az(t) + b\varphi(\sigma(t)), \\ \frac{d\sigma(t)}{dt} &= c^*z(t) + \rho\varphi(\sigma(t)), \end{aligned} \quad (1)$$

where  $A \in \mathbb{R}^{m \times m}$ ,  $b, c \in \mathbb{R}^m$ ,  $\rho \in \mathbb{R}$ ,  $z : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ ,  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , the symbol (\*) means the Hermitian conjugation.

We suppose the following assumptions are fulfilled.

**Assumption 1.** The pair  $(A, b)$  is controllable, the pair  $(A, c)$  is observable. Matrix  $A$  is Hurwitz, with

$$z_0 \triangleq \min |Re z_i| \quad (i = 1, \dots, m), \quad (2)$$

where  $z_i$  is an eigenvalue of  $A$ .

**Assumption 2.** 1) The function  $\varphi$  is  $C^1$ -smooth, and  $\Delta$ -periodic:  $\varphi(\sigma + \Delta) = \varphi(\sigma)$ ,  $\forall \sigma \in \mathbb{R}$ ;

2) The function  $\varphi$  has two simple zeros:  $0 \leq \sigma_1 < \sigma_2 < \Delta$  with

$$\varphi'(\sigma_1) > 0, \varphi'(\sigma_2) < 0; \quad (3)$$

3)

$$\int_0^\Delta \varphi(\sigma) d\sigma \leq 0. \quad (4)$$

Notice that if  $(z(t), \sigma(t))^T$  is a solution of (1) then  $(z(t), \sigma(t) + \Delta k)^T$  ( $k \in \mathbb{Z}$ ) also is a solution of (1). So (1) has a cylindric phase space.

Throughout the paper we shall use the information about the equation of the second order

$$\ddot{\sigma} + a\dot{\sigma} + \varphi(\sigma) = 0 \quad (a > 0). \quad (5)$$

(If  $\varphi(\sigma) = \sin \sigma$  it is a well-known equation of mathematical pendulum). It is equivalent to the system

$$\begin{aligned} \dot{z} &= -az - \varphi(\sigma) \quad (a > 0), \\ \dot{\sigma} &= z. \end{aligned} \quad (6)$$

System (6) has a denumerable set of equilibria. It follows from (3) that any equilibrium  $(0, \sigma_1 + \Delta k)^T$  ( $k \in Z$ ) is Lyapunov stable and any equilibrium  $(0, \sigma_2 + \Delta k)^T$  ( $k \in Z$ ) is a saddle point. System (6) has been exhaustively investigated (see for example [12, pp. 185-201] and the bibliography there).

**Lemma 1:** [12, pp. 185-201] For any  $\varphi(\sigma)$  there exists a bifurcational value  $a_{cr}$  such that for  $a > a_{cr}$  every solution of (6) converges to some equilibrium and for  $a \leq a_{cr}$  the system (6) has both converging solutions and solutions with  $z(t) = \dot{\sigma}(t) \geq \varepsilon > 0$ .

The phase portrait of (6) in case  $a > a_{cr}$  ( $\sigma_1 = 0, (0, 0)^T$  is a stable focus) is shown in Fig. 1.

System (6) is associated with a first order equation

$$F(\sigma) \frac{dF(\sigma)}{d\sigma} + aF(\sigma) + \varphi(\sigma) = 0 \quad (F = \dot{\sigma} = z). \quad (7)$$

Two separatrices  $z_1$  and  $z_2$  “going into” the saddle point  $(0, \sigma_2)$  (see Fig. 1) “merge” and form a solution  $F_0(\sigma)$  of (7). Consider solutions

$$F_k(\sigma) = F_0(\sigma + \Delta k) \quad (k \in Z). \quad (8)$$

**Lemma 2:** [11], [12, pp. 185-201] If  $a > a_{cr}$  then the solution  $F_k(\sigma)$  have the following properties:

$$\begin{aligned} P1) & F_k(\sigma_2 + \Delta k) = 0; \\ P2) & F_k(\sigma) \neq 0 \quad \text{for } \sigma \neq \sigma_2 + \Delta k; \\ P3) & F_k(\sigma) \rightarrow \pm\infty \quad \text{as } \sigma \rightarrow \mp\infty. \end{aligned} \quad (9)$$

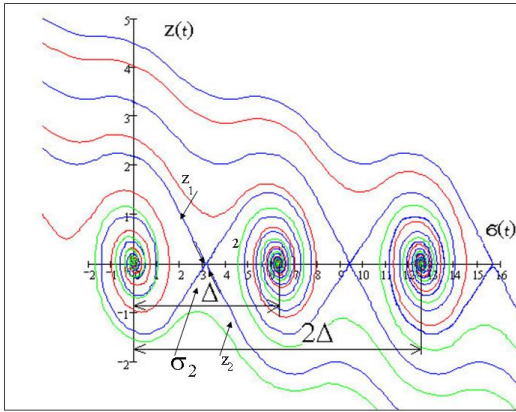


Fig. 1. A phase portrait of (6) for  $a > a_{cr}$

We reduce system (1) to the system

$$\begin{aligned} \dot{z} &= -\bar{a}z - \varkappa\varphi(\sigma) \quad (\bar{a}, \varkappa > 0), \\ \dot{\sigma} &= z. \end{aligned} \quad (10)$$

which is easily (by linear change of variable  $t$ ) transformed to the system (6) with  $a = \frac{\bar{a}}{\sqrt{\varkappa}}$ . So the equation

$$F(\sigma) \frac{dF(\sigma)}{d\sigma} + \bar{a}F(\sigma) + \varkappa\varphi(\sigma) = 0 \quad (F = \dot{\sigma} = z). \quad (11)$$

has the solutions  $F_k(\sigma)$  with the properties P1, P2, P3 for  $\frac{\bar{a}}{\sqrt{\varkappa}} > a_{cr}$ . We are going to use the solutions  $F_k(\sigma)$  in Lyapunov-type functions.

Our argument combines the NRT and KYP-lemma. So we need the transfer function of (1) from  $\varphi$  to  $-\dot{\sigma}$ :

$$K(p) = -\rho + c^*(A - pI_m)^{-1}b \quad (p \in \mathbb{C}), \quad (12)$$

where  $I_m$  is an  $m \times m$  - unit matrix.

**Theorem 1:** Suppose there exist positive numbers  $\varkappa, \varepsilon, \lambda$  such that the following conditions are fulfilled:

- 1) the matrix  $A + \lambda I_m$  is Hurwitz;
- 2)

$$2\sqrt{\frac{\lambda\varepsilon}{\varkappa}} > a_{cr}; \quad (13)$$

- 3) for  $\omega \geq 0$  the inequality

$$\pi_0(\omega, \lambda) \triangleq \varkappa \operatorname{Re} K(i\omega - \lambda) - \varepsilon |K(i\omega - \lambda)|^2 \geq 0 \quad (i^2 = -1) \quad (14)$$

is true.

Then system (1) is Lagrange stable.

*Proof:* Introduce the quadratic form

$$\begin{aligned} G_0(z, \xi) &\triangleq 2z^*H((A + \lambda I_m)z + b\xi) + \varepsilon(c^*z + \rho\xi)^2 + \\ &+ \varkappa\xi(c^*z + \rho\xi), \quad z \in \mathbb{R}^m, \xi \in \mathbb{R}. \end{aligned} \quad (15)$$

According the KYP-lemma [11], the frequency-domain inequality (14) guaranties that there exists a matrix  $H = H^*$  such that

$$G_0(z, \xi) \leq 0, \quad \forall z \in \mathbb{R}^m, \xi \in \mathbb{R}. \quad (16)$$

Consider a set of Lyapunov-type functions

$$V_k(t) \triangleq z^*(t)Hz(t) - \frac{1}{2}F_k^2(\sigma(t)), \quad (17)$$

where  $F_k$  is a solution of (11) with  $\bar{a} = 2\sqrt{\lambda\varepsilon}$ . Note that

$$\begin{aligned} G_0(z, 0) &= 2z^*H(A + \lambda I_m)z + \\ &+ \varepsilon(c^*z)^2 \leq 0, \quad z \in \mathbb{R}^m, \end{aligned} \quad (18)$$

whence we have by condition 1) that  $H > 0$  [22]. Computing  $\dot{V}(t)$  in virtue of (1) we obtain:

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &= 2z^*(t)H((A + \lambda I_m)z(t) + b\varphi(\sigma(t))) + \\ &+ 2\sqrt{\lambda\varepsilon}F(\sigma(t))\dot{\sigma}(t) + \varkappa\varphi(\sigma(t))\dot{\sigma}(t) - \\ &- \lambda F_k^2(\sigma(t)). \end{aligned} \quad (19)$$

It follows from (16) that

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &\leq -\varepsilon\dot{\sigma}^2(t) + 2\sqrt{\lambda\varepsilon}F(\sigma(t))\dot{\sigma}(t) - \\ &- \lambda F_k^2(\sigma(t)) \leq 0. \end{aligned} \quad (20)$$

Hence

$$V_k(t)e^{2\lambda t} \leq V_k(0) \quad (21)$$

The properties P1,P2,P3 imply that for any  $(z(0), \sigma(0))$  there exists  $k_0 \in N$  such that

$$V_{\pm k_0}(0) = z^*(0)Hz(0) - \frac{1}{2}F_{\pm k_0}^2(\sigma(0)) < 0. \quad (22)$$

So

$$V_{\pm k_0}(t) < 0, \quad \forall t \geq 0. \quad (23)$$

Since  $H > 0$  we conclude that

$$F_{\pm k_0}(\sigma(t)) > 2z^*(t)Hz(t) \geq 0, \quad (24)$$

which means that  $F_{\pm k_0}(\sigma(t))$  can not vanish for  $t \geq 0$ . Consequently

$$\sigma_2 - \Delta k_0 < \sigma(t) < \sigma_2 + \Delta k_0, \quad \forall t \geq 0. \quad (25)$$

Thus for any  $(z(0), \sigma(0))$  the function  $\sigma(t)$  is bounded. Since  $A$  is Hurwitz  $z(t)$  is bounded as well. So Theorem 1 is proved. ■

The proof of Theorem 1 expounds the very idea of nonlocal reduction. Though the theorem itself is only a particular case of some general assertions. Here is one of them. Let

$$\mu_1 = \inf_{\sigma \in [0, \Delta]} \varphi'(\sigma), \quad \mu_2 = \sup_{\sigma \in [0, \Delta]} \varphi'(\sigma) \quad (26)$$

**Theorem 2:** [12, p. 203] Suppose there exist positive  $\varkappa, \varepsilon, \lambda, \tau$  and  $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$  such that the requirements 1) and 2) of Theorem 1 are fulfilled, and for all  $\omega \geq 0$  the inequality

$$\begin{aligned} \pi(\omega, \lambda) \triangleq & \operatorname{Re}\{\varkappa K(i\omega - \lambda) - \tau(K(i\omega - \lambda) + \\ & + \alpha_1^{-1}(i\omega - \lambda))^*(K(i\omega - \lambda) + \alpha_2^{-1}(i\omega - \lambda))\} - \\ & - \varepsilon |K(i\omega - \lambda)|^2 \geq 0 \end{aligned} \quad (27)$$

is true. Then system (1) is Lagrange stable.

Lagrange stability is the basic property of synchronization systems. If a Lagrange stable system is monostable (every bounded solution converges) it is gradient-like. Here is a frequency-domain criterion of monostability for synchronization systems.

**Theorem 3:** [12, pp. 118-123] Suppose there exist  $\varkappa, \varepsilon, \tau, \delta > 0, \alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$  such that

$$\pi(\omega, 0) < \delta, \quad \forall \omega \geq 0. \quad (28)$$

Then for any bounded solution of (1) the following assertions are true:

$$\dot{\sigma}(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty \quad (29)$$

$$z(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty \quad (30)$$

$$\sigma(t) \rightarrow \sigma_{eq} \quad \text{as } t \rightarrow +\infty, \quad (31)$$

where  $\varphi(\sigma_{eq}) = 0$ .

The requirement of Theorem 3 is as a rule fulfilled if (27) is valid for all  $\omega \geq 0$  and some  $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$  and positive  $\varkappa, \varepsilon, \tau, \lambda$ .

Next to Lagrange stability and gradient-like behavior is the problem of cycle-slipping (the estimate of  $|\sigma(0) - \sigma_{eq}|$ ). The NRT has successfully coordinated the number of slipped cycles of low order and high order systems [23].

The NRT has also been applied for stability analysis of control systems with nonlinearities satisfying ‘‘the sector condition’’ [10]. As a result new sufficient conditions of absolute stability have been established.

The goal of this paper is to ‘‘improve’’ the NRT. In next section we shall weaken the restriction on varying parameters  $\varepsilon, \lambda$ .

### III. THE NONLOCAL REDUCTION TECHNIQUE AND PERIODIC LYAPUNOV FUNCTIONS.

In this section we combine the NRT and the method of periodic Lyapunov functions [21, p.72]. New criteria keep the advantages of both methods.

Introduce the function

$$\Phi(\sigma) = \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}, \quad (32)$$

with  $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$  and the constant

$$\nu_0 = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta \Phi(\sigma) |\varphi(\sigma)| d\sigma}. \quad (33)$$

**Theorem 4:** Suppose there exist  $\lambda \in (0, z_0), \varkappa, \varepsilon, \tau, \delta > 0, \alpha_1 = -\alpha_2$  such that the following conditions are satisfied: 1) for all  $\omega \geq 0$  the inequality

$$\pi(\omega, \lambda) \geq \delta \quad (34)$$

is true;

2)

$$4\lambda\varepsilon > a_{cr}^2 \left( \varkappa - \frac{2\sqrt{\tau\delta}}{|\nu_0|} \right); \quad (35)$$

3)  $|\nu_0| \leq 1$ .

Then (1) is Lagrange stable.

*Proof:*

**A) The KYP-lemma for an extended synchronization system**

In order to use the inequalities

$$\alpha_1 \leq \varphi'(\sigma) \leq \alpha_2, \quad (\alpha_1 \alpha_2 < 0) \quad (36)$$

we traditionally embed the system (1) into the system of higher order [11]

$$\begin{aligned} \frac{dy}{dt} &= Qy(t) + L\eta(t), \\ \frac{d\sigma}{dt} &= D^*y(t). \end{aligned} \quad (37)$$

Here  $y(t) = (z(t), \varphi(\sigma(t)))^T, \eta(t) = \frac{d}{dt}\varphi(\sigma(t))$ ,

$$Q = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D = \begin{bmatrix} c^* \\ \rho \end{bmatrix}. \quad (38)$$

Consider the quadratic form

$$\begin{aligned} G(y, \eta) &= 2y^*H((Q + \lambda I_{m+1})y + L\eta) + \delta(L^*y)^2 + \\ &+ \varepsilon(D^*y)^2 + \varkappa y^*LD^*y + \tau(D^*y - \alpha_1^{-1}\eta)(D^*y - \\ &- \alpha_2^{-1}\eta) \quad (y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}). \end{aligned} \quad (39)$$

By KYP-lemma there exists a matrix  $H = H^*$  [11] such that

$$G(y, \eta) \leq 0 \quad \forall y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}. \quad (40)$$

Let

$$H = \begin{bmatrix} H_0 & h \\ h^* & \alpha \end{bmatrix}, \quad (H_0 \in \mathbb{R}^{m \times m}, h \in \mathbb{R}^m, \alpha \in \mathbb{R}) \quad (41)$$

For  $\bar{y} = (z, 0)^T$  we have

$$G(\bar{y}, 0) = 2z^*H_0(A + \lambda I_m)z + (\varepsilon + \tau)(c^*z)^2 \quad \forall z \in \mathbb{R}^m. \quad (42)$$

The choice of  $\lambda$  guarantees that the matrix  $A + \lambda I_m$  is Hurwitz. Then the matrix  $H_0$  is positive definite [22]. So if  $\varphi(\sigma(\bar{t})) = 0$  one has

$$y^*(\bar{t})Hy(\bar{t}) > 0, \quad z(\bar{t}) \neq 0. \quad (43)$$

### B) The Lyapunov-type function

Proceeding from (35) choose a  $\varkappa_1 \in (0, \varkappa)$  such that

$$\frac{4\lambda\varepsilon}{a_{cr}^2} > \varkappa_1 \quad (44)$$

and

$$\varkappa_2 \triangleq \varkappa - \varkappa_1 < \frac{2\sqrt{\tau\delta}}{|\nu_0|}. \quad (45)$$

Introduce a Lyapunov-type function

$$V_k(t) = y^*(t)Hy(t) - \frac{1}{2}F_k^2(\sigma(t)) + \varkappa_2 \int_{\sigma_2}^{\sigma_k(t)} \Psi(\zeta)d\zeta, \quad (46)$$

where  $F_k(\zeta)$  ( $k \in Z$ ) is a solution of

$$F(\zeta) \frac{dF(\zeta)}{d\zeta} + 2\sqrt{\lambda\varepsilon}F(\zeta) + \varkappa_1\varphi(\zeta) = 0 \quad (47)$$

with properties P1–P3, and

$$\Psi(\zeta) \triangleq \varphi(\zeta) - \nu_0|\varphi(\zeta)|\Phi(\zeta). \quad (48)$$

Notice that

$$\int_0^\Delta \Psi(\zeta)d\zeta = 0, \quad (49)$$

$$\varphi(\sigma_2) = 0, \quad \varphi'(\sigma_2) < 0. \quad (50)$$

Compute the derivative of  $V_k(t)$  in virtue of (37):

$$\dot{V}_k(t) = 2y^*(t)H(Qy(t) + L\varphi(\sigma(t))) - F_k'(\sigma(t))F_k(\sigma(t))\dot{\sigma}(t) + \varkappa_2\Psi(\sigma(t))\dot{\sigma}(t). \quad (51)$$

Then

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &= 2y^*(t)H[(Q + \lambda I_{m+1})y(t) + L\varphi(\sigma(t))] - F_k'(\sigma(t))F_k(\sigma(t))\dot{\sigma}(t) + \\ &+ \varkappa_2\Psi(\sigma(t))\dot{\sigma}(t) - \lambda F_k^2(\sigma(t)) + \\ &+ 2\lambda\varkappa_2 \int_{\sigma_2}^{\sigma_k(t)} \Psi(\zeta)d\zeta. \end{aligned} \quad (52)$$

It follows from (47) and (48) that

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &= 2y^*(t)H[(Q + \lambda I_{m+1})y(t) + L\varphi(\sigma(t))] + 2\sqrt{\lambda\varepsilon}F_k(\sigma(t))\dot{\sigma}(t) + \\ &+ \varkappa_1\varphi(\sigma(t))\dot{\sigma}(t) - \varkappa_2\nu_0|\varphi(\sigma(t))|\Phi(\sigma(t))\dot{\sigma}(t) + \\ &+ 2\lambda\varkappa_2 \int_{\sigma_2}^{\sigma_k(t)} \Psi(\zeta)d\zeta - \lambda F_k^2(\sigma(t)). \end{aligned} \quad (53)$$

Then from (40) we have

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &\leq -\varepsilon\dot{\sigma}^2(t) - \delta\varphi^2(\sigma(t)) - \\ &- \tau\dot{\sigma}^2(t)\Phi^2(\sigma(t)) + 2\sqrt{\lambda\varepsilon}F_k(\sigma(t))\dot{\sigma}(t) - \lambda F_k^2(\sigma(t)) + \\ &+ \lambda\varkappa_2 \int_{\sigma_2}^{\sigma_k(t)} \Psi(\zeta)d\zeta - \nu_0\varkappa_2|\varphi(\sigma(t))|\Phi(\sigma(t))\dot{\sigma}(t), \end{aligned} \quad (54)$$

whence

$$\begin{aligned} \dot{V}_k(t) + 2\lambda V_k(t) &\leq [-\delta\varphi^2(\sigma(t)) - \tau(\dot{\sigma}(t)\Phi(\sigma(t)))^2 - \\ &- \nu_0\varkappa_2|\varphi(\sigma(t))|\Phi(\sigma(t))\dot{\sigma}(t)] + 2\lambda\varkappa_2 \int_{\sigma_2}^{\sigma(t)} \Psi(\zeta)d\zeta. \end{aligned} \quad (55)$$

The inequality (45) implies that the first summand in the right-hand part of (55) is a negative definite quadratic form of  $|\varphi(\sigma(t))|$  and  $\Phi(\sigma(t))\dot{\sigma}(t)$ . From (49) and (50) we have that

$$\int_{\sigma_2}^{\sigma(t)} \Psi(\zeta)d\zeta < 0. \quad (56)$$

Thus

$$\dot{V}_k(t) + 2\lambda V_k(t) \leq 0. \quad (57)$$

Hence

$$V_k(t)e^{2\lambda t} \leq V_k(0). \quad (58)$$

Since

$$V_k(0) = y^*(0)Hy(0) - \frac{1}{2}F_k^2(\sigma(0)) + \varkappa_2 \int_{\sigma_2}^{\sigma(0)} \Psi(\zeta)d\zeta \quad (59)$$

one can always choose a natural  $k_0 \in \mathbb{N}$  in such a way that  $V_{\pm k_0}(0) < 0$ . Then

$$V_{\pm k_0}(t) < 0, \quad \forall t \geq 0. \quad (60)$$

**C) The Lagrange stability** Let  $\bar{t}$  be such moments that

$$\sigma(\bar{t}) = \sigma_2 + \Delta l \quad (l \in Z). \quad (61)$$

Then for any  $\bar{t}$

$$y^*(\bar{t})Hy(\bar{t}) = z^*(\bar{t})H_0z(\bar{t}) \geq 0 \quad (62)$$

and

$$\int_{\sigma_2}^{\sigma(\bar{t})} \Psi(\zeta)d\zeta = 0. \quad (63)$$

It follows from (60) that

$$F_{\pm k_0}^2(\sigma(\bar{t})) \neq 0 \quad (64)$$

whence

$$\sigma_2 - \Delta k_0 < \sigma(\bar{t}) < \sigma_2 + \Delta k_0. \quad (65)$$

So for any  $(z(0), \sigma(0))^T$  there exists a  $k_0 \in \mathbb{N}$  such that (65) is true, which is equivalent to the assertion:

$$\sigma_2 - \Delta k_0 < \sigma(t) < \sigma_2 + \Delta k_0 \quad \forall t \geq 0.$$

Theorem 4 is proved.  $\blacksquare$

**Remark 1:** It is obvious that the inequality (35) is weaker than (13). More than that in case  $2\sqrt{\tau\delta} > \varkappa|\nu_0|$ , (35) is fulfilled for any  $\lambda, \varepsilon$ .

**Theorem 5:** Suppose there exist positive  $\varkappa, \tau, \varepsilon, \delta$ ,  $\lambda \in (0, z_0)$ ,  $\alpha_1 \leq \mu_1, \alpha_2 \leq \mu_2$  such that the following conditions are satisfied:

- 1) the inequality (34) is valid for all  $\omega \geq 0$ ;
- 2) for  $\varkappa_1 \in [0, \varkappa]$  the quadratic form

$$W(x, y, z) = \lambda x^2 + \varepsilon y^2 + \delta z^2 + (\varkappa - \varkappa_1)\nu y z + a_{cr}\sqrt{\varkappa_1}xy \quad (66)$$

where

$$\nu = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta |\varphi(\sigma)| d\sigma} \quad (67)$$

is positive definite.

Then system (1) is Lagrange stable.

*Proof:* It follows from condition 2) that

$$4\lambda\varepsilon\delta > a_{cr}^2 \varkappa_1 \delta + (\varkappa - \varkappa_1)^2 \nu^2 \lambda. \quad (68)$$

Let

$$\varepsilon_2 \triangleq \frac{(\varkappa - \varkappa_1)^2 \nu^2}{4\delta}. \quad (69)$$

Then

$$4\lambda\varepsilon_1 > a_{cr}^2 \varkappa_1, \quad (70)$$

where  $\varepsilon_1 \triangleq \varepsilon - \varepsilon_2$ . Introduce the equation

$$F(\zeta) \frac{dF(\zeta)}{d\zeta} + 2\sqrt{\lambda\varepsilon_1} F(\zeta) + \varkappa_1 \varphi(\zeta) = 0. \quad (71)$$

We repeat the part A) from the proof of Theorem 4, and introduce Lyapunov-type functions

$$W_k(t) = y^*(t)Hy(t) - \frac{1}{2}F_k(\sigma(t)) + \varkappa_2 \int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta) d\zeta. \quad (72)$$

Here  $\varkappa_2 \triangleq \varkappa - \varkappa_1$ ,

$$\Psi_0(\zeta) \triangleq \varphi(\zeta) - \nu|\varphi(\zeta)|, \quad (73)$$

$F_k(\zeta)$  ( $k \in Z$ ) is a solution of (71) with the properties P1–P3. It is obvious that

$$\int_0^\Delta \Psi_0(\zeta) d\zeta = 0. \quad (74)$$

It follows from (34) that for a certain matrix  $H = H^*$  the inequality (40) is valid. Then we have in virtue of (1)

$$\begin{aligned} \dot{W}_k(t) + 2\lambda W_k(t) &= 2y^*(t)H((Q + \lambda I_{m+1})y(t) + \\ &+ L\varphi(\sigma(t))) - F'_k(\sigma(t))F_k(\sigma(t))\dot{\sigma}(t) + \varkappa_2 \Psi_0(\sigma(t))\dot{\sigma}(t) - \\ &- \lambda F_k^2(\sigma(t)) + 2\lambda \varkappa_2 \int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta) d\zeta \leq -\varepsilon \dot{\sigma}^2(t) - \\ &- \delta \varphi^2(\sigma(t)) - \varkappa \dot{\sigma}(t)\varphi(\sigma(t)) - \tau \dot{\sigma}^2(t)\Phi^2(\sigma(t)) + \\ &+ 2\sqrt{\lambda\varepsilon_1} F_k(\sigma(t))\dot{\sigma}(t) + \varkappa_1 \varphi(\sigma(t))\dot{\sigma}(t) + \\ &+ \varkappa_2 \varphi(\sigma(t))\dot{\sigma}(t) - \varkappa_2 \nu |\varphi(\sigma(t))|\dot{\sigma}(t) - \lambda F_k^2(\sigma(t)) + \\ &+ 2\lambda \varkappa_2 \int_{\sigma_2}^{\sigma(t)} \Psi(\zeta) d\zeta. \end{aligned} \quad (75)$$

Since (74) implies that

$$\int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta) d\zeta \leq 0, \quad \forall \sigma \quad (76)$$

it follows from (69), (70) and (75) that

$$\begin{aligned} \dot{W}_k(t) + \lambda W_k(t) &\leq -(\varepsilon_2 \dot{\sigma}^2(t) + \delta \varphi^2(\sigma(t)) + \\ &+ \varkappa_2 \nu |\varphi(\sigma(t))|\dot{\sigma}(t)) - (\varepsilon_1 \dot{\sigma}^2(t) - 2\sqrt{\lambda\varepsilon_1} F_k(\sigma(t))\dot{\sigma}(t) + \\ &+ \lambda F_k^2(\sigma(t))) \leq 0, \quad \forall t \geq 0. \end{aligned} \quad (77)$$

Hence

$$W_k(t)e^{2\lambda t} \leq W_k(0), \quad \forall t \geq 0, \forall k \in Z. \quad (78)$$

So we can choose a natural  $k_0$  such that

$$W_{\pm k_0}(0) < 0 \quad (79)$$

and guarantee that

$$W_{\pm k_0}(t) < 0, \quad \forall t \geq 0. \quad (80)$$

The inequality (80) is analogous to the inequality (60). Thus the end of the proof is just analogous to that of Theorem 4.  $\blacksquare$

**Remark 2:** Condition 2 of Theorem 4 is weaker than the requirement (13). More than that if  $2\sqrt{\varepsilon\delta} > \varkappa|\nu|$ , then for any  $\lambda > 0$  there exists a  $\varkappa_1$  (small enough) such that condition 2 of Theorem 4 is true.

**Remark 3:** Theorems 4 and 5 exploit different integral terms in Lyapunov functions, which leads to different estimates for stability regions. The final estimate must be obtained as their union.

**Example.** Consider a PLL with the proportional integrating filter:

$$K(p) = T \frac{Tmp + 1}{Tp + 1} \quad (m \in (0, 1)) \quad (81)$$

and

$$\varphi(\sigma) = \sin(\sigma) - \beta \quad (\beta \in (0, 1)). \quad (82)$$

Then

$$|\nu_0| = \frac{2\pi\beta}{4\beta + \pi - 2\arcsin\beta + 2\beta\sqrt{1-\beta^2}} \quad (83)$$

Condition 3) of Theorem 4 is valid for  $\beta \leq 0.8$ .

Let us show that conditions 1) and 2) of Theorem 4 are less limiting than the conditions of Theorem 2. Suppose for certain  $\varkappa, \varepsilon, \tau > 0, \lambda \in (0, T^{-1})$  and  $|\alpha_1| = \alpha_2 = 1$  the inequality (27) is valid for  $\omega \in \mathbb{R}$ . That is

$$\begin{aligned} \pi(\omega, \lambda) \triangleq \tau(\omega^2 + \lambda^2) + \varkappa \operatorname{Re} K(i\omega - \lambda) - \\ - (\varepsilon + \tau)|K(i\omega - \lambda)|^2 \geq 0, \quad \forall \omega \in \mathbb{R}. \end{aligned} \quad (84)$$

From  $m \in (0, 1)$  and  $T\lambda \in (0, 1)$  it follows that

$$|K(i\omega - \lambda)|^2 \geq T^2 m^2, \quad \forall \omega \in \mathbb{R}. \quad (85)$$

Then from (84), (85) we have

$$\begin{aligned} \tau(\omega^2 + \lambda^2) + \varkappa \operatorname{Re} K(i\omega - \lambda) - (\bar{\varepsilon} + \tau)|K(i\omega - \lambda)|^2 - \\ - \varepsilon_0 T^2 m^2 \geq 0, \quad \forall \omega \in \mathbb{R}, \end{aligned} \quad (86)$$

where  $\varepsilon = \bar{\varepsilon} + \varepsilon_0$ .

So condition 1) of Theorem 4 is fulfilled with  $\delta = \varepsilon_0 T^2 m^2$ . Numbers  $\varepsilon_0$  and  $\bar{\varepsilon}$  we can choose as we wish, but it is necessary to take certain facts into consideration.

1) Suppose that

$$\varepsilon > \frac{\varkappa^2 \nu_0^2}{4T^2 m^2 \tau}. \quad (87)$$

Then we can choose  $\varepsilon_0$  in such a way that

$$\varepsilon > \varepsilon_0 > \frac{\varkappa^2 \nu_0^2}{4T^2 m^2 \tau}, \quad (88)$$

whence  $4\tau\delta > \varkappa^2\nu_0^2$  and

$$2\sqrt{\tau\delta} > \varkappa|\nu_0|. \quad (89)$$

Then condition 2) of Theorem 4 is fulfilled.

2) Suppose that

$$\varepsilon \leq \frac{\varkappa^2\nu_0^2}{4T^2m^2\tau}. \quad (90)$$

and consequently

$$\delta < \frac{\varkappa^2\nu_0^2}{4\tau}. \quad (91)$$

whence

$$\varkappa - \frac{2\sqrt{\tau\delta}}{|\nu_0|} > 0. \quad (92)$$

Condition 2) of Theorem 4 takes the form

$$4\lambda\varepsilon > a_{cr}^2\left(\varkappa - \frac{2\sqrt{\tau\varepsilon_0}mT}{|\nu_0|}\right) + 4\lambda\varepsilon_0 \quad (\varepsilon_0 < \varepsilon). \quad (93)$$

If

$$\varepsilon_0 < \min\left\{\varepsilon, \frac{a_{cr}^4\tau m^2T^2}{4\lambda^2\nu_0^2}\right\}. \quad (94)$$

and condition 2) of Theorem 1 is true then (93) is also true.

So if the conditions of Theorem 2 are fulfilled, the conditions 1) and 2) of Theorem 4 are fulfilled as well with  $\varepsilon_0$  property choosed. On the contrary, the condition 2) of Theorem 1 may be violated though the conditions of Theorem 4 are fulfilled.

Let us compare Theorem 2 and Theorem 5. If the inequality (27) is satisfied then as it follows from (86) the requirement 1) of Theorem 5 is true with  $\delta = \varepsilon_0m^2T^2$  where  $\varepsilon_0 \in (0, \varepsilon)$ , and the requirement 2) of Theorem 5 takes the form

$$4\lambda\varepsilon > a_{cr}^2\varkappa_1 + \frac{(\varkappa - \varkappa_1)^2}{\varepsilon_0m^2T^2}\nu^2\lambda + 4\lambda\varepsilon_0. \quad (95)$$

We choose

$$\varepsilon_0 = \frac{(\varkappa - \varkappa_1)}{2Tm}\nu. \quad (96)$$

Then the right-hand part of (95) have the least possible value

$$a_{cr}^2\varkappa_1 + 4\lambda\nu\frac{\varkappa - \varkappa_1}{Tm}. \quad (97)$$

Suppose the requirement 2) of Theorem 1 is violated, so that

$$4\lambda\varepsilon \leq a_{cr}^2\varkappa. \quad (98)$$

It is not difficult to verify that in case  $K(p)$  is defined by (81) the frequency-domain inequality (27) is true for all  $\omega \geq 0$  if

$$\tau + \varepsilon = \frac{\varkappa(1 - T\lambda)}{T(1 - mT\lambda)}, \quad (99)$$

where  $\tau$  can be chosen arbitrarily small. So if

$$\nu < m\frac{1 - T\lambda}{1 - mT\lambda}, \quad (100)$$

the inequality  $\varepsilon_0 < \varepsilon$  is valid for any  $\varkappa_1 \in (0, \varkappa)$ .

If  $\varkappa_1$  is small enough we have

$$4\lambda\varepsilon > a_{cr}^2\varkappa_1 + 4\lambda\nu\frac{\varkappa - \varkappa_1}{Tm}. \quad (101)$$

Thus if (100) is fulfilled, the condition 2) of Theorem 5 is satisfied though the condition 2) of Theorem 1 is violated.

## IV. CONCLUSION

In this paper, we study asymptotic behavior of periodic (pendulum-like, synchronization) Lur'e-type systems. We combine two methods previously used for stability analysis, namely, Leonov's method of nonlocal reduction and the method of periodic Lyapunov functions, introducing a novel class of Lyapunov-type functions. Using the Kalman–Yakubovich–Popov lemma, we derive new frequency–algebraic criteria for the convergence of solutions.

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