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# A new approach for the determination of the Iwan density function in modeling friction contact

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**Abstract:** Friction between interfaces seriously affects the dynamics of structures with joints. An accurate description of the friction behavior, usually given in the form of hysteresis loops, is then a prerequisite for a successful prediction of the dynamics of joint structures. **Driven by this demand, this paper proposes a new approach in the framework of the Iwan model to better simulate the nonlinear constitutive relationship of joints. The approach derives the Iwan density function from the contact pressure distribution on the joint interface, without having to assume it like traditional Iwan-type models. Following this, the corresponding force-displacement expressions can be obtained. The proposed approach has been applied to two different contact geometries: sphere-on-sphere and flat-on-flat. For the spherical contact, comparisons between the simulated results and the analytical solutions available in the literature show perfect agreement. Moreover, the effectiveness of the approach in flat-to-flat contact has been verified by comparing the simulated hysteresis loop with the experimental counterpart.**

**Keywords:** Mechanical joints; friction contact; Iwan model; density function; hysteresis.

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## 1 Introduction

22 Jointed components are indispensable elements in mechanical systems. Due to the presence of  
23 friction at joint interfaces, the dynamic behavior of jointed structures is nonlinear and shows stiffness  
24 softening and energy dissipation. The stiffness of jointed structures depends on amplitude of  
25 oscillatory displacements, while the energy dissipation caused by friction mechanism dominates the  
26 structural damping, even reaching a 90% of the total damping [1-3]. In addition, friction could  
27 induce stick-slip vibration which results in instability of mechanical systems with friction [4-7]. A  
28 good friction model can help in the design of controllers for eliminating the frictional instability.  
29 Therefore, a reliable constitutive model capable of representing the nonlinear feature of joint  
30 mechanism is a prerequisite for predictive simulations, design and optimization of joint structures in  
31 dynamics analysis [8].

32

33 Constitutive friction models should fulfill some significant requirements [9] to be effective: i) the  
34 model must be able to describe micro/gross slip behaviors and reproduce the dependence of friction  
35 damping on the amplitude of applied loads, ii) parameters of the model should be easily estimated  
36 and iii) the model must be easily integrated into a finite element code.

37

38 In the past decades, several constitutive friction models have been developed. The Iwan model was  
39 originally developed to reproduce elastic-plastic behavior of materials and then employed to simulate  
40 the friction hysteresis behavior of joint interfaces [10, 11]. Similarly, the Valanis model originating  
41 from plasticity mechanics was used to represent the friction behavior of a bolted joint [12]. The Dahl

42 model was designed to simulate a symmetrical hysteresis loop observed in bearings subjected to  
43 sinusoidal excitations with small amplitudes [13, 14]. Of course, there are other contact models [15-  
44 18].

45

46 Iwan model is widely used in the field of joint structural dynamics because of its ability to describe  
47 the observed friction phenomena and the simplicity of extraction of model parameters. Many  
48 improved Iwan models [19-30] have been derived by the original model. These Iwan-type models  
49 can be grouped into two sets, according to the different modeling schemes. The first scheme is based  
50 on a combination of the framework of Iwan model with tribological approach involving rough  
51 contact [19, 20]. This modeling scheme can be regarded to be physics-based. However, it is difficult  
52 to integrate these models into dynamic analysis processes, due to the complexity of the force-  
53 displacement formulation.

54

55 The second scheme preserves the essence of the Iwan model and is based on the consideration of  
56 improving accuracy and completeness of the model. Segalman et al. [9] developed a four-parameter  
57 Iwan model that considers a power-law relation between energy dissipation and amplitude of applied  
58 load. A truncated power-law distribution with one Dirac delta function was applied to implement the  
59 representation of microslip and energy dissipation. As a supplement, Segalman et al. [21] presented  
60 an inversion of Masing models via continuous Iwan systems to address some uncommon cases in  
61 which displacement is expressed in terms of load history instead of load in terms of displacement  
62 history. The second derivative of the displacement expression is used to derive the Iwan density

63 **function.** Song et al. [22], using the experimentally observation reported in [3], employed an  
64 additional linear elastic spring in parallel with the original Iwan model to reproduce the residual  
65 stiffness during gross slip. Based on this experimental evidence, they developed an adjusted Iwan  
66 beam element to simulate the effect of bolted joints on beam structures. Similar to Song’s work,  
67 Wang et al. [23] developed an improved Iwan model based on four-parameter Iwan model to  
68 describe both the residual stiffness during gross slip and the smooth transition of joint stiffness from  
69 microslip to gross slip conditions. Based on the four-parameter Iwan model and Song’s finding, Li et  
70 al. [24, 25] proposed a six-parameter Iwan model to model lap joints. They used a truncated power-  
71 law distribution with two Dirac delta functions to consider two phenomena: the residual stiffness  
72 during the gross slip regime and the energy dissipation during the microslip regime. They compared  
73 simulations with experimental results, which show that the six-parameter Iwan model has good  
74 reproduction of hysteresis phenomena of lap joints. Brake [26] developed an improved Iwan model  
75 including “pinning” behavior to describe friction behavior at a bolted joint interface, in which the  
76 pinning stiffness is obtained analytically in accordance with Hertz theory [27]. This improvement  
77 can capture the contact between the bolt shank and the hole **when the sliding distance between**  
78 **contact interfaces is greater than the gap between the bolt shank and the hole.** Rajaei et al. [28]  
79 developed a generalized Iwan model, different from the abovementioned Iwan-type models. They  
80 considered the effect of the variation of normal load on tangential recovery force in two ways: the  
81 variations of distribution function of the critical sliding force and stiffness. The effects of these  
82 variations were observed on the hysteresis loops measured on a beam with frictional contact support.  
83 Recently, Li et al. [29, 30] proposed a modified Iwan model including a normal linear spring with

84 “detachment” to simulate the friction contact behavior between turbine blade and underplatform  
85 damper. This model can represent both normal load variations and tangential microslip. In Ref. [30],  
86 they conducted a preliminary exploration of the Iwan density function and obtained analytical force-  
87 displacement expressions for line contact.

88

89 The effective modeling of joints depends on understanding and reproducing the basic physics  
90 associated with a jointed interface. Although those abovementioned Iwan-type models can describe  
91 the observed phenomena in some experiments, an essential fact - namely the effect of normal load  
92 distribution on tangential friction behavior - has been hardly considered. Substantially, the contact  
93 pressure distribution determines the motion state of a portion of contact area, either stick or slip [31].  
94 In practical applications, the contact pressure distribution is influenced by the geometry of contact  
95 bodies as well as roughness of surfaces which may be the main reason for the difference among some  
96 measured power-law relations of energy dissipation on the amplitude of applied force [32].

97

98 This paper presents a novel approach that aims to directly calculate the Iwan density function from a  
99 known or measured contact pressure distribution in the framework of the Iwan model and to further  
100 better describe the friction hysteresis behavior of the joint surface. Two different contact problems:  
101 spherical contact and flat contact, are treated separately. Furthermore, the effectiveness of the  
102 approach is verified by comparison with analytical solutions and experimental results. The paper is  
103 organized as follows. Section 2 describes the paradigm behind the proposed modeling approach and  
104 introduces the corresponding parameter estimation method. The proposed model is applied to two

105 different contact geometries. [Section 3](#) shows the case of a sphere-on-sphere contact. Results of the  
106 simulation, summarized by hysteresis loops, are compared with the analytical solution given by  
107 Mindlin [31]. [Section 4](#) deals with a flat-on-flat contact and this case is used to validate the model by  
108 comparing simulated results with experimental hysteresis loops. [Section 5](#) closes the paper  
109 discussing the overall approach and presenting the main conclusions.

<b>Nomenclature</b>			
$k_t$	tangential contact stiffness	$G$	shear modulus
$n$	number of Jenkins elements	$T_m$	amplitude of tangential force
$f_i^*$	critical sliding force on the $i^{\text{th}}$ element	$W$	dissipated energy per cycle
$\varphi$	density function	$r$	distance from the contact center
$\Delta$	range of critical sliding force	$n_{eq}$	normal force per contact width
$\mu$	friction coefficient	$\gamma$	proportional coefficient
$N$	normal load	$p_m$	mean contact pressure
$T$	tangential friction force	$w$	width of the plate
$\delta$	tangential relative displacement	$h$	thickness of the plate
$\delta_m$	amplitude of relative displacement	$lb$	lower bound of the integral
$p$	distribution function of contact pressure	$ub$	upper bound of the integral
$x, y$	spatial coordinates	$\alpha$	residual stiffness coefficient
$a$	size of contact area	$T_{\text{norm}}$	normalized tangential force
$t, t'$	tangential sliding stress (traction)	$\delta_{\text{norm}}$	normalized relative displacement
$p_0$	maximum contact pressure	DF	density function
$\vartheta$	Poisson's ratio	SD	sliding direction



111

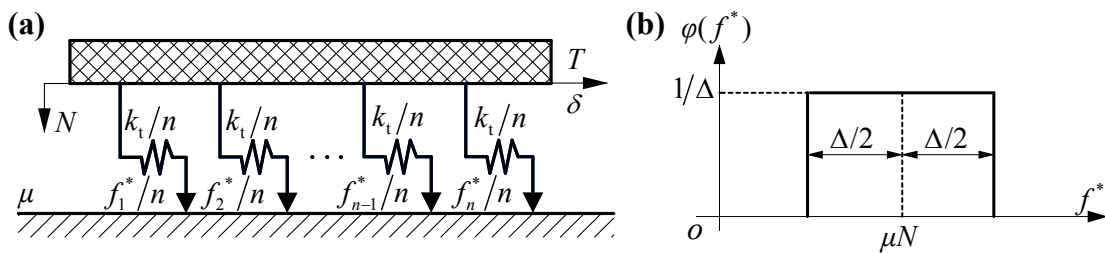
## 2 Modeling approach

112 In this section, the Iwan model is briefly reviewed and the importance of the Iwan density function  
 113 (DF) is emphasized. Then we put forward a novel approach that can explicitly determine the Iwan  
 114 DF from the pressure distribution on the contact surface. This approach gives a physics-based  
 115 explanation of the DF and does not introduce new parameters.

### 2.1 The density function of Iwan model

117 The Iwan model [10] consists of Jenkins elements [33] in a parallel as shown in Fig. 1(a). A Jenkins  
 118 element is an ideal piecewise unit which can reproduce either slip or stick. Each element is composed  
 119 of a linear spring whose stiffness is  $k_t/n$  and a Coulomb slider with a critical sliding force  $f_i^*/n$ .  
 120 The stiffness  $k_t$  is the total tangential stiffness,  $f_i^*$  is the critical sliding force on the  $i^{th}$  element  
 121 and  $n$  is the number of Jenkins elements. The critical sliding force is the tangential force on a single  
 122 element at the onset of sliding. It should be noted that the critical sliding force is typically different  
 123 for each element. According to [10], the critical sliding force on each element can be represented by  
 124 a density function  $\varphi(f^*)$  where “ $\varphi(f^*)df^*$  is the fraction of total number of elements having  $f^*$   
 125  $\leq f_i^* \leq f^* + df^*$ ”. The original Iwan model assumes a uniform DF  $\varphi(f^*) = 1/\Delta$ , as indicated in  
 126 Fig. 1(b), where  $\mu$  is the friction coefficient,  $N$  the normal load and  $\Delta$  the range of  $f^*$ .

127



128 **Fig. 1.** (a) Scheme of the Iwan model, (b) a uniform density function of the critical sliding force.

129

130 The total tangential force results from two groups of elements: those elements which are in slip or  
131 yield state and the elements which are unyielded or in stick state. For a small-amplitude monotonic  
132 loading case, the force-displacement relation (also called backbone function) is defined as

$$T(\delta) = \int_0^{k_t \delta} f^* \varphi(f^*) df^* + k_t \delta \int_{k_t \delta}^{\infty} \varphi(f^*) df^*. \quad (1)$$

133 where  $T$  is the tangential force and  $\delta$  the relative displacement. Increasing the relative  
134 displacement increases the tangential force. All the elements will be in a slip state once the tangential  
135 force becomes larger than the limit force associated to a given friction coefficient,  $\mu N$ . From Eq. (1),  
136 it can be seen that the DF is of great importance, which directly relates the total tangential force to  
137 the relative displacement.

138

139 For a cyclic load, the tangential force during unloading is,

$$T(\delta) = \int_0^{\frac{k_t(\delta_m - \delta)}{2}} -f^* \varphi(f^*) df^* + \int_{\frac{k_t(\delta_m - \delta)}{2}}^{k_t \delta_m} [f^* - k_t(\delta_m - \delta)] \varphi(f^*) df^* + k_t \delta \int_{k_t \delta_m}^{\infty} \varphi(f^*) df^*, \quad (2)$$

140 where  $\delta_m$  is the maximum displacement reached at the end of the loading phase. The expression of  
141 the tangential force for reloading can be deduced in a similar way. Eq. (2) is based on the Masing's  
142 hypothesis [9, 21] and further details can be found in Ref. [10, 11].

143

144 The original Iwan model [10] uses three parameters (stiffness, total “yield force”, and the range of  
145 the yield force) to describe the relationship between the force and the displacement. The ratio of the

146 force range over the yield force is related to the DF of the critical sliding force for which Iwan  
147 assumes a uniform distribution. This ratio is usually identified by matching with experimental data  
148 and changes the shape of hysteresis curves during micro slip. **Therefore, it can be seen that the DF**  
149 **plays a key role in the Iwan model. In the next section, a novel approach is proposed that can directly**  
150 **derive the Iwan DF from the contact pressure distribution on the joint surface, differently from what**  
151 **is done in the Iwan-type models where the DF is assumed a priori regardless of the pressure**  
152 **distribution.**

## 153 **2.2 Modeling process**

154 Under tangential loads the contact surface shows two regions: a slip region in which corresponding  
155 contact pairs on contact surfaces undergo a relative motion and a stick region in which the relative  
156 motion is not allowed. As an example, on a sphere-on-sphere contact the slip region is the annular  
157 periphery of the contact area, where the contact pressure is lower and is not able to constrain the  
158 relative motion between contact pairs. Therefore, it is evident that the contact pressure plays an  
159 important role in defining the motion state of contact pairs. If the DF of the critical sliding force is  
160 related to pressure distribution, the obtained DF has an explicit physical significance and is no more  
161 a sheer parameter in the model.

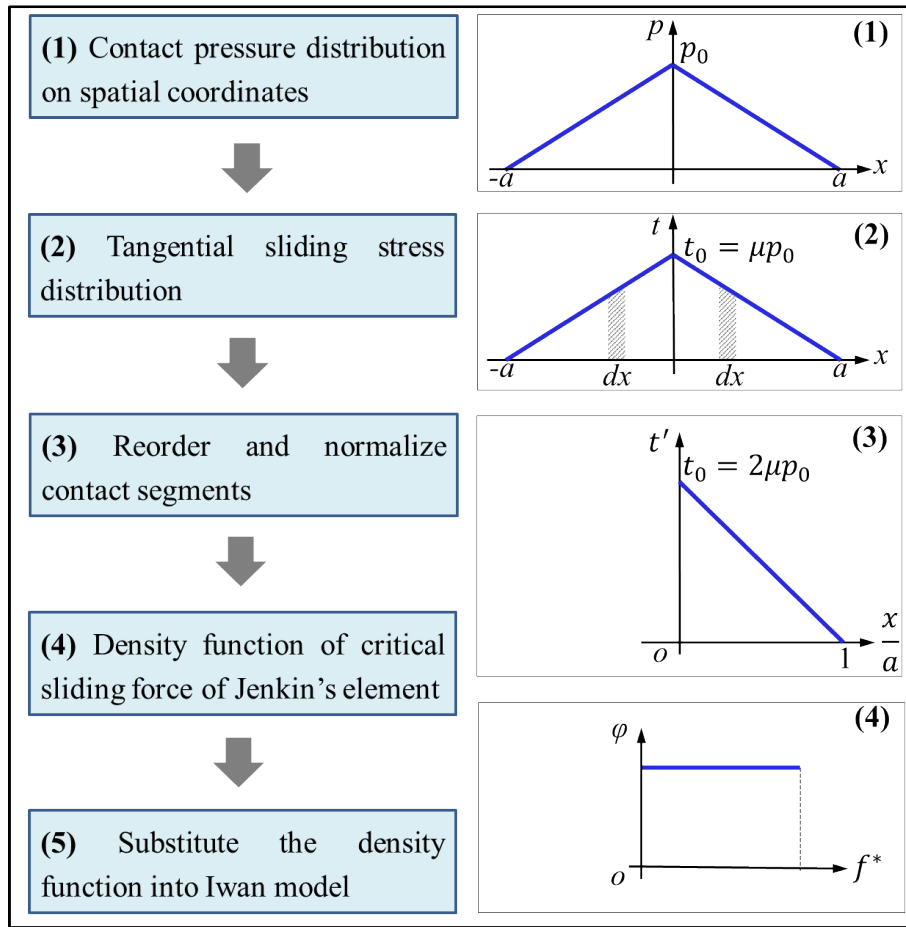
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163 **Fig. 2** illustrates in five steps the proposed approach. Step (1) assumes a contact pressure distribution  
164  $p(x)$  from known analytical or semi-analytical solutions, measurements or empirical formulations,

$$p = p(N, x, a), \quad |x| \leq a, \quad (3)$$

165 where  $a$  and  $N$  denote the size of contact area and the normal load, respectively. During the whole

166 process the contact pressure distribution is assumed independent of the relative motion.



167

168 **Fig. 2.** Flow chart of the proposed modeling approach and an example of a "triangle" pressure  
 169 distribution.

170

171 Step (2) builds the distribution function of the tangential sliding stress  $t(x)$  (also denoted "traction")  
 172 under the assumption that the Coulomb law between pressure  $p(x)$  and traction  $t(x)$  holds at  
 173 infinitesimal level. In the case of slip on the whole contact area the distribution function of traction  
 174 becomes,

$$t(x) = \mu p(N, x, a), \quad |x| \leq a. \quad (4)$$

175 Eq. (4) cannot simulate the microslip condition. The density function  $\varphi(f^*)$  determines the

176 distribution of the critical value (or threshold) of the sliding force. This threshold defines the slip and  
177 stick regions in the contact area, which in turn determine the real tangential force.

178

179 To illustrate the procedure a simple triangular distribution of contact pressure, symmetrical about the  
180  $x$  axis with  $p_0$  the maximum pressure, has been considered, see Fig. 2. In step (3), contact segments  
181 with the same tangential pressure have been sorted in descending order, to simplify the following  
182 calculation. The sorting process does not change the total sliding force and the sorted traction  
183 distribution  $t'$  is

$$t'(x) = 2\mu p(N, x, a), \quad 0 \leq x \leq a. \quad (5)$$

184 The traction distribution  $t'(x)$  in the spatial domain is related to the distribution of traction  $f^*$  on  
185 the Jenkins elements

$$f^* = a \cdot t'(x). \quad (6)$$

186 From a statistical viewpoint,  $\varphi(f^*)df^*$  is the probability that the critical sliding force falls within  
187 the range  $f^*$  and  $f^* + df^*$ , namely the occurrence of the event  $f^*$  in the spatial domain

$$\varphi(f^*)df^* = \frac{dx}{a}. \quad (7)$$

188 Then, the distribution density  $\varphi(f^*)$  can be defined as the cotangent of the curve of the critical  
189 sliding force

$$\varphi(f^*) = \frac{dx}{a \cdot df^*}. \quad (8)$$

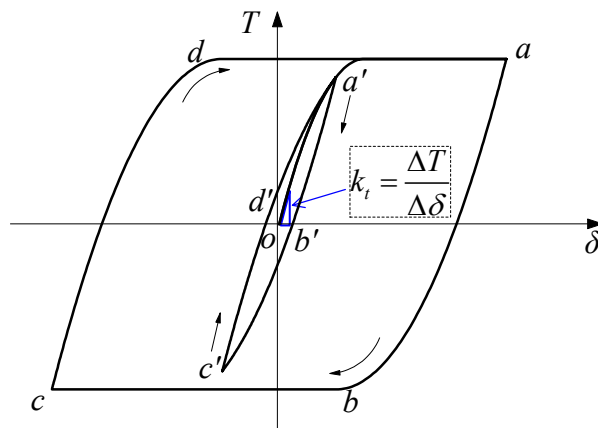
190 Step (5) uses the obtained DF into the Iwan model and gets the tangential force for monotonic and  
191 cyclic loading cases by Eqs. (1) and (2).

192 **2.3 Parameter estimation method**

193 Compared with the original Iwan model, the proposed modeling approach does not involve  
194 parameters related to the DF form, that is defined a priori. Therefore, the proposed model uses only  
195 two parameters, related to the tangential stiffness and the friction coefficient. These parameters need  
196 to be identified from experimental results.

197

198 It is well known that the tangential stiffness can be easily estimated through the force-displacement  
199 curve, denoted as hysteresis loop, see Fig. 3. The slope of the curve in the initial state (“stick” state)  
200 is the tangential stiffness, while, when the contact reaches a gross slip state, the ratio of tangential to  
201 normal force is the friction coefficient.



202

203 **Fig. 3.** Typical hysteresis loops for micro-slip and gross slip cases, and schematic of contact  
204 stiffness.

205

### 3 Analytical validation in a sphere-on-sphere contact

206

207 In this section, the proposed approach is used to simulate the tangential force against the relative  
208 displacement in the friction contact between two elastic spheres. Results predicted by the model are  
209 compared with Mindlin analytical solution [31, 34] to validate the approach.

210

#### 3.1 Mindlin analytical solution

212 Mindlin [31, 34] studied the friction contact of two elastic spheres pressed by a normal force  $N$  in  
213 which the contact area is circular with radius  $a$ . The contact area and pressure distribution on the  
214 surface are given by Hertz theory [27]. Mindlin results show that the contact area is divided into slip  
215 and stick region, the slip region being at the outer radius. The slip region increases with the  
216 tangential load. According to Mindlin, for a monotonic tangential loading case, the displacement-  
217 force relation on the contact surface is given as

$$\delta = \frac{3(2 - \nu)\mu N}{16Ga} \left[ 1 - \left( 1 - \frac{T}{\mu N} \right)^{2/3} \right] \quad (9)$$

218 where  $\nu$  and  $G$  denote Poisson's ratio and shear modulus, respectively.

219

220 For a cyclic tangential loading case, the relationship between the tangential force and the relative  
221 displacement for the unloading and reloading are given as

$$\delta = \begin{cases} \frac{3(2-\vartheta)\mu N}{16Ga} \left[ 2 \left( 1 - \frac{T_m - T}{2\mu N} \right)^{2/3} - \left( 1 - \frac{T_m}{\mu N} \right)^{2/3} - 1 \right], & \dot{T} < 0, \\ -\frac{3(2-\vartheta)\mu N}{16Ga} \left[ 2 \left( 1 - \frac{T_m + T}{2\mu N} \right)^{2/3} - \left( 1 - \frac{T_m}{\mu N} \right)^{2/3} - 1 \right], & \dot{T} > 0, \end{cases} \quad (10)$$

222 where  $T_m$  is the maximum tangential force before reversal.

223 The curve depicted according to Eq. (10) can form a hysteresis loop. The area enclosed by the loop  
 224 represents the dissipated energy per cycle

$$W = \frac{9(2-\vartheta)\mu^2 N^2}{10Ga} \left\{ \left[ 1 - \left( 1 - \frac{T_m}{\mu N} \right)^{5/3} \right] - \frac{5T_m}{6\mu N} \left[ 1 + \left( 1 - \frac{T_m}{\mu N} \right)^{2/3} \right] \right\}. \quad (11)$$

225 For a small  $T_m/\mu N$ , Eq. (11) reduces to

$$W = \frac{(2-\vartheta)T_m^3}{36Ga\mu N}. \quad (12)$$

226 The dissipated energy is proportional to the cubic power of the maximum tangential load. It should  
 227 be noted that this equation does not consider gross slip case and is valid only for small tangential  
 228 loads.

229

### 230 3.2 Modeling sphere-on-sphere friction contact

231 According to the proposed approach the contact pressure distribution is assumed a priori, and this  
 232 pressure is used to obtain the DF of the critical sliding force. For two elastic spheres in contact, the  
 233 pressure distribution is given by Hertz theory [27], as shown in Fig. 4(a),

$$p(r) = p_0 \sqrt{1 - (r/a)^2}, \quad 0 \leq r \leq a, \quad (13)$$

234 where  $p(r)$  is the contact pressure,  $r$  the distance from the center of contact area, and  $p_0$  the  
 235 maximum contact pressure  $p_0 = 3N/(2\pi a^2)$ .

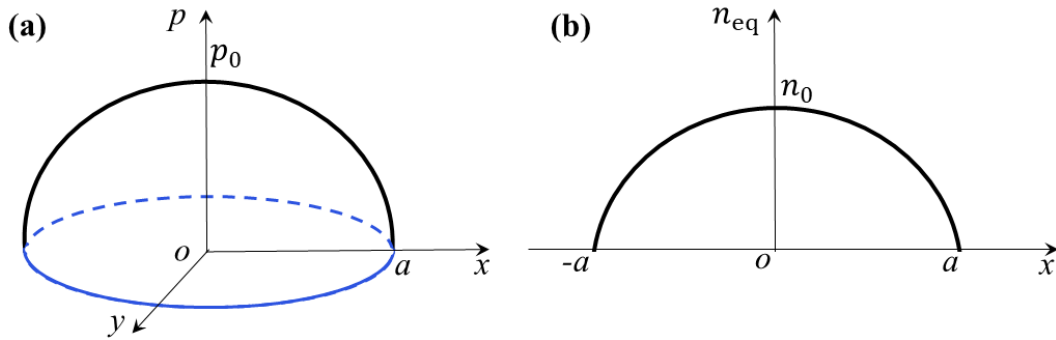


236

237 The two-dimensional distribution is converted to an equivalent one-dimensional distribution along  
238 the  $x$  axis. This step is needed to describe the pressure distribution (given in spatial coordinates) in  
239 the Jenkins elements coordinates. To achieve this conversion, the contact pressure is integrated along  
240 the  $y$  axis to give the normal force per contact width

$$n_{\text{eq}}(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2} dy = \frac{\pi p_0}{2a}(a^2 - x^2), \quad (14)$$

241 In this process, the circular contact area becomes a one-dimensional contact region in the domain  
242  $-a \leq x \leq a$ . Fig. 4(b) shows the normal force per contact width whose maximum value is  $n_0 = \pi p_0$   
243  $a/2$ .



244

245 **Fig. 4.** Contact pressure distribution of two elastic spheres (a) on two-dimensional space, (b) on one-  
246 dimensional space.

247

248 The tangential sliding stress distribution in the slip case can be obtained by Coulomb law (see the  
249 step (2) in Fig. 2),

$$t(x) = \mu n_{\text{eq}}(x) = \frac{\pi \mu p_0}{2a}(a^2 - x^2), \quad (-a \leq x \leq a). \quad (15)$$

250 Due to the symmetrical distribution of  $t(x)$ , contact segments with the same sliding stress can be  
251 sorted giving

$$t'(x) = \frac{\pi\mu p_0}{a}(a^2 - x^2), \quad (0 \leq x \leq a). \quad (16)$$

252 Coordinates in the contact region are normalized, according to the step (3) in Fig. 2, and the sliding  
253 stress distribution is written as

$$t'(x/a) = \pi\mu p_0 a \left[ 1 - \left( \frac{x}{a} \right)^2 \right], \quad \left( 0 \leq \frac{x}{a} \leq 1 \right). \quad (17)$$

254 The critical sliding force in the normalized contact coordinate is then

$$f^*(x/a) = a \cdot t'(x/a) = \pi\mu p_0 a^2 \left[ 1 - \left( \frac{x}{a} \right)^2 \right], \quad \left( 0 \leq \frac{x}{a} \leq 1 \right). \quad (18)$$

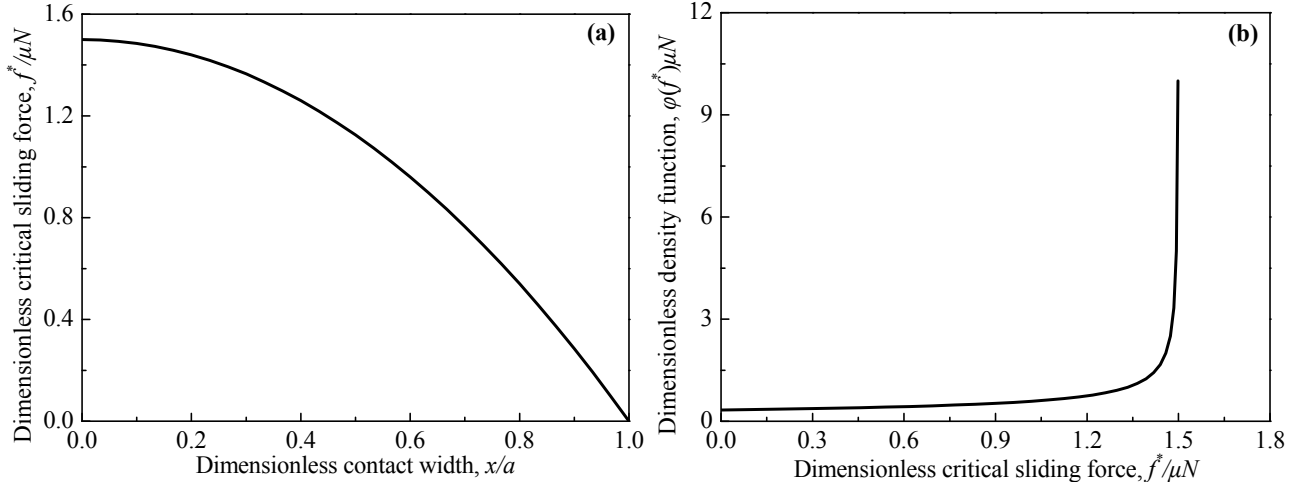
255 Fig. 5(a) plots the dimensionless critical sliding force  $f^*/\mu N$  vs. the dimensionless contact width  
256  $x/a$ . Solving Eq. (18) for  $x/a$ , considering that  $p_0 = 3N/(2\pi a^2)$ , the following relationship is  
257 obtained

$$\frac{x}{a} = \sqrt{1 - \frac{2f^*}{3\mu N}}, \quad \left( 0 \leq \frac{x}{a} \leq 1 \right). \quad (19)$$

258 Finally, the DF  $\varphi(f^*)$  of the critical sliding force  $f^*(x/a)$  is carried out according to its  
259 definition

$$\varphi(f^*) = \left| \frac{d(x/a)}{df^*} \right| = \frac{1}{\sqrt{3\mu N(3\mu N - 2f^*)}}. \quad (20)$$

260 Fig. 5(b) plots the dimensionless density function  $\varphi(f^*)\mu N$  vs. the dimensionless critical sliding  
261 force  $f^*/\mu N$ .



262

263 **Fig. 5.** (a) Dimensionless critical sliding force  $f^*/\mu N$  vs. dimensionless contact width  $x/a$ ; (b)

264 Dimensionless density function  $\varphi(f^*)\mu N$  vs. dimensionless critical sliding force  $f^*/\mu N$ .

265

266 Substituting Eq. (20) into Eq. (1) yields the tangential force,

$$T = \mu N - \frac{(3\mu N - 2k_t\delta)^{3/2}}{3\sqrt{3\mu N}}. \quad (21)$$

267 For a cyclic loading, the expression of the tangential force is obtained according to the Masing's

268 hypothesis,

$$T(\delta) = \begin{cases} -T(\delta_m) + 2T\left(\frac{\delta_m + \delta}{2}\right), & \delta > 0 \\ T(\delta_m) - 2T\left(\frac{\delta_m - \delta}{2}\right), & \delta < 0 \end{cases}. \quad (22)$$

### 269 3.3 Comparison with analytical solution

270 The proposed approach has been applied to two identical elastic spheres in contact and the results

271 have been compared with the Mindlin's analytical solution. Material properties, friction coefficient,

272 contact radius and normal load used in the simulation are listed in Table 1.

273

**Table 1** Analytical model parameters

Shear modulus	Poisson's ratio	Friction coefficient	Contact radius	Normal load
27GPa	0.33	0.5	2mm	500N

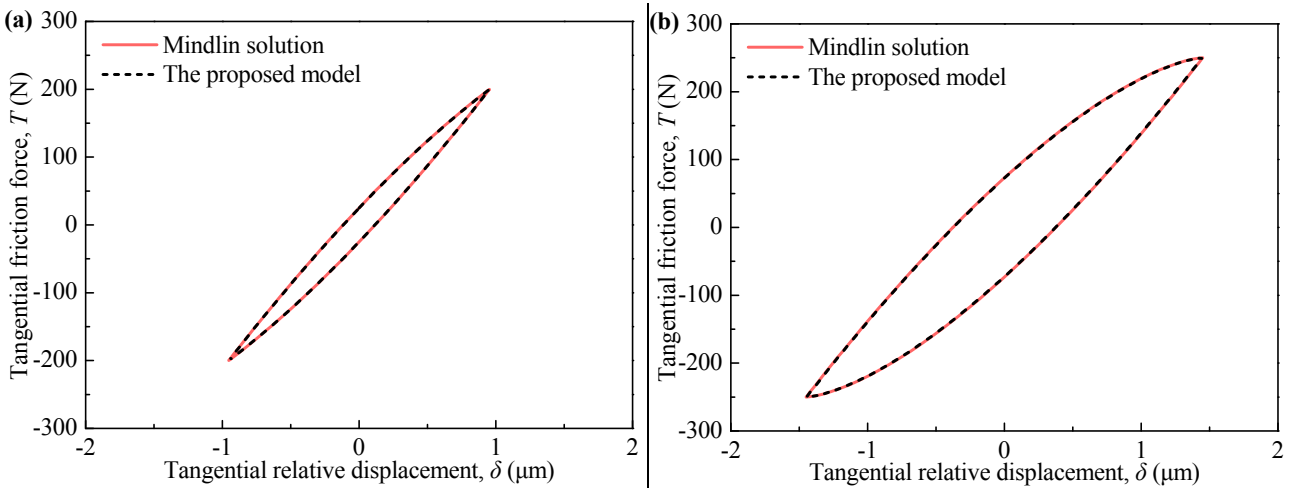
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275 The tangential stiffness to be used in the model is the slope of force-displacement curve when the  
 276 body are at rest and start moving. To be consistent, this stiffness has been computed using the  
 277 Mindlin solution

$$k_t = \lim_{\delta \rightarrow 0} \frac{\partial T}{\partial \delta} = \lim_{\delta \rightarrow 0} \frac{8Ga}{2 - \vartheta} \left[ 1 - \frac{16Ga\delta}{3(2 - \vartheta)\mu N} \right]^{1/2} = \frac{8Ga}{2 - \vartheta} \quad (23)$$

278

279 A harmonic displacement  $\delta$  was applied to one of the contact bodies to compute the tangential force  
 280  $T$ . Two cases were studied: a small relative displacement for which the sliding condition is not  
 281 reached (maximum tangential force  $T_m < \mu N = 200\text{N}$ ) and a large relative displacement up to full  
 282 sliding conditions (maximum tangential force  $T_m = \mu N = 250\text{N}$ ). Fig. 6 shows that the proposed  
 283 method is equivalent to the Mindlin's analytical solution; as can be easily verified by substituting Eq.  
 284 (23) into Eq. (21).



285

286 **Fig. 6.** Comparison of hysteresis loops predicted by the proposed model with Mindlin solution, (a)  
 287  $T_m = 200\text{N}$ ; (b)  $T_m = 250\text{N}$ .

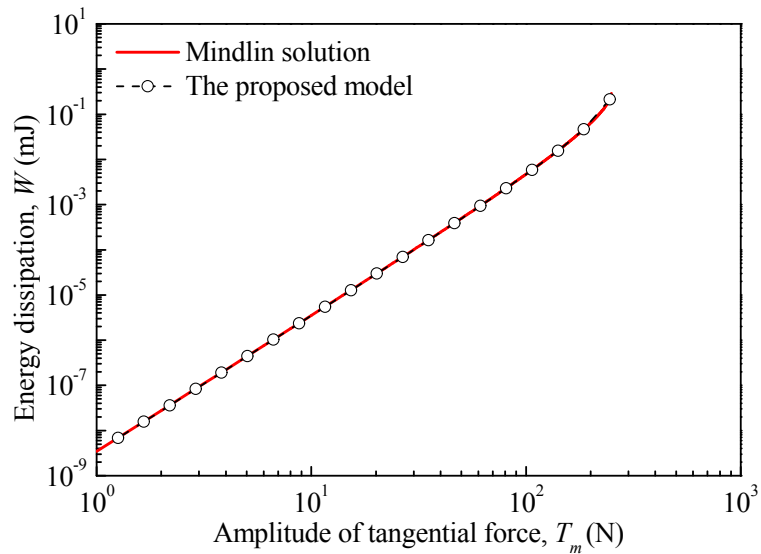
288

289 **Fig. 7** compares the dissipated energy per cycle and confirms the equivalence of the two methods.

290 The proposed model can also be used to simulate friction behavior for several contact geometries. In

291 the next section, two flat contact bodies will be modeled with the proposed approach and compared

292 with experimental data.



293

294 **Fig. 7.** Comparison of predicted energy dissipation per cycle with Mindlin's solution.

295

296

## 4 Modeling friction for lap joint plates

297 Lap joints are unavoidable in assembled structures such as bolted and riveted joints. Friction contact  
298 at the joint interfaces has a significant effect on the dynamic behavior of assembled structures, in  
299 which non-linearities originate at the contact. Therefore, the development of a reliable friction model  
300 for flat-on-flat contact can contribute to accurately predict the forced-responses of these structures. In  
301 this section, the proposed approach is used to simulate contact behavior in lap joints and then  
302 validated by comparison with measured results.

303

### 304 4.1 Representation of friction between lap joints

305 It is challenging to obtain a distribution function of contact pressure for flat-on-flat contact because  
306 of asperities and waviness that affect the real contact area. Moreover, cyclic loading modifies the  
307 contact because surfaces change their topography and pressure is redistributed on the contact area.  
308 These phenomena, more evident in flat contact than in point contact, could limit the application of  
309 the proposed model.

310

311 To overcome this limit, a quadratic function with a variable parameter  $\omega$  was assumed to describe the  
312 distribution function of the contact pressure. This variable parameter was denoted ‘proportional  
313 coefficient’ and defined as the ratio between the peak pressure  $p_0$  and the mean pressure  $p_m$

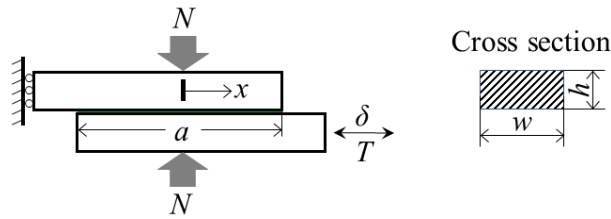
$$\gamma = \frac{p(0)}{N/aw}. \quad (24)$$

314 This quadratic function can represent three typical distribution forms widely employed to  
315 approximate real cases, namely “concave”, “convex”, and “uniform” distribution [13]. According to

316 the proposed approach, a unified friction model including the variable parameter is developed for  
 317 flat-on-flat contact. The optimal choice of this parameter can be inferred from the measured  
 318 hysteresis loops.

319

320 Fig. 8 shows two plates pressed together by a normal load  $N$  and excited by an oscillating tangential  
 321 displacement  $\delta$ . The nominal contact area length is  $a$ . The cross section of the plate is a rectangle of  
 322 width  $w$  and thickness  $h$ . The normal load along the transverse direction is assumed to be uniformly  
 323 distributed.



324

325

**Fig. 8.** Lap joint plates model and cross section of the plate

326

327 In this paper, we define the quadratic function assuming that i) the real contact area is the nominal  
 328 contact area and ii) the extreme value of the quadratic function is in the center of the contact area

$$p(x) = \frac{N}{aw} \left[ \frac{12(1-\gamma)x^2}{a^2} + \gamma \right], \quad \left( x \leq \left| \frac{a}{2} \right| \right) \quad (25)$$

329 where  $\gamma$  is the proportional coefficient varying in the range  $[0, 1.5]$ . For the case  $\gamma < 1$ , Eq. (25) is a

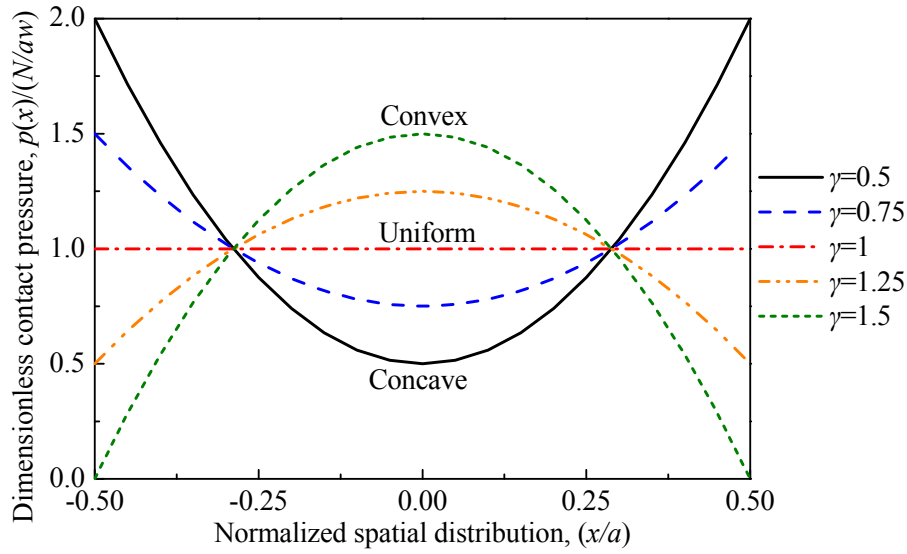
330 concave distribution function, while for the case  $\gamma > 1$ , it is a convex distribution function.

331 Proportional coefficient  $\gamma = 1$  means that the distribution of contact pressure is constant. Fig. 9

332 shows dimensionless contact pressure,  $p(x)aw/N$ , against the normalized contact coordinate,  $x/a$ ,

333 for different proportional coefficients. For a real lap joint, the corresponding pressure distribution

334 function is approximately represented by selecting a suitable proportional coefficient.



335

336 **Fig. 9.** Dimensionless contact pressure distribution,  $p(x)aw/N$ , vs. normalized contact coordinate,

337  $x/a$ , for different proportional coefficients.

338

339 According to the proposed model, section 2, the tangential sliding stress distribution can be derived

340 as

$$t(x) = \mu p(x) = \frac{\mu N}{aw} \left[ \frac{12(1-\gamma)x^2}{a^2} + \gamma \right], \quad \left( x \leq \left| \frac{a}{2} \right| \right). \quad (26)$$

341 Then, the distribution function is reordered, and the spatial coordinate normalized,

$$t'(2x/a) = \frac{2\mu N}{aw} \left[ 3(1-\gamma) \left( \frac{2x}{a} \right)^2 + \gamma \right], \quad \left( 0 \leq \frac{2x}{a} \leq 1 \right). \quad (27)$$

342 The critical sliding force in the normalized contact coordinate is

$$f^*(2x/a) = \frac{aw}{2} \cdot t' \left( \frac{2x}{a} \right) = \mu N \left[ 3(1-\gamma) \left( \frac{2x}{a} \right)^2 + \gamma \right]. \quad (28)$$

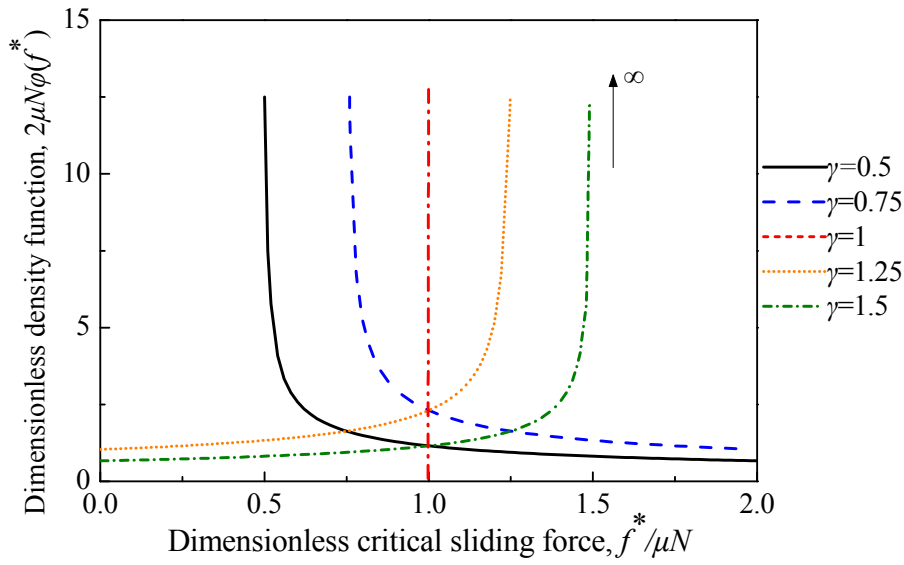
343 According to the definition of the Iwan DF in Eq. (8), it can be obtained by deriving the normalized

344 contact coordinate with respect to the critical sliding force,



$$\varphi(f^*) = \left| \frac{d(2x/a)}{df^*} \right| = \frac{1}{2\mu N \sqrt{3(1-\gamma) \left( \frac{f^*}{\mu N} - \gamma \right)}}. \quad (29)$$

345 Fig. 10 shows a group of dimensionless density functions of critical sliding force  $\varphi(f^*)$  with  
 346 different proportional coefficients. For the concave distribution, the DF goes to infinity when the  
 347 critical sliding force approaches its minimum,  $\gamma\mu N$ , whereas for the convex distribution, the DF  
 348 goes to infinity when the critical sliding force approaches its maximum,  $\gamma\mu N$ .



349  
 350 Fig. 10 Dimensionless density function,  $2\mu N\varphi(f^*)$ , vs. dimensionless critical sliding force,  $f^*/\mu N$   
 351 , for different proportional coefficients.

352  
 353 The force-displacement expression is obtained from integral in Eq. (1)

$$T(\delta) = \int_{lb}^{k_t\delta} f^* \varphi(f^*) df^* + k_t\delta \int_{k_t\delta}^{ub} \varphi(f^*) df^*. \quad (30)$$

354 setting the proper upper and lower bounds that are different between the concave and convex  
 355 distributions and are listed in Table 2. Details can be found in Appendix A.

356

357 **Table 2.** Upper and lower bounds of the concave and convex distributions, and corresponding force-  
 358 displacement expressions

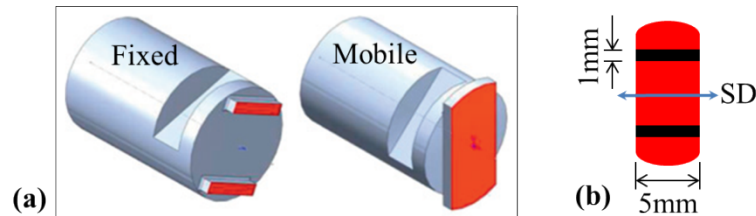
Concave distribution ( $\gamma < 1$ )	Convex distribution ( $\gamma > 1$ )
$lb = \gamma\mu N, ub = (3 - 2\gamma)\mu N$	$lb = (3 - 2\gamma)\mu N, ub = \gamma\mu N$
$T(\delta) = k_t\delta, \left(\delta \leq \frac{\gamma\mu N}{k_t}\right)$	$T(\delta) = k_t\delta, \left(\delta \leq \frac{(3 - 2\gamma)\mu N}{k_t}\right)$
$T(\delta) = k_t\delta + \frac{2(\gamma\mu N - k_t\delta)}{3} \sqrt{\frac{k_t\delta - \gamma\mu N}{3(1 - \gamma)\mu N}}$	$T(\delta) = \mu N - \frac{2(\gamma\mu N - k_t\delta)}{3} \sqrt{\frac{k_t\delta - \gamma\mu N}{3(1 - \gamma)\mu N}}$
$\left(\frac{\gamma\mu N}{k_t} < \delta \leq \frac{(3 - 2\gamma)\mu N}{k_t}\right)$	$\left(\frac{(3 - 2\gamma)\mu N}{k_t} < \delta \leq \frac{\gamma\mu N}{k_t}\right)$
$T(\delta) = \mu N, \left(\delta > \frac{(3 - 2\gamma)\mu N}{k_t}\right)$	$T(\delta) = \mu N, \left(\delta > \frac{\gamma\mu N}{k_t}\right)$

359

#### 360 4.2 Model validation

361 Hysteresis loops measured on flat-on-flat contacts have been used to extract the contact parameters  
 362 (namely tangential stiffness  $k_t$ , friction coefficient  $\mu$ , residual stiffness coefficient  $\alpha$  and  
 363 proportional coefficient  $\gamma$ ) and to validate the proposed model. Details of the test rig and  
 364 experimental operation can be found in [35, 36]. Fig. 11 shows the specimens, denoted to as Fixed  
 365 and Mobile respectively, used in the experimental tests and the corresponding contact area.

366



367 **Fig. 11.** (a) Specimens and contact surface of fretting tests, (b) contact area in black and sliding  
 368 direction (SD).

369

370 Three cases, with different normal loads 87 N, 164 N and 254 N, were tested. Fig.12 shows typical  
371 measured hysteresis loops, in blue solid line, during ten periods. The three stages ( I ) stick, ( II )  
372 micro-slip, and (III) gross slip are indicated in the loops. The slope of the stick portion is regarded as  
373 the tangential stiffness. The friction coefficient is extracted from the ratio of tangential force over  
374 normal force at the onset of the gross slip stage. The residual stiffness coefficient is expressed as the  
375 ratio of the slope of the gross slip portion over the tangential stiffness. The proportional coefficient,  
376 used to effectively reproduce micro-slip, can be estimated with the best fit on the measured loop. All  
377 estimated parameters are listed in Table 3. Results illustrate the three normal loads share the same  
378 proportional coefficient in this experiment. Fig. 12 shows the comparison between the simulated and  
379 the experimental hysteresis loops with the proposed model. It can be seen that simulations match  
380 well with experimental results. Figure 12(a) shows that there are slight differences between the  
381 simulation and the experiment when the load is reversed. This is due to the inconsistent tangential  
382 stiffness during the loading and unloading phases, which sometimes occurs in the test. Figure 12(b  
383 and c) shows that there are two uncommon bulges in the measured hysteresis loops. The first one  
384 occurred at the transition from the stuck to the micro-slip state. This may be caused by local  
385 interactions between the asperities on contact surfaces. The second occurred at the end of the gross  
386 slip stage. This phenomenon has been observed in many tests [35, 37, 38], however, the physical  
387 reason for it is not yet fully understood. Two possible reasons are “velocity effect” and interactions at  
388 contact edges. These uncommon behaviors observed in the experiment were not captured by the  
389 proposed method.

390

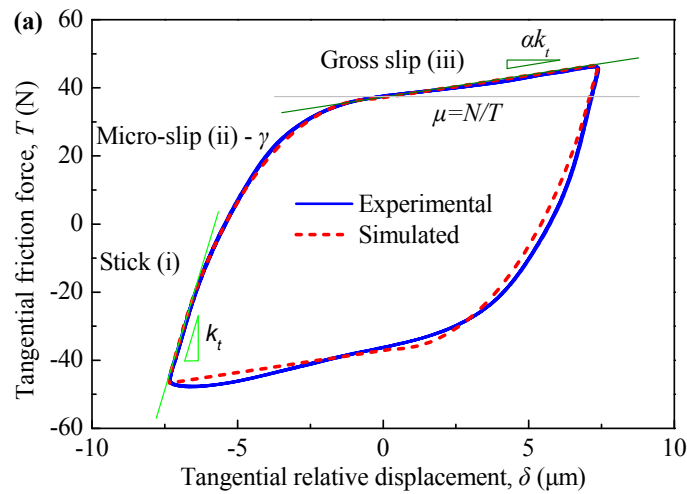
391 Due to the existence of waviness and roughness of contact surfaces, it is difficult to analytically  
392 obtain the contact pressure distribution of flat-on-flat contact. Besides, the dispersion of contact  
393 pressure distribution by measurement may be significant. Even so, the obtained Iwan DF based on a  
394 quadratic distribution depending on a proportional coefficient can give satisfactory results. This is a  
395 more practical application of the proposed method.

396

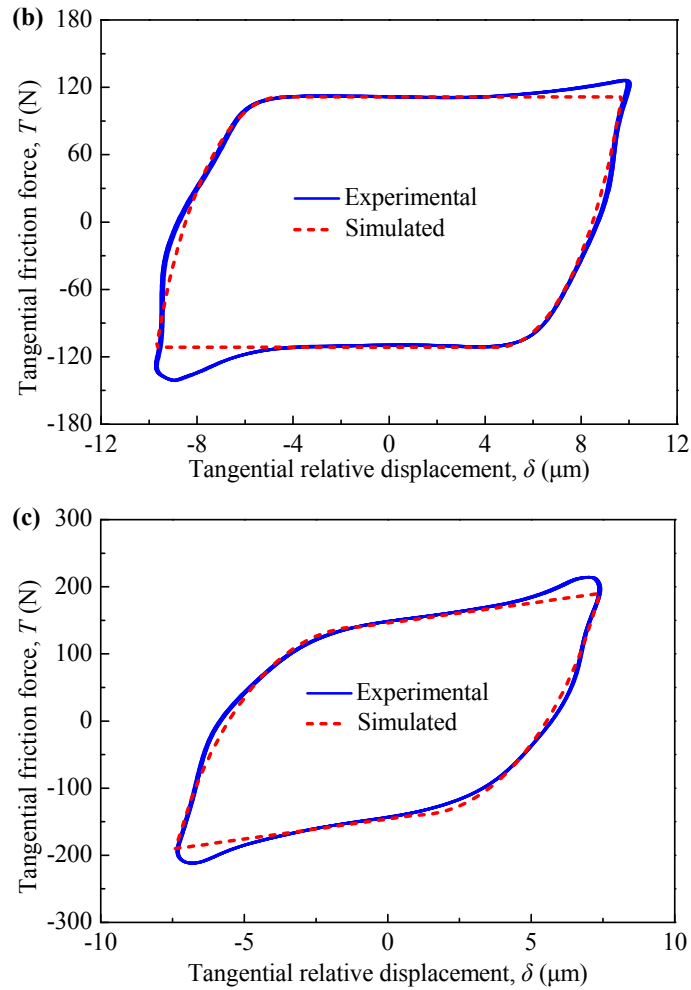
397 **Table 3** Applied normal loads and estimated contact parameters in tests

Normal load, $N$ (N)	Tangential stiffness, $k_t$ (N/ $\mu\text{m}$ )	Residual stiffness coefficient, $\alpha$	Friction coefficient, $\mu$	Proportional coefficient, $\gamma$
87	30.5	0.0427	0.48	0.2
164	113.7	0	0.68	0.2
254	133.7	0.0445	0.65	0.2

398



399



400

401

402 **Fig. 12.** Comparison between numerical and experimental hysteresis loops for two normal loads, (a)

403  $N=87$  N, (b)  $N=164$ N, (c)  $N=254$ N.

404

### 405 4.3 Effect of the proportional coefficient

406 The proposed contact model introduces a proportional coefficient whose range varies in the range [0,

407 1.5]. The effect of this proportional coefficient is to change the shape of the hysteresis loop, then

408 modelling different contacts. To study the effect that the proportional coefficient has on the

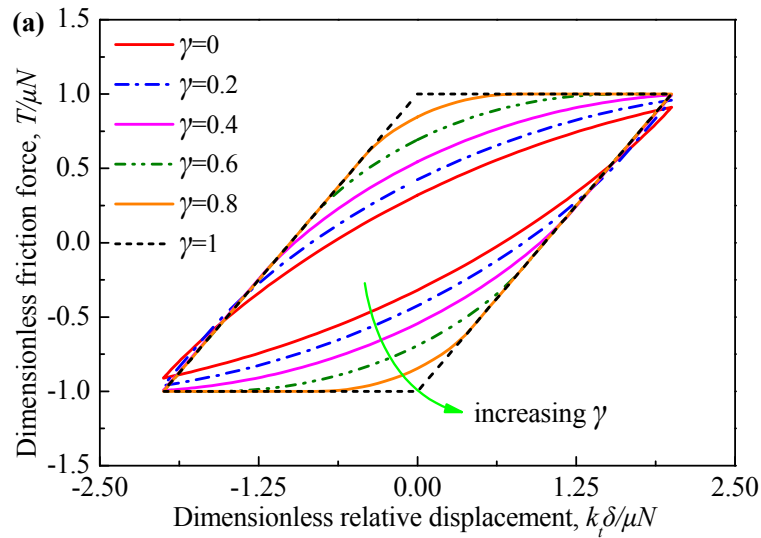
409 hysteresis loops a normalization procedure is introduced. The relative displacement and tangential

410 friction force are normalized as  $\delta_{\text{norm}} = \delta / \delta_{\text{max\_stick}} = k_t \delta / \mu N$  and  $T_{\text{norm}} = T / \mu N$ . The obtained

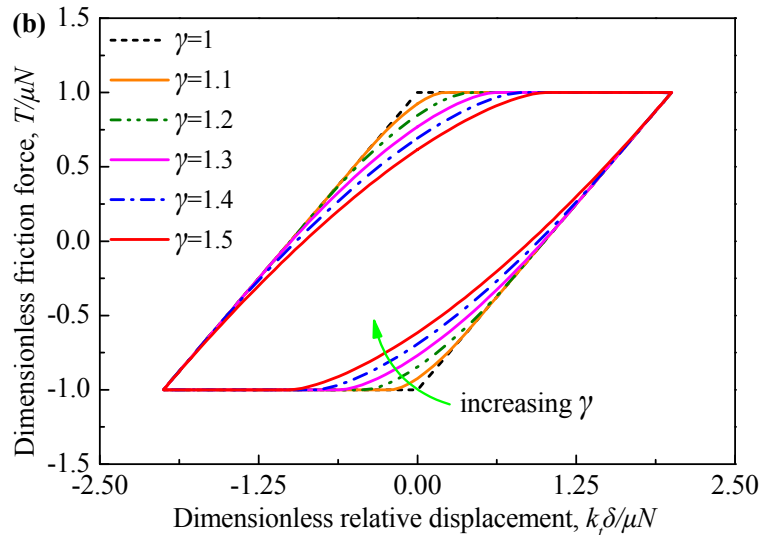
411 dimensionless backbone function is then

$$T_{\text{norm}} = \begin{cases} \delta_{\text{norm}} + \frac{2(\gamma - \delta_{\text{norm}})}{3} \sqrt{\frac{\delta_{\text{norm}} - \gamma}{3(1 - \gamma)}}, & 0 \leq \gamma \leq 1 \\ 1 - \frac{2(\gamma - \delta_{\text{norm}})}{3} \sqrt{\frac{\delta_{\text{norm}} - \gamma}{3(1 - \gamma)}}, & 1 < \gamma \leq 1.5 \end{cases}. \quad (31)$$

412 Fig. 13 shows a group of dimensionless hysteresis loops with different proportional coefficients and  
 413 the same others contact parameters. The imposed displacement is a sinusoidal motion with a  
 414 normalized amplitude of 2 and a frequency of 1 Hz. It is evident that the proportional coefficient  
 415 controls the micro-slip region and the global shape of hysteresis loops. When  $\gamma < 1$ , the predicted  
 416 micro-slip effect gradually weakens as the proportional coefficient increases. While it is opposite  
 417 when  $\gamma > 1$ .



418



419

420 **Fig. 13.** Evolution of hysteresis loops (dimensionless friction force  $T/\mu N$  vs. dimensionless relative  
421 displacement  $k_t\delta/\mu N$ ) with the increase of proportional coefficient  $\gamma$ , (a)  $\gamma$  in the range  $[0, 1]$ ; (b)  $\gamma$   
422 in the range  $[1, 1.5]$ .

423

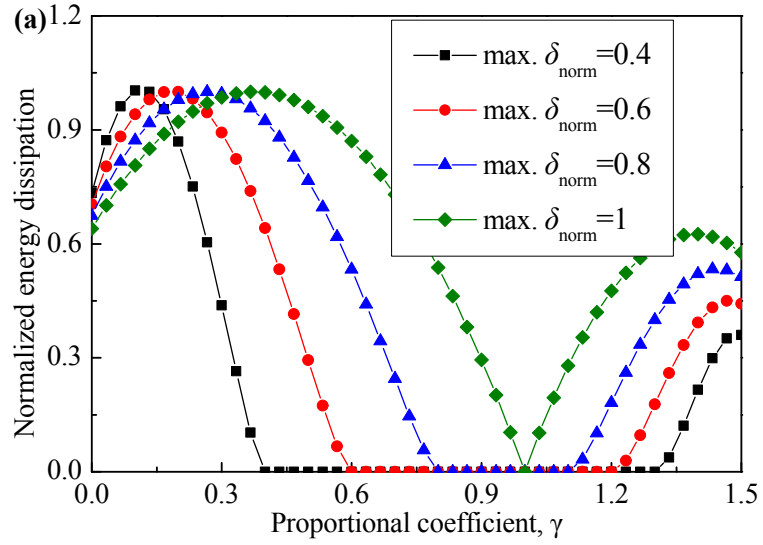
424 Correspondingly, the proportional coefficient has an influence on the dissipated energy per cycle.

425 **Fig. 14** depicts the evolution of the normalized energy dissipated (enclosed area divided by its  
426 maximum value) with increasing the proportional coefficient under different displacement  
427 amplitudes. The normalized energy dissipated is not monotonic and shows dependence on  
428 amplitudes of displacement. For a relatively small displacement amplitude, there is a bandwidth of  
429 the proportional coefficient at which the corresponding model predicts zero energy dissipation. In  
430 this case, the stuck regime dominates the motion of contact surfaces. And the smaller the  
431 displacement amplitude, the wider this bandwidth. While for a relatively larger displacement  
432 amplitude, the energy dissipation shows a maximum when  $\gamma=1$ . As the proportional coefficient  
433 increases, the normalized energy dissipated increases first and then drops slightly. In addition, the  
434 difference in the dissipated energy among different proportional coefficient decreases with the  
435 increasing proportional coefficient. This is due to the gradual dominance of the gross slip regime.

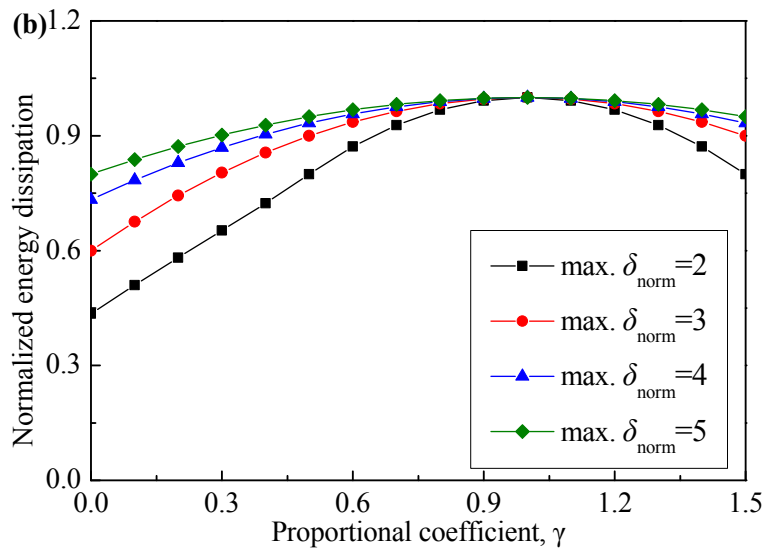
436

437 From a certain point of view, the proposed method is a generalized Iwan model, which starts with the  
438 contact pressure distribution and shows how to derive the Iwan DF. In applications where it is  
439 difficult to obtain the pressure distribution function, the proportional coefficient, which indirectly  
440 reflects the pressure distribution, can be easily estimated from experimental results.

441



442



443 **Fig. 14.** Normalized energy dissipation (enclosed area divided by its maximum value) vs.  
444 proportional coefficient  $\gamma$  under different displacement amplitudes.

445



446

## 5 Conclusions

447 This work presents a new friction modeling approach based on the framework of the original Iwan's  
448 model. The aim of this new approach is to better describe the relationship between the friction force  
449 and the relative displacements. The proposed approach was employed to simulate the friction  
450 behavior of a sphere-on-sphere contact in which the pressure distribution has been assumed as in the  
451 Hertzian contact. The approach was validated by comparison with the Mindlin's analytical solution.  
452 In addition, the approach was applied to a flat-on-flat contact of lap joints where the pressure  
453 distribution is known with great approximation. A proportional coefficient is introduced, which  
454 controls the micro-slip region and the global shape of hysteresis loops.

455

456 The application of this method to contacts with different normal loads showed a great variation of the  
457 contact parameters such as contact stiffness, friction coefficient, and residual stiffness. On the other  
458 hand, the proportional coefficient remained the same or showed a non-appreciable variation. The  
459 main outcome of this paper is that even if the pressure distribution is not known, for the specific  
460 contact, it can be replaced by the density function whose distribution is well represented with the  
461 proportional coefficient. Experimental evidence has shown that the proportional coefficient, that is  
462 related to the density function distribution, remains constant for contacts with the same geometry but  
463 different contact conditions. The method described in this paper is then useful for improving the  
464 prediction performance of dynamics of jointed structures in many engineering applications and for  
465 the optimal design of components of assemblies, and to simulate wear between contact surfaces.

466

467

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472

### **Declaration of interests**

474 The authors declare that they have no conflicts of competing interests.

475

476

## Appendix A

477 Considering the upper and lower bounds (the integral limits in Eq. (1)) of critical sliding force are  
 478 different between the concave and convex distributions, the corresponding force-displacement  
 479 expressions are derived respectively. For the concave distribution, the critical sliding force falls  
 480 within the range  $[\gamma\mu N, (3 - 2\gamma)\mu N]$ . When the relative displacement is lower than  $\gamma\mu N/k_t$ , all  
 481 Jenkins elements are in stick state. That is, the recovery force behaves linearly with respect to  
 482 relative displacement,

$$T(\delta) = k_t\delta, \quad \left(0 < \delta \leq \frac{\gamma\mu N}{k_t}\right) \quad (\text{A.1})$$

483 When the relative displacement falls with the range  $[\gamma\mu N/k_t, (3 - 2\gamma)\mu N/k_t]$ , the recovery force is  
 484 obtained by substituting the DF into Eq. (1),

$$\begin{aligned} T(\delta) &= \int_{\gamma\mu N}^{k_t\delta} f^* \varphi(f^*) df^* + k_t\delta \int_{k_t\delta}^{(3-2\gamma)\mu N} \varphi(f^*) df^* \\ &= k_t\delta + \frac{2(\gamma\mu N - k_t\delta)}{3} \sqrt{\frac{k_t\delta}{\mu N - \gamma}} \left( \frac{\gamma\mu N}{k_t} < \delta \leq \frac{(3-2\gamma)\mu N}{k_t} \right). \end{aligned} \quad (\text{A.2})$$

485 When the relative displacement is larger than  $(3 - 2\gamma)\mu N/k_t$ , all Jenkins elements are in slip state  
 486 and the recovery force equals  $\mu N$ .

487

488 For the convex distribution, the critical sliding force falls within the range  $[(3 - 2\gamma)\mu N, \gamma\mu N]$ .

489 When the relative displacement is lower than  $(3 - 2\gamma)\mu N/k_t$ , the recovery force is

$$T(\delta) = k_t\delta, \quad \left(0 < \delta \leq \frac{(3-2\gamma)\mu N}{k_t}\right). \quad (\text{A.3})$$

490 When the relative displacement falls with the range  $[(3 - 2\gamma)\mu N/k_t, \gamma\mu N/k_t]$ , the recovery force is

$$\begin{aligned} T(\delta) &= \int_{(3-2\gamma)\mu N}^{k_t\delta} f^* \varphi(f^*) df^* + k_t\delta \int_{k_t\delta}^{\gamma\mu N} \varphi(f^*) df^* \\ &= \mu N - \frac{2(\gamma\mu N - k_t\delta)}{3} \sqrt{\frac{k_t\delta}{\mu N} - \gamma}, \quad \left( \frac{(3-2\gamma)\mu N}{k_t} < \delta \leq \frac{\gamma\mu N}{k_t} \right). \end{aligned} \tag{A.4}$$

491 Similarly, when the relative displacement is larger than  $\gamma\mu N/k_t$ , the model is in gross slip state and

492 the recovery force equals  $\mu N$ . For cyclic loadings, the force-displacement relation can be obtained by

493 Masing hypothesis.

494

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