Correction to: Material description of fluxes in terms of differential forms

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Correction to: "Material description of fluxes in terms of differential forms"

"Dedicated to Prof. David Steigmann in recognition of his contributions"

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Although the final result presented in Equation (34) of our work [?] is correct, the proof of Equation (34) contains an error. The text starting immediately after Equation (33) with "When a metric tensor $\boldsymbol{g}$..." and ending immediately before Equation (36) with "... in the alternative notation" should be replaced with the text below.

Correction to the Proof of Equation (34)

Let us assume that the space $\mathcal{S}$ is equipped with a metric tensor $\boldsymbol{g}$, i.e., a symmetric and positive-definite tensor field valued in $[T\mathcal{S}]^0_2$, defining the scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as $\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$. The metric $\boldsymbol{g}$ induces the

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musical isomorphisms $\flat : T\mathcal{S} \to T^\star\mathcal{S} : \boldsymbol{v} \mapsto \flat(\boldsymbol{v}) \equiv \boldsymbol{v}^\flat$, which maps a vector $\boldsymbol{v}$ with components v^c to a covector $\boldsymbol{v}^\flat$ with components $g_{ac}v^c$, and its inverse $\sharp : T^\star\mathcal{S} \to T\mathcal{S} : \boldsymbol{\alpha} \mapsto \sharp(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}^\sharp$, which maps a covector $\boldsymbol{\alpha}$ with components α_c to a vector $\boldsymbol{\alpha}^\sharp$ with components $g^{ac}\alpha_c$, where g^{ac} are the components of the inverse of the matrix $[\![g_{ab}]\!]$ of $\boldsymbol{g}$. The isomorphism $\sharp$ and the metric tensor $\boldsymbol{g}$ induce the scalar product of covectors $\boldsymbol{\alpha}.\boldsymbol{\beta} = \boldsymbol{g}(\boldsymbol{\alpha}^\sharp, \boldsymbol{\beta}^\sharp) = \boldsymbol{\alpha}(\boldsymbol{\beta}^\sharp)$.

The $(n-1)$-dimensional tangent bundle Ts of the hypersurface s determines a 1-dimensional sub-bundle of $T^\star\mathcal{S}$ containing the annihilators of Ts, i.e., the covectors $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}\boldsymbol{u} \equiv \boldsymbol{\nu}(\boldsymbol{u}) = 0$, for every $\boldsymbol{u} \in Ts$. Moreover, using the scalar product of covectors, we can define the *unit normal covector* $\boldsymbol{n}$ to the hypersurface s as the annihilating covector such that $\|\boldsymbol{n}\|^2 = \boldsymbol{n}.\boldsymbol{n} = 1$.

The integral (33) of an $(n-1)$-form $\boldsymbol{\omega}$ on the hypersurface s can be expressed in terms of the axial vector field $\boldsymbol{w}$ of $\boldsymbol{\omega}$ with respect to the volume form $\boldsymbol{\mu}$, i.e., $\boldsymbol{w}$ is such that $\boldsymbol{\iota_w}\boldsymbol{\mu} = \boldsymbol{\omega}$. If we introduce the *axial projector* $\boldsymbol{a} = \boldsymbol{n}^\sharp \otimes \boldsymbol{n}$ (in components, $a^a{}_b = n^a n_b$) and the *transverse projector* $\boldsymbol{t} = \boldsymbol{i} - \boldsymbol{n}^\sharp \otimes \boldsymbol{n}$ (in components, $t^a{}_b = \delta^a{}_b - n^a n_b$, where $\boldsymbol{i}$ is the spatial identity tensor, it holds that $\boldsymbol{i} = \boldsymbol{a} + \boldsymbol{t}$ and that any vector field $\boldsymbol{w}$ can be decomposed as

$$\boldsymbol{w} = \boldsymbol{i}\boldsymbol{w} = (\boldsymbol{a}+\boldsymbol{t})\boldsymbol{w} = \boldsymbol{a}\boldsymbol{w} + \boldsymbol{t}\boldsymbol{w} = \boldsymbol{w}_a + \boldsymbol{w}_t, \tag{C1}$$

where $\boldsymbol{w}_a = \boldsymbol{a}\boldsymbol{w} = (\boldsymbol{n}\boldsymbol{w})\boldsymbol{n}^\sharp$ and $\boldsymbol{w}_t = \boldsymbol{t}\boldsymbol{w} = \boldsymbol{w} - (\boldsymbol{n}\boldsymbol{w})\boldsymbol{n}^\sharp$ are the axial and the transverse component of $\boldsymbol{w}$, respectively. By construction, $\boldsymbol{w}_t$ is an element of the tangent bundle of the $(n-1)$-dimensional manisold $s \subset \mathcal{S}$. Hence, due to linearity, the $(n-1)$-form $\boldsymbol{\omega} = \boldsymbol{\iota_w}\boldsymbol{\mu}$ can be written as

$$\boldsymbol{\omega} = \boldsymbol{\iota_w}\boldsymbol{\mu} = \boldsymbol{\iota}_{(\boldsymbol{w}_a+\boldsymbol{w}_t)}\boldsymbol{\mu} = \boldsymbol{\iota}_{\boldsymbol{w}_a}\boldsymbol{\mu} + \boldsymbol{\iota}_{\boldsymbol{w}_t}\boldsymbol{\mu}. \tag{C2}$$

Let now $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}\} \subset Ts$ be a set of linearly indepdendent vectors spanning Ts. Since $\boldsymbol{w}_t$ can be expressed as a linear combination of $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}$, we obtain

$$(\boldsymbol{\iota}_{\boldsymbol{w}_t}\boldsymbol{\mu})(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}) = \boldsymbol{\mu}(\boldsymbol{w}_t, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}) = 0. \tag{C3}$$

Comparing Eq. (C3) with the definition of $\boldsymbol{\omega}$ in Eq. (C2), we find

$$\boldsymbol{\omega}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}) = (\boldsymbol{\iota_w}\boldsymbol{\mu})(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}) = (\boldsymbol{\iota}_{\boldsymbol{w}_a}\boldsymbol{\mu})(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}), \tag{C4}$$

and, since (C4) must hold true for all $(n-1)$-tuples $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-1}\} \subset Ts$, we can write

$$\boldsymbol{\omega} = \boldsymbol{\iota_w}\boldsymbol{\mu} \equiv \boldsymbol{\iota}_{\boldsymbol{w}_a}\boldsymbol{\mu}, \tag{C5}$$

i.e., only the axial component of $\boldsymbol{w}$, which is the componet parallel to the vector $\boldsymbol{n}^\sharp$ associated with the normal covector $\boldsymbol{n}$ to the hypersurface s, contributes to $\boldsymbol{\omega}$. Finally, by exploiting the result $\boldsymbol{w}_a = (\boldsymbol{n}\boldsymbol{w})\boldsymbol{n}^\sharp = (\boldsymbol{w}\boldsymbol{n})\boldsymbol{n}^\sharp$ and the linearity of the interior product, Eq. (C5) becomes

$$\boldsymbol{\omega} = \boldsymbol{\iota_w}\boldsymbol{\mu} \equiv \boldsymbol{\iota}_{\boldsymbol{w}_a}\boldsymbol{\mu} = (\boldsymbol{w}\boldsymbol{n})\boldsymbol{\iota}_{\boldsymbol{n}^\sharp}\boldsymbol{\mu} = (\boldsymbol{w}\boldsymbol{n})\boldsymbol{\alpha}, \tag{34 corr.}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\iota}_{\boldsymbol{n}^\sharp}\boldsymbol{\mu} \tag{1}$$

is the $(n-1)$-form induced on the hypersurface s by the volume form $\boldsymbol{\mu}$ and the metric $\boldsymbol{g}$. Therefore, on the basis of these results, the flux of an extensive quantity q across the hypersurface s can be expressed in the alternative notation [...]

References

1. S. Federico, A. Grillo, R. Segev, Material description of fluxes in terms of differential forms, Continuum Mechanics and Thermodynamics, **28**(1-2), 379-390 (2016) https://doi.org/10.1007/s00161-015-0437-2