HILBERT FUNCTIONS OF SCHEMES OF DOUBLE AND REDUCED POINTS

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Abstract. It remains an open problem to classify the Hilbert functions of double points in $\mathbb{P}^2$. Given a valid Hilbert function H of a zero-dimensional scheme in $\mathbb{P}^2$, we show how to construct a set of fat points $Z \subseteq \mathbb{P}^2$ of double and reduced points such that H_Z, the Hilbert function of Z, is the same as H. In other words, we show that any valid Hilbert function H of a zero-dimensional scheme is the Hilbert function of a set a positive number of double points and some reduced points. For some families of valid Hilbert functions, we are also able to show that H is the Hilbert function of only double points. In addition, we give necessary and sufficient conditions for the Hilbert function of a scheme of a double points, or double points plus one additional reduced point, to be the Hilbert function of points with support on a star configuration of lines.

1. Introduction

Throughout this paper, k will denote an algebraically closed field of characteristic zero. Let $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ be a finite set of reduced points with associated homogeneous ideal $I_X = I_{P_1} \cap \cdots \cap I_{P_s} \subseteq R = k[x_0, x_1, x_2]$. Given positive integers $m_1, \ldots, m_s$, we let $Z = m_1P_1 + \cdots + m_sP_s$ denote the scheme defined by the homogeneous ideal $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$. We refer to Z as a *set of fat points*. We call m_i the *multiplicity* of the point P_i; when $m_i = 2$, we sometimes call P_i a *double point*. Given a set of fat points Z, the *support* of Z is the set $\mathrm{Supp}(Z) = \{P_1, \ldots, P_s\}$.

Information about the set of fat points Z is encoded into its Hilbert function. Recall that the *Hilbert function* of Z is the function $H_Z : \mathbb{N} \to \mathbb{N}$ defined by

$$i \mapsto \dim_k(R/I_Z)_i = \dim_k R_i - \dim_k(I_Z)_i$$

where R_i, respectively $(I_Z)_i$, denotes the i-th graded piece of R, respectively $(I_Z)_i$ (see Chapter 5 of [16] for a comprehensive introduction to Hilbert functions). It is then natural to ask if one can characterize what functions are the Hilbert function of a set of fat points. A complete characterization of the Hilbert functions of reduced points (i.e., all the $m_i = 1$) was first described by Geramita, Maroscia, and Roberts [12]. However, even in the case that all the fat points are double points, a characterization of the Hilbert functions remains elusive (see, for example, the surveys of Gimigliano [13] and Harbourne [14]). In this paper, we contribute to this open problem by showing that every Hilbert function of a collection of reduced points in $\mathbb{P}^2$ is also the Hilbert function of a collection of double points and reduced points in $\mathbb{P}^2$. In specific cases, we can give a sufficient

2000 *Mathematics Subject Classification.* 14M05, 13D40; 13H15; 14N20.

Key words and phrases. fat points, star configuration points, Hilbert functions.

Version: June 19, 2019.

condition for a numerical function to be the Hilbert function of a scheme consisting only of double points.

To further describe our results, we introduce some additional notation. One way to study the Hilbert function of H_Z is to study its *first difference function* (sometimes called the Castelnuovo function) which is given by

$$\Delta H_Z(i) = H_Z(i) - H_Z(i-1) \text{ for all } i \geq 0, \text{ where } H_Z(-1) = 0.$$

When Z is a zero-dimensional scheme in $\mathbb{P}^2$, it can be shown (see Remark 2.2) that all but a finite number of values of $\Delta H_Z(i)$ are zero. Furthermore, if $\Delta H_Z(i) = 0$ for all $i \geq \sigma + 1$, and if we write $\Delta H_Z = (h_0, \ldots, h_\sigma)$ to encode all the non-zero values of ΔH_Z, then there is an $0 < \alpha \leq \sigma$ such that

(*a*) $h_i = i + 1$ if $0 \leq i < \alpha$, and
(*b*) $h_i \geq h_{i+1}$ if $\alpha \leq i \leq \sigma$.

We call $\Delta H = (h_0, \ldots, h_\sigma)$ a *valid Hilbert function* of a zero-dimensional scheme in $\mathbb{P}^2$ if ΔH satisfies conditions (*a*) and (*b*). Ideally, we want to answer the following question:

Question 1.1. *Let $\Delta H = (h_0, \ldots, h_\sigma)$ be a valid Hilbert function. Write $\sum \Delta H = \sum_{i=0}^{\sigma} h_i$ as $\sum \Delta H = 3d + r$ with $r \in \{0, 1, 2\}$. Does there exist a set Z of d double points and r reduced points in $\mathbb{P}^2$ such that $\Delta H_Z = \Delta H$?*

Note that a scheme Z with d double points and r reduced points in $\mathbb{P}^2$ will have $\deg(Z) = 3d + r$. Furthermore, it is known that $H_Z(i) = \deg(Z)$ for $i \gg 0$. This explains why we require $\sum \Delta H = 3d + r$. If we could answer this question, we could determine if a valid Hilbert function is the Hilbert function of a set of double points. Thus, the above question is quite difficult.

We can ask a weaker question by simply asking if any set of double points and reduced points can be constructed:

Question 1.2. *Let $\Delta H = (h_0, \ldots, h_\sigma)$ be a valid Hilbert function. Can one always find integers d and r where d is positive and $r \geq 0$ with $\sum \Delta H = 3d + r$ such that H is the Hilbert function of a set Z of d double points and r simple points in $\mathbb{P}^2$?*

Note that if we allow $d = 0$ and $r = \sum \Delta H$, then the above question is simply asking if ΔH is the Hilbert function of r reduced points, which follows from Geramita, Maroscia, and Roberts [12]. We can now view Question 1.1 as asking if the d in Question 1.2 can be taken to be the maximum allowed value. Ideally, when trying to answer Question 1.2, we want to make d as large as possible.

One of the main results of this paper (Theorem 3.1) will give us a tool to answer to Question 1.2. Specifically, starting with a set of double and reduced points on a collection of general lines in $\mathbb{P}^2$, we describe how to "merge" three reduced points to make a new scheme with one new double point and three fewer reduced points. Moreover, this procedure does not change the Hilbert function. The results of Cooper, Harbourne, and Teitler [8] are the crucial ingredient to prove that our new configuration has the correct Hilbert function. By reiterating this process, in a controlled fashion, Construction 3.5 shows how to start from a valid Hilbert function ΔH and create a set Z of double and simple points with $\Delta H = \Delta H_Z$. Our answer to Question 1.2 is given in Theorem 3.9 where we find a d, that depends only on ΔH, such that we can construct a set of d double points and $(\sum \Delta H) - 3d$ reduced points whose Hilbert function is H. In fact, for all $1 \leq d' \leq d$, we can find a scheme of d' double points and $(\sum \Delta H) - 3d'$ reduced points with Hilbert function H (see Corollary 3.10). Moreover, in Theorem 3.11 we give a condition on a valid Hilbert function H that guarantees that H is the Hilbert function of only double points.

We then focus on the special cases that

$$\Delta H = (\underbrace{1, 2, 3, \ldots, t}_{t}, \underbrace{t+1, \ldots, t+1}_{t}) \text{ or } \Delta H = (\underbrace{1, 2, 3, \ldots, t}_{t}, \underbrace{t+1, \ldots, t+1}_{t}, 1)$$

In these cases, our construction produces $\binom{t+1}{2}$ double points, respectively $\binom{t+1}{2}$ double points and one reduced points. In the first case, the support of the points are the $\binom{t+1}{2}$ points of intersection of t general lines in $\mathbb{P}^2$. This fact is equivalent to the statement that the points in the support are a *star-configuration* of points in $\mathbb{P}^2$; star configurations are widely studied, e.g. see [4, 5]. We prove (see Theorem 4.4) that this configuration is the *only* configuration of $\binom{t+1}{2}$ double points in $\mathbb{P}^2$ with $\Delta H_Z = \Delta H$. In the second case, we show (see Theorem 4.6) a similar result by showing again that there is only one configuration of $\binom{t+1}{2}$ double points and one reduced point that has $\Delta H_Z = \Delta H$.

We conclude our paper with some final comments related to how well our construction performs, i.e., given a known valid Hilbert function of t double points, how many double points does our procedure produce for the same valid Hilbert function. In the case that the support of points is in generic position, we derive an asymptotic estimate.

Acknowledgements. The computer algebra system CoCoA [1] played an integral role in this project. The authors thank the hospitality of the Università di Catania and McMaster University where part of this work was carried out. Carlini and Catalisano were supported by GNSAGA of INDAM and by Miur (Italy) funds. Guardo thanks FIR-UNICT 2014 and GNSAGA-INDAM for supporting part of the visit to McMaster University. Guardo's work has also been supported by the Università degli Studi di Catania, "Piano della Ricerca 2016/2018 Linea di intervento 2". Van Tuyl's research was supported by NSERC Discovery Grant 2014-03898.

2. Preliminaries

We begin with a review of the relevant background; we continue to use the notation and definitions given in the introduction.

Definition 2.1. A sequence $\Delta H = (h_0, h_1, \ldots, h_\sigma)$ is a *valid Hilbert function of a set of points in* $\mathbb{P}^2$ if there is an $0 < \alpha \leq \sigma$ such that

(a) $h_i = i + 1$ if $0 \leq i < \alpha$, and
(b) $h_i \geq h_{i+1}$ if $\alpha \leq i \leq \sigma$.

Note that the indexing of ΔH begins with 0.

Remark 2.2. It can be shown that $H : \mathbb{N} \to \mathbb{N}$ is a Hilbert function of a set of points in $\mathbb{P}^2$ if and only if $\Delta H(i) = H(i) - H(i-1)$ is a valid Hilbert function. More precisely, it was first shown by Geramita, Maroscia, and Roberts [12] that H is the Hilbert function of a reduced set of points in $\mathbb{P}^2$ if and only if ΔH is the Hilbert function of an artinian quotient of $k[x, y]$. Using Macaulay's theorem which classifies all Hilbert functions, one can determine all possible Hilbert functions of artinian quotients of $k[x, y]$. In particular, one can show that the Hilbert functions of artinian quotients of $k[x, y]$ must satisfy the conditions of being a valid Hilbert function. The work of Geramita, Maroscia, and Roberts implies that if $Z \subseteq \mathbb{P}^2$ is any set of fat points, then ΔH_Z must satisfy the conditions (a) and (b) given above.

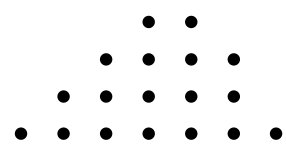
Definition 2.3. Let $\Delta H = (1, 2, \ldots, \alpha, h_\alpha, \ldots, h_\sigma)$ be a valid Hilbert function of a set of points in $\mathbb{P}^2$. We define the *conjugate of* ΔH, denoted $\Delta H^\star$, to the be tuple

$$\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star) \text{ where } h_i^\star = \#\{j \mid h_j \geq i\}.$$

Remark 2.4. The definition above is reminiscent of the conjugate of a partition. Recall that a tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a *partition* of an integer s if $\sum_{i=1}^r \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every i. The *conjugate* of λ is the tuple $\lambda_i^\star = \#\{j \mid \lambda_j \geq i\}$. Note that ΔH is not a partition, but $\Delta H^\star$ is a partition. Furthermore, $h_1^\star = \sigma + 1$ since there are $\sigma + 1$ non-zero entries in ΔH.

Example 2.5. Given a valid Hilbert function ΔH, it is convenient to represent ΔH pictorially. That is, we make $\sigma + 1$ columns of dots, where we place h_i dots in the i-th column. For example, if $\Delta H = (1, 2, 3, 4, 4, 3, 1)$, then we can represent ΔH pictorially as:

$\Delta H =$



The tuple $\Delta H^\star = (7, 5, 4, 2)$ can be read directly off of this diagram; specifically, it is the number of dots in each row reading from bottom to top.

Given a valid Hilbert function ΔH, one can use $\Delta H^\star$ to construct a set of reduced points $X \subseteq \mathbb{P}^2$ such that $H_X = H$ by building a suitable k-configuration. We present a specialization of this idea; an example appears as the first step of Example 3.6.

Theorem 2.6. *Let $\Delta H = (1, 2, \ldots, \alpha, h_\alpha, \ldots, h_\sigma)$ be a valid Hilbert function with $\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star)$. Let $\ell_1, \ldots, \ell_\alpha$ be α lines in $\mathbb{P}^2$ such that no three lines meet at a point. For $i = 1, \ldots, \alpha$, let $X_i \subseteq \ell_i$ be any set of $h_i^\star$ points such that $X_i \cap \ell_j = \emptyset$ for all $i \neq j$. If $X = X_1 \cup \cdots \cup X_\alpha$, then $H_X = H$.*

Proof. We sketch out the main ideas. Our hypotheses on the X_i's implies that no point of X is of the form $\ell_i \cap \ell_j$ with $i \neq j$. One can verify that $h_1^\star > h_2^\star > \cdots > h_\alpha^\star$. But then X is a k-configuration of type $(h_\alpha^\star, \ldots, h_1^\star)$ as first defined by Roberts and Roitman [17] (and later generalized by Geramita, Harima, and Shin [11]). In particular, one can use [17, Theorem 1.2] to compute the Hilbert function of X to show that it is the same as H. □

We now recall some crucial results from the work of Cooper, Harbourne, Teitler [8]. We have specialized their definitions to $\mathbb{P}^2$.

Definition 2.7 ([8, Definition 1.2.5])**.** Let $Z = m_1P_1 + m_2P_2 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^2$. Fix a sequence $\ell_1, \ldots, \ell_n$ of lines in $\mathbb{P}^2$, not necessarily distinct.

(a) Define the fat point schemes $Z_0, \ldots, Z_n$ by $Z_0 = Z$ and $Z_j = Z_{j-1} : \ell_j$ for $1 \leq j \leq n$. That is, Z_j is the scheme defined by $I_{Z_{j-1}} : \langle L_j \rangle$ if L_j is the linear form defining ℓ_j and I_{Z_j} is the ideal defining Z_j.

(b) The sequence $\ell_1, \ldots, \ell_n$ *totally reduces* Z if $Z_n = \emptyset$ is the empty scheme. This statement is equivalent to the property that for each fat point m_iP_i, there are at least m_i indices $\{j_1, \ldots, j_{m_i}\}$ such that each ℓ_{j_k} passes through P_i.

(c) We associate with Z and the sequence $\ell_1, \ldots, \ell_n$ an integer vector

$$\mathbf{d} = \mathbf{d}(Z; \ell_1, \cdots, \ell_n) = (d_1, \ldots, d_n),$$

where $d_j = \deg(\ell_j \cap Z_{j-1})$, the degree of the scheme theoretic intersection of ℓ_j with Z_{j-1}. We refer to $\mathbf{d}$ as the *reduction vector* for Z induced by the sequence $\ell_1, \ldots, \ell_n$. We will say that $\mathbf{d}$ is a *full reduction vector* for Z if $\ell_1, \ldots, \ell_n$ totally reduces Z.

Remark 2.8. If Z is a fat point scheme, and if $P_{i_1}, \ldots, P_{i_j}$ are all the points in the support of Z that lie on the line ℓ, then $\deg(\ell \cap Z) = m_{i_1} + \cdots + m_{i_j}$, i.e., the sum of the multiplicities of the points lying on $\ell \cap Z$. The scheme $Z : \ell$ is the scheme that we obtain by reducing the multiplicities of $P_{i_1}, \ldots, P_{i_j}$ by one (or removing the point if its multiplicity is 1), and leaving the other multiplicities alone.

Example 2.9. Consider three non-collinear points P_1, P_2, P_3 and the set of fat points $Z = 3P_1 + 3P_2 + 2P_3$. Let $\ell_1 = \ell_2$, ℓ_3 and ℓ_4 be the lines through P_1P_2, P_1P_3, and P_2P_3, respectively. Then a full reduction vector for this scheme is $(6, 4, 3, 2)$. The pictures below show how to build this vector. For example, in Figure 1, the line ℓ_1 passes through P_1 and P_2. Since the multiplicity of P_1 is three, and the same for P_2, we have $d_1 = 3 + 3 = 6$. We then reduce the multiplicity of P_1 and P_2 by one, as in Figure 2. Then $d_2 = 2 + 2 = 4$. The

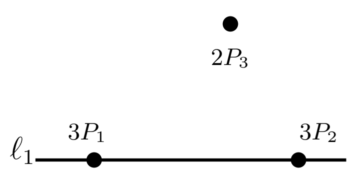


FIGURE 1.

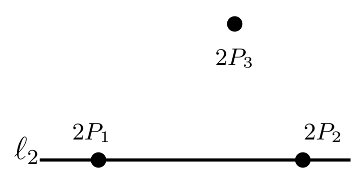


FIGURE 2.

line ℓ_3 in Figure 3 passes through one point of multiplicity one and one of multiplicity two, thus giving $d_3 = 3$. Note that when we reduce each multiplicity, the point P_1 is removed. In the last step, we use the line ℓ_4 to get $d_4 = 2$ as in Figure 4.

The next result is another specialization of Cooper, Harbourne and Teitler [8].

Theorem 2.10. *Let $Z = Z_0$ be a fat point scheme in $\mathbb{P}^2$ with full reduction vector $\mathbf{d} = (d_1, \ldots, d_n)$. If $d_1 > d_2 > \cdots > d_n$, then H_Z only depends on $\mathbf{d}$.*

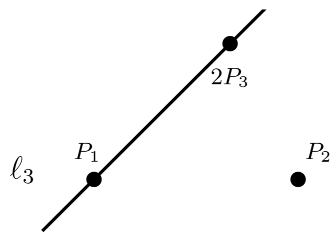


Figure 3.

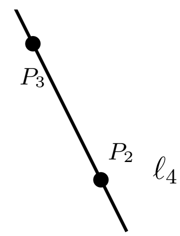


Figure 4.

Proof. From our assumption, we have $d_i - d_{i+1} \geq 1$ for all $i = 1, \ldots, n$. Then $d_i - d_{i+p} \geq p$ for all i. This implies that $\mathbf{d} = (d_1, \ldots, d_n)$ is a GMS vector (see Definition 2.2.1 in [8]). Then Theorem 2.2.2 and Section 2.3 of [8] imply

$$H_Z(t) = \sum_{i=0}^{n-1} \binom{\min\{t-i+1, d_{i+1}\}}{1},$$

that is, H_Z can be computed directly from $\mathbf{d}$. □

3. An operation that preserves the Hilbert function

In this section, we first show that under certain conditions, we can degenerate a fat point scheme Z consisting of double points and reduced points to make a new fat point scheme Z' consisting of one additional double point, and three less reduced points. Furthermore, the two schemes will have the same Hilbert function. Note that degeneration techniques have been successfully used in other situations, e.g. see [6, 7].

By repeatedly applying this procedure, we can do the following. Let ΔH be a valid Hilbert function of a set of points. Theorem 2.6 implies that there is a set of reduced points X with Hilbert function ΔH that satisfies the hypotheses of our procedure given below. We can then remove three points from X and add a double point to make a set Z of fat points with the same Hilbert function as ΔH. We can continue this procedure (provided the hypotheses of our procedure are still satisfied) to build sets of fat points consisting of double and reduced points that have Hilbert function ΔH. This procedure will then allow us to give an answer to Question 1.2.

We now state and prove the main step in our procedure.

Theorem 3.1. *Let $\ell_1, \ldots, \ell_n$ be n lines in $\mathbb{P}^2$ such that no three lines meet at a point. Let $P_{i,j} = \ell_i \cap \ell_j$ for $1 \leq i < j \leq n$. Suppose that Z is a set of double points and reduced points in $\mathbb{P}^2$ that satisfies the following conditions:*

(*a*) $\mathrm{Supp}(Z) \subseteq \bigcup_{i=1}^n \ell_i$, *i.e., all the points in the support lie on the lines ℓ_i.*
(*b*) *If $2P$ is a double point of Z, then $P = P_{i,j}$ for some $i < j$, i.e., all double points of Z lie at an intersection point of two ℓ_p's.*
(*c*) *If Q is a reduced point of Z and $Q \in \ell_i$, then $Q \neq \ell_i \cap \ell_p$ for $p \neq i$, i.e., the reduced points do not lie at an intersection point.*
(*d*) *If*

$$d_j = \deg(Z_{j-1} \cap \ell_j)$$

for $j = 1, \ldots, n$, then $d_1 > d_2 > \cdots > d_n$, where $Z_j = Z_{j-1} : \ell_i$ and we set $Z_0 = Z$.

(e) *There exist i, j with $i < j$ such that Z contains two reduced points $Q_1, Q_2 \in \ell_i$ and a reduced point $R \in \ell_j$, but $2P_{i,j}$ is not a double point of Z.*

Let Z' be the set of double and reduced points obtained by adding the double point $2P_{i,j}$ to Z, and removing the reduced points $\{R, Q_1, Q_2\}$. Then Z and Z' have the same Hilbert function.

Proof. We begin by observing that $\mathrm{Supp}(Z')$ is also contained in $\bigcup_{i=1}^{n} \ell_i$ by our construction since the only point we added to the support is $P_{i,j}$. This observation and (a) imply that the lines $\ell_1, \ldots, \ell_n$ totally reduce both Z and Z'. Indeed, if Q is a reduced point of Z, respectively Z', then it lies on some distinct ℓ_i by (c). If $2P$ is a double point of Z, or Z', then $P = \ell_i \cap \ell_j$ for some $i < j$ by (b) (or by the construction of Z'), so there are at least two ℓ_p's that pass through $2P$. It follows from the equivalent statement in Definition 2.7 that the ℓ_i's totally reduce Z and Z'.

To finish the proof, we claim it is enough to show that

$$\deg(Z_{j-1} \cap \ell_j) = \deg(Z'_{j-1} \cap \ell_j) \text{ for all } 1 \le j \le n.$$

Indeed, if this fact is true, then part (d) and Theorem 2.10 imply that the Hilbert function of Z and Z' are the same.

To verify the claim, we first observe that our change from Z to Z' only effects the points on the lines ℓ_i and ℓ_j, and consequently, could only effect the value of $\deg(Z'_{p-1} \cap \ell_p)$ for $p = i$ and j. In the computation of $\deg(Z_{i-1} \cap \ell_i)$ we get a contribution of two from each reduced point Q_1 and Q_2. Those two points do not contribute to $\deg((Z'_{i-1} \cap \ell_i)$ since we have removed them, but the fat point $2P_{i,j}$ (which is not in Z) contributes two to the degree. The other points of Z_{i-1} and Z'_{i-1} on ℓ_i remain the same, so they contribute equally to the degree. So $\deg(Z_{i-1} \cap \ell_i) = \deg(Z'_{i-1} \cap \ell_i)$.

When we compute $\deg(Z_{j-1} \cap \ell_j)$ we get a contribution of one from R. This point does not contribute to $\deg(Z'_{j-1} \cap \ell_j)$ since it was removed. We, however, get a contribution of one from $P_{i,j}$. (The multiplicity of $P_{i,j}$ was dropped from two to one when we formed Z'_i.) As we mentioned above, the other points on ℓ_j contribute the same. So, again we have $\deg(Z_{j-1} \cap \ell_j) = \deg(Z'_{j-1} \cap \ell_j)$. This completes the proof. □

Remark 3.2. We note that the hypothesis of Theorem 3.1 are sufficient conditions which allow us, for example, to apply the results of [8]. Consider condition (d) on the degrees d_i. If some of the inequalities do not hold, then the conclusion might be false. Let Z be a set of five points supported on the union of the lines ℓ and ℓ', namely two points on the former and three on the latter. Thus $\Delta H_Z = (1, 2, 2)$. If we set $\ell_1 = \ell$ and $\ell_2 = \ell'$, we get $d_1 = 2$ and $d_2 = 3$, and condition (c) is not satisfied. Indeed, the resulting set Z' is such that $\Delta H_{Z'} = (1, 2, 1, 1)$. Hence the two Hilbert functions are not equal.

Remark 3.3. There are also counterexamples when the degrees d_i are not all distinct. Consider, for example, the complete intersection of a cubic $\ell_1 \cup \ell_2 \cup \ell_3$ with the union of two distinct lines. This set of six points lie on a conic, but applying our construction leads to a scheme of one double point and three simple points not lying on a conic. That is,

$d_1 = d_2 = d_3 = 2$, and our construction does not preserve the Hilbert function in degree two.

Remark 3.4. When we construct Z', then Z' will also satisfy the hypotheses $(a)-(d)$ of Theorem 3.1. If Z' also satisfies (e), then we can add another double point, and so on, until hypothesis (e) is no longer satisfied.

We expand upon the above remark. Given a valid Hilbert function ΔH with $\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star)$, we present a construction based upon Theorem 3.1 which will allow us to produce a set Z of double and simple points such that $\Delta H = \Delta H_Z$. The rough idea behind our construction is to start with a set of reduced points with the correct Hilbert function, and then, in a controlled fashion, repeatedly replace three reduced points with a double point, and use Theorem 3.1 to show that the Hilbert function does not change after each iteration. Below, we will use the notation $n_+ = \max\{n, 0\}$.

Construction 3.5.

INPUT: *A valid Hilbert function ΔH with $\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star)$.*
OUTPUT: *A scheme Z of double points and reduced points in $\mathbb{P}^2$ with $H_Z = H$.*

STEP 0. *Let $\ell_1, \ldots, \ell_\alpha$ and $P_{i,j}$ be as in Theorem 3.1. Let Z_0 be a set reduced points of $\mathbb{P}^2$ with $H_{Z_0} = H$ as constructed in Theorem 2.6 with $Z_0 \subseteq \bigcup_{i=1}^{\alpha} \ell_i$ such that $|Z_0 \cap \ell_i| = h_i^\star$. Continue to* STEP 1.

For $n \geq 1$:

STEP n. *Set $\mathbf{h_n} = ((h_n^\star - (n-1))_+, \ldots, (h_\alpha^\star - (n-1))_+)$ and*

$$s_n = \#\{k \mid n+1 \leq k \leq \alpha \text{ and } h_k^\star \geq n\}.$$

If $s_n = 0$, then return Z_{n-1}. Otherwise, let

$$t_n = \min\left\{\left\lfloor \frac{(h_n^\star - (n-1))_+}{2} \right\rfloor, s_n\right\}.$$

Remove $2t_n$ points on ℓ_n and one point on each ℓ_j for $j = n+1, \ldots, n+t_n$ from Z_{n-1}, and add the double points $2P_{n,j}$ where $P_{n,j} = \ell_n \cap \ell_j$, for $j = n+1, \ldots, n+t_n$. Let Z_n denote the resulting scheme. Continue to STEP $n+1$.

Proof. We verify that Construction 3.5 produces the stated output. First, note that the construction will stop at (or before) STEP α because $s_\alpha = 0$.

We now show that the scheme Z_n of double and reduced points construced at STEP n has the property that $H_{Z_n} = H$. We proceed by induction on n. If $n = 0$, then the set of reduced points Z_0 has the property $H_{Z_0} = H$ by Theorem 2.6.

So, now assume that $n \geq 1$, and that Z_{n-1} satisfies the induction hypothesis. We first observe that the value $(h_i^\star - (n-1))_+$ in the tuple $\mathbf{h_n} = ((h_n^\star - (n-1))_+, \ldots, (h_\alpha^\star - (n-1))_+)$ counts the minimal possible number of reduced points on ℓ_i for $i = n, \ldots, \alpha$ after constructing Z_{n-1}. This is because in each of STEP 1 through $n-1$, we used at most one reduced point on ℓ_i when constructing Z_j with $0 \leq j \leq n-1$.

If $s_n = 0$, then our procedure terminates with Z_{n-1}, which, by induction, is a set of double and reduced points with $H_{Z_{n-1}} = H$. Now suppose that $s_n > 0$, i.e., there exists a $n+1 \leq k \leq \alpha$ such that $h_k^\star \geq n$, or equivalently, $h_k^\star - (n-1) \geq 1$. In fact, because $h_1^\star > h_2^\star > \cdots > h_n^\star > \cdots > h_k^\star > \cdots > h_\alpha^\star$, we can assume $h_{n+j}^\star - (n-1) \geq 1$ for $j = 1, \ldots, s_n$, and also $h_n^\star - (n-1) \geq 2$. Consequently,

$$t_n = \min\left\{ \left\lfloor \frac{(h_n^\star - (n-1))_+}{2} \right\rfloor, s_n \right\} \geq 1.$$

From our description of the entries of $\mathbf{h_n}$, it follows that we can find $2t_n$ reduced points on ℓ_n and a reduced point on each of the lines $\ell_{n+1}, \ldots, \ell_{n+t_n}$ in Z_{n-1}. For each $j = 1, \ldots, t_n$, we remove two reduced points from ℓ_n and one reduced point from ℓ_{n+j}, and replace these three reduced points with the double $2P_{n,n+j}$ where $P_{n,n+j} = \ell_n \cap \ell_{n+j}$. By repeatedly applying Theorem 3.1, each time we add a new double point and remove the reduced points, the new scheme has the same Hilbert function. Consequently, the scheme Z_n produced by STEP n satisfies $H_{Z_n} = H_{Z_{n-1}} = H$, as desired. □

Example 3.6. We illustrate Construction 3.5 with the valid Hilbert function $\Delta H = (1,2,3,4,2)$. In this case $\Delta H^\star = (5,4,2,1)$. Fix four general lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4, i.e., no three of the lines meet at a point. If we place five points on ℓ_1, four points on ℓ_2, two points on ℓ_3, and one point on ℓ_4, as in the Figure 5, then the set of reduced points Z_0 has Hilbert function $\Delta H_X = \Delta H$ (by Theorem 2.6). The construction of Z_0 is STEP 0 of Construction 3.5.

For STEP 1, we let $\mathbf{h_1} = (5,4,2,1)$ and $s_1 = 3$. Since $s_1 \neq 0$, we let $t_1 = \min\{\lfloor \frac{5}{2} \rfloor, 3\} = 2$. We remove $2 \cdot 2 = 4$ points from ℓ_1, and 1 point from ℓ_2 and 1 point from ℓ_3, and we add the double points $2P_{1,2}$ and $2P_{1,3}$ to make the scheme Z_1 as in Figure 6. The double points are denoted with a 2 in the figure. Note that by Theorem 3.1 this scheme has the same Hilbert function as Z_0. Roughly speaking, we are "merging" two points on ℓ_1 with a third point on ℓ_2 (or ℓ_3) to make the double point $2P_{1,2}$ (or $2P_{1,3}$).

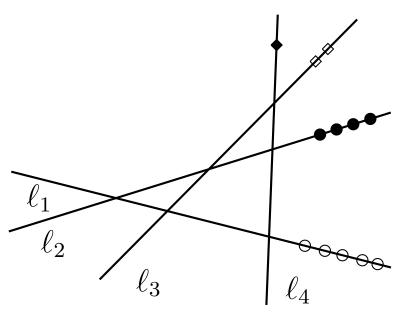


FIGURE 5. The set Z_0 with $H_{Z_0} = H$

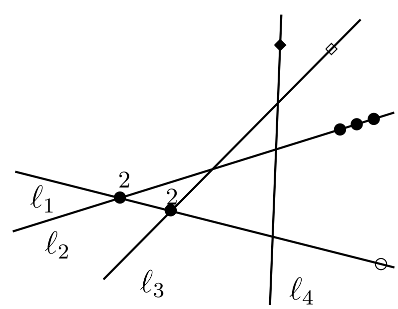


FIGURE 6. The scheme Z_1 with $H_{Z_1} = H$

Moving to STEP 2, we set $\mathbf{h_2} = ((4-1)_+, (2-1)_+, (1-1)_+) = (3,1,0)$ and $s_2 = 1$. (Note the j-th entry of $\mathbf{h_2}$ is a lower bound on the number of reduced points on ℓ_{j+1} for $j = 2,3,4$.) Because $s_2 \neq 0$, we let $t_2 = \min\{\lfloor \frac{3}{2} \rfloor, 1\} = 1$. We remove two points from ℓ_2 and one point from ℓ_3 from Z_1, but add the double point $2P_{2,3}$ to form the scheme Z_2. Again, Theorem 3.1 implies that this construction of double and reduced points has the same Hilbert function as Z_1, and consequently, Z. See Figure 7 for an illustration of Z_2.

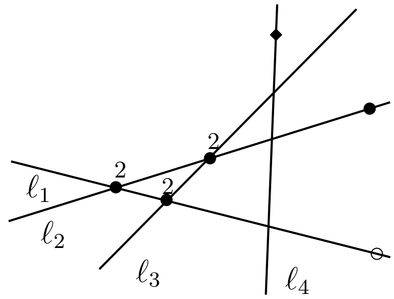


FIGURE 7. The scheme Z_2 with $H_{Z_2} = H$

Finally, at STEP 3 we have $\mathbf{h_3} = ((2-2)_+, (1-2)_+) = (0,0)$ and $s_3 = 0$. Our construction terminates and returns the scheme Z_2, which consists of three double points and three reduced points, and $\Delta H_{Z_2} = \Delta H = (1,2,3,4,2)$.

Example 3.7. Construction 3.5 may not produce a set of just double points, even if ΔH is known to be the valid Hilbert function of a set of double points. For example, consider the scheme Z of two double points. We have $\Delta H_Z = (1,2,2,1)$. So, $\Delta H = (1,2,2,1)$ is a valid Hilbert function of a set of double points. However, Construction 3.5 cannot be used to detect this fact. In particular, since $\Delta H^\star = (4,2)$ Construction 3.5 returns only one double point (and two simple points on ℓ_1, and an other simple point on ℓ_2).

Example 3.8. Consider the valid Hilbert function $\Delta H = (\underbrace{1,\ldots,1}_{\sigma+1})$. Construction 3.5 applied to this ΔH stops at the beginning of STEP 1 by producing $\sigma+1$ reduced points on a line ℓ_1. A valid Hilbert function of this type is the only time Construction 3.5 terminates at STEP 1. Note that it can be shown that if $\Delta H = (1,\ldots,1)$, then the only zero-dimensional scheme Z with $\Delta H_Z = \Delta H$ is precisely a set of reduced points on a line. If $\Delta H = (1,2,\ldots)$, then Construction 3.5 will produce a scheme Z with at least one double point.

As noted in the last remark, if $\Delta H = (1,2,\ldots)$, then our procedure produces a scheme with at least one double point. We are actually interested in producing a scheme with the largest possible number of double points. We can determine the number of double points produced by Construction 3.5 directly from ΔH, as shown in the next result which gives an answer to Question 1.2.

Theorem 3.9. *Let ΔH be a valid Hilbert function with $\Delta H^\star = (h_1^\star,\ldots,h_\alpha^\star)$. Set*

$$d = \sum_{i=1}^{\alpha-1} \min\left\{ \left\lfloor \frac{(h_i^\star - (i-1))_+}{2} \right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^\star \geq i\} \right\}.$$

Then there is a set of fat points Z of d double points and $r = (\sum \Delta H) - 3d$ reduced points with $\Delta H_Z = \Delta H$.

Proof. We first note that s_i in Construction 3.5 equals $\#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^\star \geq i\}$ Also, we have $s_1 \geq s_2 \geq \cdots \geq s_{\alpha-1} \geq s_\alpha = 0$. Let $j = \max\{i \mid s_i \neq 0\}$. So, for

$i = 1, \ldots, j$,

$$\min\left\{\left\lfloor\frac{(h_i^\star-(i-1))_+}{2}\right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^\star \geq i\}\right\}.$$

is the number of double points added to Z_{i-1} at STEP i. For $i = j+1, \ldots, \alpha$, Construction 3.5 has terminated, and no new double points are added. This fact is captured in the summation via the fact that all the summands

$$\min\left\{\left\lfloor\frac{(h_i^\star-(i-1))_+}{2}\right\rfloor, \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^\star \geq i\}\right\} = 0$$

for $i = j+1, \ldots, \alpha$. □

We record some corollaries:

Corollary 3.10. *Let $\Delta H = (1, 2, \ldots, \alpha, h_\alpha, \ldots, h_\sigma)$ be a valid Hilbert function, and let d be as in Theorem 3.9.*

(i) *The integer d satisfies $\min\left\{\left\lfloor\frac{\sigma+1}{2}\right\rfloor, \alpha-1\right\} \leq d \leq \binom{\alpha}{2}$.*

(ii) *For every integer $1 \leq e \leq d$, there exists a scheme Z of e double points and $(\sum \Delta H) - 3e$ reduced points with $H_Z = H$.*

(iii) *Let $e = \min\left\{\lfloor\frac{\sigma+1}{2}\rfloor, \alpha-1\right\}$. Then there exists a scheme Z with e double points and $(\sum \Delta H) - 3e$ reduced points.*

Proof. For (i), the support of each double point produced by Construction 3.5 has the form $\ell_i \cap \ell_j$. Since there are only α lines $\ell_1, \ldots, \ell_\alpha$ used in this construction, the outputted scheme can have at most $\binom{\alpha}{2}$ double points, thus giving the upper bound. For the lower bound, note that in the computation of d using Theorem 3.9, when the index is $i = 1$, we have $h_1^\star = \sigma + 1$ and $\alpha - 1 = \#\{2 \leq k \leq \alpha \text{ and } h_k^\star \geq 1\}$, i.e., the lower bound is the first term in the sum.

The proof of (iii) will follow from (i) and (ii) since $e \leq d$. For (ii), notice that Construction 3.5 adds one double point at a time, and when it finishes, we have d double points. Since $e \leq d$, we use this procedure again, but changing our stopping criterion so the procedure terminates when we have e double points. □

Recall that one of the fundamental problems about Hilbert functions of double points in $\mathbb{P}^2$ is to classify what functions are the Hilbert functions of double (or more generally, fat) points. We can now contribute to this problem by identifying some new functions as the Hilbert functions of double points.

Theorem 3.11. *Let ΔH be a valid Hilbert function, and let $\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star)$. Suppose that*

$$\frac{(h_i^\star-(i-1))_+}{2} = \#\{k \mid i+1 \leq k \leq \alpha \text{ and } h_k^\star \geq i\} \text{ for } i = 1, \ldots \alpha - 1.$$

Then ΔH is Hilbert function of a set of double points in $\mathbb{P}^2$.

Proof. We prove this by showing that Construction 3.5 terminates with a scheme of only double points. At STEP 1, observe that the vector $\mathbf{h}_1 = (h_1^\star, \ldots, h_\alpha^\star)$ has the property that $h_i^\star$ is the exact number of reduced points on ℓ_i. The hypotheses imply that when Construction 3.5 executes STEP 1, the value of t_1 is given by

$$t_1 = \frac{(h_1^\star)_+}{2} = s_1.$$

But this means that all the reduced points on ℓ_1 and one reduced point on each of $\ell_{1+1}, \ldots, \ell_{1+s_1=\alpha}$ are removed to form $2t_1$ double points. In particular, no reduced points are left on ℓ_1 and $\mathbf{h}_2 = ((h_2^\star - 1)_+, \ldots, (h_\alpha^\star - 1)_+)$ has the property that $(h_i - 1)_+^\star$ is exactly the number of reduced points remaining on ℓ_i. (In the general, case, $(h_i^\star - 1)_+$ is only a lower bound for the number of reduced points that remain).

Proceeding by induction on n, note that at STEP n, the tuple $\mathbf{h_n} = ((h_n^\star - (n - 1))_+, \ldots, (h_\alpha^\star - (n-1))_+)$ counts exactly the number of reduced points on ℓ_i for $i = n, \ldots, \alpha$ after constructing Z_{n-1}. This is because in each of STEP 1 through $n-1$, we used at exactly one reduced point on ℓ_i when constructing Z_j with $0 \leq j \leq n-1$. The hypotheses imply when Construction 3.5 executes STEP n we get

$$t_n = \frac{(h_n^\star - (n-1))_+}{2} = s_n.$$

So, all the remaining reduced points on ℓ_n and one reduced point on each of $\ell_{n+1}, \ldots, \ell_{n+s_n}$ are removed to form $2t_n$ double points. In particular, no reduced points are left on ℓ_n and $\mathbf{h}_{n+1} = ((h_{n+1}^\star - n)_+, \ldots, (h_\alpha^\star - (n))_+)$ has the property that $(h_i - n)_+^\star$ is the exactly the number of reduced points remaining on ℓ_i.

When the algorithm terminates, $\mathbf{h_n} = (0, \ldots, 0)$, that is, no reduced points remain. □

Example 3.12. Consider the valid Hilbert function

$$\Delta H = (1,2,3,4,5,6,2,2,1,1) \Leftrightarrow \Delta H^\star = (10,7,4,3,2,1).$$

Then $\Delta H^\star$ statisfies the hypothesis of Theorem 3.11 since

$$\begin{aligned}
\frac{h_1^\star}{2} &= 5 = \#\{k \mid 2 \leq k \leq 6 \text{ and } h_k^\star \geq 1\} \\
\frac{(h_2^\star - 1)_+}{2} &= 3 = \#\{k \mid 3 \leq k \leq 6 \text{ and } h_k^\star \geq 2\} \\
\frac{(h_3^\star - 2)_+}{2} &= 1 = \#\{k \mid 4 \leq k \leq 6 \text{ and } h_k^\star \geq 3\} \\
\frac{(h_i^\star - (i-1))_+}{2} &= 0 = \#\{k \mid i+1 \leq k \leq 6 \text{ and } h_k^\star \geq i\} \text{ for } i \geq 4.
\end{aligned}$$

Then ΔH is the Hilbert function of $(\sum \Delta H)/3 = 9$ double points. Indeed, if $\ell_1, \ldots, \ell_6$ are six general lines such that no three meet at a point, Construction 3.5 will produce the scheme of nine double points

$$Z = 2P_{1,2} + 2P_{1,3} + 2P_{1,4} + 2P_{1,5} + 2P_{1,6} + 2P_{2,3} + 2P_{2,4} + 2P_{2,5} + 2P_{3,4}$$

where $P_{i,j} = \ell_i \cap \ell_j$ with Hilbert function $H_Z = H$.

The following corollary is used in the next section.

Corollary 3.13. *Let ΔH be a valid Hilbert function with $\Delta H^\star = (h_1^\star, \ldots, h_\alpha^\star)$. If*

$$\frac{h_i^\star - (i-1)}{2} = \alpha - i \text{ for } i = 1, \ldots \alpha,$$

then there exists a scheme Z in $\mathbb{P}^2$ of only double points with $\Delta H = \Delta H_Z$.

Proof. We have

$$\Delta H^\star = (\underbrace{2\alpha - 2, \ldots, \alpha+1, \alpha, \alpha-1}_{\alpha}).$$

Now apply Theorem 3.11. □

4. Special configurations

In this section, we will examine two valid Hilbert functions:

$$\Delta H_1 = (\underbrace{1,2,3,\ldots,t}_{t}, \underbrace{t+1,\ldots,t+1}_{t}), \text{ and } \Delta H_2 = (\underbrace{1,2,3,\ldots,t}_{t}, \underbrace{t+1,\ldots,t+1}_{t}, 1).$$

In the first case, we will show that only the sets Z of $\binom{t}{2}$ double points with support on a star configuration have $\Delta H_Z = \Delta H_1$. In the second case, we will show that the only sets Z of $\binom{t}{2}$ double points and one simple point that have $\Delta H_Z = \Delta H_2$ are sets of double points with support on a star configuration plus one extra point on one of the lines that define the star configuration.

We start by collecting together some required tools. The first result we need is the following theorem (see [3] or [9]).

Theorem 4.1. *Let $X \subset \mathbb{P}^2$ be a zero-dimensional subscheme, and assume that there is a t such that $\Delta H_X(t-1) = \Delta H_X(t) = d$. Then the degree t components of I_X have a GCD, say F, of degree d. Furthermore, the subscheme W of X lying on the curve defined by F (i.e., I_W is the saturation of the ideal (I_X, F)) has Hilbert function whose first difference is given by the truncation $\Delta H_W(i) = \min\{\Delta H_X(i), d\}$.*

Now we recall some well known results about star configurations. Given any linear form $L \in R$, we let ℓ denote the corresponding line in $\mathbb{P}^2$. Let $\ell_1, \ldots, \ell_{t+1}$ be a set of $t+1$ distinct lines in $\mathbb{P}^2$ that are three-wise linearly independent (general linear forms). In other words, no three lines meet at a point. A *star configuration* of $\binom{t+1}{2}$ points in $\mathbb{P}^2$ is formed from all pairwise intersections of the $t+1$ linear forms.

Geramita, Harbourne, and Migliore have computed the Hilbert function of double points whose support is a star configuration. Specifically,

Theorem 4.2 ([10, Theorem 3.2])**.** *In $\mathbb{P}^2$, let t be a positive integer, let X be a star configuration of $\binom{t+1}{2}$ points, and let $Z = 2X$ be a set of double points whose support is X. Then the first difference of the Hilbert function of Z is*

$$\Delta H_Z(i) = \begin{cases} i+1 & \text{if } 0 \le i \le t \\ t+1 & \text{if } t+1 \le i \le 2t-1 \\ 0 & \text{if } i \ge 2t. \end{cases}$$

Remark 4.3. We point out that the above result can be generalized to $\mathbb{P}^n$.

4.1. **Double points on a star configuration.** We consider the valid Hilbert function

$$\Delta H = (\underbrace{1,2,3,\ldots,t}_{t},\underbrace{t+1,\ldots,t+1}_{t}).$$

By Corollary 3.13 with $t=\alpha-1$, we already know in this case that Construction 3.5 gives a set of only double points. In this subsection, we will prove that the configuration produced by our construction is a set of double points lying on star configuration, and furthermore, this is the only set of double points whose first difference is equal to ΔH. Since for $t \leq 2$ all is trivial, we will assume that $t \geq 3$.

Theorem 4.4. *Let $t \geq 3$ be an integer, and let $Z \subset \mathbb{P}^2$ be a set of $\binom{t+1}{2}$ double points. Then the sequence*

$$\Delta H = (\underbrace{1,2,3,\ldots,t}_{t},\underbrace{t+1,\ldots,t+1}_{t}) \tag{4.1}$$

is the first difference of the Hilbert function of Z if and only if Z is a set of double points whose support is a star configuration.

Proof. ($\Leftarrow$) This follows from Theorem 4.2.

($\Rightarrow$) Suppose that Z is a set of $\binom{t+1}{2}$ double points such that the first difference of Hilbert function is given by (4.1), i.e.,

$$\Delta H_Z = \begin{array}{cccccccccccc}
 & & & & & & \bullet & \bullet & \bullet & \cdots & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
 & & & & & & & \vdots & & & \\
 & & & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
 & & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
 & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
0 & 1 & 2 & 3 & & & t & & & & 2t-1
\end{array} \quad t+1 \text{ rows} \tag{4.2}$$

We want to prove that the support of Z must be a star configuration constructed from $t+1$ general lines.

Let I_Z be the ideal of Z. We shall sometimes refer to $(I_Z)_t$ as *the linear system of all the plane curves of degree t containing Z*, since this is what the forms in $(I_Z)_t$ correspond to from a geometrical point of view. From (4.1) we note that the smallest degree in a minimal set of generators of I_Z is $t+1$, the largest degree is $2t$, and there is only one curve, say $\mathcal{C} = \{F=0\}$, in the linear system defined by $(I_Z)_{t+1}$. Moreover I_Z does not have new minimal generators until the degree $2t$.

Because $(I_Z)_{2t}$ contains all the forms of type $F \cdot (x,y,z)^{t-1}$, then the linear system $(I_Z)_{2t}$ cannot be composed with a pencil, see [18, pg. 26]. We recall that a linear system is composed with a pencil if any of its elements is of the type $\phi_1 \cdot \phi_2 \cdots \phi_n$ where the forms ϕ_i are of the type $c_1\psi_1 + c_2\psi_2$ for scalars c_i and forms ψ_i. Moreover, since I_Z is generated in degrees $\leq 2t$, the base locus of $(I_Z)_{2t}$ is exactly Z. Hence, $(I_Z)_{2t}$ has no fixed components. By Bertini's Theorem (for example, see [15]), the general curve of the linear system $(I_Z)_{2t}$ is integral (irreducible and reduced). Thus we may assume that $(I_Z)_{2t} = (G_1, \dots, G_r)$ where $r = \dim_k(I_Z)_{2t}$ and each $\mathcal{G}_i = \{G_i = 0\}$ is an integral curve.

For each curve $\mathcal{G}_i$, consider the intersection $\mathcal{G}_i \cap \mathcal{C}$. Since each $\mathcal{G}_i$ is integral and $\deg \mathcal{G}_i > \deg \mathcal{C}$, we have $\deg(\mathcal{G}_i \cap \mathcal{C}) = \deg \mathcal{G}_i \cdot \deg \mathcal{C}$. Now each point of Z is a double point of both $\mathcal{G}_i$ and $\mathcal{C}$. So, the degree of the scheme $\mathcal{G}_i \cap \mathcal{C}$ at each point of Z is at least 4. Hence $\deg(\mathcal{G}_i \cap \mathcal{C}) \geq 4\binom{t+1}{2} = 2t(t+1) = \deg \mathcal{G}_i \cdot \deg \mathcal{C} = \deg(\mathcal{G}_i \cap \mathcal{C})$. It follows that $\mathcal{G}_i$ and $\mathcal{C}$ only intersect at the points of Z, and that for each point P in the support of Z, the degree of the scheme $\mathcal{G}_i \cap \mathcal{C}$ at P is exactly 4.

Now observe that the curve $\mathcal{C}$ is not integral. In fact, $\mathcal{C}$ has $\binom{t+1}{2}$ double points, but an integral curve of degree $t+1$ has at most $\binom{t}{2}$ double points.

We claim that $\mathcal{C}$ totally reduces by distinct lines, that is, if $a\mathcal{L}$ is an irreducible component of $\mathcal{C}$ (i.e., the polynomial defining $\mathcal{L}$ is irreducible) of multiplicity a, we will show that $\mathcal{L}$ is a line and $a = 1$.

Let P_1 be a general point on $\mathcal{L}$. Since F vanishes on P_1, the first difference of the Hilbert function of $Z + P_1$ is the following

$$\Delta H_{Z+P_1}(i) = \begin{cases} i+1 & \text{if } 0 \leq i \leq t \\ t+1 & \text{if } t+1 \leq i \leq 2t-1 \\ 1 & \text{if } i = 2t \\ 0 & \text{if } i > 2t. \end{cases}$$

To see why this is the case, when we add a point to Z, we have to add a point to the diagram in (4.2). There are only two places to put a point and maintain a valid Hilbert function: 1) put a point where $i = t+1$ or 2) put a point where $i = 2t$. In the first case we get $\Delta H_{Z+P_1}(t+1) = t+2$, and so we do not have curves in the linear system defined by $(I_Z)_{t+1}$. But this is a contradiction since P_1 is a point of $\mathcal{C}$. So 2) must hold. We will prove that $\mathcal{L}$ is a common component for the curves of the linear system $(I_{Z+P_1})_{2t}$.

Recall that each $\mathcal{G}_i$ intersects $\mathcal{L}$ only at the points of Z. If d denotes the degree of $\mathcal{L}$, then the degree of $\mathcal{G}_i \cap a\mathcal{L}$ is $2tad$. Now, consider a curve $\mathcal{T} = \{T = 0\}$ with $T \in (I_{Z+P_1})_{2t}$. We have that $\deg(\mathcal{T} \cap a\mathcal{L}) \geq 2tad + 1$. However $\deg \mathcal{T} = 2t$ and $\deg a\mathcal{L} = ad$. So, by Bezout's Theorem, $\mathcal{L}$ is a common component for every curve of the linear system $(I_{Z+P_1})_{2t}$.

Now look at $Z + P_1 + P_2$ where P_2 is another general point on $\mathcal{L}$. Since $\mathcal{L}$ is a common component for every curve of $(I_{Z+P_1})_{2t}$, then the first difference of the Hilbert function of $Z + P_1 + P_2$ is given by

$$\Delta H_{Z+P_1+P_2}(i) = \begin{cases} i+1 & \text{if } 0 \leq i \leq t \\ t+1 & \text{if } t+1 \leq i \leq 2t-1 \\ 1 & \text{if } i = 2t \\ 1 & \text{if } i = 2t+1 \\ 0 & \text{if } i > 2t+1. \end{cases}$$

So, using Theorem 4.1 with $d = 1$, we get that $Z + P_1 + P_2$ has a subscheme of degree $2t+2$ lying on a line ℓ. But Z imposes independent conditions to the curves of degree $2t-1$, hence Z has at most t double points with support on a line, and so $P_1, P_2 \in \ell$ and $\deg(\mathcal{C} \cap \ell) = 2t+2$. Since $2t+2 > t+1 = \deg \mathcal{C} \cdot \deg \ell$, then the line ℓ is a component of $\mathcal{C}$. Now observe that P_1 and P_2 are generic points on $\mathcal{L}$, so $\mathcal{L}$ must be the line ℓ.

It follows that every irreducible component of $\mathcal{C}$ is a line, and thus

$$\mathcal{C} = a_1\ell_1 + \cdots + a_v\ell_v,$$

where the ℓ_i are lines and $a_1 + \cdots + a_v = t+1$. Now we will show that $a_i = 1$ for all i, that is, $\mathcal{C}$ is a union of $t+1$ distinct lines. First observe that no a_i can be bigger than 2. Indeed, if $a_i > 2$, then the curve $\mathcal{C} \setminus \ell_i$ would be a curve of degree t containing Z; this contradicts the fact that $\mathcal{C}$ is the curve of minimal degree containing Z. Hence, by this observation, or simply by recalling that $\deg(\mathcal{G}_i \cap \mathcal{C})$ in every $P \in Z$ is exactly 4, the irreducible components of $\mathcal{C}$ are simple or double lines. After relabeling, we can assume

$$\mathcal{C} = 2\ell_1 + \cdots + 2\ell_s + \ell_{s+1} + \cdots + \ell_{2s+r},$$

where $2s + r = t + 1$.

We observe that the points of Z lying on the simple lines can lie only on the intersection with other simple lines. So there are at most $\binom{r}{2}$ such points. Since $\deg(\mathcal{G}_i \cap \mathcal{C}) = 4$ for every $P \in Z$, the points of Z on the double lines cannot lie on the intersections with other lines. Moreover, since Z imposes independent conditions to the curves of $(I_Z)_{2t}$, on each line $\mathcal{L}_i$ we have at most t double points, and so the number of points of Z on the double lines is at most st. It follows that at most $st + \binom{r}{2}$ points of Z lie on $\mathcal{C}$, that is,

$$|Z| = \binom{t+1}{2} \leq st + \binom{r}{2}.$$

Because $t + 1 = 2s + r$, we have

$$\binom{2s+r}{2} \leq s(2s + r - 1) + \binom{r}{2},$$

and from here we get $rs = 0$. If $r = 0$, then we get

$$\mathcal{C} = 2\ell_1 + \cdots + 2\ell_s \quad \text{where} \quad 2s = t + 1,$$

and Z must have t points on each line ℓ_i. By Bezout's Theorem, it follows that the curves of degree $2t - 1$ through Z have the lines ℓ_i as fixed components. Removing these s lines from Z we remain with a scheme Z' of $|Z| = \binom{t+1}{2}$ simple points and we get

$$\dim_k(I_Z)_{2t-1} = \dim_k(I_{Z'})_{2t-1-s} \geq \binom{2t-1-s+2}{2} - \binom{t+1}{2} = \frac{5t^2 - 4t - 1}{8}.$$

But $(I_Z)_{2t-1}$ is not defective, hence

$$\dim_k(I_Z)_{2t-1} = \binom{2t-1+2}{2} - 3\binom{t+1}{2} = \frac{t^2 - t}{2}.$$

But $\frac{t^2-t}{2} \not\geq \frac{5t^2-4t-1}{8}$ for any t, so we get a contradiction. Therefore, $s = 0$ and $\mathcal{C}$ is a union of $t+1$ distinct lines. It follows that the support of Z is a star configuration of $t+1$ lines. □

Remark 4.5. It is easy to see that if ΔH_Z is of type (4.1), Construction 3.5 gives a set of double points on a star configuration. For instance, if $\Delta H = (1, 2, 3, 4, 4, 4)$, then $\Delta H^\star = (6, 5, 4, 3)$. Step 0 and the final step of Construction 3.5 are given in Figure 8, respectively Figure 9. In Figure 9, the three reduced points near the intersection of ℓ_i and ℓ_j should be viewed as one double point at $\ell_i \cap \ell_j$.

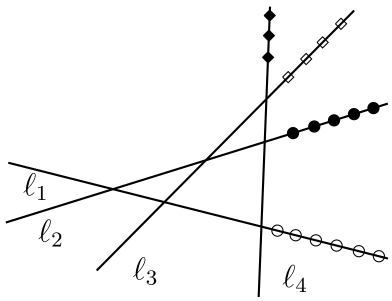


FIGURE 8. Initial setup of Construction

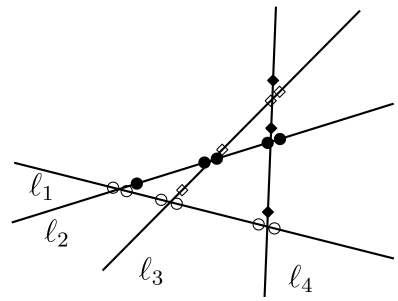


FIGURE 9. Output of Construction

4.2. **Double points on a star configuration plus one point.** In this section we will investigate when a scheme of one simple point union $\binom{t+1}{2}$ double points has the same Hilbert function of one simple point union double points with support on a star configuration.

Theorem 4.6. *Let $t \geq 3$ be an integer, let $Z \subset \mathbb{P}^2$ be a set of $\binom{t+1}{2}$ double points, and let P be a simple point. Then the sequence*

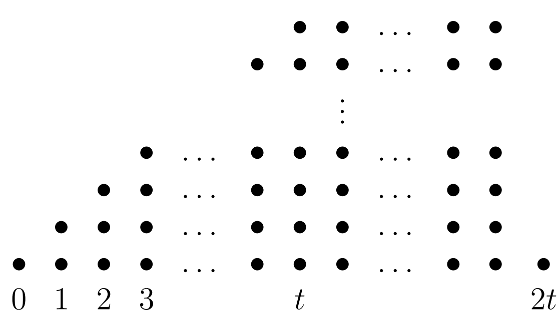
$$\Delta H = (\underbrace{1, 2, 3, \dots, t}_{t}, \underbrace{t+1, \dots, t+1}_{t}, 1)$$

is the first difference of the Hilbert function of $Z + P$ if and only if Z is a set of double points whose support is a star configuration of $t+1$ lines and P lies on one of those lines.

Proof. ($\Leftarrow$) One can compute the Hilbert function from [10, Theorem 3.2].

($\Rightarrow$) Suppose that $Z + P$ is a set of $\binom{t+1}{2}$ double points and a simple point, such that the first difference of the Hilbert function looks like

(4.3) $\Delta H_{Z+P} =$

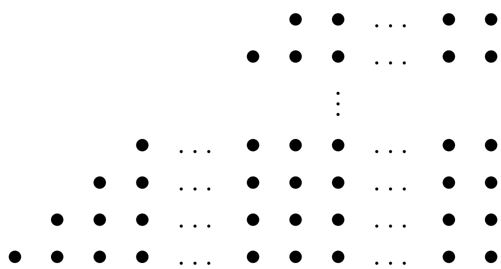


$t+1$ rows

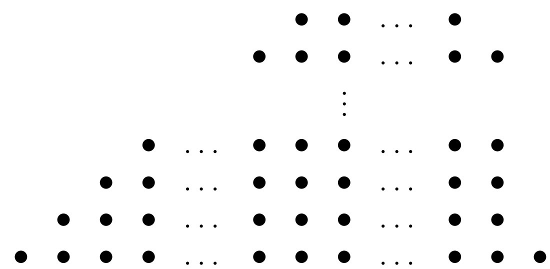
Our goal is to show that the support of Z must be a star configuration of $t+1$ lines and a point P that lies on one of those lines.

Consider only the scheme Z. The first difference of the Hilbert function of Z can only have one of the following two forms:

(4.4)



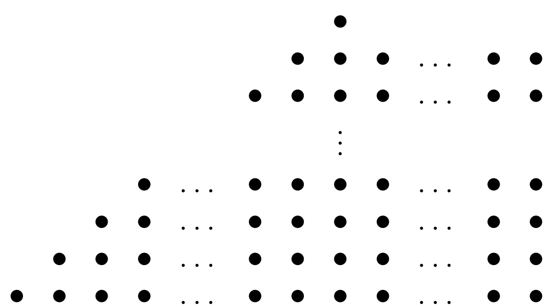
or



To see why, note that ΔH_Z is constructed from ΔH_{Z+P} by removing exactly one point. The two cases represent the only two ways to remove a point from (4.3) and still have a valid Hilbert function.

If ΔH_Z is of the first type, then by Theorem 4.1, the support of Z is a star configuration and the curve $\mathcal{C}$ is given by the product of the $t+1$ lines of the star configuration. If P does not lie on a line of the star configuration, then $P \notin \mathcal{C}$, and thus $(I_{Z+P})_{t+1} = 0$. Hence the first difference Hilbert function of $Z+P$ would have type

$\Delta H_{Z+P} =$



which is different from (4.3). So P lies on a line of $\mathcal{C}$, and the conclusion follows.

It suffices to show that the second case cannot occur. So assume for a contradiction that ΔH_Z is given by the second diagram in (4.4). Note that the smallest degree in a minimal set of generators of I_Z is $t+1$ and there is only one curve, say $\mathcal{C} = \{F = 0\}$, in the linear system defined by $(I_Z)_{t+1}$.

Now we will show that the linear system $(I_{Z+P})_{2t}$ has no fixed components. Suppose for a contradiction that $\mathcal{T}$ is a fixed irreducible component of $(I_{Z+P})_{2t}$ and let $Q \in \mathcal{T}$ be a generic point. Since $\dim_k(I_{Z+P+Q})_{2t} = \dim_k(I_{Z+P})_{2t}$ we have

$$\Delta H_{Z+P+Q}(i) = \begin{cases} i+1 & \text{if } i \leq t \\ t+1 & \text{if } t+1 \leq i \leq 2t-1 \\ 1 & \text{if } i = 2t \\ 1 & \text{if } i = 2t+1 \\ 0 & \text{if } i \geq 2t+2. \end{cases}$$

By Theorem 4.1 with $d = 1$, we have that $Z+P+Q$ has a subscheme W of degree $2t+2$ lying on a line ℓ. But W cannot be contained in Z, since $(I_Z)_{2t}$ cannot have a scheme of degree $2t+2$ on a line. So the scheme W is the intersection of ℓ with t points of Z plus

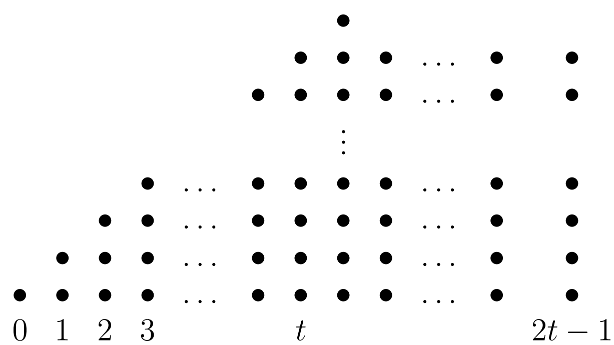
P and Q. But Q is generic on $\mathcal{T}$, thus $\mathcal{T}$ should be the line ℓ. Now consider the scheme $W \setminus Q \subset \ell$ which is the union of t points of Z plus P.

Since the scheme $W \setminus Q$ has degree $2t+1$, the point P cannot give independent conditions to the curves of the linear system $(I_{Z+P})_{2t-1}$. If ΔH_Z resembles the second case of (4.4), then ΔH_{Z+P} cannot be of type (4.3), since in this case P would impose independent conditions to the curves of $(I_{Z+P})_{2t-1}$, a contradiction. Thus, the linear system $(I_{Z+P})_{2t}$ does not have fixed components.

Moreover, since $(I_{Z+P})_{2t}$ contains forms of the type $F \cdot (x,y,z)^{t-1}$, then it cannot be composed with a pencil, see [18, pg.26]. Using Bertini's Theorem (see [15]), the general curve of $(I_{Z+P})_{2t}$ is reduced and irreducible. Let $\mathcal{G}$ be such a general integral curve. We have that $\deg \mathcal{G} \cdot \deg \mathcal{C} = 2t(t+1)$. But since $\deg(\mathcal{G} \cap \mathcal{C}) \geq 4\frac{t(t+1)}{2} + 1$, the curve $\mathcal{G}$ would contain a component of $\mathcal{C}$, thus giving a contradiction. $\square$

Finally we show that if we add a multiplicity "on the top"of (4.2), a statement similar to that of Theorem 4.6 does not hold. More precisely let $t \geq 3$ be an integer, and let $Z \subset \mathbb{P}^2$ be a set of $\binom{t+1}{2}$ double points. Suppose that P is a simple point such that the first difference Hilbert function of $Z+P$ has the form

(4.5) $\Delta H_{Z+P} =$



$t+1$ rows

Obviously, if the support of Z is a star configuration of $t+1$ lines and P is a generic simple point, then the first difference of the Hilbert function of $Z+P$ is of type (4.5), but the converse is not true, as we show in the following example.

Example 4.7. Consider a scheme Z of $\binom{t+1}{2}$ double points, whose support, except one double point, say $2Q$, are the points of a star configuration of the $t+1$ general lines $\ell_1, \ldots, \ell_{t+1}$. More precisely, the points of Z lie on the intersections $\ell_i \cap \ell_j$, for all $i \neq j$, except for $(i,j) = (1,2)$. Let the simple point P be a general point on ℓ_1 and suppose Q is a general point on ℓ_2 (see Figure 10). Note that in Figure 10, all the points of intersection are double points.

In order to prove that the first difference of the Hilbert function of $Z+P$ is of type (4.5), it is enough to prove that $\dim_k(I_{Z+P})_{t+2} = 2$ and $(I_{Z+P})_{2t-1}$ is not defective. By Bezout's Theorem, the t lines $\ell_2, \ldots, \ell_{t+1}$ are fixed components for the curves of the two linear systems $(I_{Z+P})_{t+2}$ and $(I_{Z+P})_{2t-1}$. Hence we have

$$\dim_k(I_{Z+P})_{t+2} = \dim_k(I_X)_2 \text{ and } \dim_k(I_{Z+P})_{2t-1} = \dim_k(I_X)_{t-1},$$

where X is a union of t simple points on the line ℓ_1 and the point $Q \in \ell_2$. Since $2 < t$, in order to compute $\dim(I_X)_2$, we may remove the line ℓ_1 and we get $\dim_k(I_{Z+P})_{t+2} =$

$\dim_k(I_Q)_1 = 2$. Since X imposes independent conditions to the curve of degree $t-1$ we have

$$\dim_k(I_X)_{t-1} = \binom{t-1+2}{2} - t - 1 = \binom{t}{2} - 1,$$

which is the expected dimension of $(I_{Z+P})_{2t-1}$, and we are done.

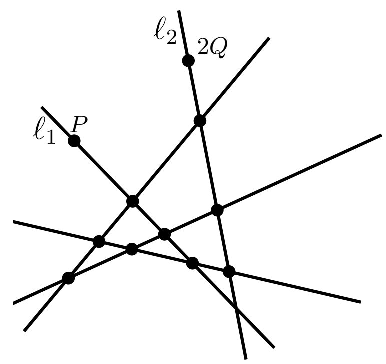


FIGURE 10. All but one double point on a star configuration plus one double point and one simple point

5. Final remarks

In this paper we presented an algorithm that, given a valid Hilbert function H for a zero-dimensional scheme, will produce a scheme consisting of double and simple points having Hilbert function H.

We know that for some special H (for example, see Theorem 3.11) our algorithm will produce a set consisting of *only* double points. Furthermore, in Section 4 we showed that for one family of valid Hilbert functions H, not only does our algorithm produce a scheme with the maximal possible number of double points, our algorithm produces the only possible configuration of double points with this Hilbert function.

However, there are H for which our algorithm does not perform well. Consider, for example, when H is the Hilbert function of double points with collinear support; our algorithm will produce a set with just *one* double point! Thus it is natural to ask how well our algorithm performs.

The major obstacle in answering this question is the following: given a Hilbert function H of a degree $3t$ zero-dimensional scheme, we do not know the maximal number of double points that the scheme can possess. Of course t gives an upper bound, but this bound might not be sharp. Ideally we would like to compare this unknown number with the number of double points that our algorithm produces for H and possibly make some asymptotic estimate.

This problem will be the object of further investigations, but we can already present an interesting result. Consider the scheme consisting of t, generic double points and let H be its Hilbert function. It is well known (e.g., see [2]) that, with the exceptions $t = 2$ and

$t = 5$, $H(i) = \min\left\{\binom{i+2}{2}, 3t\right\}$. The following result describes the asymptotic behavior of our algorithm for this H.

Proposition 5.1. *Let H be the Hilbert function of t generic double points, and let $s(t)$ be the number of double points produced by our algorithm with input H. Then*

$$\lim_{t\to+\infty} \frac{s(t)}{t} = \frac{3}{4}.$$

Proof. For each positive integer t, we choose b and $0 \le \epsilon \le b+1$ such that $3t = \binom{b+2}{2} + \epsilon$; note that b is uniquely determined by t and viceversa. We consider the case b odd, and a similar argument applies in the case b even. With this notation we have that, for $t \ge 6$,

$$\Delta H = (1, 2, \ldots, b+1, \epsilon).$$

Moreover, it is easy to see that, applying our algorithm to ΔH produces the same result when applying our algorithm to the length $b+1$ sequence

$$\Delta H_1 = (1, 2, \ldots, \frac{b+1}{2}, \underbrace{\frac{b+3}{2}, \ldots, \frac{b+3}{2}}_{\frac{b+1}{2}}).$$

As shown is Section 4, we obtain a set of

$$s(b) = \frac{1}{8}(b+1)(b+3)$$

double points. By a change of variables and using the bound $\epsilon \le b+1$ we get

$$\lim_{t\to+\infty} \frac{s(t)}{t} = \lim_{d\to+\infty} \frac{s(b)}{\frac{1}{3}\binom{b+2}{2} + \frac{\epsilon}{3}} = \frac{3}{4},$$

and this complete the proof. □

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