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Tangible Tools in Mathematics for Engineering Students: Experimental Activity at Politecnico di Torino

(Article begins on next page)

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Tangible tools in Mathematics for Engineering students:

an experimental activity at Politecnico di Torino

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Abstract. In the paper we present the activities of the Maths lab at Politecnico di Torino that we tested for the first time in the academic year 2018-19. These were proposed to undergraduate Engineering students in first year with the aim to integrate active learning into our traditional courses. The activities took place in classes of at most 25 students. The work was organized in small groups, mainly following a problem solving approach. A central role was given to artefacts and technological applications. In the paper we discuss the main motivations of our lab project, two specific activities and some data collected by means of a questionnaire in order to examine the interplay among tangible and virtual tools.

1. Introduction

We think of a Maths lab as a place where one can see, touch and practise mathematical objects. It is a place where one is encouraged to develop a mathematical view on reality. This is something experienced in education, especially in primary and middle school (Sarama and Clements, 2016), but there is a common, unexpressed opinion that Maths lab fits only for children and teenagers. In 2003, U.M.I. (Unione Matematica Italiana, the Italian Mathematical Society) gave a sort of definition of Maths lab for secondary school: *A mathematics laboratory is not intended as opposed to a classroom, but rather as a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people (students and teachers), structures (classrooms, tools, organisation and management), ideas (projects, didactical planning and experiments)* (Anichini et al., 2004). Our idea is that Maths lab is good for all ages students and this is why we are developing a project for undergraduate students in Engineering.

The Maths lab, hereafter called LA.M.PO. (Laboratorio di Matematica del Politecnico di Torino), was first introduced in the academic year 2018-19. In this paper we present the first outcomes of the project and some data collected during the year, with a special emphasis on the role given to tangible and virtual tools.

The project centered on LA.M.PO. aims to blend active learning into traditional courses. While planning our activities, we considered some characterizing features of our students and of our courses.

- First-year classes are very large: there are about 250 students in each class. These numbers imply that it is difficult to organize cooperative and active learning, even if this would be particularly fruitful for 19-year-old students. The interaction between students and the instructor is necessarily weak. In the cognitive process, large classes are an obstacle for reaching upper categories (analyse, create), and for most students, knowledge

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is limited to the first categories (remember, understand, and, partially, apply and analyse) (Anderson and Kratwohl, 2001; Alcock and Simpson, 2001).

- 19-year-old students are becoming adult and invite us to undertake an approach that tends to andragogy (Knowles, 1980). In this respect, a special focus is given to motivation to learning maths. They are enrolled in an Engineering program, but they mostly study fundamental subjects during their first-year courses (Mathematical Analysis, Chemistry, Computer Science, Linear Algebra, Physics) and do not see a direct application of the contents of the courses in the various fields of engineering they have chosen.

The effort to develop a study of mathematics connected to the real world led us to propose tools that help not only its visualization but also its exploration. This is not a trivial task. On one hand students have easy access to digital tools (e.g. GeoGebra, Wolfram Alpha) and they actually use them, even if not explicitly required by the teachers. On the other hand, they don't recognize mathematical objects around them. We then proposed different tangible tools in LA.M.PO.'s activities. Though the use of tangible models to introduce mathematical concepts may be natural in Engineering curricula (Bartolini Bussi et al., 2010), we realize it is rare, in Italy at least. In this sense, the work of Emma Castelnuovo is an everlasting font of inspiration (Castelnuovo, 1972; Castelnuovo, 2008): updating her ideas to our context is a challenge for our lab.

The article is organized in the following way: Section 2 briefly describes LA.M.PO.'s activities, Section 3 focuses on the use of tangible and virtual tools by means of the description of two activities, Section 4 presents students' feedback by means of data collected through a questionnaire, and conclusions are driven in Section 5.

2. LA.M.PO.'s activities

We designed our activities with these features: i) they have some engineering applications; ii) they can be developed by using contents studied in first-year courses; iii) they can be investigated by means of tangible and/or digital tools. This approach not only combines mathematics and technology, but it does so in a way that is grounded in students' everyday experiences and interest. The six activities are briefly described below.

Title	**Lab 1. Experimenting mathematical language with origami**
Description	The aim of this activity is to make students think about mathematical language and propositional logic by means of origami folding. This allows students to find out known theorems on flatness, a property that is very interesting in view of technological applications.
Tangible/virtual models	Origami/ Some videos by Brigham Young University
Applications	Solar lens, self-folding robot, temporary covering
Reference	Serre, S., & Spreafico, M.L. (2018). An Origami Project for Exploring the Learning of Mathematical Logic. In Lang,R., Bolitho,M., & You, Z., *Proceedings of 7OSME: 7th International Meeting on Origami in Science, Mathematics and Educations*, pp. 225-240. Hertfordshire, UK: Tarquin.

Title	**Lab 2. Mathematics and the parsimony of the Universe: minimum problems in Nature**
Description	Nature is parsimonious. One of the big achievements of the last century mathematics is having understood that this criterion allows to explain many natural phenomena. We propose some mathematical problems coming from applications (brachistochrone, catenary, isoperimetric problem…) that are formulated as minimum problems. We show

	that the expected solution is not always the correct one, also by also using concrete experiments.
Tangible/virtual models	Chain, wooden model of the line and brachistochrone
Applications	Light reflection and refraction, pendulum
Reference	Hildebrandt, S., & Tromba, A. (2012). *The parsimonious Universe.* New York: Springer.

Title	**Lab 3. The Mole Antonelliana and integral calculus**
Description	We propose a synthesis of the main results of integral calculus in one variable by means of different problems that one can pose in front of the Mole Antonelliana described as a developable surface.
Tangible/virtual models	3D printed vaults, translucid models, graphic tablet/ Geogebra, Tinkercad, C
Applications	Computation of areas and volumes of architectonic models
Reference	Pavignano, M., Cumino, C., Spreafico, M.L., & Zich, U. (2017). Paper models for the understanding of architectural shape: research, design, testing and teaching in a dialogue between Geometry and Representation. In Di Luggo, A., Giordano, P., Florio, R., Papa, L.M., Rossi, A., Zerlenga, O., Barba, S., Campi, M., & Cirafici, A. (Eds.), *Territori e frontiere della rappresentazione. UID-Unione Italiana Disegno. Atti del 39° Convegno Internazionale dei Docenti delle discipline della Rappresentazione,* pp. 1279-1286. Roma: Gangemi.

| Title	**Lab 4. Matrix, the internet heart**
Description	What does connect matrices with the internet? We try to explain it in a simple way and discover one of the algorithms that underlie our searches on the internet.
Tangible/virtual models	Not used
Applications	Internet search engines
Reference	Ishii, H., & Tempo, R. (2014). The PageRank problem, multiagent consensus, and web aggregation. *IEEE Control System Magazine, 34,* 34-53.

Title	**Lab 5. Conics and mechanisms**
Description	We trace conics with different methods: mathematical machines, origami and software. The aim is to discover some interesting properties of these curves that are useful for technological applications.
Tangible/virtual models	Origami models, 3D prints of conic sections, mathematical machines, mirascope/ Geogebra, Tinkercad, GIM
Applications	Car lights, hanging bridges, antennas
Reference	Bartolini Bussi, M., & Maschietto, M. (2006). *Macchine Matematiche.* Milano: Springer-Verlag. http://www.macchinematematiche.org

Title	**Lab 6. Mathematics and geographic maps**
Description	By means of experiments also, we want to introduce students to two problems related to geographical maps: the choice of a projection in order to represent the Earth's surface (which cannot be developed) on a plane and its coloring problem.
Tangible/virtual models	Inflatable globe, maps/ http://metrocosm.com/compare-map-projections.html, https://thetruesize.com/
Applications	Geographical maps
Reference	McCleary, J. (1995). *Geometry from a differentiable viewpoint.* Cambridge: Cambridge University Press. https://www.youtube.com/watch?v=kIID5FDi2JQ, https://www.youtube.com/watch?v=NgbK43jB4rQ&t=712s

For each activity we opened a call to all first-year students accepting up to 75 registrations and dividing them into three groups. Globally, 196 students participated to at least one the activities proposed. Students didn't have any formal incentive for participating to the activities.

Each group worked for three hours. As an alternative to our traditional frontal lectures, we started with the introductions of participants and instructors in order to create a climate of openness and collaboration between the students and between the students and the instructors (Holt, 2010). After a presentation of the topics, we posed a problem and we proposed to students to work in small groups while we were going around interacting with them. We encouraged them to exchange opinions, discuss strategies and cooperate in order to find solutions among peers (Hausmann et al., 2004). In fact, it is well known that the mechanisms developed during collaborative sessions are effective. To facilitate this process and to support conjectures formulation, we gave them tangible tools (origami, 3D models, etc.). Then we collected and commented remarks, conjectures, solutions to exercises and systematized the outputs of the groups' work. This final step is very important because it allows the instructor to observe students' cognitive process and get to the solution of the posed problems by using ideas and errors of the students.

At the end of each activity, we proposed a short evaluation questionnaire to participants, and, in some occasions, we also collected interviews. At the end of the whole project, we asked students to fill in an on-line questionnaire to get a global feedback on the project and some suggestions. We report and discuss some data in Section 4.
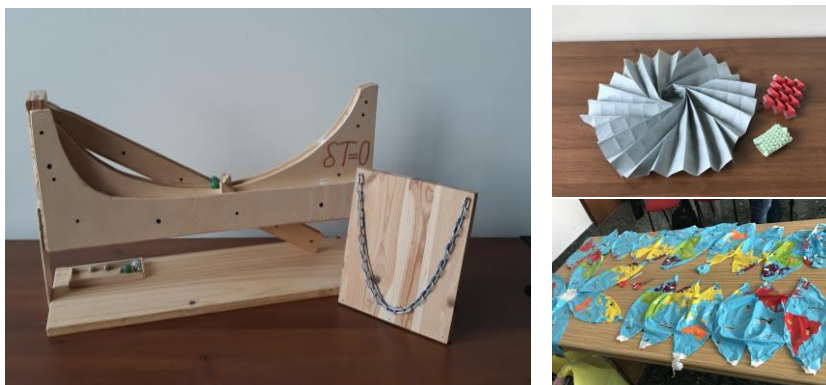

Fig.1. Examples of tangible tools. Left: A wooden model of a brachistochrone and a catenary made by Luca Lussardi (Lab 2). Top-right: solar lens and other origami tools applied to technology (Lab 1); bottom-right: inflatable globes (Lab 6). Photo credits by Maria Luisa Spreafico.

3. Tangible and virtual models: discussion of two labs

3.1 Lab 3: The Mole Antonelliana and integral calculus

This lab activity is inspired by (Pavignano et al., 2017) and is adapted to students who have already studied Riemann integrals in one variable. The Mole Antonelliana is the symbol of Turin: since our students come from all over Italy, they are curious about it. We explain the origin of the activity which was originated in the framework of a collaboration with architects. The toy problem we propose is to determine the area of the surface to be covered by a roofing system and the length of a wire for lighting the interior. In particular, we ask students to guess the ratio between the area of the roof and that of the floor.
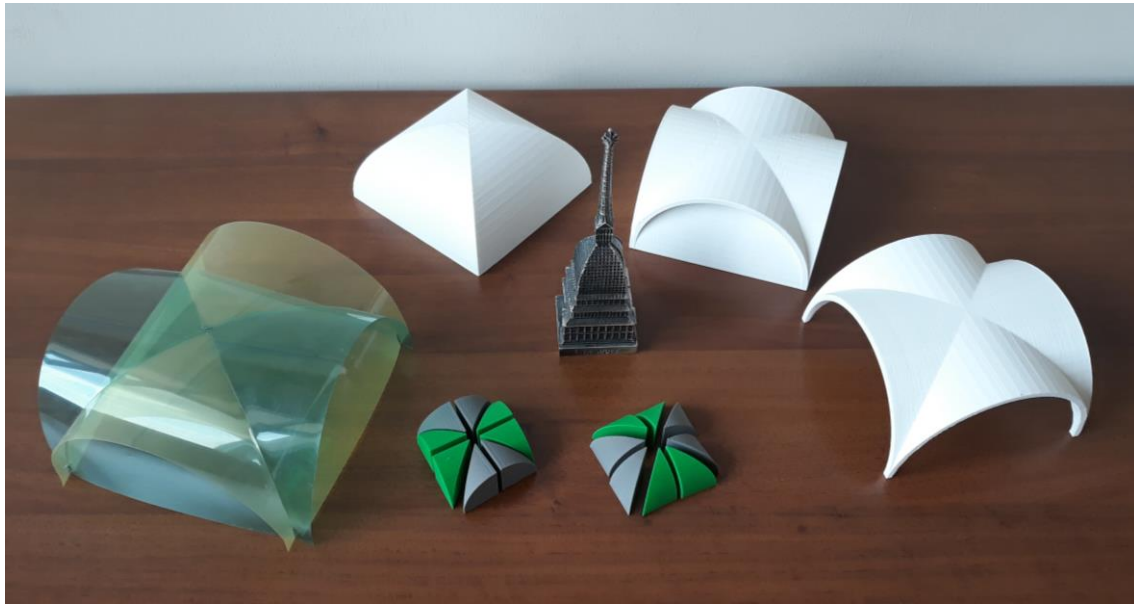
Fig.2. Tangible tools used in Lab 3: models of vaults obtained with a 3D printer (white, and green-grey), acetate sheets and an iron model of the Mole Antonelliana. Photo credits by Maria Luisa Spreafico.

We then ask students to describe the Mole geometrically. We show them the 3D printed model of a cloister vault obtained from the intersections of two cylinders and propose this as an approximation of the lower part of the Mole (see white models in Fig.2). An analogous 3D printed model is composed of two semi-circular cylinders, each one divided into four wedges. These can be reassembled in order to form a groin vault and a cloister vault (see green and grey models in Fig.2). Here we explain how the 3D printed models can be obtained. We have chosen to use Tinkercad, that is freely available on the web (https://www.tinkercad.com). We suggest students try it themselves and perhaps print their models in the Politecnico labs where some 3D printers are available. Another possibility would be to use GeoGebra, known by most students, but this software is better for second year students who have studied the functions of two variables. For students the task of using a geometric description in order to print a model is an intriguing one.

Going back to the problem posed, since guessing the ratio is not easy, we propose a simplified version of the task to be discussed in groups of 5-6 persons. A very rough approximation of the lower part of the Mole is a pyramid: students can work out computations on the pyramid by simply using the Pythagorean theorem.

In order to give to a better approximation of the roof, we then provide each group with two acetate sheets which have been cut so that they can be assembled to make a model analogous to the 3D printed models (see the acetate model, left in Fig. 2). This shows that the surface can be developed. The material of the acetate sheet can easily be flattened. This hands-on step is very effective: one could also show a virtual surface that displays its own flattening, but this would not be as compelling as pressing the acetate on the table with one's own hand. In this case even a very good virtual tool as GeoGebra is not as effective as passing from the plane sheet to the 3D cylinder and vice-versa.

At this point, students themselves see that the curve intersection of the two cylinder lies on a plane and it looks like the sine function (Fig.3 shows a "tangible proof" of these facts): they can now easily compute the vault's area on the basis of the model's measurements by computing a

definite integral. They realize that the ratio between the roof and the floor areas is 2, which is usually much higher than their initial guesses.

Fig.3. Flattened acetate sheets with evidenced sine function and cylinders intersecting in a plane. Photo credits by Maria Luisa Spreafico.

Then we move on to the computation of the length of a wire that is attached to the "diagonals" of the roof, namely to the intersections of the two cylinders. The students still don't know the formula to compute the length of a curve, so we work it out together. They can now apply it to the sine function, getting to the problem of finding a primitive of $\sqrt{1 + cos^2x}$, that is an elliptic integral. We briefly discuss the fact that since an elementary primitive cannot be found, the integral should be numerically computed.

The instructor suggests students reconstruct formulas for computing the approximated definite integrals explained in the lectures. Since students are learning the software C in the Computer Science course, they are invited to write a code to compute it. If there is not enough time the instructor briefly presents the code (see Fig.4). The workshop ends with an estimate of the length of the wire required to light the Mole.

Students react positively to the suggestion of creating 3D printed models and programming. Virtual tools and software are pervasive in this lab activity but they mainly realized before and after class: 3D models are prepared by some elder students before the class and are proposed to first year students as a continuation of the lab activity, and complete writing of the C-code usually goes out the lab's time.
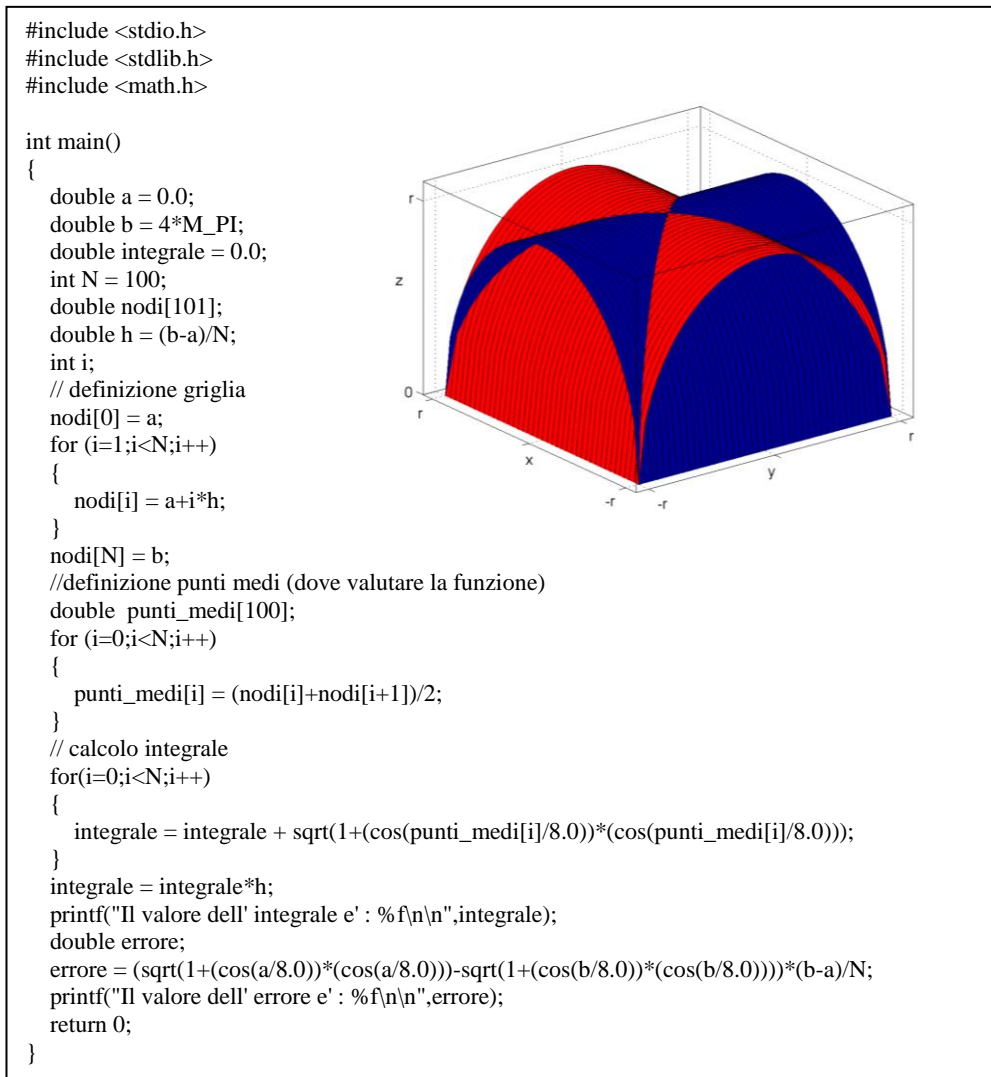
Questions and ideas for future work

Tangible tools offer a concrete experience while most digital tools require an a priori knowledge of the software used and therefore more time. Which are the best digital tools that can be used for several tasks as representation, computation, 3D printing?

As we already remarked students positively react to the possibility to produce 3D models. How can one plan other activities which produce a real object?

In our lab activity the development of the surface is of particular interest. This could be a starting point for a hands-on activity on surfaces, where students can practise constructing surfaces with different properties, such as ruled and revolution surfaces, through real objects.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

int main()
{
    double a = 0.0;
    double b = 4*M_PI;
    double integrale = 0.0;
    int N = 100;
    double nodi[101];
    double h = (b-a)/N;
    int i;
    // definizione griglia
    nodi[0] = a;
    for (i=1;i<N;i++)
    {
        nodi[i] = a+i*h;
    }
    nodi[N] = b;
    //definizione punti medi (dove valutare la funzione)
    double  punti_medi[100];
    for (i=0;i<N;i++)
    {
        punti_medi[i] = (nodi[i]+nodi[i+1])/2;
    }
    // calcolo integrale
    for(i=0;i<N;i++)
    {
        integrale = integrale + sqrt(1+(cos(punti_medi[i]/8.0))*(cos(punti_medi[i]/8.0)));
    }
    integrale = integrale*h;
    printf("Il valore dell' integrale e' : %f\n\n",integrale);
    double errore;
    errore = (sqrt(1+(cos(a/8.0))*(cos(a/8.0)))-sqrt(1+(cos(b/8.0))*(cos(b/8.0))))*(b-a)/N;
    printf("Il valore dell' errore e' : %f\n\n",errore);
    return 0;
}
```


Fig.4. The C-code for computing the elliptic integral and a virtual representation of the intersection of two cylinders made with MATLAB.

3.2 Lab 5: Conics and Mechanisms

The aim of this lab activity is to investigate definitions and construction of conics and to show some of their technological applications.

The first part of the lab activity consists in introducing conics as conic sections. Different points of view and models are proposed, and parabola is taken as the main example. We compare analytic description with geometrical representation and tangible with virtual tools.

At first, we distribute 3D models of cones cut with planes (printed with Tinkercad) and we look at the same intersections with a construction on GeoGebra. With 3D models students tangibly explore the intersections: for example, they can lay the sectioned cones down on a squared sheet, draw the contour and write the equation of the traced curve. On the other hand, GeoGebra gives a dynamic visualization of these intersections by means of a parametric description: the slider continuously modifies the intersecting plane position, and the eccentricity of the conic sections obtained is simultaneously checked.

Fig.5. Tangible tools used in Lab 5: a mirascope, a "water" model for conics, 3D printed models of conic sections, a parabola drawer with tightened thread. Photo credits by Maria Luisa Spreafico.

We also show that the equation of a parabola can be obtained intrinsically by means of Euclidean geometry techniques.
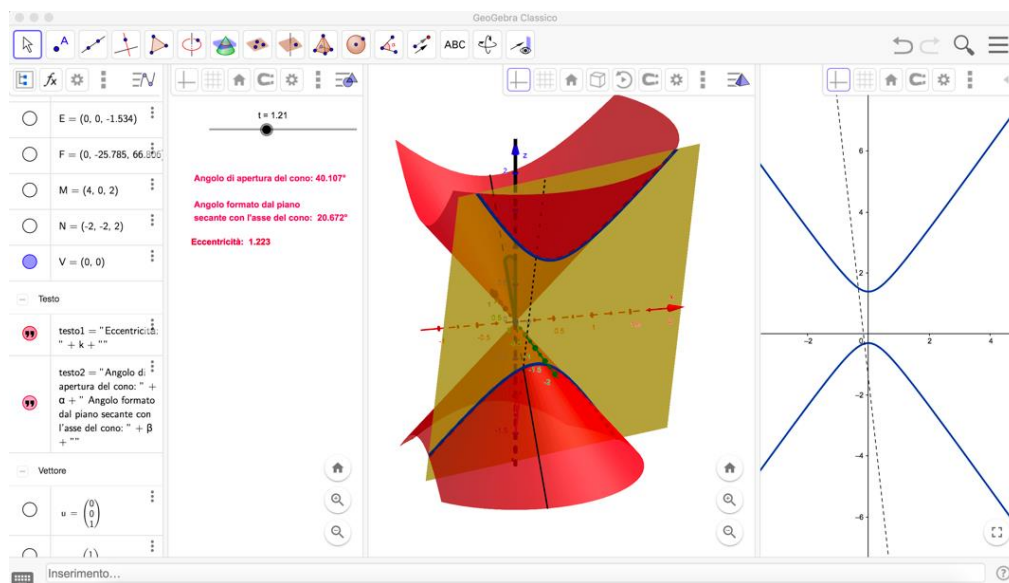
Fig.6. A GeoGebra dynamic visualization of the conic sections. From left to right: the first column collects data of points and equations; the second column contains the slider, which makes it possible to move the intersecting plane, and shows the eccentricity related to each position; the third column represents the 3D mutual position between the plane and the cone, and their intersection; the last right column visualizes the conic section on the plane. The hyperbolic sections can be also seen in the red 3D model in Fig.5.

Next, students are invited to work in small groups: by using a ribbon and nails they manually construct curves on a foamboard table. We propose they investigate the following problem: which kind of curve can one draw using a ribbon and 1, 2 or 3 nails? The three cases correspond to a circle, an ellipse and a union of arcs of ellipses, that students confuse with an egg.

Students are then invited to build a handcrafted mathematical machine (see the white board in Fig.5): a parabola drawer with tightened thread, in the spirit of Maschietto (2005). Here we start explaining how to construct the machine and students have to understand and explain why the obtained curve is a parabola. An interesting aspect of drawing curves with machines is that the force used to keep the thread taut is directly connected to the shape of these curves. The resistance of the thread has a mathematical interpretation: this force is used in order to keep the pencil along the sliding vertical line so that the distances between pencil and focus and between pencil and horizontal line are the same. We repeat and observe the building process of the machine also by using GeoGebra, see Fig.7 for a brief description.
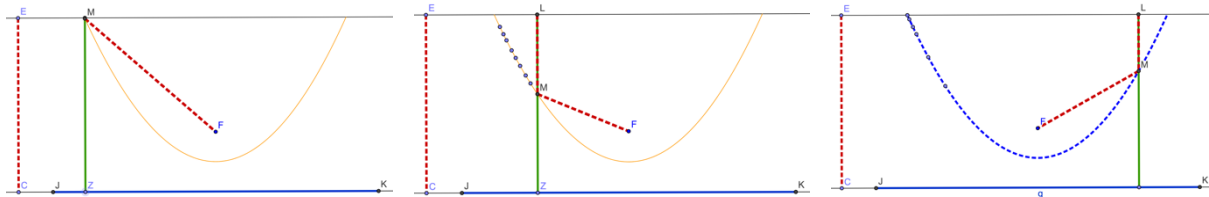
Fig. 7. A GeoGebra description of the machine drawing a parabola. In each figure the final result is shown in orange to support the description; students start with a white board. The green line represents a vertical rigid tool moving along the horizontal line JK. The red ribbon has length equal to EC and the pencil points in M. In this way, whenever the rigid tool moves, we get MZ=MF. From the left: initial position of the machine; in the centre: the point Z moves to the right and some points M are marked by GeoGebra along the left arc of the parabola; on the right: moving the point Z along JK over and over again, one gets the trace of the whole parabola arc.

In order to explain the relevance of machine drawing curves for technology, we mention robots that track specific curves with their mechanical arms. Gim is a useful tool to simulate mechanisms: it a registered software for kinematic analysis of planar and spatial mechanisms used in mechanical engineering. We then show some videos running Gim files.

In the second part of the lab activity, we analyze the focal properties of the parabola by giving simple application to antennas and car lights. Students are divided into groups to explore a mirascope (the black lenticular object in Fig.5), a funny illusion maker based on focal properties: we ask them to explain how it works. They open it and move the small frog inside it, and they understand that the illusion depends on the parabola's focal properties. This challenge is appreciated as students work with their peers and they can use some theoretical concepts in a concrete case.

We close the lab mentioning that parabolas also occur in the description of the shape of some suspended bridges.

Questions and ideas for future work

Students appreciate investigating 3D models of conic sections and moving the dynamic corresponding GeoGebra representation, at the same time. In the future it could be interesting to verify whether the simultaneous use of digital and tangible tools produces a better understanding of maths concepts.

Nevertheless, it remains to be understood how different representations of conics can be linked. This is also an explicit request from the students. In the spirit of the lab, we may tell students that this is not an easy task: it needs time and effort to formalize the problem. At the same time, we

can propose some experiences to materialize the different representations. For example, for the parabola, they can connect the 3D model of a parabolic section to the equation using synthetic geometry, the trace of the drawing machine to the equation of parabola in coordinates by drawing a coordinate system on the white board, and follow the equations of the parabolic sections on GeoGebra.

An open discussion has come out as to whether the parabola is the only curve with a symmetry axis with the property of concentrating rays parallel to the axis in a single point, namely the focus. This suggests investigating this problem analytically in the next editions of the lab and to construct new tangible tools: billiard tables with different contours and in particular with a parabolic contour.

In our opinion, in both activities presented, the most incisive part is the dynamic nature of the models: in Lab 3 composing and decomposing the vaults by means of the wedges and of the acetate sheets, and in Lab 5 the construction and use of the parabola drawer. The 3D printed models of vaults and conic sections are not very effective as objects themselves, but they become very interesting as far as Tinkercad (or possibly other programs) can be used in order to construct them (Baki et al., 2011; Bartolini Bussi et al., 2010): virtual tools are particularly effective as they have an inherent dynamic feature .

One may ask whether the use of virtual environments makes tangible tools more meaningful. Our experience in the lab starts the other way round: from objects. Objects exist in their own right, but we emphasize the need to go beyond the object alone: this can be done by means of virtual tools and mathematical concepts. Analysis of objects which are similar, and problems related to them requires on the one hand abstraction, conceptualization, formulas and on the other hand, software which allows visualization and emphasizes the role of parameters.

4. Some data and students' feedback

In order to test our activities, we collected data regarding the following four categories: the profile of the participants and comparison with general data from Politecnico di Torino; technical aspects of the organization; methodology; suggestions for improvement of LA.M.PO.. There are four types of data: data extracted from registrations, from approval questionnaires filled in after each activity, from interviews and from a concluding on-line questionnaire.

Very briefly, we can state that students who attended the lab represent the Politecnico population in terms of their distribution among different Engineering degree courses, with a proportionally slightly larger group of students enrolled in Mathematics and Physics curricula.

We collected 235 approval questionnaires filled in at the end of each activity and we got very positive feedback. We asked students to express their level of agreement on the following statements: *the subject is interesting, the activity is useful, my background is adequate in order to attend the activity* and *the instructors are clear.* The satisfaction percentages (sum of answers *somewhat agree* and *strongly agree*) were 99, 96, 93 and 99 percent respectively.

The most interesting data come out from the 38 on-line questionnaires collected at the end of the year. In this case our aim was to receive a feedback about methodology and future activities.

The cornerstone of our Lab is to propose an interactive and concrete approach to mathematics, allowing students to work with both tangible and virtual models and to discover maths inside technological applications (Castelnuovo, 2008). For this reason, one section of the questionnaire was devoted to these topics and it is the only one that we discuss in this snapshot paper.
We focus on virtual and tangible tools: 84% of students agreed (*somewhat agree* or *strongly agree*) with the fact that *the use of objects was useful for the comprehension of the topic* (question 7) and 92% agreed (*somewhat agree* and *strongly agree*) with the fact that *the use of virtual models was useful for the comprehension of the topic* (question 8).

Next, we wanted to compare the importance given to the use of models with other aspects which characterize the activities (questions 12-17). The results can be seen in Fig.8.
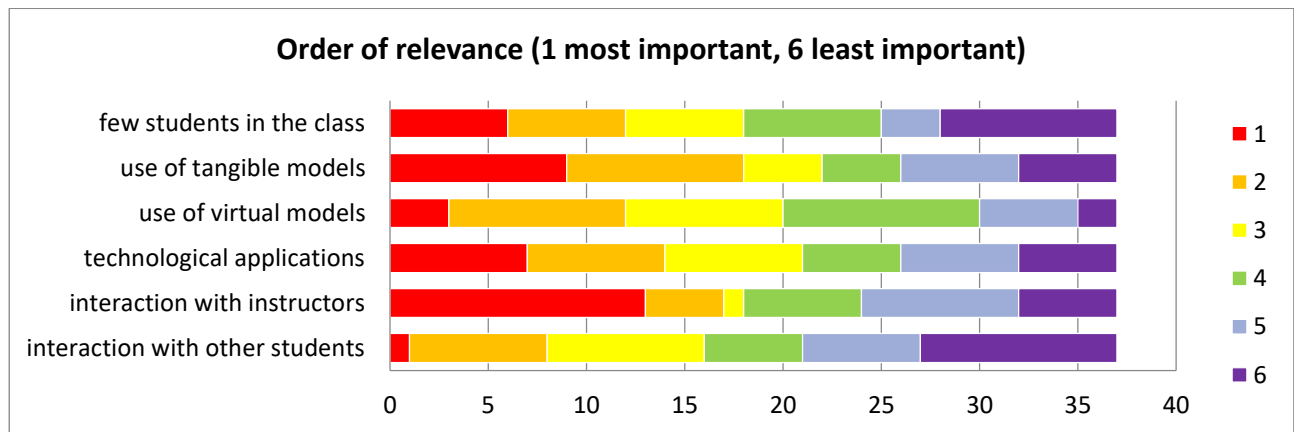

Fig.8. Characterising aspects of LA.M.PO.'s activities in order of relevance. We asked students to *Put in order of importance, from 1 to 6, the following aspects of laboratory activities (1 is the most important and 6 the least important).*

As Fig.8 shows, the sum of scores 1 and 2 suggests that the most important aspects for students are the *use of tangible models* and the *interaction with the instructor,* followed by the *technological applications*. This confirms that students understood and appreciated the emphasis given to models and technological applications. Moreover, they considered interaction with the instructors to be fundamental: this is a natural counterpart of the extremely large classes of the first year courses.

A surprising datum is the low score obtained by the *interaction with the other students*. There may be different reasons for this. On one hand, each activity lasted only three hours and the time for building up collective learning was not enough. On the other hand, the classrooms were not suited for working in groups. In the future we will have a new classroom with groups of desks that should facilitate interaction: we will verify whether a better room organization will give better results on this side.

The responses are connected to the subsequent question in which we asked students to indicate aspects to be improved. A series of suggestions were given: *number of the activities, engineering applications, links with course content, solutions to the exercises, more maths contents, use of objects, use of software, material for personal study* (question 10). Results are summarized in Fig.9.
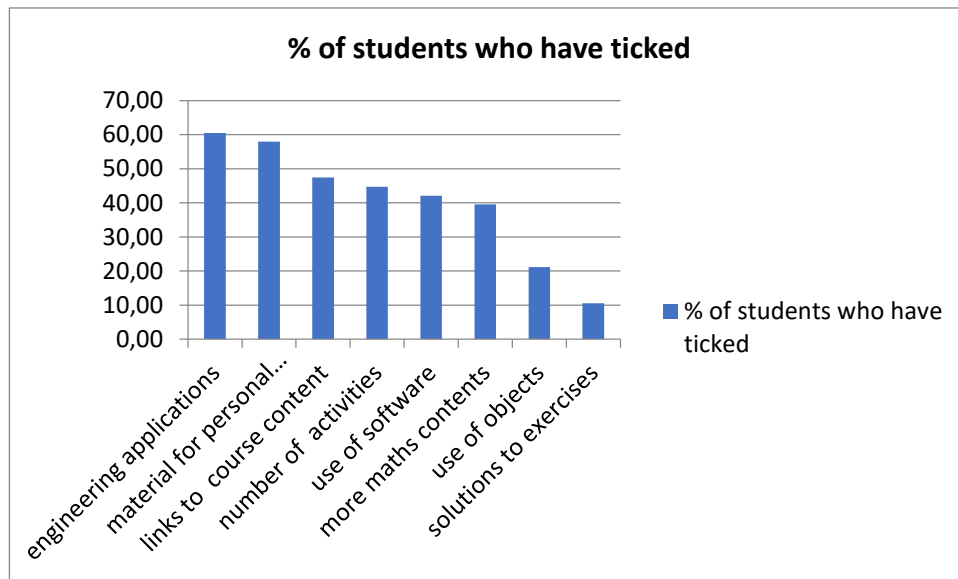
Fig.9. Students' suggestions for improving the lab.

Most ticks collected were on *engineering applications* (61%). Students considered *engineering applications* both as a characterizing aspect and as something to be strengthened. This confirms the fact that, for almost-adult students, motivation is fundamental in the learning process, as pointed out by Knowles (1980). On the other hand, engineering applications are a source of inspiration for the use of both tangible and virtual tools. In response to this, we are already working to introduce new examples and models taken from engineering.

Finally, we turn to the crucial terms *use of software* and *use of objects*. We used the words *software* and *objects* which are not the same as *virtual models* and *tangible models*: we consider them as a sort of approximation close to students' common language. The *use of software* was ticked by the 42% of students answering the questionnaire whereas the *use of objects* only by 21%. Since in the previous question the *use of objects* was indicated as a relevant aspect, we conclude that our emphasis on objects was sufficient. Actually, software had a minor role in the lab activities with respect to objects. There are two main reasons for this choice: introducing software takes a long time and students often use it themselves anyway. On the other hand, we think that the relationship between the *use of software* or, more generally, *virtual models*, and the use of objects, or, more generally, *tangible models,* was particularly fruitful in Lab 3 and Lab 5, as discussed in the previous section.

The lab activities took place in a great atmosphere of participation, so we had the opportunity to collect some final interviews. We report some of them here.

Student A: "It would be interesting to enrich courses with more practical applications or to start from them and then move on to theory".

Student B: "The topics are very interesting. They stimulate creativity and lateral thinking".

Student C: "The method of the lab could be used with visually impaired students".

The last comment is particularly interesting for us as it confirms our idea of experimenting the lab approach in an inclusive way.

5. Conclusions

In this paper we have presented the first outcomes of a Maths lab project for engineering students. Its characterizing elements represent an active learning approach and the integration of tangible and virtual models into traditional teaching.

In order to strengthen the use of technological applications to motivate and illustrate mathematical concepts, we have begun to interact with colleagues from other departments and, hopefully, some of them will participate in our future activities. New topics to be treated are: the shapes of suspended bridges as functions of one variable, different mechanisms for drawing curves, instruments for cartography. These topics naturally go along with the use of virtual tools and artefacts commonly employed in engineering that are present in the labs and the historical collections of Politecnico di Torino. We strongly believe that starting from concrete objects and instruments can really motivate students to study maths. Real objects are already interesting: the possibility to touch, turn, modify them encourages exploration and the development of ideas. Such manipulation helps in reading their virtual representations. Moreover, different aspects of maths can help understanding origin, scope and design of objects: theoretical tools, software, graphic and 3D representations can merge in a mathematical view of reality.

Students who freely chose to attend activities evaluated them very positively, but they are a small group. We wish to make more students test this approach as we are convinced that it can be fruitful for our almost-adult students. This is confirmed by the demand for material for personal study (ticked by 58% of interviewed students, see Fig.9) and by the agreement of 90% interviewed students to the statement *The approach taken in the lab is inspiring for the maths' study*. We are particularly interested in investigating in which sense our approach is inspiring for the study of maths. This is significant for us, as one of our challenges is to support a positive and active attitude towards mathematics. Students participating in the lab seem to be ready to actively contribute to their knowledge building process and we are stimulated by the idea of evolving from being teachers to becoming facilitators in this process.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Figures references

All tables were made by the authors and all photos were taken by the author Maria Luisa Spreafico.