Dimensional Reduction of the Kirchhoff-Plateau Problem

DIMENSIONAL REDUCTION OF THE KIRCHHOFF-PLATEAU PROBLEM

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Abstract. We obtain the Plateau problem with elastic boundary as a variational limit of the Kirchhoff-Plateau problem when the cross section of the boundary rod vanishes. The boundary is a framed curve that can sustain bending and twisting.

Keywords: Kirchhoff-Plateau problem, dimensional reduction.

2010 Mathematics Subject Classification: 49J45, 49Q05, 74K10.

1. Introduction

The classical Plateau problem is a milestone in the field of Calculus of Variations and it has given rise to a great variety of beautiful generalizations in Mathematics. It consists, in its simplest form, in finding a surface of least area passing through a given closed line. Surprisingly enough, much less is known about the problem when the line is not fixed but elastic, or even when the latter is replaced by a 3D object like a rod, giving then rise to the so-called *Kirchhoff-Plateau problem*, in which also the elastic energy associated to the rod has to be minimized and may have contributions from bending and twisting. In this case, also the concept of "passing through" must be properly reformulated.

The Plateau problem with elastic boundary curve, a precursor of the Kirchhoff-Plateau problem, has been investigated only in recent years. The first existence results were given by Bernatzky [2] and Bernatzky and Ye [3] who employed the theory of currents, but their elastic energy fails to satisfy the physical requirement of invariance under superposed rigid transformations. Furthermore, a strong hypothesis is used to avoid self-contact. Giomi and Mahadevan in [11] investigated the bifurcation from the flat state and provide also numerical examples. The Kirchhoff-Plateau problem, where a 3-dimensional elastic rod plays the role of the boundary of the soap film, was first formulated by Giusteri et al. [12] where the authors derive general equilibrium and linear stability conditions by considering the first and second variations of the energy functional. Stability properties of flat circular solutions under various conditions regarding the material properties of the rod have been investigated also by Chen and Fried [8], Biria and Fried [6, 7], and Hoang and Fried [17]. The first existence result for the Kirchhoff-Plateau problem has been given by Giusteri et al. [13] where in particular the authors showed the existence of a set of minimal area employing the recent approach by De Lellis et al. [10] which used a concept of contact between film and rod, called *spanning*, introduced by Harrison and Pugh [16]. This can be also generalized to complicated topologies of the rod, like knots, and even to links (see Bevilacqua et al. [4] and [5]). Since it is clear that, in order to avoid self-intersection and to represent a proper physical problem, the rod must be thin enough, a natural question raises: under what assumptions the Kirchhoff-Plateau problem "tends" to Plateau problem with elastic boundary curve when the rod shrinks to a line? The first attempt is clearly to look for simple cases and the first one, the linear twisted case sketched below, already shows that in the linear case the elastic energy tends to zero as a certain power of the radius of the cross-section of the rod, so that a dimensional reduction is necessary. However, the linear case is unsatisfactory in many aspects, the most apparent one being the fact that the displacement involved there is not volume-preserving. It's therefore better to use a finite elasticity example (we used the simplest one in nonlinear elasticity) and it turns out that, while the energy scales in the same way as in the linear case, the limiting curve may retain some "memory" of the twisting, showing that more than the simple image of the curve must be given in order to model the associated elastic energy. The same conclusion comes also from another experiment: suppose to force a regular elastic curve to stay in a plane. Then every deformed curve will have zero torsion, but some energy associated to it will be present and will be compensated by the reaction of the plane, so that an energy associated to torsion is perfectly conceivable.

The aim of this work is then to perform a dimensional reduction of the classical Kirchhoff-Plateau problem where the limiting curve is the midline of the rod. The paper is organized as follows. In section 2 we sketch the physical motivations that support the fact that the limit curve can sustain also a twisting energy. Then, in sections 3 and 4 we define the rigorous setting of the problem and we state the Γ-convergence result. Section 5 contains the proof of the main theorem. The result follows from lower semicontinuity of the elastic energy, from the pointwise convergence of the weight of the rod and from lower and upper estimate for the soap film energy.

2. Physical motivation

This section is dedicated to some physical observations about the dimensional reduction process.

2.1. **Linear elasticity.** Let us consider a rod in space, which can sustain bending and torsion within the linear theory of elasticity. If we split the total stored energy of the rod into the corresponding contributions E_b and E_t, we may find in the literature (Gurtin [15]) their expressions as

$$E_b = \frac{\pi}{8} E\varepsilon^4 k^2 \qquad E_t = \frac{\pi}{4} G\varepsilon^4 \theta^2,$$

where E is the Young's modulus, G is the shear modulus, ε is the radius of the section of the rod which can be thought, for simplicity, as a circular section, k is the curvature of the rod and $\theta = \omega L$, where ω is the twist density and it is due to the applied couple and L is the length of the rod. Since these quantities are energy per unity length, in order to get the total stored energy we have to integrate along the length of the rod:

$$E_{\text{rod}} = \int_0^L (E_b + E_t)\, ds = \frac{\pi}{8} L E \varepsilon^4 k^2 + \frac{\pi}{4} L G \varepsilon^4 \theta^2 \tag{2.1}$$

The expression (2.1) suggests that if we wish to obtain something not trivial as Γ-limit we should consider

$$E^{\varepsilon}_{\text{rod}} = \int_0^L \frac{1}{\varepsilon^4}(E_b + E_t)\, ds.$$

Notice that the rescaled density

$$\frac{1}{\varepsilon^4}(E_b + E_t) = \frac{\pi}{8} E k^2 + \frac{\pi}{4} G \theta^2$$

does not depend on ε.

2.2. **Non-linear elasiticity.** Let us now model our rod as a cylinder made of an isotropic, hyperelastic, incompressible material assuming for simplicity a neo-Hookean strain energy function ψ

$$\psi(\mathsf{F}) = \frac{\mu}{2}\left(\operatorname{tr}\mathsf{C} - 3\right),$$

where F is the deformation gradient and $\mathsf{C} = \mathsf{F}\mathsf{F}^{\mathrm{T}}$ and μ is the shear modulus. The material can sustain bending and torsion but we want to take into account only its torsion. The cylinder has axial length L and external radius ε in the fixed reference configuration Ω_0 described by the cylindrical coordinates (R, Θ, Z). The deformation mapping $\chi : \Omega_0 \to \mathbb{R}^3$ brings the material point represented by $\boldsymbol{X} = \boldsymbol{X}(R, \Theta, Z)$ to the point represented by $\boldsymbol{x} = \boldsymbol{x}(r, \theta, z) = \chi(\boldsymbol{X})$, where (r, θ, z) are the coordinates in the deformed configuration. Hence, following Truesdell and Noll [20], the deformation of a cylinder subjected to a finite torsion rate γ is given by

$$r = R, \qquad \theta = \Theta + \gamma Z, \qquad z = Z,$$

where its gradient can be easily computed as

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma R \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking into account the constraint of incompressibility ($\det \mathsf{F} = 1$), Piola stress tensor and the Cauchy stress tensor are given by

$$\mathsf{S} = \mu \mathsf{F} - p \mathsf{F}^{-\mathrm{T}} \qquad \mathsf{T} = \mathsf{S}\mathsf{F}^{\mathrm{T}} = \mu \mathsf{F}\mathsf{F}^{\mathrm{T}} - p\mathsf{I}, \tag{2.2}$$

where p is the Langrangian multiplier which enforces the incompressibility. The balance equation is given by

$$\operatorname{div}\mathsf{T} = 0 \quad \text{in } \Omega = \chi(\Omega_0),$$

where $\Omega_0 = \{0 < R < \varepsilon, 0 < \Theta < 2\pi, 0 < Z < L\}$. It turns out that p is only radial, so by assuming zero stress on the lateral surface it is not difficult to see that the total stored energy in the cylinder is given by

$$E_{\rm cyl} = \frac{\mu L \pi \gamma^2}{4}\varepsilon^4 \,.$$

As before, the area contribution is proportional to ε^4. So, the non-linear case is in agreement with the linear one, so we obtain that the simplest rescaling of the functional is

$$E^{\varepsilon}_{\rm cyl} = \frac{1}{\varepsilon^4}\int_{\Omega_0} \psi(\mathsf{F}) R \; dR \, d\Theta \, dZ. \tag{2.3}$$

Moreover, an interesting fact happens: when the cylinder has a finite radius, clearly every orthogonal reference system with $\mathbf{e}_3$ as axis is rotated by a fixed angle which is proportional to the value of the Z coordinate, but this feature remains unaltered also in the limit, when the cylinder becomes the Z-axis. This is in accordance with the notion of *framed curve* and the well-known fact that the torsion is not continuous with respect to uniform convergence. For example, helices on the finite cylinders have finite and strictly positive torsion independent of the radius of the cylinder, while their limit is a straight line and therefore has zero torsion. Notice that we can rewrite (2.3) as

$$E^{\varepsilon}_{\rm cyl} = \int_0^L \left(\frac{1}{\varepsilon^4}\int_0^{2\pi}\int_0^{\varepsilon} \psi(\mathsf{F}) R \, dR \, d\Theta \right) dZ \tag{2.4}$$

As in the linear case we notice that the rescaled density

$$\frac{1}{\varepsilon^4}\int_0^{2\pi}\int_0^{\varepsilon} \psi(\mathsf{F}) R \, dR \, d\Theta = \frac{\mu}{4}\pi\gamma^2$$

does not depend on ε.

3. Mathematical preliminaries

3.1. **Plateau problem.** In this paragraph we recall how to solve the Plateau problem using one of the most recent result based on a new notion of spanning condition which dates back to Harrison and Pugh [16] and has been also developed by De Lellis et al. [10]. In particular, we will follow the notation of De Lellis et al. [10] choosing a suitable class of loops, particularly appropriate for the Kirchhoff-Plateau problem (see Giusteri et al. [13]). Let H be a closed set in $\mathbb{R}^3$. We denote by $\mathcal{C}_H$ the set of all smooth embeddings $\gamma\colon S^1 \to \mathbb{R}^3 \setminus H$ which are not homotopic to a constant in $\mathbb{R}^3 \setminus H$. Given $K \subset \mathbb{R}^3$ we say that K *spans* H if $K \subset \mathbb{R}^3 \setminus H$ is relatively closed in $\mathbb{R}^3 \setminus H$ and $K \cap \gamma(S^1) \neq \emptyset$ for every γ in $\mathcal{C}_H$. Then, in De Lellis et al. [10] the following theorem is proved.

Theorem 3.1. *Assume that there exists $S \subset \mathbb{R}^3$ such that $\mathcal{H}^2(S) < +\infty$ and that spans $\mathbb{R}^3 \setminus H$. Then the problem*

$$\min\{\mathcal{H}^2(S) : S \textit{ spans } \mathbb{R}^3 \setminus H\}$$

has a solution[1].

3.2. **Geometry of closed curves.** In this paragraph we recall some classical notions of the theory of curves and knots. Let $L>0$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \colon [0,L] \to \mathbb{R}^3$ be two continuous and closed curves. We define their *linking number* as

$$\mathrm{Link}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{4\pi} \int_0^L \int_0^L \frac{\boldsymbol{x}_1(s) - \boldsymbol{x}_2(t)}{|\boldsymbol{x}_1(s) - \boldsymbol{x}_2(t)|^3} \cdot \boldsymbol{x}_1'(s) \times \boldsymbol{x}_2'(t)\, ds dt.$$

It turns out that the linking number is always integer. We say that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are *isotopic*, and we use the notation $\boldsymbol{x}_1 \simeq \boldsymbol{x}_2$, if there exists an open neighborhood N_1 of $\boldsymbol{x}_1([0,L])$, an open neighborhood N_2 of $\boldsymbol{x}_2([0,L])$ and a continuous map $\Phi \colon N_1 \times [0,1] \to \mathbb{R}^3$ such that $\Phi(N_1, \tau)$ is homeomorphic to N_1 for all τ in $[0,1]$ and

$$\Phi(\cdot, 0) = \text{Identity}\,, \quad \Phi(N_1, 1) = N_2\,, \quad \Phi(\boldsymbol{x}_1([0,L]), 1) = \boldsymbol{x}_2([0,L])\,.$$

Roughly speaking, two closed curves are isotopic if and only if they have the same knot type. Following Gonzalez et al. [14], we define the *minimal global radius of curvature* of a closed curve $\boldsymbol{x} \in W^{1,p}([0,L];\mathbb{R}^3)$, with $p>1$, by

$$\Delta(\boldsymbol{x}) = \inf_{s \in [0,L]} \inf_{\sigma, \tau \in [0,L] \setminus \{s\}} R(\boldsymbol{x}(s), \boldsymbol{x}(\sigma), \boldsymbol{x}(\tau))$$

where $R(x,y,z)$ denotes the radius of the smallest circle containing x,y,z, with the convention $R(x,y,z) = +\infty$ if x,y,z are collinear. The global radius of curvature has been introduced in order to have a precise way to say if a tubular neighborhood of a curve has self-intersections. More precisely, if $r>0$ we define the *r-tubular neighborhood of $\boldsymbol{x}$* by

$$U_r(\boldsymbol{x}) = \bigcup_{s \in [0,L]} B_r(\boldsymbol{x}(s)).$$

Accordingly to Ciarlet and Nečas [9] we say that $U_r(\boldsymbol{x})$ is *not self-intersecting* if for any $p \in \partial U_r(\boldsymbol{x})$ there exists a unique $s \in [0,L]$ such that $\|p - \boldsymbol{x}(s)\| = r$. The following result holds true (see Gonzalez et al. [14]).

Lemma 3.2. *Let $\boldsymbol{x} \in W^{1,p}([0,L];\mathbb{R}^3)$ be a closed curve and let $r>0$. Then $\Delta(\boldsymbol{x}) \geq r$ if and only if $U_r(\boldsymbol{x})$ is not self-intersecting. In particular, if $\Delta(\boldsymbol{x}) > 0$ then $\boldsymbol{x}$ is simple, that is $\boldsymbol{x} \colon [0,L) \to \mathbb{R}^3$ is injective.*

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[1] It is possibile to prove that we obtain a solution in the class of Almgren minimal sets. We do not enter in details since we will not make use of them. We refer to Almgren [1] for a precise definition. Here we just stress that they seem to be the best model for soap films because they have, thanks to a celebrated theorem by Taylor [19], the singularities of Plateau type.

4. Setting of the problem and main result

4.1. **The rod.** Let $L > 0$, let $p \in (1, +\infty)$ and let $\kappa_1, \kappa_2, \omega \in L^p([0,L])$. We let $w = (\kappa_1, \kappa_2, \omega) \in L^p([0,L];\mathbb{R}^3)$. Let $\boldsymbol{x}_0, \boldsymbol{t}_0, \boldsymbol{d}_0 \in \mathbb{R}^3$ with $\boldsymbol{t}_0 \perp \boldsymbol{d}_0$ and $|\boldsymbol{t}_0| = |\boldsymbol{d}_0| = 1$. Denote by $\boldsymbol{x}[w] \in W^{2,p}([0,L];\mathbb{R}^3)$ and $\boldsymbol{t}[w], \boldsymbol{d}[w] \in W^{1,p}([0,L];\mathbb{R}^3)$ the unique solution of the Cauchy problem

$$\begin{cases} \boldsymbol{x}[w]'(s) = \boldsymbol{t}[w](s)\,, \\ \boldsymbol{t}[w]'(s) = \kappa_1(s)\boldsymbol{d}[w](s) + \kappa_2(s)\boldsymbol{t}[w](s) \times \boldsymbol{d}[w](s)\,, \\ \boldsymbol{d}[w]'(s) = \omega(s)\boldsymbol{t}[w](s) \times \boldsymbol{d}[w](s) - \kappa_1(s)\boldsymbol{t}[w](s) \\ \boldsymbol{x}[w](0) = \boldsymbol{x}_0,\ \boldsymbol{t}[w](0) = \boldsymbol{t}_0,\ \boldsymbol{d}[w](0) = \boldsymbol{d}_0. \end{cases}$$

It is easy to prove that $\boldsymbol{t}[w](s) \perp \boldsymbol{d}[w](s)$ and $|\boldsymbol{t}[w](s)| = |\boldsymbol{d}[w](s)| = 1$ for any $s \in [0,L]$ which means that the frame

$$(\boldsymbol{t}[w](s), \boldsymbol{d}[w](s), \boldsymbol{t}[w](s) \times \boldsymbol{d}[w](s))$$

is an orthonormal frame in $\mathbb{R}^3$ for any $s \in [0,L]$. For any $s \in [0,L]$ let $\mathcal{A}(s) \subset \mathbb{R}^2$ be compact and simply connected such that $B_\eta(\mathbf{0}) \subset \mathcal{A}(s) \subset B_\nu(\mathbf{0})$ for any $s \in [0,L]$ and for some $\eta, \nu > 0$. For any $\varepsilon > 0$ let

$$\Omega_\varepsilon = \big\{(s, \zeta_1, \zeta_2) : s \in [0,L] \text{ and } (\zeta_1, \zeta_2) \in \varepsilon\mathcal{A}(s)\big\}.$$

For any $w \in L^p([0,L];\mathbb{R}^3)$ let then

$$\boldsymbol{p}_\varepsilon[w] \colon \Omega_\varepsilon \to \mathbb{R}^3, \quad \boldsymbol{p}_\varepsilon[w](s, \zeta_1, \zeta_2) = \boldsymbol{x}[w](s) + \zeta_1 \boldsymbol{d}[w](s) + \zeta_2 \boldsymbol{t}[w](s) \times \boldsymbol{d}[w](s)\,. \tag{4.1}$$

Moreover, we set

$$\Lambda_\varepsilon[w] = \boldsymbol{p}_\varepsilon[w](\Omega_\varepsilon).$$

4.2. **The constraints.** Before defining the energy it is convenient to fix the constraints. The fact that the midline is a closed curve can be readily expressed by

(C1) $\boldsymbol{x}[w](L) = \boldsymbol{x}[w](0) = \boldsymbol{x}_0$.

The closure of the midline is supplemented with the closure of the tangent vectors

(C2) $\boldsymbol{t}[w](L) = \boldsymbol{t}[w](0) = \boldsymbol{t}_0$

and the assignment of the other clamping condition

(C3) $\boldsymbol{d}[w](0) = \boldsymbol{d}_0$.

To prescribe how many times the ends of the rod are twisted before being glued together we close up the curve $\boldsymbol{x}[w] + \tau\boldsymbol{d}[w]$, for $\tau > 0$ fixed and small enough, defining, as in Schuricht [18],

$$\tilde{\boldsymbol{x}}_\tau[w](s) = \begin{cases} \boldsymbol{x}[w](s) + \tau\boldsymbol{d}[w](s) \\ \qquad \text{if } s \in [0,L] \\ \boldsymbol{x}[w](L) + \tau(\cos(\varphi_w(s-L))\boldsymbol{d}[w](L) + \sin(\varphi_w(s-L))\boldsymbol{t}[w](L) \times \boldsymbol{d}[w](L)) \\ \qquad \text{if } s \in [L, L+1] \end{cases}$$

where $\varphi_w \in [0, 2\pi)$ is the unique angle between $\boldsymbol{d}_0$ and $\boldsymbol{d}[w](L)$ such that $\varphi_w - \pi$ has the same sign as $\boldsymbol{d}_0 \times \boldsymbol{d}[w](L) \cdot \boldsymbol{t}_0$. We trivially identify $\boldsymbol{x}[w]$ with its extension $\boldsymbol{x}[w](s) = \boldsymbol{x}(L)$ for any $s \in [L, L+1]$ and therefore we require that

(C4) $\mathrm{Link}(\boldsymbol{x}[w], \tilde{\boldsymbol{x}}_\tau[w]) = L_0$ for some fixed $L_0 \in \mathbb{Z}$.

To encode the knot type of the midline we fix a continuous mapping $\boldsymbol{\ell}\colon [0, L] \to \mathbb{R}^3$ such that $\boldsymbol{\ell}(L) = \boldsymbol{\ell}(0)$ and require that

(C5) $\boldsymbol{x}[w] \simeq \boldsymbol{\ell}$.

Finally, in order to prevent self-intersections also in the limit we require that

(C6) $\Delta(\boldsymbol{x}[w]) \geq \Delta_0$, for some prescribed $\Delta_0 > 0$.

We denote by V the set of all constraints, namely

$$V = \big\{w \in L^p([0, L]; \mathbb{R}^3) : \text{(C1)–(C6) hold true}\big\}.$$

It turns out that V is weakly closed in $L^p([0, L]; \mathbb{R}^3)$ (see Gonzalez et al. [14] and Schuricht [18]).

Remark 4.1. *Following Ciarlet and Nečas* [9] *the non-interpenetration of matter can be enforced through the global injectivity condition*

$$\int_{\Omega_\varepsilon} \det D\boldsymbol{p}_\varepsilon[w]\, ds d\zeta_1 d\zeta_2 \leq \mathcal{L}^3(\boldsymbol{p}_\varepsilon[w](\Omega_\varepsilon))$$

which follows from (C6) and Lemma 3.2 if ε is small enough.

4.3. **The energy contributions and the convergence result.** In what follows, without loss of generality $\varepsilon \in (0, 1)$. First, let $f\colon \mathbb{R}^3 \times [0, L] \to \mathbb{R} \cup \{+\infty\}$ be bounded from below such that:

(a) $f(\cdot, s)$ is continuous and convex for any s in $[0, L]$;
(b) $f(a, \cdot)$ is measurable for any $a \in \mathbb{R}^3$;
(c) $f(a, s) \geq c_1|a|^p + c_2$ for some $c_1 > 0$ and c_2 in $\mathbb{R}$.

Taking into account the discussion in section 2 we define the elastic energy of the bounding rod $E^{\mathrm{el}}\colon V \to \mathbb{R} \cup \{+\infty\}$ as

$$E^{\mathrm{el}}(w) = \int_0^L f(w(s), s)\, ds.$$

The second energy contribution we want to take into account is the weight of the rod. Let $\Omega = \Omega_1$ and let $\rho \in L^\infty(\Omega)$ with $\rho \geq 0$ be the mass density function and $\boldsymbol{g}$ be the gravitational acceleration. Let us define $E^{\mathrm{g}}_\varepsilon\colon V \to \mathbb{R} \cup \{+\infty\}$ as

$$E^{\mathrm{g}}_\varepsilon(w) = \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} \rho(s, \zeta_1, \zeta_2) \boldsymbol{g} \cdot \boldsymbol{p}_\varepsilon[w](s, \zeta_1, \zeta_2)\, ds d\zeta_1 d\zeta_2$$

where $\boldsymbol{p}_\varepsilon$ is defined as in (4.1). The last contribution is the soap film energy. We define $E_\varepsilon^{\rm sf}\colon V \to \mathbb{R} \cup \{+\infty\}$ as

$$E_\varepsilon^{\rm sf}(w) = 2\sigma \inf\left\{\mathcal{H}^2(S) : S \subset \mathbb{R}^3 \text{ and } S \text{ spans } \Lambda_\varepsilon[w]\right\},$$

where $\sigma > 0$ is a constant called *surface tension.*

Let $\rho_0\colon [0, L] \to \mathbb{R}$ be given by

$$\rho_0(s) = \lim_{(\xi_1,\xi_2)\to(0,0)} \rho(s, \xi_1, \xi_2)$$

For any $w \in V$ we let

$$E_0(w) = \int_0^L f(w(s), s)\, ds + \int_0^L |\mathcal{A}(s)|\rho_0(s)\boldsymbol{g} \cdot \boldsymbol{x}[w](s)\, ds \\ + 2\sigma \inf\left\{\mathcal{H}^2(S) : S \subset \mathbb{R}^3 \text{ spans } \boldsymbol{x}[w]([0, L])\right\}.$$

Remark 4.2. *We observe that the functional E_0 is an energy on framed curves: when dealing with minimization of E_0 we are finding a soap film spanning an elastic and heavy boundary curve.*

We are ready to state our main result.

Theorem 4.3. *For any $\varepsilon > 0$ let $E_\varepsilon\colon V \to \mathbb{R} \cup \{+\infty\}$ be given by*

$$E_\varepsilon(w) = E^{\rm el}(w) + E_\varepsilon^{\rm g}(w) + E_\varepsilon^{\rm sf}(w).$$

Let (ε_h) be a sequence with $\varepsilon_h \to 0$ as $h \to +\infty$ and let (w_h) be a sequence in V with $E_{\varepsilon_h}(w_h) \le c$ for some $c > 0$. Then, up to a subsequence, $w_h \rightharpoonup w$ in $L^p([0, L]; \mathbb{R}^3)$ and $w \in V$. Moreover, the family $\{E_\varepsilon\}_{\varepsilon>0}$ Γ-converges to E_0 as $\varepsilon \to 0^+$ with respect to the weak topology of $L^p([0, L]; \mathbb{R}^3)$, namely:

(a) *for any sequence (ε_h) with $\varepsilon_h \to 0$, for any $w \in V$ and for any sequence (w_h) in V with $w_h \rightharpoonup w$ in $L^p([0, L]; \mathbb{R}^3)$ we have*

$$E_0(w) \le \liminf_{h\to+\infty} E_{\varepsilon_h}(w_h); \tag{4.2}$$

(b) *for any $w \in V$ there is a sequence (ε_h) with $\varepsilon_h \to 0$ and a sequence $(\bar{w}_h)$ in V with $\bar{w}_h \rightharpoonup w$ in $L^p([0, L]; \mathbb{R}^3)$ such that*

$$E_0(w) \ge \limsup_{h\to+\infty} E_{\varepsilon_h}(\bar{w}_h). \tag{4.3}$$

Proof. The compactness statement is Proposition 5.1. Inequality (4.2) follows combining (5.1), (5.2) and (5.3) with the subadditivity of the liminf operator. Next, for any $w \in V$ let $w_h = w$. Of course $\bar{w}_h \rightharpoonup w$ in $L^p([0, L]; \mathbb{R}^3)$. Inequality (4.3) follows easily combining (5.2) and (5.5) with the superadditivity of the limsup operator. □

As a standard consequence of Theorem 4.3 we have the next result.

Corollary 4.4. *Let (ε_h) be such that $\varepsilon_h \to 0$ as $h \to +\infty$. For any $h \in \mathbb{N}$ let $w_h \in V$ be such that*

$$E_{\varepsilon_h}(w_h) = \min_V E_{\varepsilon_h}. \tag{4.4}$$

Then up to a subsequence $w_h \rightharpoonup w_0$ in $L^p([0,L];\mathbb{R}^3)$ and

$$E_0(w_0) = \min_V E_0.$$

5. Proof of the convergence result

Fix a sequence $\varepsilon_h \to 0$ as $h \to +\infty$.

Proposition 5.1. (compactness) *Let (w_h) be a sequence in V with $E_{\varepsilon_h}(w_h) \le c$ for some $c > 0$. Then, up to a subsequence, $w_h \rightharpoonup w$ in $L^p([0,L];\mathbb{R}^3)$ and $w \in W$.*

Proof. Since $f(a,s) \ge c_1|a|^p + c_2$ we can say that $\|w_h\|_p$ is bounded. Then, up to a subsequence, $w_h \rightharpoonup w$ in $L^p([0,L];\mathbb{R}^3)$. Moreover, $w \in V$ since V is weakly closed in $L^p([0,L];\mathbb{R}^3)$ and this yields the conclusion. □

Proposition 5.2. (lower semicontinuity of the elastic energy) *Let $w \in W$. Then for any sequence (w_h) in V with $w_h \rightharpoonup w$ in $L^p([0,L];\mathbb{R}^3)$ we have*

$$\int_0^L f(w(s),s)\,ds \le \liminf_{h\to+\infty} E^{\mathrm{el}}(w_h). \tag{5.1}$$

Proof. Inequality (5.1) follows easily by the fact that V is weakly closed in $L^p([0,L];\mathbb{R}^3)$ and the functional

$$u \mapsto \int_0^L f(u(s),s)\,ds$$

is weakly lower semicontinuous in $L^p([0,L];\mathbb{R}^3)$ because $f(\cdot,s)$ is convex. □

The study of the weight term is easy since the weak convergence $w_h \rightharpoonup w$ implies the uniform convergence of the midlines.

Proposition 5.3. (convergence of the weight) *For any $w \in W$ and for any sequence (w_h) in V with $w_h \rightharpoonup w$ in $L^p([0,L];\mathbb{R}^3)$ we have*

$$\lim_{h\to+\infty} E^{\mathrm{g}}_{\varepsilon_h}(w_h) = \int_0^L |\mathcal{A}(s)|\rho_0(s)\boldsymbol{g}\cdot\boldsymbol{x}[w](s)\,ds. \tag{5.2}$$

Proof. By the change of variables $\zeta_i = \varepsilon_h\eta_i$, $i = 1,2$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon_h^2}\int_{\Omega_{\varepsilon_h}} \rho(s,\zeta_1,\zeta_2)\boldsymbol{g}\cdot\boldsymbol{p}_{\varepsilon_h}[w_h](s,\zeta_1,\zeta_2)\,dsd\zeta_1 d\zeta_2 \\
&\quad= \frac{1}{\varepsilon_h^2}\int_{\Omega_{\varepsilon_h}} \rho(s,\zeta_1,\zeta_2)\boldsymbol{g}\cdot(\boldsymbol{x}[w_h](s) + \zeta_1\boldsymbol{d}[w_h](s) + \zeta_2\boldsymbol{t}[w_h](s)\times\boldsymbol{d}[w_h](s))\,dsd\zeta_1 d\zeta_2 \\
&\quad= \int_{\Omega} \rho(s,\varepsilon_h\eta_1,\varepsilon_h\eta_2)\boldsymbol{g}\cdot(\boldsymbol{x}[w_h](s) + \varepsilon_h\eta_1\boldsymbol{d}[w_h](s) + \varepsilon_h\eta_2\boldsymbol{t}[w_h](s)\times\boldsymbol{d}[w_h](s))\,dsd\eta_1 d\eta_2.
\end{aligned}$$

Passing to the limit as $h \to +\infty$, using the fact that $\boldsymbol{x}[w_h] \to \boldsymbol{x}[w]$ uniformly on $[0, L]$ and applying the Dominated Convergence Theorem we conclude. $\square$

The main difficulty is to pass to the limit in the soap film part of the energy. First of all we need the following Lemma whose proof requires minor modifications of the proof of Theorem 2 in De Lellis et al. [10] (see also Lemma 3.4 in Giusteri et al. [13]).

Lemma 5.4. *Let (Λ_h) be a sequence of closed subsets of $\mathbb{R}^3$ converging in the Hausdorff topology to a closed set $\Lambda \neq \emptyset$. For any $h \in \mathbb{N}$ let $S_h \subset \mathbb{R}^3$ be such that*

$$\mathcal{H}^2(S_h) = \min\left\{\mathcal{H}^2(S) : S \text{ spans } \Lambda_h\right\}.$$

Let $\mu_h = \mathcal{H}^2 \llcorner S_h$. Then, up to a subsequence, $\mu_h \overset{*}{\rightharpoonup} \mu$ and

$$\mu \geq \mathcal{H}^2 \llcorner S_\infty$$

where $S_\infty = \operatorname{spt}(\mu) \setminus \Lambda$ is a countably $\mathcal{H}^2$-rectifiable set.

Proposition 5.5. (lower estimate for the soap film energy) *For any $w \in W$ and for any sequence (w_h) in V with $w_h \rightharpoonup w$ in $L^p([0, L]; \mathbb{R}^3)$ we have*

$$2\sigma \inf\{\mathcal{H}^2(S) : S \text{ spans } \boldsymbol{x}[w]([0, L])\} \leq \liminf_{h \to +\infty} E^{\mathrm{sf}}_{\varepsilon_h}(w_h). \tag{5.3}$$

Proof. Let $S_h \subset \mathbb{R}^3$ be such that

$$\mathcal{H}^2(S_h) = \min\{\mathcal{H}^2(S) : S \text{ spans } \Lambda_{\varepsilon_h}[w_h]\}.$$

Since $\boldsymbol{x}[w_h] \to \boldsymbol{x}[w]$ uniformly on $[0, L]$ we easily deduce that $\Lambda_{\varepsilon_h}[w_h] \to \boldsymbol{x}[w]([0, L])$ in the Hausdorff topology. Let $\mu_h = \mathcal{H}^2 \llcorner S_h$. Then, using Lemma 5.4 we can say that, up to a subsequence, $\mu_h \overset{*}{\rightharpoonup} \mu$ and

$$\mu \geq \mathcal{H}^2 \llcorner S_\infty$$

where $S_\infty = \operatorname{spt}(\mu) \setminus \boldsymbol{x}[w]([0, L])$ is a countably $\mathcal{H}^2$-rectifiable set. We now prove that S_∞ spans $\boldsymbol{x}[w]([0, L])$. Let $\gamma \in \mathcal{C}_{\boldsymbol{x}[w]([0,L])}$. The key point is to show that for any $r > 0$ such that $U_{2r}(\gamma) \subset \mathbb{R}^3 \setminus \boldsymbol{x}[w]([0, L])$, there exists $M = M(r) > 0$ such that, for h large enough,

$$\mathcal{H}^2(S_h \cap U_r(\gamma)) \geq M. \tag{5.4}$$

The proof of (5.4) can be done as in the proof of Lemma 3.5 in Giusteri et al. [13]: indeed the proof uses essentially the fact that $\Lambda_{\varepsilon_h}[w_h] \to \boldsymbol{x}[w]([0, L])$ in the Hausdorff topology. Using (5.4) we can show that S_∞ spans $\boldsymbol{x}[w]([0, L])$. Assume by contradiction that there exists $\gamma \in \mathcal{C}_{\boldsymbol{x}[w]([0,L])}$ with $\gamma(S^1) \cap S_\infty = \emptyset$ and take $r > 0$ as before. We then find that $\mu(U_r(\gamma)) = 0$ and, therefore, that

$$\lim_h \mathcal{H}^2(S_h \cap U_r(\gamma)) = 0$$

which contradicts (5.4). Finally, we obtain

$$\begin{aligned}\liminf_h 2\sigma \inf\{\mathcal{H}^2(S) : S \text{ spans } \Lambda_{\varepsilon_h}[w_h]\} \\ &= 2\sigma \liminf_h \mathcal{H}^2(S_h) \\ &\geq 2\sigma \mathcal{H}^2(S_\infty) \\ &\geq 2\sigma \inf\{\mathcal{H}^2(S) : S \text{ spans } \boldsymbol{x}[w]([0,L])\}\end{aligned}$$

and this yields the conclusion. □

Now we prove the upper estimate.

Proposition 5.6. (upper estimate for the soap film energy) *For any $w \in V$ we have*

$$2\sigma \inf\{\mathcal{H}^2(S) : S \textit{ spans } \boldsymbol{x}[w]([0,L])\} \geq \limsup_{h\to+\infty} E^{\mathrm{sf}}_{\varepsilon_h}(w). \tag{5.5}$$

Proof. Without loss of generality, we can assume that

$$\inf\{\mathcal{H}^2(S) : S \text{ spans } \boldsymbol{x}[w]([0,L])\} < +\infty.$$

otherwise (5.5) becomes trivial. Let $S_\infty \subset \mathbb{R}^3$ be such that

$$\mathcal{H}^2(S_\infty) = \min\{\mathcal{H}^2(S) : S \text{ spans } \boldsymbol{x}[w]([0,L])\}$$

We have to construct $S_h \subset \mathbb{R}^3$ which spans $\Lambda_{\varepsilon_h}[w]$ and such that $\mathcal{H}^2(S_h) \leq \mathcal{H}^2(S_\infty)$. The idea is to look S_∞ outside $\Lambda_{\varepsilon_h}[w]$. Let

$$S_h = S_\infty \setminus \Lambda_{\varepsilon_h}[w].$$

We claim that S_h spans $\Lambda_{\varepsilon_h}[w]$. This is straightforward since for any $\gamma \in \mathcal{C}_{\Lambda_{\varepsilon_h}[w]}$ we have $\gamma(S^1) \cap (S_\infty \setminus \Lambda_{\varepsilon_h}[w]) \neq \emptyset$. Of course we have $\mathcal{H}^2(S_h) \leq \mathcal{H}^2(S_\infty)$. As a consequence,

$$\begin{aligned}\limsup_{h\to+\infty} E^{\mathrm{sf}}_{\varepsilon_h}(w) &\leq 2\sigma \limsup_{h\to+\infty} \mathcal{H}^2(S_h) \\ &\leq 2\sigma \mathcal{H}^2(S_\infty) \\ &= 2\sigma \inf\{\mathcal{H}^2(S) : S \text{ spans } \boldsymbol{x}[w]([0,L])\}\end{aligned}$$

and this ends the proof. □

Remark 5.7. *It is likely that the condition*

$$S \textit{ spans } \boldsymbol{x}[w]([0,L]), \quad \boldsymbol{x}[w] \in W^{2,p}([0,L];\mathbb{R}^3),$$

carries some more informations on its boundary. For instance, is it true that S assumes $\boldsymbol{x}[w]([0,L])$ as a boundary in a more classical sense? We do not know the answer, and this seems to be challenging since very few results are known about boundary regularity for Plateau problem.

Acknowledgements

The work has been partially supported by INdAM-GNFM (Bevilacqua, Marzocchi) and INdAM-GNAMPA (Lussardi).

References

[1] F. J. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Memoirs of the Amer. Math. Soc. 165 (1976), i–199.

[2] F. Bernatzki, *On the existence and regularity of mass-minimizing currents with an elastic boundary*, Ann. Global Anal. Geom. 15 (1997), 379–399.

[3] F. Bernatzki and R. Ye, *Minimal surfaces with an elastic boundary*, Ann. Global Anal. Geom. 19 (2001), 1–9.

[4] G. Bevilacqua, L. Lussardi, A. Marzocchi, *Soap film spanning electrically repulsive elastic protein links*, Proceedings of School & Research Workshop *Mathematical Modeling of Self-Organizations in Medicine, Biology and Ecology: from micro to macro*, Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur. 96 (2018), suppl. 3, A1, 13pp.

[5] G. Bevilacqua, L. Lussardi, A. Marzocchi, *Soap film spanning an elastic link*, to appear on Quart. Appl. Math. (DOI: 10.1090/qam/1510).

[6] A. Biria and E. Fried, *Buckling of a soap film spanning a flexible loop resistant to bending and twisting*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 470 (2014), 20140368.

[7] A. Biria and E. Fried, *Theoretical and experimental study of the stability of a soap film spanning a flexible loop*, Int. J. Engrg. Sci. 94 (2015), 86–102.

[8] Y.-C. Chen and E. Fried, *Stability and bifurcation of a soap film spanning a flexible loop*, J. Elasticity 116 (2014), 75–100.

[9] P.G. Ciarlet and J. Nečas, *Injectivity and self-contact in nonlinear elasticity*, Arch. Rational Mech. Anal. 97 (1987), 171–188.

[10] C. De Lellis, F. Ghiraldin, F. Maggi, *A direct approach to Plateau's problem*, J. Eur. Math. Soc. 19 (2017), 2219–2240.

[11] L. Giomi and L. Mahadevan, *Minimal surfaces bounded by elastic lines*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 468 (2012), 1851–1864.

[12] G.G. Giusteri, P. Franceschini and E. Fried, *Instability paths in the Kirchhoff-Plateau problem*, J. Nonlinear Sci. 26 (2016), 1097–1132.

[13] G.G. Giusteri, L. Lussardi, E. Fried, *Solution of the Kirchhoff-Plateau problem*, J. Nonlinear Sci. 27 (2017), 1043–1063.

[14] O. Gonzalez, J. H. Maddocks, F. Schuricht and H. von der Mosel, *Global curvature and self-contact of nonlinearly elastic curves and rods*, Calc. Var. 14 (2002), 29–68.

[15] M. E. Gurtin, *An introduction to continuum mechanics*, Academic press, vol. 158, 1982.

[16] J. Harrison, H. Pugh, *Existence and soap film regularity of solutions to Plateau's problem*, Adv. Calc. Var. 9, (2016), 357–394.

[17] T. M. Hoang and E. Fried, *Influence of a spanning liquid film on the stability and buckling of a circular loop with intrinsic curvature or intrinsic twist density*, Math. Mech. Solids 23 (2018), 43–66.

[18] F. Schuricht, *Global injectivity and topological constraints for spatial nonlinearly elastic rods*, J. Nonlinear Sci. 12 (2002), 423–444.

[19] J. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) 103 (1976), no. 3, 489–539.

[20] C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*, Springer-Verlag Berlin Heidelberg, 2004.

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