# Leonov's method of nonlocal reduction for pointwise stability of phase systems

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***Abstract*—In this paper we go on with the analysis of the asymptotic behavior of Lur'e–type systems with periodic nonlinearities and infinite sets of equilibria. It is well known by now that this class of systems can not be efficiently investigated by the second Lyapunov method with the standard Lur'e–Postnikov function ("a quadratic form plus an integral of the nonlinearity"). So several new methods have been elaborated within the framework of Lyapunov direct method. The nonlocal reduction technique proposed by G.A. Leonov in the 1980s is based on the comparison principle. The feedback system is reduced to a low-order system with the same nonlinearity and known asymptotic behavior. Its trajectories are injected into Lyapunov function of the original system. In this paper we develop the method of nonlocal reduction. We propose a new Lyapunov–type function which involves both the trajectories of the comparison system and a modified Lur'e–Postnikov function. As a result a new frequency–algebraic criterion ensuring the convergence of every solution to some equilibrium point is obtained.**

***Index Terms*—Nonlinear system, periodic nonlinearity, Lyapunov–type function, point–wise stability.**

## I. Introduction

The paper is devoted to asymptotic behavior of Lur'e-type systems with periodic nonlinearities and infinite sets of equilibria. Such systems are usually called pendulum-like systems or synchronization systems [1], [2]. This class of systems involves models of damped pendulums, electric motors, power generators, vibrational units, and various synchronization circuits such as phase and frequency locked loops [3]–[6]. Stability and oscillations of synchronization systems have been investigated in many published works (see [8] and bibliography there).

Because of specific character of these systems standard Lyapunov function such as a quadratic form plus the integral of the nonlinearity (Lur'e–Postnikov function) is of no good here. That is why special methods have been elaborated within the framework of Lyapunov direct method.The most efficient proved to be the method of periodic Lyapunov functions and the method of nonlocal reduction.

The method of periodic Lyapunov functions has been proposed in [9]. The method uses a modified Lur'e–Postnikov function. By special procedure (Bakaev–Guzh technique) the original periodic nonlinearity is substituted by a periodic nonlinearity with a zero mean value.

The method of nonlocal reduction have been developed by G.A.Leonov [10]–[15]. It is based on the "comparison principle" [16]. A low-order comparison system with the same nonlinearity as the original one and well-known asymptotic behavior is exploited. Its trajectories are used to construct Lyapunov-type functions for the original system. Each of the two methods has its own advantages.

In this paper we combine the two aforementioned methods and the Kalman–Yakubovich–Popov (KYP) lemma [13], obtaining new "frequency–algebraic" stability criteria. Along with Lagrange stability, previously studied in [17], we establish a new criterion of "pointwise" stability, that is, convergence of every solution to one of the equilibria.

## II. The statement of the problem

We consider the system of indirect control with periodic nonlinearity:

$$\begin{aligned} \frac{dz(t)}{dt} &= Az(t) + b\varphi(\sigma(t)), \\ \frac{d\sigma(t)}{dt} &= c^* z(t) + \rho\varphi(\sigma(t)). \end{aligned} \tag{1}$$

Here $A \in \mathbb{R}^{m\times m}$, $b, c \in \mathbb{R}^m$, $\rho \in \mathbb{R}$, $z : \mathbb{R}_+ \to \mathbb{R}^m$, $\sigma : \mathbb{R}_+ \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, the symbol (*) means the Hermitian conjugation.

We assume that matrix $A$ is Hurwitz, the pair $(A, b)$ is controllable and the pair $(A, c)$ is observable. The function $\varphi$

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is $C^1$-smooth, and $\Delta$–periodic:

$$\varphi(\sigma+\Delta)=\varphi(\sigma),\ \forall\sigma\in\mathbb{R}. \tag{2}$$

It has two zeros: $0\le\sigma_1<\sigma_2<\Delta$ such that

$$\varphi'(\sigma_1)>0,\ \varphi'(\sigma_2)<0. \tag{3}$$

We suppose for the definiteness that

$$\int_0^\Delta \varphi(\sigma)d\sigma\le 0. \tag{4}$$

Notice that if $(z(t),\sigma(t))^T$ is a solution of (1) then $(z(t),\sigma(t)+\Delta k)^T$ $(k\in Z)$ also is a solution of (1). So (1) has a cylindric phase space.

System (1) has a denumerable set of Lyapunov stable and Lyapunov unstable equlibria $\Lambda=\{(0,\sigma_j+\Delta k)^T : j=1,2,k\in Z\}$. Its asymptotic behavior is described by two types of stability: Lagrange stability (every solution is bounded) and point-wise stability (every solution converges). The latter is often characterized as gradient–like behavior.

In this paper we firstly establish a frequency-algebraic criterion for Lagrange stability and then add the conditions that guarantee the point-wise convergence of solutions.

## III. Lagrange stability

Consider the equation of the second order

$$\ddot\sigma+a\dot\sigma+\varphi(\sigma)=0\quad(a>0). \tag{5}$$

In case $\varphi(\sigma)=\sin\sigma-\beta$ $(\beta\in(0,1))$ it is the equation of mathematical pendulum. It also describes the phase-locked loop with the integrating lowpass filter. It can serve as the simplistic mathematical model for the synchronous machine.

The equation (5) is equivalent to the system

$$\begin{aligned}&\dot z=-az-\varphi(\sigma)\quad(a>0),\\&\dot\sigma=z.\end{aligned} \tag{6}$$

which has been exhaustively investigated (see for example [14, pp. 185-201] and the bibliography there). Further we shall use the results of this investigation.

Namely for any $\varphi(\sigma)$ there exists a bifurcation value $a_{cr}$ such that for $a>a_{cr}$ every solution of (6) converges and for $a\le a_{cr}$ the system (6) has solutions with $z(t)=\dot\sigma(t)\ge\varepsilon>0$. It must be noted that points $(0,\sigma_2+\Delta k)$ $(k\in Z)$ are saddle points for (6). They have two separatrices that "go into" them as $t\to+\infty$.

Consider the first order equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma}+aF(\sigma)+\varphi(\sigma)=0\quad(F=\dot\sigma=z). \tag{7}$$

associated with (6). If $a>a_{cr}$ it has a solution $F_0(\sigma)$ with the following properties:

$$F_0(\sigma_2)=0, \tag{8}$$

$$F_0(\sigma)\ne0,\quad\text{if}\quad\sigma\ne\sigma_2, \tag{9}$$

$$F_0(\sigma)\to\pm\infty\quad\text{as}\quad\sigma\to\mp\infty. \tag{10}$$

This solution "consists" of the two separatrices "going into" $(0,\sigma_2)$.

Introduce the constants

$$\mu_1\triangleq\inf_{\sigma\in[0,\Delta)}\varphi'(\sigma),\ \mu_2\triangleq\sup_{\sigma\in[0,\Delta)}\varphi'(\sigma)\quad(\mu_1,\mu_2<0), \tag{11}$$

$$\nu\triangleq\frac{\int_0^\Delta\varphi(\sigma)d\sigma}{\int_0^\Delta|\varphi(\sigma)|\,d\sigma}. \tag{12}$$

$$\lambda_0\triangleq\min_{i=1,\dots,m}|Re\,\lambda_i|\quad(i=1,\dots,m), \tag{13}$$

where $\lambda_i$ is an eigenvalue of the matrix $A$.

Introduce the transfer function of the linear part of (1)

$$K(p)=-\rho+c^*(A-pI_m)^{-1}b\quad(p\in\mathbb{C}), \tag{14}$$

where $I_m$ is an $m\times m$ - unit matrix.

**Theorem 1:** Suppose there exist positive $\tau,\varepsilon,\delta,$ $\lambda\in(0,z_0)$, $\alpha_1\le\mu_1,\alpha_2\le\mu_2$ such that the following conditions are satisfied:
1) the inequality

$$\begin{gathered}Re\{K(i\omega-\lambda)-\tau(K(i\omega-\lambda)+\\+\alpha_1^{-1}(i\omega-\lambda))^*(K(i\omega-\lambda)+\alpha_2^{-1}(i\omega-\lambda))\}-\\-\varepsilon|K(i\omega-\lambda)|^2\ge\delta\end{gathered} \tag{15}$$

is valid for all $\omega\ge0$;
2) for $\varkappa_1\in[0,1]$ the quadratic form

$$\begin{aligned}&P(x,y,z)=\lambda x^2+\varepsilon y^2+\delta z^2+(1-\varkappa_1)\nu yz+\\&+a_{cr}\sqrt{\varkappa_1}xy\end{aligned} \tag{16}$$

is positive definite.
Then system (1) is Lagrange stable.

*Proof:*
Consider the system

$$\begin{aligned}\frac{dy}{dt}&=Qy(t)+L\eta(t),\\\frac{d\sigma}{dt}&=D^*y(t).\end{aligned} \tag{17}$$

where $y(t)=(z(t),\varphi(\sigma(t)))^T$, $\eta(t)=\frac{d}{dt}\varphi(\sigma(t))$,

$$Q=\begin{bmatrix}A&b\\0&0\end{bmatrix},L=\begin{bmatrix}0\\1\end{bmatrix},D=\begin{bmatrix}c^*\\\rho\end{bmatrix}. \tag{18}$$

Introduce the quadratic form

$$\begin{gathered}G(y,\eta)=2y^*H((Q+\lambda I_{m+1})y+L\eta)+\delta(L^*y)^2+\\+\varepsilon(D^*y)^2+\varkappa y^*LD^*y+\tau(D^*y-\alpha_1^{-1}\eta)(D^*y-\\-\alpha_2^{-1}\eta)\qquad(y\in\mathbb{R}^{m+1},\eta\in\mathbb{R}).\end{gathered} \tag{19}$$

By Kalman-Yakubovich-Popov lemma the inequality (15) guaranties that there exists a matrix $H=H^*$ [13] such that

$$G(y,\eta)\le0,\qquad\forall y\in\mathbb{R}^{m+1},\eta\in\mathbb{R}. \tag{20}$$

Let

$$H=\begin{bmatrix}H_0&h\\h^*&\alpha\end{bmatrix}\quad(H_0\in\mathbb{R}^{m\times m},h\in\mathbb{R}^m,\alpha\in\mathbb{R}). \tag{21}$$

Then for $\bar{y} = (z, 0)^T$ it is true that

$$G(\bar{y}, 0) = 2z^* H_0(A+\lambda I_m)z + (\varepsilon+\tau)(c^* z)^2 \ \forall z \in \mathbb{R}^m. \quad (22)$$

Since the pair $(A + \lambda I_m, b)$ is controllable the matrix $H_0$ is positive definite [18]. So if $\varphi(\sigma(\bar{t})) = 0$ one has

$$y^*(\bar{t}) H y(\bar{t}) > 0, \quad z(\bar{t}) \neq 0. \quad (23)$$

Consider the condition 2) of the Theorem. It implies that

$$\varepsilon > \frac{a_{cr}^2 \varkappa_1}{4\lambda} + \frac{(1-\varkappa_1)^2 \nu^2}{4\delta}. \quad (24)$$

Let $\varepsilon = \varepsilon_1 + \varepsilon_2$, where

$$\varepsilon_2 \triangleq \frac{(1-\varkappa_1)^2 \nu^2}{4\delta}. \quad (25)$$

Then

$$\varepsilon_1 > \frac{a_{cr}^2 \varkappa_1}{4\lambda}. \quad (26)$$

System

$$\begin{array}{l} dotz = 2\sqrt{\lambda \varepsilon_1} z + \varkappa_1 \varphi(\sigma), \\ \dot{\sigma} = z. \end{array} \quad (27)$$

by linear change of variable $t$ can be transformed to the system (6) with $a = 2\sqrt{\frac{\lambda \varepsilon_1}{\varkappa_1}}$. So the equation

$$F(\sigma) \frac{dF(\sigma)}{d\sigma} + 2\sqrt{\lambda \varepsilon_1} F(\sigma) + \varkappa_1 \varphi(\sigma) = 0. \quad (28)$$

in virtue of (26) has the solutions $F_0(\sigma)$ with the properties (8), (9), (10).

Introduce the Lyapunov function:

$$W(t) = y^*(t) H y(t) \quad (29)$$

with matrix $H$ from (20) and the Lyapunov functions

$$V_k(t) = W(t) - \frac{1}{2} F_k^2(\sigma(t)) + \varkappa_2 \int_{\sigma_2}^{\sigma_k(t)} \Psi_0(\zeta) d\zeta, \quad (30)$$

with $\varkappa_2 = 1 - \varkappa_1$,

$$\Psi_0(\zeta) \triangleq \varphi(\zeta) - \nu |\varphi(\zeta)|, \quad (31)$$

and $F_k(\sigma) = F_0(\sigma + \Delta k)$ $(k \in Z)$.

It is obvious that

$$\int_0^{\Delta} \Psi_0(\zeta) d\zeta = 0. \quad (32)$$

In virtue of system (17)

$$\begin{array}{l} \dot{V}_k(t) + 2\lambda V_k(t) = 2y^*(t) H((Q + \lambda I_{m+1}) y(t) + \\ + L\varphi(\sigma(t))) - F_k'(\sigma(t)) F_k(\sigma(t)) \dot{\sigma}(t) + \varkappa_2 \Psi_0(\sigma(t)) \dot{\sigma}(t) - \\ - \lambda F_k^2(\sigma(t)) + 2\lambda \varkappa_2 \displaystyle\int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta) d\zeta. \end{array} \quad (33)$$

Now we can apply the inequality (20)

$$\begin{array}{l} \dot{V}_k(t) + 2\lambda V_k(t) \leq -\varepsilon \dot{\sigma}^2(t) - \\ - \delta \varphi^2(\sigma(t)) - \dot{\sigma}(t) \varphi(\sigma(t)) + \\ + 2\sqrt{\lambda \varepsilon_1} F_k(\sigma(t)) \dot{\sigma}(t) + \varkappa_1 \varphi(\sigma(t)) \dot{\sigma}(t) + \\ + \varkappa_2 \varphi(\sigma(t)) \dot{\sigma}(t) - \varkappa_2 \nu |\varphi(\sigma(t))| \dot{\sigma}(t) - \lambda F_k^2(\sigma(t)) + \\ + 2\lambda \varkappa_2 \displaystyle\int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta) d\zeta. \end{array} \quad (34)$$

It follows from (4) and (32) that

$$\int_{\sigma_2}^{\sigma} \Psi_0(\zeta) d\zeta \leq 0, \quad \forall \sigma. \quad (35)$$

Then

$$\begin{array}{l} \dot{V}_k(t) + \lambda V_k(t) \leq -(\varepsilon_2 \dot{\sigma}^2(t) + \delta \varphi^2(\sigma(t)) + \\ + \varkappa_2 \nu |\varphi(\sigma(t))| \dot{\sigma}(t)) - (\varepsilon_1 \dot{\sigma}^2(t) - 2\sqrt{\lambda \varepsilon_1} F_k(\sigma(t)) \dot{\sigma}(t) + \\ + \lambda F_k^2(\sigma(t))), \quad \forall t \geq 0, \end{array} \quad (36)$$

whence in virtue of (25)

$$\dot{V}_k(t) + 2\lambda V_k(t) \leq 0 \quad \forall t \geq 0. \quad (37)$$

Hence

$$V_k(t) e^{2\lambda t} \leq V_k(0), \quad \forall t \geq 0, \forall k \in Z. \quad (38)$$

Notice that

$$V_k(0) = y^*(0) H y(0) - \tfrac{1}{2} F_k^2(\sigma(0)) + \varkappa_2 \int_{\sigma_2}^{\sigma(0)} \Psi(\zeta) d\zeta \quad (39)$$

The property (10) of $F_0(\sigma)$ implies that one can always choose a natural $k_0 \in \mathbb{N}$ in such a way that

$$V_{\pm k_0}(0) < 0. \quad (40)$$

Then

$$V_{\pm k_0}(t) < 0, \quad \forall t \geq 0. \quad (41)$$

The inequality (41) gives the opportunity to prove the Lagrange stability of (1).

Suppose $\bar{t}$ is such that

$$\sigma(\bar{t}) = \sigma_2 + \Delta K, \quad K \in Z. \quad (42)$$

Then $\varphi(\sigma(\bar{t})) = 0$,

$$\int_{\sigma_2}^{\sigma(\bar{t})} \Psi_0(\zeta) d\zeta = 0 \quad (43)$$

and it follows from (23) that

$$y^*(\bar{t}) H y(\bar{t}) = z^*(\bar{t}) H_0 z(\bar{t}) \geq 0. \quad (44)$$

The inequality (41) implies that

$$F_{\pm k_0}^2(\sigma(\bar{t})) \neq 0 \quad (45)$$

whence

$$\sigma_2 - \Delta k_0 < \sigma(\bar{t}) < \sigma_2 + \Delta k_0. \quad (46)$$

Hence for any $z(0), \sigma(0)$ there exist a $k_0 \in \mathbb{N}$ such that

$$\sigma_2 - \Delta k_0 < \sigma(t) < \sigma_2 + \Delta k_0, \quad \forall t \geq 0. \quad (47)$$

Theorem 1 is proved. ■

## IV. Point–wise stability

**Theorem 2:** Suppose there exist positive $\tau, \varepsilon, \delta$, $\lambda \in (0, \lambda_0)$, $\alpha_1 \le \mu_1, \alpha_2 \le \mu_2$ such that

$$\tau \alpha_1^{-1} \alpha_2^{-1} = 0, \tag{48}$$

$$\tau(\alpha_1^{-1} + \alpha_2^{-1})\rho \le 0 \tag{49}$$

and all the conditions of Theorem 1 are satisfied. Then the following relations are true

$$\lim_{t \to +\infty} \dot{\sigma}(t) = 0, \tag{50}$$

$$\lim_{t \to +\infty} z(t) = 0, \tag{51}$$

$$\lim_{t \to +\infty} \varphi(\sigma(t)) = 0, \tag{52}$$

$$\lim_{t \to +\infty} \sigma(t) = q, \tag{53}$$

where $\varphi(q) = 0$.

*Proof:* Consider separately a Lyapunov function $W(t)$ introduced by (29). In virtue of system (17) one has

$$\frac{dW}{dt} + 2\lambda W(t) = 2y^*(t)H[(Q + \lambda I_{m+1})y(t) + L\varphi(\sigma(t))]. \tag{54}$$

Since the frequency–domain inequality (15) is fulfilled there exists a matrix $H = H^*$ such that the inequality (20) is valid, whence

$$\frac{dW}{dt} + 2\lambda W(t) + \varepsilon \dot{\sigma}^2(t) + \dot{\sigma}(t)\varphi(\sigma(t)) + \delta\varphi^2(\sigma(t)) \le 0. \tag{55}$$

It follows from (48) that the quadratic form $G(y, \eta)$ is linear with respect to $\eta$. Since $G(y, \eta)$ is nonnegative for all $y, \eta$, we conclude that

$$2HL = \tau(\alpha_1^{-1} + \alpha_2^{-1})D. \tag{56}$$

Taking into account (21) we get

$$2h = \tau(\alpha_1^{-1} + \alpha_2^{-1})c, \; 2\alpha = \tau(\alpha_1^{-1} + \alpha_2^{-1})\rho. \tag{57}$$

Then

$$\begin{aligned} W(t) = z^*(t)H_0 z(t) + \tau(\alpha_1^{-1} + \alpha_2^{-1})c^* z(t)\varphi(\sigma(t)) + \\ + \tfrac{1}{2}(\alpha_1^{-1} + \alpha_2^{-1})\rho\varphi^2(\sigma(t)). \end{aligned} \tag{58}$$

The relations (55) and (58) together with the equations (17) imply:

$$\begin{aligned} &\tfrac{dW}{dt} + (2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\dot{\sigma}(t)\varphi(\sigma(t)) - \\ &-\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1})\rho\varphi^2(\sigma(t)) + 2\lambda z^*(t)H_0 z(t) + \\ &+\varepsilon\dot{\sigma}^2(t) + \delta\varphi^2(\sigma(t)) \le 0, \end{aligned} \tag{59}$$

whence

$$\begin{aligned} &\tfrac{d}{dt}\{W(t) + (2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\int_{\sigma(0)}^{\sigma(t)} \varphi(\zeta)d\zeta\} + \\ &+\varepsilon\dot{\sigma}^2(t) + \delta\varphi^2(\sigma(t)) \le \lambda\tau(\alpha_1^{-1} + \alpha_2^{-1})\rho\varphi^2(\sigma(t)) - \\ &-2\lambda z^*(t)H_0 z(t). \end{aligned} \tag{60}$$

Notice that since $H_0$ is positive definite and the inequality (49) is true the first summand in the right part of (60) is negative. Then one can deduce from (60) that

$$\begin{aligned} &\varepsilon\int_0^t \dot{\sigma}^2(\tau)d\tau + \delta\int_0^t \varphi^2(\sigma(\tau))d\tau \le W(0) - W(t) - \\ &-(2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\int_{\sigma(0)}^{\sigma(t)} \varphi(\zeta)d\zeta. \end{aligned} \tag{61}$$

All the conditions of Theorem 1 are fulfilled here. So every solution of system (17) is bounded for $t \in \mathbb{R}_+$. It follows that right part of (61) is also bounded, for $t \in \mathbb{R}_+$. Thus

$$\int_0^\infty \dot{\sigma}^2(t)dt < +\infty, \; \int_0^\infty \varphi^2(\sigma(t))dt < +\infty. \tag{62}$$

It is easy to establish that the relations (50)– (53) follow from (62) [14]. Theorem 2 is proved. ■

## V. Conclusion

In this paper we combine two efficient methods for stability investigation of Lur'e-type pendulum-like systems. Using the Kalman-Yakubovich-Popov lemma and a novel Lyapunov–type function, we obtain a new frequency-algebraic criterion ensuring the converges of every solution to an equilibrium.

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