New criteria for gradient–like behavior of synchronization systems with distributed parameters

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***Abstract*—This paper is concerned with stability properties of a Lur'e system obtained by interconnection of a general linear time-invariant block (possibly, infinite-dimensional) and a periodic nonlinearity. Such systems usually have multiple equilibria. In the paper, two new frequency-algebraic stability criteria are established by using. Popov's method of "a priori integral indices", Leonov's method of nonlocal reduction and the Bakaev-Guzh technique.**

***Index Terms*—Lagrange stability, gradient-like behavior, frequency-domain methods**

I. Introduction

In this paper, we examine the asymptotic behavior of infinite dimensional control systems with periodic nonlinearities and multiple equilibria. Among systems with periodic nonlinearities are mathematical pendulum, mechanical systems, electrical machines [1], vibration units [2], synchronization circuits [3]–[5]. Such systems are called synchronization or pendulum-like systems.

The aforementioned systems often cannot be adequately described my ordinary differential equations. Such are systems with delays, for instance. In this paper we consider a class of systems described by integro-differential Volterra equations.

The stability problems for synchronization systems differ essentially from those for the systems of the single equilibrium. The two most significant problems for pendulum-like systems are Lagrange stability (every solution is bounded) and gradient-like behavior (every solution converges to a certain equilibrium). It turned out that standard methods are of no good for stability investigation of pendulum-like systems. New methods have been elaborated in the framework of the traditional ones (see [6] and bibliography therein).

In this paper we examine stability of infinite-dimension pendulum-like system by the technique stemming from the Popov's method of "a priori integral indices" [7], [8]. We combine the Popov's method with Leonov's nonlocal reduction principle [9], [10] and "Bakaev-Guzh technique" [6], [10], so that the Popov's functionals involve trajectories of low-order stable comparison system and a modified nonlinearity with zero value other the period. As a result we derive new frequency-algebraic stability criteria for pendulum-like systems.

II. The statement of the problem

Consider an integro-differential equation

$$\dot{\sigma}(t) = b(t) + \rho\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau, \quad (1)$$

Here $h \geq 0$; $\varphi : \mathbb{R} \to \mathbb{R}$; $\gamma, b : [0, +\infty) \to \mathbb{R}$, The solution of (1) is uniquely determined by the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t) \in \mathbb{C}[-h, 0]. \quad (2)$$

We assume that the function $b(t)$ is continuous, the function $\gamma(\cdot)$ is piece–wise continuous and

$$\gamma(t)e^{rt},\, b(t)e^{rt} \in L_2[0, +\infty) \quad (r > 0). \quad (3)$$

The function $\varphi(\sigma)$ is $\mathbb{C}^1$-smooth and Δ–periodic: $\varphi(\sigma) = \varphi(\sigma + \Delta)$. It also has two simple zeros $0 \leq \sigma_1 < \sigma_2 < \Delta$ with $\varphi'(\sigma_1) > 0$, $\varphi'(\sigma_2) < 0$. Without loss of generality we assume that

$$\int_0^\Delta \varphi(\zeta)\,d\zeta \leq 0. \quad (4)$$

Equation (1) is a special case of Lur'e system. We shall need the transfer function of the linear part of (1) from $\xi = \varphi(\sigma)$ to $(-\dot{\sigma})$:

$$K(p) = -\rho e^{-ph} + \int_0^\infty \gamma(t) e^{-pt}\, dt \quad (p \in \mathbb{C}). \tag{5}$$

System (1) is said to be *Lagrange stable* if any its solution is bounded. System (1) is said to be *gradient–like* if every its solution converges to an equilibrium:

$$\dot{\sigma}(t) \xrightarrow[t\to\infty]{} 0, \quad \sigma(t) \xrightarrow[t\to\infty]{} \sigma_{eq}, \quad \varphi(\sigma_{eq}) = 0. \tag{6}$$

The *Lagrange stability* is the basic asymptotic property for pendulum–like systems since it is often possible to prove the *dichotomy* property of the system: every solution is either unbounded or convergences. In this paper we shall first establish the conditions ensuring that (1) is Lagrange stable. Then for an important particular case the conditions for gradient–like behavior will be obtained. We use here two special techniques, destined for Lur'e systems with periodic nonlinearities. The first is the *Bakaev-Guzh procedure*. According the procedure we substitute the original periodic nonlinearity by a periodic nonlinearity with zero value over the period.

The other technique is the Leonov's nonlocal reduction method [11]. It exploits the properties of a special *comparison system*. As a comparison system, we use the second oder system

$$\begin{aligned} \dot{z} &= -az - \varphi(\sigma) \quad (a > 0), \\ \dot{\sigma} &= z, \end{aligned} \tag{7}$$

which has been exhaustively investigated (see [9], [10] and references therein). Equation (7) has Lyapunov stable equilibria $(0, \sigma_1 + \Delta k)$ and saddle-point equilibria $(0, \sigma_2 + \Delta k)$ $(k = 0, \pm 1, \ldots)$. It has a bifurcation value a_{cr} such that if $a > a_{cr}$ every solution of (7) converges to some equilibrium.

In this case the first order equation

$$F(\sigma)\frac{dF}{d\sigma} + aF(\sigma) + \varphi(\sigma) = 0 \quad (F = \dot{\sigma} = z), \tag{8}$$

associated with (7), has solutions $F_k(\sigma)\,(k \in \mathbb{Z})$ such that

$$\begin{gathered} F_k(\sigma_2 + \Delta k) = 0, \quad F_k(\sigma) \neq 0 \quad \forall \sigma \neq \sigma_2 + \Delta k, \\ F_k(\sigma) \xrightarrow[\sigma \to \mp\infty]{} \pm\infty. \end{gathered} \tag{9}$$

The solution $F_k(\sigma)$ is produced by two separatrices which "go in" at the point $(0, \sigma_2 + \Delta k)$

III. Lagrange stability

Introduce the constants

$$\mu_1 \triangleq \inf_{\sigma\in[0,\Delta)} \varphi'(\sigma);\; \mu_2 \triangleq \sup_{\sigma\in[0,\Delta)} \varphi'(\sigma)\,(\mu_1\mu_2 < 0) \tag{10}$$

and the function

$$\Phi(\sigma) \triangleq \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}, \tag{11}$$

with $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$. Denote

$$\nu_0 = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta |\varphi(\sigma)|\, \Phi(\sigma)\, d\sigma}. \tag{12}$$

Theorem 1: Suppose there exist $\varepsilon, \tau, \delta > 0, \lambda \in (0, \frac{r}{2})$ $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$ such that the following conditions are true:

$$\begin{gathered} \pi(\omega,\lambda) \triangleq Re\{K(\imath\omega - \lambda) - \tau(K(\imath\omega - \lambda) + \\ +\alpha_1^{-1}(\imath\omega - \lambda))^*(K(\imath\omega - \lambda) + \alpha_2^{-1}(\imath\omega - \lambda))\} - \\ -\varepsilon|K(\imath\omega - \lambda)|^2 - \delta \geq 0, \; \forall\omega \geq 0, \end{gathered} \tag{13}$$

where the symbol $(^*)$ means the complex conjugation;

$$4\lambda\varepsilon > a_{cr}^2\left(1 - \frac{2\sqrt{\tau\delta}}{|\nu_0|}\right); \tag{14}$$

$$\max_{\sigma\in[0,\Delta)} \Phi(\sigma) \leq \frac{1}{|\nu_0|}. \tag{15}$$

Then (1) is Lagrange stable.

Proof: We use the standard scheme of Popov's method [6]. Let $\sigma(t)$ be an arbitrary solution of (1), $\eta(t) = \varphi(\sigma(t))$. Determine the functions $(T > 1)$:

$$v(t) \triangleq \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } t \in [0,1], \\ 1, & \text{if } t < 1; \end{cases} \tag{16}$$

$$\eta_T(t) \triangleq \begin{cases} v(t)\eta(t), & \text{if } t \leq T, \\ 0, & \text{if } t > T; \end{cases} \tag{17}$$

$$\zeta_T(t) \triangleq \rho\eta_T(t-h) - \int_0^t \gamma(t-\tau)\eta_T(\tau)\, d\tau; \tag{18}$$

Let $[f]^\mu(t) \triangleq f(t)e^{\mu t} \quad (\mu \in \mathbb{R})$. Then

$$[\zeta_T]^\lambda(t) = \rho e^{\lambda h}[\eta_T]^\lambda(t-h) - \int_0^t [\gamma]^\lambda(t-\tau)[\eta_T]^\lambda(\tau)\, d\tau. \tag{19}$$

Denote the set of all $\sigma_2 + \Delta k$ $(k \in Z)$ by S. Let

$$\Sigma \triangleq \{T : T > 1,\, \sigma(T) \in S\}. \tag{20}$$

If Σ is bounded then the function $\sigma(t)$ is bounded as well.

Suppose Σ is not bounded. Consider the functionals

$$\begin{aligned} R_T \triangleq & \int_0^\infty \{[\eta_T]^\lambda[\zeta_T]^\lambda + \varepsilon([\zeta_T]^\lambda)^2 + \delta([\eta_T]^\lambda)^2 + \\ & +\tau([\zeta_T]^\lambda - \alpha_1^{-1}\eta_{T,\lambda})([\zeta_T]^\lambda - \alpha_2^{-1}\eta_{T,\lambda})\}\, dt \quad (T \in \Sigma). \end{aligned} \tag{21}$$

where

$$\eta_{T,\lambda}(t) \triangleq \frac{d}{dt}([\eta_T]^\lambda) - \lambda[\eta_T]^\lambda \quad (T \in \Sigma, t \neq 0, T). \tag{22}$$

Notice that

$$\mathfrak{F}([\zeta_T]^\lambda)(\imath\omega) = -K(\imath\omega - \lambda)\mathfrak{F}([\eta_T]^\lambda)(\imath\omega), \tag{23}$$

$$\mathfrak{F}\left(\frac{d}{dt}[\eta_T]^\lambda\right) = \imath\omega\mathfrak{F}[\eta_T]^\lambda(\imath\omega), \tag{24}$$

where $\mathfrak{F}(f)(\imath\omega)$ stands for Fourier–transform of function f. Then in virtue of Plancherel theorem one has

$$R_T = -\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\pi(\omega,\lambda). \tag{25}$$

It follows from (13) that

$$R_T \le 0 \quad \forall T \in \Sigma. \tag{26}$$

On the other hand

$$R_T \ge I_T + I_{1T} \quad (T \in \Sigma). \tag{27}$$

where

$$\begin{aligned} I_T = \int\limits_0^T \{&\delta(\varphi(\sigma(t)))^2 + \varepsilon\dot\sigma^2(t) + \dot\sigma(t)\varphi(\sigma(t)) + \\ &+\tau\dot\sigma^2(t)\Phi^2(\sigma(t)\}e^{2\lambda t}\,dt \end{aligned} \tag{28}$$

and the integrals I_{1T} are uniformly bounded. Then

$$I_T \le C \quad (T \in \Sigma), \tag{29}$$

where C does not depend on T.

Let us choose $\varkappa \in (0,1)$ such that

$$\frac{4\lambda\varepsilon}{a_{cr}^2} > \varkappa > 1 - \frac{2\sqrt{\tau\delta}}{|\nu_0|}. \tag{30}$$

Let

$$I_T = J_{1T} + J_{2T}, \tag{31}$$

where

$$\begin{aligned} J_{1T} &\triangleq \int\limits_0^T \{(1-\varkappa)\varphi(\sigma(t))\dot\sigma(t) + \tau\dot\sigma^2(t)\Phi^2(\sigma(t)) + \\ &+\delta\varphi^2(\sigma(t))\}e^{2\lambda t}\,dt, \\ J_{2T} &\triangleq \int\limits_0^T \{\varkappa\varphi(\sigma(t))\dot\sigma(t) + \varepsilon\dot\sigma^2(t)\}e^{2\lambda t}\,dt. \end{aligned} \tag{32}$$

In order to apply Bakaev–Guzh procedure introduce the function

$$\Psi(\sigma) = \varphi(\sigma) - \nu|\varphi(\sigma)|\Phi(\sigma). \tag{33}$$

Then

$$\begin{aligned} J_{1T} = \int\limits_0^T \{&(1-\varkappa)\nu|\varphi(\sigma(t))|\dot\sigma(t)\Phi(\sigma(t)) + \tau\dot\sigma^2(t)\Phi^2(\sigma(t)) + \\ &+\delta\varphi^2(\sigma(t))\}e^{2\lambda t}\,dt + (1-\varkappa)\int\limits_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt \end{aligned} \tag{34}$$

The first addend in the right–hand part of (34) is positive definite in virtue right-hand part of (30). Consider the second summand:

$$\int\limits_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt = e^{2\lambda T}\int\limits_{\sigma(T_0)}^{\sigma(T)}\Psi(\zeta)\,d\zeta, \tag{35}$$

where $T_0 \in [0,T]$. Since

$$\int\limits_0^{\Delta}\Psi(\zeta)\,d\zeta = 0, \Psi(\sigma_1) = \Psi(\sigma_2) = 0,\ \Psi(\zeta)\varphi(\zeta) \ge 0 \tag{36}$$

we conclude that

$$\int\limits_{\sigma(T_0)}^{\sigma(T)}\Psi(\zeta)\,d\zeta \ge 0 \quad (T \in \Sigma). \tag{37}$$

Then it follows then from (29) and (31) that

$$J_{2T} \le C_1, \quad (T \in \Sigma), \tag{38}$$

where C_1 does not depend on T.

We shall apply the nonlocal reduction technique now and consider the equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma} + 2\sqrt{\frac{\lambda\varepsilon}{\varkappa}}F(\sigma) + \varphi(\sigma) = 0. \tag{39}$$

It follows from left-hand part of (30) that (39) has solutions $F_k(\sigma)$ with the properties (9). Note that $\hat F_k = \sqrt{\frac{\varkappa}{2}}F_k$ is a solution of the equation

$$\hat F(\sigma)\hat F'(\sigma) + \sqrt{2\lambda\varepsilon}\hat F + \frac{\varkappa}{2}\varphi(\sigma) = 0. \tag{40}$$

Inject $\hat F_k$ into J_{2T}:

$$\begin{aligned} J_{2T} = \int\limits_0^T \{&G(\dot\sigma(t), \varphi(\sigma(t)), \hat F_k(\sigma(t))\hat F_k'(\sigma(t))) - \\ &-\tfrac{1}{4\varepsilon_1}(\varkappa\varphi(\sigma(t)) + 2\hat F_k(\sigma(t))\hat F_k'(\sigma(t)))^2 + \\ &+2\lambda\hat F_k^2(\sigma(t))\}e^{2\lambda t}\,dt - \hat F_k^2(\sigma(T)))e^{2\lambda T} + F_k^2(\sigma(0)), \end{aligned} \tag{41}$$

where

$$G(x,y,z) = (\sqrt{\varepsilon}x + \frac{\varkappa}{2\sqrt{\varepsilon}}y + \frac{1}{\sqrt{\varepsilon}}z)^2. \tag{42}$$

Then

$$J_{2T} \ge -\hat F_k^2(\sigma(T))e^{2\lambda T} + \hat F_k^2(\sigma(0)). \tag{43}$$

Since $\hat F_k$ is a solution of (40) the first summand in right–hand part of (43) is equal to zero. Then it follows from (43) and (38) that

$$\hat F_k^2(\sigma(t))e^{2\lambda t} \ge \hat F_k^2(\sigma(0)) - C_1 \text{ for } \sigma(t) = \sigma_2 + \Delta l\ (l,k \in Z). \tag{44}$$

Let us choose the number $k_0 \in N$ so large that

$$\sigma_2 - \Delta k_0 < \sigma(0) < \sigma_2 + \Delta k_0, \tag{45}$$

and

$$\hat F_{\pm k_0}^2(\sigma(0)) > C_1. \tag{46}$$

The inequality (46) implies that

$$\sigma_2 - \Delta k_0 < \sigma(t) < \sigma_2 + \Delta k_0. \tag{47}$$

Theorem 1 is proved. ■

IV. Gradient–like behavior

In this section we assume that $b(t)$ and $\gamma(t)$ have piece–wise continuous derivatives and

$$\dot{b}(t)e^{rt}, \dot{\gamma}(t)e^{rt}, b(t)e^{rt} \in L_2[0, +\infty) \quad (r > 0). \tag{48}$$

Theorem 2: Let $h = 0$. Suppose all the conditions of (1) are fulfilled and besides

$$\alpha_1^{-1}\alpha_2^{-1} = 0, \tag{49}$$

$$\rho(\alpha_2^{-1} - \alpha_1^{-1}) \le 0. \tag{50}$$

Then system (1) is gradient–like.

Proof: Consider the functionals

$$\begin{aligned} J_{3T} = \int_0^T \{\bar{\varkappa}\eta(t)\dot{\sigma}(t) + \varepsilon\dot{\sigma}^2(t) + \delta\eta^2(t)\}e^{2\lambda t}\,dt + \\ +\tau\int_0^T (\dot{\sigma}^2(t) + (\alpha_2^{-1} + \alpha_1^{-1})\ddot{\sigma}(t)\eta(t))e^{2\lambda t}\,dt, \end{aligned} \tag{51}$$

where $\bar{\varkappa} = 1 + 2\lambda\tau(\alpha_2^{-1} + \alpha_1^{-1})$ Let us demonstrate that for all $T > 1$ the following estimate is true:

$$J_{3T} \le C_1, \tag{52}$$

where C_1 does not depend on T.

We shall use here functions $\eta_T(t), \zeta_T(t)$ defined by formulas (17), (18), for all $T > 1$. Let $\alpha_1^{-1} = 0$, whence $\rho\tau\alpha_2^{-1} \le 0$. Determine the functions $\bar{\zeta}_T(t) = \zeta_T(t) - \rho\eta_T(t)$ and consider the functionals

$$\begin{aligned} R_{1T} \triangleq \int_0^T \{(1 + \lambda\tau\alpha_2^{-1}[\eta_T]^\lambda(t)[\zeta_T]^\lambda(t) + \\ +(\tau + \varepsilon)([\zeta_T]^\lambda(t))^2 + \delta([\eta_T]^\lambda(t))^2 + \\ +\tau\alpha_2^{-1}(\frac{d}{dt}[\zeta_T]^\lambda(t))[\eta_T]^\lambda(t)\}\,dt \end{aligned} \tag{53}$$

Note that

$$\begin{aligned} \mathfrak{F}(\frac{d}{dt}[\zeta_T]^\lambda(t))(\imath\omega) = -\imath\omega(K(\imath\omega - \lambda) + \\ +\rho)\mathfrak{F}([\eta_T]^\lambda(\imath\omega). \end{aligned} \tag{54}$$

Then in view of Plancherel theorem we have

$$\begin{aligned} R_{1T} = \frac{1}{2\pi}\int_{\infty}^{+\infty} Re\{(1 + \lambda\tau\alpha_2^{-1})K(\imath\omega - \lambda) - (\tau + \varepsilon)|K(\imath\omega - \\ -\lambda)|^2 + \delta + \imath\omega\tau\alpha_2^{-1}(K(\imath\omega - \lambda)|\mathfrak{F}([\eta_T]^\lambda)(\imath\omega)|^2\,d\omega \end{aligned} \tag{55}$$

whence in virtue of frequency-domain condition it follows that

$$R_T \le 0 \tag{56}$$

On the other hand

$$R_{1T} \ge J_{3T} - \tau\alpha_2^{-1}\rho\int_0^T \frac{([\eta_T]^\lambda(t))}{dt}[\eta_T]^\lambda(t)\,dt + J_{4T} \tag{57}$$

where J_{4T} is uniformly bounded. The inequalities (56) and (57) imply (52).

The case $\alpha_2^{-1} = 0$ can be treated in the same way as the previous one.

Introduce the functional

$$\begin{aligned} J_{5T} \triangleq \int_0^T \{\bar{\varkappa}\eta(t)\dot{\sigma}(t) + \varepsilon\dot{\sigma}^2(t) + \delta\eta^2(t) + \\ +\tau(\dot{\sigma}^2(t) + (\alpha_1 + \alpha_2)\ddot{\sigma}(t)\eta(t))\}\,dt, \end{aligned} \tag{58}$$

$$J_{5T} = J_{3\hat{T}} \tag{59}$$

where $\hat{T} \in [0, T]$. Then

$$J_{5T} < C_1 \quad \forall T > 1. \tag{60}$$

From (58) we have

$$\begin{aligned} J_{5T} = \varkappa\int_{\sigma(0)}^{\sigma(T)} \varphi(\sigma)\,d\sigma + \varepsilon\int_0^T \dot{\sigma}^2(t)\,dt + \delta\int_0^T \dot{\eta}^2(t)\,dt + \\ +\tau\int_0^T (\dot{\sigma}^2(t) - (\alpha_1^{-1} + \alpha_2^{-1})\dot{\sigma}(t)\dot{\eta}(t))\,dt + \\ +\tau(\alpha_1^{-1} + \alpha_2^{-1}))\dot{\sigma}(T)\dot{\eta}(T) - \dot{\sigma}(0)\dot{\eta}(0)). \end{aligned} \tag{61}$$

Since all the conditions of Theorem 1 are fulfilled every solution $\sigma(t)$ is bounded on $[0, +\infty)$, which implies together with (61) and (60) that

$$\dot{\sigma}(t), \varphi(\sigma(t)) \in L_2[0, +\infty). \tag{62}$$

It is now easy to show [6] that (62) entails the relations (6). ■

V. Conclusion

For a class of infinite dimensional synchronization systems with multiple equilibria new frequency–algebraic stability criteria are proposed. The criteria are obtained by combining Leonov's idea of nonlocal reduction and Bakaev–Guzh technique with method of a priori integral indices.

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