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Hardy spaces on weighted homogeneous trees

Laura Arditti, Anita Tabacco and Maria Vallarino

Abstract We consider an infinite homogeneous tree $\mathcal{V}$ endowed with the usual metric d defined on graphs and a weighted measure μ. The metric measure space $(\mathcal{V}, d, \mu)$ is nondoubling and of exponential growth, hence the classical theory of Hardy spaces does not apply in this setting. We construct an atomic Hardy space $H^1(\mu)$ on $(\mathcal{V}, d, \mu)$ and investigate some of its properties, focusing in particular on real interpolation properties and on boundedness of singular integrals on $H^1(\mu)$.

Key words: Hardy spaces; homogeneous trees; exponential growth.

1 Introduction

Let $\mathcal{V}$ be an infinite homogeneous tree of order $q+1$ endowed with the usual distance d defined on a graph (see Section 2 for the precise definitions). Fix a doubly-infinite geodesic g in $\mathcal{V}$ and define a mapping $N : g \to \mathbb{Z}$ such that

$$|N(x) - N(y)| = d(x, y) \qquad \forall x, y \in g\,. \tag{1}$$

We define the level function $\ell : \mathcal{V} \to \mathbb{Z}$ as

$$\ell(x) = N(x') - d(x, x'),$$

where x' is the only vertex in g such that $d(x, x') = \min\{d(x, z) : z \in g\}$. Let μ be the measure on $\mathcal{V}$ defined by

Laura Arditti · Anita Tabacco · Maria Vallarino
Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Dipartimento di Eccellenza 2018-2022, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino Italy
e-mail: laura.arditti@polito.it, anita.tabacco@polito.it, maria.vallarino@polito.it

$$\int_{\mathcal{V}} f \, d\mu = \sum_{x \in \mathcal{V}} f(x) q^{\ell(x)} \tag{2}$$

for every function f defined on $\mathcal{V}$. Then μ is a weighted counting measure. We shall show in Subsection 2.1 that the space $(\mathcal{V}, d, \mu)$ is nondoubling and it is of exponential growth. In particular on such space the classical Calderón–Zygmund theory does not hold.

Hebisch and Steger [8] developed a new Calderón–Zygmund theory which can be applied also to nondoubling metric measure spaces and showed that such a theory can be applied to the space $(\mathcal{V}, d, \mu)$. In particular they proved that there exists a family of appropriate sets in $\mathcal{V}$, which are called Calderón–Zygmund sets, which replace the family of balls in the classical Calderón–Zygmund theory. We mention also that some properties of the space $(\mathcal{V}, d, \mu)$ were investigated in more detail in [1].

The purpose of this work is to develop a theory of Hardy spaces on $(\mathcal{V}, d, \mu)$, which is a natural development of the Calderón–Zygmund theory introduced in [8]. Following the classical atomic definition of Hardy spaces [5], for each p in $(1, \infty]$ we define an atomic Hardy space $H^{1,p}(\mu)$. Atoms are functions supported in Calderón–Zygmund sets, with vanishing integral and satisfying a certain size condition. We shall prove that all the spaces $H^{1,p}(\mu)$, $p \in (1, \infty]$, coincide and we simply denote by $H^1(\mu)$ this atomic Hardy space.

We then find the real interpolation spaces between $H^1(\mu)$ and $L^q(\mu)$, $q \in (1, \infty]$. The interpolation results which we prove are the analogue of the classical interpolation results (see [7, 11, 13, 14]), but the proofs are different. Indeed, in the classical setting the maximal characterization of the Hardy space is used to obtain the interpolation results, while the Hardy space $H^1(\mu)$ introduced in this paper has an atomic definition.

Further, we show that a singular integral operator whose kernel satisfies an integral Hörmander condition, extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$. As a consequence of this result, we show that spectral multipliers of a distinguished Laplacian $\mathcal{L}$ and the first order Riesz transform associated to $\mathcal{L}$ extend to bounded operators from $H^1(\mu)$ to $L^1(\mu)$.

It would be also interesting to characterize the dual space of $H^1(\mu)$ and to obtain complex interpolation results involving $H^1(\mu)$, its dual and the $L^q(\mu)$-spaces. This will be the object of further investigations.

All the results described above may be considered as an analogue of the classical theory of Hardy spaces.

The classical Hardy space [5, 6, 15] was introduced in $(\mathbb{R}^n, d, m)$, where d is the Euclidean metric and m denotes the Lebesgue measure and more generally on a space of homogeneous type, i.e. a metric measure space (X, d, μ) where the doubling condition is satisfied, i.e., there exists a constant C such that

$$\mu\big(B(x, 2r)\big) \le C\, \mu\big(B(x, r)\big) \qquad \forall x \in X, \qquad \forall r > 0. \tag{3}$$

Extensions of the theory of Hardy spaces have been considered in the literature on various metric measure spaces which do not satisfy the doubling condition (3). The literature on this subject is huge and we shall only cite here some contributions [3, 4, 12, 16] which are strictly related to our work.

In particular, we mention that Celotto and Meda [4] studied various Hardy spaces on a homogeneous tree $\mathcal{V}$ endowed with the metric d and the counting measure, which is not the measure μ that we consider here. Their theory is useful to study the boundedness of singular integral operators related to the standard Laplacian defined on trees which is self-adjoint with respect to the counting measure and not to the measure μ. The theory we develop here instead is useful to study singular integral operators related to a distinguished Laplacian self-adjoint on $L^2(\mu)$ (see Subsection 4.3).

We mention that in [12, 16] the authors used the Calderón–Zygmund theory introduced by Hebisch and Steger in [8] to construct Hardy spaces on some solvable Lie groups of exponential growth and studied their properties. Our work can be thought as a counterpart in a discrete setting of the results in [16], and some of our proofs are strongly inspired by it.

Positive constants are denoted by C; these may differ from one line to another, and may depend on any quantifiers written, implicitly or explicitly, before the relevant formula.

2 Weighted homogeneous trees

In this section we introduce the infinite homogeneous tree and we define a distance d and and a measure μ on it. We show that the corresponding metric measure space $(\mathcal{V}, d, \mu)$ does not satisfy the doubling property. We then introduce a family of sets, called trapezoids, which will be fundamental in the construction of Hardy spaces.

Definition 1 An infinite homogeneous tree of order $q + 1$ is a graph $T = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ denotes the set of edges, with the following properties:

(i) T is connected and acyclic;
(ii) each vertex has exactly $q + 1$ neighbours.

On $\mathcal{V}$ we can define the distance $d(x, y)$ between two vertices x and y as the length of the shortest path between x and y. We also fix a doubly-infinite geodesic g in T, that is a connected subset $g \subset \mathcal{V}$ such that

(i) for each element $v \in g$ there are exactly two neighbours of v in g;
(ii) for every couple (u, v) of elements in g, the shortest path joining u and v is contained in g.

We define a mapping $N : g \to \mathbb{Z}$ such that

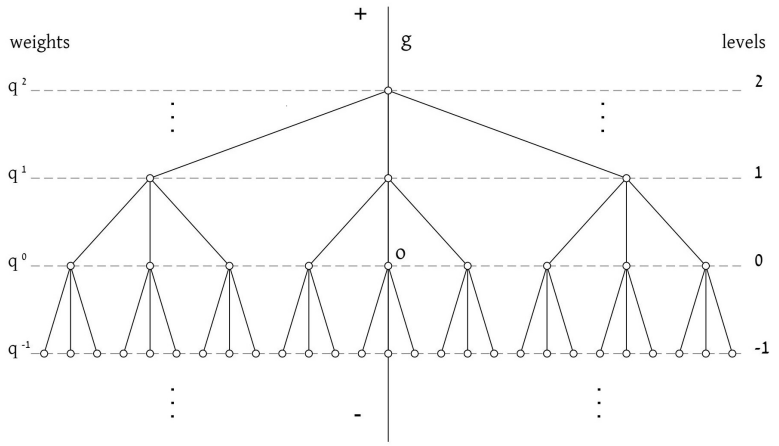

Fig. 1 Representation of the measure μ ($q = 3$)

$$|N(x) - N(y)| = d(x, y) \qquad \forall x, y \in g\,. \tag{4}$$

This corresponds to the choice of an origin $o \in g$ (the only vertex for which $N(o) = 0$) and an orientation for g; in this way we obtain a numeration of the vertices in g. We define the level function $\ell : \mathcal{V} \to \mathbb{Z}$ as

$$\ell(x) = N(x') - d(x, x')$$

where x' is the only vertex in g such that $d(x, x') = \min\{d(x, z) : z \in g\}$. For $x, y \in \mathcal{V}$ we say that y *lies above* x if

$$\ell(x) = \ell(y) - d(x, y).$$

In this case we also say that x *lies below* y.

Let μ be the measure on $\mathcal{V}$ such that for each function $f : \mathcal{V} \to \mathbb{C}$

$$\int_{\mathcal{V}} f\, d\mu = \sum_{x \in \mathcal{V}} f(x) q^{\ell(x)}. \tag{5}$$

Then μ is a weighted counting measure such that the weight of a vertex depends only on its level and the weight associated to a certain level is given by q times the weight of the level immediately underneath (see Figure 1).

2.1 Doubling and local doubling properties

Observe that the space $(\mathcal{V}, d, \mu)$ exhibits exponential volume growth. Indeed given $x_0 \in \mathcal{V}$ and $r \geq 1$ consider the sphere $S_r(x_0) = \{x \in \mathcal{V} : d(x, x_0) = r\}$ and the closed ball $B_r(x_0) = \{x \in \mathcal{V} : d(x, x_0) \leq r\}$. A direct computation shows that for $r \geq 1$ their measures are given by:

$$\mu(S_r(x_0)) = q^{\ell(x_0)+r-1}(1+q) \qquad \text{and} \qquad \mu(B_r(x_0)) = q^{\ell(x_0)}\,\frac{q^{r+1}+q^r-2}{q-1}.$$

We notice that they depend on the level of the center x_0 and grow exponentially with respect to the radius r. As a consequence we can prove the following.

Proposition 1 *The space $(\mathcal{V}, d, \mu)$ is not doubling but it is locally doubling.*

Proof Fix $x_0 \in \mathcal{V}$ and notice that

$$\lim_{r\to\infty} \frac{\mu(B_{2r}(x_0))}{\mu(B_r(x_0))} = \lim_{r\to\infty} \frac{q^{\ell(x_0)}\,\frac{q^{2r+1}+q^{2r}-2}{q-1}}{q^{\ell(x_0)}\,\frac{q^{r+1}+q^r-2}{q-1}} = \lim_{r\to\infty} q^r = +\infty.$$

Thus the doubling property (3) fails.

Instead, we show that $(\mathcal{V}, d, \mu)$ is locally doubling. Indeed, fix $x_0 \in \mathcal{V}$ and $R > 0$ and consider $r \leq R$; one has

$$\begin{aligned}\mu(B_{2r}(x_0)) &= q^{\ell(x_0)}\,\frac{q^{2r+1}+q^{2r}-2}{q-1} \leq q^{\ell(x_0)}\,\frac{\frac{q^{2R+1}+q^{2R}-2}{q-1}}{\frac{q^{R+1}+q^R-2}{q-1}}\,\frac{q^{r+1}+q^r-2}{q-1}\\ &= C_R\,\mu(B_r(x_0))\end{aligned}$$

with $C_R = \frac{q^{2R+1}+q^{2R}-2}{q^{R+1}+q^R-2} > 0$ independent of x_0 and r. □

2.2 Admissible trapezoids and Calderón–Zygmund sets

In this subsection we introduce the notion of trapezoid and recall the definition and the main properties of the admissible trapezoids introduced in [8].

Definition 2 We call trapezoid a set of vertices $S \subset \mathcal{V}$ for which there exist $x_S \in \mathcal{V}$ and $a, b \in \mathbb{R}_+$ such that

$$S = \{x \in \mathcal{V} : x \text{ lies below } x_S\,, a \leq \ell(x_S) - \ell(x) < b\}\,. \tag{6}$$

In the following we will refer to x_S as the root node of the trapezoid. Among all trapezoids we are mostly interested in those where a and b are related by particular conditions, as specified in the following definitions.

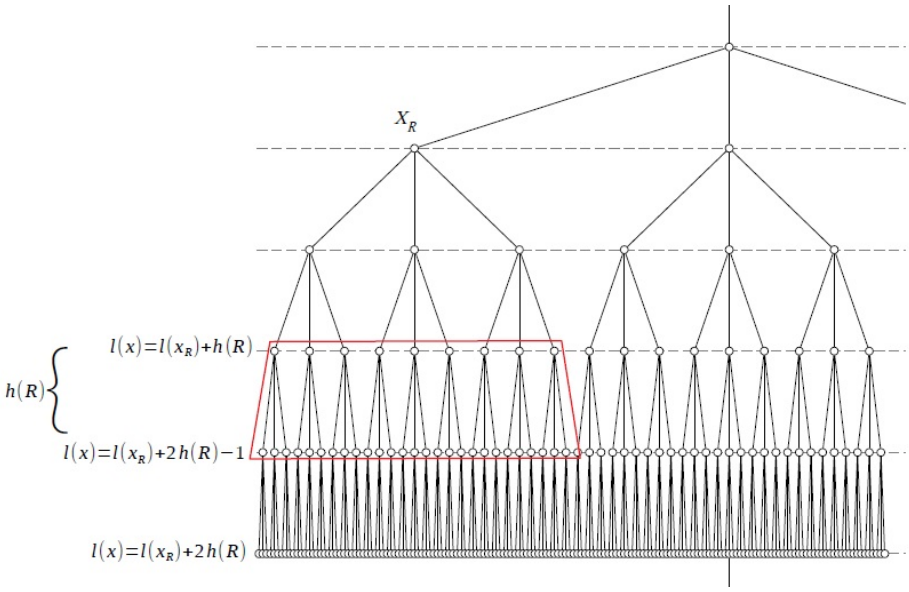

Fig. 2 Representation of an admissible trapezoid with $h(R) = 2$ ($q = 3$)

Definition 3 A trapezoid $R \subset \mathcal{V}$ is an admissible trapezoid if and only if one of the following occurs:

(i) $R = \{x_R\}$ with $x_R \in \mathcal{V}$, that is R consists of a single vertex ;
(ii) $\exists x_R \in \mathcal{V}$, $\exists h \in \mathbb{N}^+$ such that

$$R = \{x \in \mathcal{V} : x \text{ lies below } x_R\,, h \leq \ell(x_R) - \ell(x) < 2h\}\,.$$

We set $h(R) = 1$ in the first case and $h(R) = h$ in the second case. In both cases $h(R)$ can be interpreted as the height of the admissible trapezoid, which coincides with the number of levels spanned by R (see Figure 2).

Definition 4 We call width of the admissible trapezoid R the quantity

$$w(R) = q^{\ell(x_R)}.$$

We have that:

$$\mu(R) = h(R)q^{\ell(x_R)} = h(R)w(R). \tag{7}$$

We now introduce the family of Calderón–Zygmund sets. They are trapezoids, even if not of admissible type; they consist of suitable enlargements of admissible trapezoids, constructed according to the following definition.

Definition 5 Given an admissible trapezoid R, the envelope of R is the set

$$\tilde{R} = \left\{ x \in \mathcal{V} : x \text{ lies below } x_R \,,\, \frac{h}{2} \le \ell(x_R) - \ell(x) < 4h \right\} \tag{8}$$

and we set $h(\tilde{R}) = h(R)$. The envelope of an admissible trapezoid is also called a Calderón–Zygmund set.

Proposition 2 *Let R be an admissible trapezoid. Then:*

$$\mu(\tilde{R}) \le 4\mu(R). \tag{9}$$

Proof In the degenerate case one has $R = \{x_R\} = \tilde{R}$ and then $\mu(\tilde{R}) = \mu(R)$. In the nondegenerate case

$$\begin{aligned}
\mu(\tilde{R}) &= \sum_{\ell=\ell(x_R)-4h+1}^{\lfloor \ell(x_R)-\frac{h}{2} \rfloor} \sum_{x\in\tilde{R}:\ell(x)=\ell} q^{\ell} = \sum_{\ell=\ell(x_R)-4h+1}^{\lfloor \ell(x_R)-\frac{h}{2} \rfloor} q^{\ell} q^{\ell(x_R)-\ell} \\
&\le q^{\ell(x_R)} \left(\ell(x_R) - \tfrac{h}{2} - \ell(x_R) + 4h \right) \\
&\le 4\mu(R),
\end{aligned}$$

which concludes the proof. □

Proposition 3 *Let R_1 and R_2 be two admissible trapezoids. If*

$$R_1 \cap R_2 \neq \emptyset \text{ and } w(R_1) \ge w(R_2),$$

then

$$R_2 \subset \tilde{R_1}.$$

Proof The only nontrivial case is when neither R_1 nor R_2 is composed of a single vertex. Let x_{R_1} and x_{R_2} be the two root nodes of R_1 and R_2, respectively. Then

$$w(R_1) = q^{\ell(x_{R_1})} \ge q^{\ell(x_{R_2})} = w(R_2) \qquad \Longrightarrow \qquad \ell(x_{R_1}) \ge \ell(x_{R_2}).$$

Moreover, since $R_1 \cap R_2 \neq \emptyset$, x_{R_2} is below x_{R_1} and so is every vertex of R_2. In the following we denote $h_1 = h(R_1)$ and $h_2 = h(R_2)$. Let $\hat{x} \in R_1 \cap R_2 \neq \emptyset$. Then we obtain the following constraints:

$$\begin{cases} \ell(x_{R_2}) - 2h_2 + 1 \le \ell(\hat{x}) \le \ell(x_{R_1}) - h_1 \\ \ell(x_{R_1}) - 2h_1 + 1 \le \ell(\hat{x}) \le \ell(x_{R_2}) - h_2 \end{cases} \Longrightarrow \begin{cases} \ell(x_{R_1}) - \ell(x_{R_2}) \ge h_1 - 2h_2 + 1 \\ \ell(x_{R_1}) - \ell(x_{R_2}) \le 2h_1 - h_2 - 1 \,. \end{cases}$$

Let $x \in R_2$; then x lies below x_{R_1}. Moreover

$$\begin{aligned}
\ell(x_{R_1}) - \ell(x) &= \left[\ell(x_{R_1}) - \ell(x_{R_2}) \right] + \left[\ell(x_{R_2}) - \ell(x) \right] \\
&\le 2\left[2h_1 - h_2 - 1 \right] + \left[2h_2 - 1 \right] < 4h_1 .
\end{aligned}$$

$$\begin{aligned}\ell(x_{R_1}) - \ell(x) &= \left[\ell(x_{R_1}) - \ell(x_{R_2})\right] + \left[\ell(x_{R_2}) - \ell(x)\right] \\ &\geq \frac{1}{2}\left[h_1 - 2h_2 + 1\right] + \left[h_2\right] > \frac{h_1}{2}.\end{aligned}$$

So $\frac{h_1}{2} \leq \ell(x_{R_1}) - \ell(x) < 4h_1$, $\forall x \in R_2$ and this shows that $R_2 \subset \tilde{R}_1$. □

Proposition 4 *Given a Calderón–Zygmund set $\tilde{R}$, we have that for all $z \in \tilde{R}$*

$$\tilde{R} \subset B(z, 8h(\tilde{R}))\,.$$

Proof Fix a point $z \in \tilde{R}$. Every vertex $y \in \tilde{R}$ has distance $d(z, y) \leq 8h(\tilde{R}) - 2 < 8h(\tilde{R})$. Indeed, starting from z it is possible to reach y passing through at most $\left[(4h(\tilde{R}) - 1)\right] 2 = 8h(\tilde{R}) - 2$ edges, moving from z to the root node of the trapezoid and then from the root node to y. □

Definition 6 Given a Calderón–Zygmund set $\tilde{R}$, we define the set

$$\tilde{R}^* = \left\{x \in \mathcal{V} : d(x, \tilde{R}) < h(\tilde{R})/4\right\}\,. \tag{10}$$

It is easy to see that there exists a positive constant C such that for every Calderón–Zygmund set $\tilde{R}$

$$\mu(\tilde{R}^*) \leq C\mu(\tilde{R})\,. \tag{11}$$

See [1, p.75] for a proof of this fact.

3 The maximal function

In this section we define two maximal functions and describe a way to construct a covering of their level sets which will be useful in the sequel.

Definition 7 Given $f : \mathcal{V} \to \mathbb{C}$, we define the maximal function M as

$$Mf(x) = \sup_{R : x \in R} \frac{1}{\mu(R)} \int_R |f|\, d\mu$$

where the supremum is taken over all admissible trapezoids R containing x.

Consider a function $f \in L^p(\mu)$ and let $\lambda > 0$. We are interested in constructing a covering of the level set

$$\Omega_{\lambda^p} = \{x \in \mathcal{V} : M(|f|^p)(x) > \lambda^p\}\,.$$

Define S_0 as the family of all admissible trapezoids R such that

$$\int_R |f|^p\, d\mu \geq \lambda^p \mu(R).$$

Since S_0 is countable, we can introduce an ordering in S_0. All trapezoids in S_0 have bounded measure and bounded width, because $\forall R \in S_0$ we have

$$w(R) = \frac{\mu(R)}{h(R)} \leq \mu(R) \leq \frac{1}{\lambda^p} \|f\|_{L^p}^p \ .$$

So it is possible to choose in S_0 a trapezoid R_0 of largest width (in case of ties, we choose that trapezoid of largest width which occurs earliest in the ordering). Then we proceed inductively:

(i) S_{i+1} is the family of all admissible trapezoids $R \in S_i$ disjoint from $R_0, \dots, R_i$;
(ii) R_{i+1} is the trapezoid of largest width in S_{i+1} which occurs earliest in the ordering.

Let $R \in S_0$. Then by construction R intersects some R_i with $w(R_i) \geq w(R)$.

Indeed, there exists a number $j \in \{0, 1, 2, \dots\}$ such that $R \in S_j$ and $R \notin S_{j+1}$, i.e. in the previous construction there exists a step j in which one of the following occurs:

1. either R is the trapezoid of largest width that occurs earliest in the ordering, and then R is selected and $R_j = R$, so that $R \cap R_i \neq \emptyset$ for $i = j$;
2. or R is not the trapezoid of largest width that occurs earliest in the ordering and it intersects R_j. Then R is not in S_i $\forall i \geq j + 1$ and $R \cap R_i \neq \emptyset$ for $i = j$.

To ensure that there is some j with the stated property it is sufficient to avoid that S_0 can contain an infinite number of trapezoids with the same width that do not intersect each other. This possibility is excluded observing that:

$$\sum_i \mu(R_i) \leq \frac{1}{\lambda^p} \sum_i \int_{R_i} |f|^p \, d\mu \leq \frac{1}{\lambda^p} \int_{\mathcal{V}} |f|^p \, d\mu = \frac{1}{\lambda^p} \|f\|_{L^p} < \infty,$$

while if there was among the R_i's an infinite number of trapezoids with constant width w we would have

$$\sum_i \mu(R_i) \geq \sum_{n=1}^{\infty} w = \infty.$$

In conclusion,

$$\forall R \in S_0, \quad \exists i : R \cap R_i \neq \emptyset \text{ and } w(R_i) \geq w(R).$$

By Proposition 3, this implies $R \subset \tilde{R}_i$. We set $E := \bigcup_i \tilde{R}_i$. We have that $\Omega_{\lambda^p} \subset E$ and

$$\mu(E) \leq \sum_i \mu(\tilde{R}_i) \leq 4 \sum_i \mu(R_i) \leq \frac{4 \|f\|_{L^p}}{\lambda^p}. \tag{12}$$

Definition 8 Given $f : \mathcal{V} \to \mathbb{C}$ and a Calderón–Zygmund set $\tilde{Q}$, we define the maximal function $M_{\tilde{Q}}$ as follows

$$M_{\tilde{Q}}(f)(x) = \sup_{R \subset \tilde{Q} \,:\, x \in R} \mu(R)^{-1} \int_R |f| \, d\mu \qquad \forall x \in \tilde{Q},$$

where the supremum is taken over all admissible trapezoids R containing x and contained in $\tilde{Q}$. When $x \notin \tilde{Q}$ we set $M_{\tilde{Q}}(f)(x) = 0$.

Consider a function $f \in L^p(\mu)$ with support contained in a Calderón–Zygmund set $\tilde{Q}$, and let $\lambda > 0$. We define

$$\Omega_{\tilde{Q},\lambda^p} = \{x \in \mathcal{V} : M_{\tilde{Q}}(|f|^p)(x) > \lambda^p\}\,.$$

Arguing as before, we can show that there exists a family of pairwise disjoint admissible trapezoids $\{R_i\}$ such that $R_i \subset \tilde{Q}$, $\sum_i \mu(R_i) \leq \frac{1}{\lambda^p} \|f\|_{L^p}$ and $\Omega_{\tilde{Q},\lambda^p} \subset \bigcup_i \tilde{R}_i$.

4 Hardy spaces

In this section we define atomic Hardy spaces replacing balls with Calderón–Zygmund sets in the classical definition of atoms.

Definition 9 A function a is a $(1,p)$-*atom*, for $p \in (1,\infty]$, if it satisfies the following properties:

(i) a is supported in a Calderón–Zygmund set $\tilde{R}$;

(ii) $\|a\|_{L^p} \leq \mu(\tilde{R})^{1/p-1}$;

(iii) $\int_{\mathcal{V}} a \,\mathrm{d}\mu = 0$.

Observe that a $(1,p)$-atom is in $L^1(\mu)$ and it is normalized in such a way that its L^1-norm does not exceed 1.

Definition 10 The Hardy space $H^{1,p}(\mu)$ is the space of all functions h in $L^1(\mu)$ such that $h = \sum_j \lambda_j\, a_j$, where a_j are $(1,p)$-atoms and λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$. We denote by $\|h\|_{H^{1,p}}$ the infimum of $\sum_j |\lambda_j|$ over all decompositions $h = \sum_j \lambda_j\, a_j$, where a_j are $(1,p)$-atoms.

The space $H^{1,p}(\mu)$ endowed with the norm $\|\cdot\|_{H^{1,p}}$ is a Banach space.

4.1 Equivalence of spaces $H^{1,p}(\mu)$ for $p \in (1,\infty]$

It easily follows from the above definitions that $H^{1,\infty}(\mu) \subseteq H^{1,p}(\mu)$, whenever $p \in (1,\infty)$. Actually we shall prove that $H^{1,\infty}(\mu) = H^{1,p}(\mu)$, for every $p \in (1,\infty)$. To show this fact we first prove a preliminary result.

Proposition 5 *Let a be a $(1,p)$-atom, where $p \in (1,\infty)$. Then a is in $H^{1,\infty}(\mu)$ and there exists a positive constant C_p, which depends only on p, such that*

$$\|a\|_{H^{1,\infty}} \leq C_p\,.$$

Proof Let a be a $(1,p)$-atom supported in a Calderón–Zygmund set $\tilde{Q}$. We define $b := \mu(\tilde{Q})\, a$.

Let α be a positive number such that $\alpha > 2[24q(1+4^p)]^{1/(p-1)}$.

We shall prove that for all $n \in \mathbb{N}$ there exist functions a_{j_ℓ}, h_{j_n} and admissible sets $\tilde{R}_{j_\ell}$, $j_\ell \in \mathbb{N}^\ell$, $\ell = 0, ..., n$, such that

$$b = \sum_{\ell=0}^{n-1} 4(6q)^{1/p} \alpha^{\ell+1} \sum_{j_\ell} \mu(\tilde{R}_{j_\ell})\; a_{j_\ell} + \sum_{j_n \in \mathbb{N}^n} f_{j_n}\,, \tag{13}$$

where the following properties are satisfied:

(i) a_{j_ℓ} is a $(1,\infty)$-atom supported in the Calderón–Zygmund set $\tilde{R}_{j_\ell}$;
(ii) f_{j_n} is supported in R_{j_n} and $\int f_{j_n}\, \mathrm{d}\mu = 0$;
(iii) $\left(\frac{1}{\mu(\tilde{R}_{j_n})} \int_{\tilde{R}_{j_n}} |f_{j_n}|^p \, \mathrm{d}\mu\right)^{1/p} \le 2^{1-1/p}\, (6q)^{1/p}\, (1+4^p)^{1/p} \alpha^n$;
(iv) $|f_{j_n}(x)| \le |b(x)| + 4(6q)^{1/p} n\, \alpha^n\,, \qquad \forall x \in \tilde{R}_{j_n}$;
(v) $\sum_{j_n} \mu(\tilde{R}_{j_n}) \le\; 4^{n+1}\, [2^{p-1}\, (6q)\, (1+4^p)]^n \alpha^{-np}\, \|b\|_{L^p}^p$.

We first suppose that the decomposition (13) exists and we show that $a \in H^{1,\infty}(\mu)$. Set $F_n = \sum_{j_n} f_{j_n}$. We prove that $F_n \in L^1(\mu)$ and that its L^1-norm tends to zero when n tends to ∞. Indeed, by Hölder's inequality

$$\|F_n\|_{L^1} \le \sum_{j_n \in \mathbb{N}^n} \|f_{j_n}\|_{L^1} \le \sum_{j_n \in \mathbb{N}^n} \mu(\tilde{R}_{j_n})^{1/p'}\, \|f_{j_n}\|_{L^p}\,,$$

where p' is the conjugate exponent of p. Now by (iii) and (v) we have that

$$\begin{aligned} \|F_n\|_{L^1} &\le \sum_{j_n \in \mathbb{N}^n} \mu(\tilde{R}_{j_n})^{1/p'} \mu(\tilde{R}_{j_n})^{1/p}\, 2^{1-1/p}\, (6q)^{1/p}\, (1+4^p)^{1/p} \alpha^n \\ &\le 4^{n+1}\, [2^{p-1}\, 6q\, (1+4^p)]^n \alpha^{-np}\, \|b\|_{L^p}^p\, 2^{1-1/p}\, (6q)^{1/p}\, (1+4^p)^{1/p} \alpha^n\,, \end{aligned}$$

which tends to zero when n tends to ∞, since $\alpha > 2[24q(1+4^p)]^{1/(p-1)}$.

This shows that the series $\sum_{\ell=0}^{n-1} 4(6q)^{1/p} \alpha^{\ell+1} \sum_{j_\ell} \mu(\tilde{R}_{j_\ell})\; a_{j_\ell}$ converges to b in $L^1(\mu)$. Moreover by (v) we deduce that

$$\begin{aligned} \sum_{\ell=0}^{\infty} 4(6q)^{1/p} \alpha^{\ell+1} \sum_{j_\ell} \mu(\tilde{R}_{j_\ell}) &\le \sum_{\ell=0}^{\infty} 4(6q)^{1/p} \alpha^{\ell+1} 4^{\ell+1}\, [2^{p-1}\, (6q)\, (1+4^p)]^\ell \alpha^{-\ell p}\, \|b\|_{L^p}^p \\ &\le C_p\, \|b\|_{L^p}^p \le C_p\, \mu(\tilde{Q}), \end{aligned}$$

where C_p depends only on p.

It follows that b is in $H^{1,\infty}(\mu)$ and $\|b\|_{H^{1,\infty}} \le C_p\, \mu(\tilde{Q})$. Thus $a = \mu(\tilde{Q})^{-1}\, b$ is in $H^{1,\infty}(\mu)$ and $\|a\|_{H^{1,\infty}} \le C_p$, as required.

It remains to prove that the decomposition (13) exists. We prove it by induction on n.

Step $n = 1$**.** Define

$$\Omega_{\tilde{Q},\alpha^p} = \{x \in \mathcal{V} : M_{\tilde{Q}}(|b|^p)(x) > \alpha^p\}\,.$$

If $\Omega = \emptyset$, then

$$\mu(\{x_R\})^{-1}|b(x_R)|^p\,\mu(\{x_R\}) \le \alpha^p\,, \qquad \forall x_R \in \mathcal{V}\,.$$

It follows that $\|b\|_{L^\infty} \le \alpha$ and we have $b = \alpha\mu(\tilde{Q})a_{j_0}$, so that and (13) is satisfied with the $(1,\infty)$-atom $a_{j_0} = \alpha^{-1}\mu(\tilde{Q})^{-1}b$ and $f_{j_1} = 0$ for every $j_1 \in \mathbb{N}$.

If $\Omega \neq \emptyset$, then we construct a family of trapezoids $R_i, i \in \mathbb{N}$, and the corresponding Calderón–Zygmund sets $\tilde{R}_i, i \in \mathbb{N}$, as in Section 3. We then define $U_i = \tilde{R}_i \setminus (\cup_{j<i}\tilde{R}_j)$, and $h_i = b\chi_{U_i}$. One can show as in [8, p.43]

$$\int_{U_i} |h_i|^p\,d\mu \le 6q\alpha^p\,\mu(\tilde{R}_i)\,. \tag{14}$$

See also [1, p.74] for a detailed proof of the previous inequality.

We now define

$$f_i = h_i - \mu(R_i)^{-1}\int h_i d\mu\,\chi_{R_i}\,, \qquad g = b - \sum_i f_i\,.$$

Notice that f_i is supported in $\tilde{Q}\cap\tilde{R}_i$ and g is supported in $\tilde{Q}$. The average of f_i vanishes by construction. Moreover, for every $x \in \tilde{R}_i$ we have that

$$\begin{aligned}
|f_i(x)| &\le |h_i(x)| + \mu(R_i)^{-1}\int |h_i|d\mu\chi_{R_i}\\
&\le |h_i(x)| + \mu(R_i)^{-1}\Big(\int |h_i|^p d\mu\Big)^{1/p}\mu(\tilde{R}_i)^{1/p'}\\
&\le |b(x)| + (6q)^{1/p}4\,\alpha\,,
\end{aligned}$$

where we have applied (14) and Proposition 2. It follows that

$$\|f_i\|_{L^p}^p \le 2^{p-1}\|h_i\|_{L^p}^p + 6q4^p\alpha^p\mu(\tilde{R}_i) \le 2^{p-1}6q\alpha^p(1+4^p)\mu(\tilde{R}_i)\,.$$

Moreover,

$$\sum_i \mu(\tilde{R}_i) \le 4\sum_i \mu(R_i) \le 4\frac{1}{\alpha^p}\sum_i\int_{R_i}|b(x)|^p d\mu(x) \le 4\frac{1}{\alpha^p}\|b\|_{L^p}^p\,.$$

This implies that

$$\sum_i \|f_i\|_{L^p}^p \le 2^{p-1}\,6q\alpha^p(1+4^p)4\frac{1}{\alpha^p}\|b\|_{L^p}^p\,.$$

We now estimate the function g. If x is a vertex in the complement of $\tilde{Q}$, then obviously $g(x) = 0$. If x is a vertex in $\tilde{Q}\cap(\bigcup_i\tilde{R}_i)^c$, then $g(x) = b(x)$ and $M_{\tilde{Q}}(|b|^p)(x) \le \alpha^p$. Thus $|g(x)| = |b(x)| \le \alpha$. Let us now take $x \in \tilde{Q}\cap(\bigcup_i\tilde{R}_i)$

and let i_0 be the unique index such that $x \in U_{i_0}$. We distinguish two different cases. If $x \notin \bigcup_i R_i$, then $f_{i_0}(x) = h_{i_0}(x) = b(x)$ and $f_i(x) = 0$ for every $i \neq i_0$, so that $g(x) = b(x) - b(x) = 0$. If $x \in \bigcup_i R_i$, let i_1 be the unique index such that $x \in R_{i_1}$. When $i_1 = i_0$ we have that $f_i(x) = 0$ for every $i \neq i_0$ and

$$g(x) = b(x) - f_{i_0}(x) = b(x) - b(x) + \mu(R_{i_0})^{-1} \int h_{i_0} d\mu,$$

so that $|g(x)| \leq (6q)^{1/p} 4\alpha$. When $i_1 \neq i_0$ we have that $f_i(x) = 0$ for every $i \neq i_0, i_1$ and

$$g(x) = b(x) - f_{i_0}(x) - f_{i_1}(x) = b(x) - b(x) + \mu(R_{i_1})^{-1} \int h_{i_1} d\mu,$$

so that $|g(x)| \leq (6q)^{1/p} 4\alpha$.

In conclusion,

$$b(x) = (6q)^{1/p} 4\alpha \mu(\tilde{Q}) a(x) + \sum_{j_1 \in \mathbb{N}} f_{j_1}(x),$$

where $a = (6q)^{-1/p} 4^{-1} \alpha^{-1} \mu(\tilde{Q})^{-1} g$ is a $(1, \infty)$-atom supported in $\tilde{Q}$ and all properties (i)-(v) are satisfied.

Inductive step. Suppose that a decomposition

$$b = \sum_{\ell=0}^{n-1} 4(6q)^{1/p} \alpha^{\ell+1} \sum_{j_\ell} \mu(\tilde{R}_{j_\ell})\ a_{j_\ell} + \sum_{j_n \in \mathbb{N}^n} f_{j_n},$$

holds, where properties (i)-(v) are satisfied. We shall prove that a similar decomposition of b holds with $n+1$ in place of n. To do so, we decompose each function f_{j_n} by following the same construction applied above to b with respect to α^{n+1}. We omit the details. □

The following theorem is now an easy consequence of Proposition 5.

Theorem 1 *For any $p \in (1, \infty)$, $H^{1,p}(\mu) = H^{1,\infty}(\mu)$ and the norms $\|\cdot\|_{H^{1,p}}$ and $\|\cdot\|_{H^{1,\infty}}$ are equivalent.*

In the sequel we denote by $H^1(\mu)$ the space $H^{1,\infty}(\mu)$ and we define $\|\cdot\|_{H^1} = \|\cdot\|_{H^{1,\infty}}$.

4.2 Real interpolation properties of $H^1(\mu)$

In this subsection we study the real interpolation of $H^1(\mu)$ and the $L^p(\mu)$ spaces. We first recall some notation of the real interpolation of normed spaces, focusing on the K-method. For the details see [2].

Given two compatible normed spaces A_0 and A_1, for any $t > 0$ and for any $a \in A_0 + A_1$ we define

$$K(t,a;A_0,A_1)=\inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1}\,:\,a=a_0+a_1,\ a_i\in A_i\}\,.$$

Take $q\in[1,\infty]$ and $\theta\in(0,1)$. The *real interpolation space* $\big[A_0,A_1\big]_{\theta,q}$ is defined as the set of the elements $a\in A_0+A_1$ such that

$$\|a\|_{\theta,q}=\begin{cases}\Big(\int_0^\infty\big[t^{-\theta}\,K(t,a;A_0,A_1)\big]^q\,\frac{\mathrm{d}t}{t}\Big)^{1/q} & \text{if } 1\le q<\infty\\ \|t^{-\theta}\,K(t,a;A_0,A_1)\|_\infty & \text{if } q=\infty\,,\end{cases}$$

is finite. The space $\big[A_0,A_1\big]_{\theta,q}$ endowed with the norm $\|a\|_{\theta,q}$ is an exact interpolation space of exponent θ.

We refer the reader to [10] for an overview of the real interpolation results which hold in the classical setting. Our aim is to prove the same results in our context. Note that in our case a maximal characterization of $H^1(\mu)$ is not avalaible, so that we cannot follow the classical proofs but we shall only use the atomic definition of $H^1(\mu)$ to prove the results.

We shall first estimate the K functional of L^p-functions with respect to the couple of spaces $(H^1(\mu),L^{p_1}(\mu))$, $1<p_1\le\infty$.

Lemma 1 *Suppose that $1<p<p_1\le\infty$ and let $\theta\in(0,1)$ be such that $\frac{1}{p}=1-\theta+\frac{\theta}{p_1}$. Let f be in $L^p(\mu)$. The following hold:*

(i) for every $\lambda>0$ there exists a decomposition $f=g^\lambda+b^\lambda$ in $L^{p_1}(\mu)+H^1(\mu)$ such that

(i') $\|g^\lambda\|_{L^\infty}\le C\,\lambda$ and, if $p_1<\infty$, then $\|g^\lambda\|^{p_1}_{L^p_1}\le C\,\lambda^{p_1-p}\,\|f\|^p_{L^p}$;

(i") $\|b^\lambda\|_{H^1}\le C\,\lambda^{1-p}\,\|f\|^p_{L^p}$;

(ii) for any $t>0$, $K(t,f;H^1(\mu),L^{p_1}(\mu))\le C\,t^\theta\,\|f\|_{L^p}$;

(iii) $f\in[H^1(\mu),L^{p_1}(\mu)]_{\theta,\infty}$ and $\|f\|_{\theta,\infty}\le C\,\|f\|_{L^p}$.

Proof Let f be in $L^p(\mu)$. We first prove (i). Given a positive λ, let

$$\Omega_{\lambda^p}=\{x\in\mathcal{V}:M(|f|^p)(x)>\lambda^p\}\,.$$

Let $\{R_i\}$ be the collection of trapezoids constructed as in Section 3. We now define $U_i=\tilde R_i\setminus(\cup_{j<i}\tilde R_j)$ and $h_i=f\chi_{U_i}$,

$$f_i=f-\mu(R_i)^{-1}\int h_i\,d\mu\chi_{R_i}\,,\qquad b^\lambda=\sum_i f_i\qquad\text{and}\qquad g^\lambda=f-b^\lambda\,.$$

Arguing as we did in the proof of Proposition 5 we can show that

$$\|g^\lambda\|_{L^\infty}\le C\,\lambda\qquad\text{and}\qquad\frac{1}{\mu(R_i)}\int_{R_i}|h_i|^p\,d\mu\le C\,\lambda^p\,.$$

If $p_1<\infty$, then

$$\|g^\lambda\|_{L^{p_1}}^{p_1} \le \int_{(\bigcup \tilde{R}_i)^c} |f|^{p_1} d\mu + \sum_i \int_{\tilde{R}_i} |f - \sum_j f_j|^{p_1} d\mu = I + II\,.$$

To estimate I we notice that $(\cup_i \tilde{R}_i)^c \subset \Omega_{\lambda^p}^c$, so that

$$I \le \int_{(\bigcup \tilde{R}_i)^c} |f|^{p_1-p} |f|^p d\mu \le C\lambda^{p_1-p} \|f\|_{L^p}^p\,.$$

To estimate II we first observe that given i and $x \in \tilde{R}_i$ there exists only two indeces $i_0 \le i$ and i_1 such that $x \in U_{i_0}$ and $x \in R_{i_1}$. If $i_0 = i_1$, then

$$f(x) - \sum_j f_j(x) = f(x) - f_{i_0}(x) = \mu(R_{i_0})^{-1} \int h_{i_0} d\mu,$$

so that $|f(x) - \sum_j f_j(x)| \le C\lambda$. If $i_0 \ne i_1$, then

$$f(x) - \sum_j f_j(x) = f(x) - f_{i_0}(x) - f_{i_1}(x) = \mu(R_{i_1})^{-1} \int h_{i_1} d\mu,$$

so that $|f(x) - \sum_j f_j(x)| \le C\lambda$. It follows that

$$II \le C \sum_i \int_{\tilde{R}_i} \lambda^{p_1} d\mu \le C\lambda^{p_1} \sum_i \mu(\tilde{R}_i) \le C\lambda^{p_1-p} \|f\|_{L^p}^p\,.$$

In conclusion, $\|g^\lambda\|_{L^{p_1}}^{p_1} \le C\lambda^{p_1-p} \|f\|_{L^p}^p$.

We now prove that b^λ is in $H^{1,p}(\mu)$. Indeed, for any i, f_i is supported in $\tilde{R}_i$, has vanishing integral and

$$\|f_i\|_{L^p} \le C\|h_i\|_{L^p} + C\mu(R_i)^{-1} \mu(\tilde{R}_i)^{1/p} \int |h_i| d\mu \mu(\tilde{R}_i)^{1/p} \le C\lambda\,.$$

This shows that $f_i \in H^{1,p}(\mu) = H^1(\mu)$ and $\|f_i\|_{H^1} \le C\,\lambda\,\mu(\tilde{R}_i)$. Since $b^\lambda = \sum_i f_i$, b^λ is in $H^1(\mu)$ and

$$\|b^\lambda\|_{H^1} \le C\,\lambda \sum_i \mu(\tilde{R}_i) \le C\,\lambda\,\frac{\|f\|_{L^p}^p}{\lambda^p},$$

as required.

We now prove (ii). Fix $t > 0$. For any positive λ, let $f = g^\lambda + b^\lambda$ be the decomposition of f in $L^{p_1}(\mu) + H^1(\mu)$ given by (i). Thus

$$
\begin{aligned}
K(t,f;H^1(\mu),L^{p_1}(\mu)) &\le \inf_{\lambda>0}\left(\|b^\lambda\|_{H^1}+t\,\|g^\lambda\|_{L^{p_1}}\right)\\
&\le C\,\inf_{\lambda>0}\left(\lambda^{1-p}\,\|f\|_{L^p}^p+t\,\lambda^{1-p/p_1}\|f\|_{L^p}^{p/p_1}\right)\\
&= C\,\|f\|_{L^p}^{p/p_1}\,\inf_{\lambda>0}G(t,\lambda),
\end{aligned}
$$

where $G(t,\lambda)=\lambda^{1-p}\,\|f\|_{L^p}^{p(1-1/p_1)}+t\,\lambda^{1-p/p_1}$. Since

$$
\partial_\lambda G(t,\lambda)=\lambda^{-p}\left[(1-p)\,\|f\|_{L^p}^{p(1-1/p_1)}+(1-p/p_1)t\,\lambda^{-p/p_1+p}\right],
$$

we have that if $p_1<\infty$, then

$$
\inf_{\lambda>0}G(t,\lambda)=G\big(t,C_p\|f\|_{L^p}\,t^{p_1/p-pp_1}\big)=C_p\,\|f\|_{L^p}^{1-p/p_1}\,t^{\frac{p_1(p-1)}{p(p_1-1)}}\,.
$$

If $p_1=\infty$, then

$$
\inf_{\lambda>0}G(t,\lambda)=G\big(t,C_p\|f\|_{L^p}\,t^{-1/p}\big)=C_p\,\|f\|_{L^p}\,t^{1-1/p}\,.
$$

It follows that

$$
K(t,f;H^1,L^{p_1})\le C_p\,\|f\|_{L^p}\,t^\theta\,,
$$

proving (ii). This implies that $\|t^{-\theta}\,K(t,f;H^1(\mu),L^{p_1}(\mu))\|_{L^\infty}\le C_p\,\|f\|_{L^p}$, so that $f\in[H^1(\mu),L^{p_1}(\mu)]_{\theta,\infty}$ and $\|f\|_{\theta,\infty}\le C_p\|f\|_{L^p}$, as required in (iii). □

Following closely the proof of [16, Theorem] we deduce from Lemma 1 the following result.

Theorem 2 *Let $1<p<p_1\le\infty$ and $\theta\in(0,1)$ be such that $\frac{1}{p}=1-\theta+\frac{\theta}{p_1}$. Then*

$$
\left[H^1(\mu),L^{p_1}(\mu)\right]_{\theta,p}=L^p(\mu)\,.
$$

4.3 Boundedness of singular integrals on $H^1(\mu)$

In this subsection we prove that integral operators whose kernels satisfy a suitable integral Hörmander condition are bounded from $H^1(\mu)$ to $L^1(\mu)$.

Theorem 3 *Let T be a linear operator which is bounded on $L^2(\mu)$ and admits a locally integrable kernel K off the diagonal that satisfies the condition*

$$
\sup_{\tilde R}\sup_{y,\,z\in\tilde R}\int_{(\tilde R^*)^c}|K(x,y)-K(x,z)|\,d\mu(x)<\infty\,, \tag{15}
$$

where the supremum is taken over alla Calderón-Zygmund sets $\tilde R$ and $\tilde R^*$ is defined as in Definition 6. Then T extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$.

Proof Using (15), by [8, Theorem 1.2] it is easy to prove that the operator T is of weak type $(1,1)$. Then it is enough to show that there exists a constant C such that $\|Ta\|_{L^1} \le C$ for any $(1,\infty)$-atom a.

Let a be a $(1,\infty)$-atom supported in the Calderón–Zygmund set $\tilde{R}$. Recall that $\tilde{R} \subset B(x_R, 8h(\tilde{R}))$, and $\tilde{R}^*$ denote the dilated set $\{x \in \mathcal{V} : \ d(x,\tilde{R}) < h(\tilde{R})/4\}$. We estimate the integral $\int_{\mathcal{V}} |Ta| d\mu$.

We first estimate the integral on $\tilde{R}^*$ by the Cauchy-Schwarz inequality and the size estimate of the atom:

$$\begin{aligned}\int_{R^*} |Ta| d\mu &\le \|Ta\|_{L^2}\, \mu(\tilde{R}^*)^{1/2} \le C\, \|\!|T|\!\|_{L^2\to L^2}\, \|a\|_{L^2}\, \mu(\tilde{R})^{1/2} \\ &\le C\, \|\!|T|\!\|_{L^2\to L^2}\,. \end{aligned} \tag{16}$$

We consider the integral on the complementary set of $\tilde{R}^*$ by using the fact that a has vanishing integral:

$$\begin{aligned}\int_{\tilde{R}^{*c}} |Ta| d\mu &\le \int_{(\tilde{R}^*)^c} \Big| \int_{\tilde{R}} K(x,y)\, a(y) d\mu(y)\Big| d\mu(x) \\ &= \int_{(\tilde{R}^*)^c} \Big| \int_{\tilde{R}} [K(x,y) - K(x,x_R)]\, a(y) d\mu(y)\Big| d\mu(x) \\ &\le \int_{(\tilde{R}^*)^c} \int_{\tilde{R}} |K(x,y) - K(x,x_R)|\, |a(y)| d\mu(y) d\mu(x) \\ &= \int_{\tilde{R}} |a(y)| \Big(\int_{(\tilde{R}^*)^c} |K(x,y) - K(x,x_R)| d\mu(x)\Big) d\mu(y) \\ &\le \|a\|_{L^1} \sup_{y\in\tilde{R}} \int_{(\tilde{R}^*)^c} |K(x,y) - K(x,x_R)| d\mu(x) \\ &\le C\,, \end{aligned}$$

as required. □

Remark: The previous result applies to singular integral operators associated with the Laplacian $\mathcal{L}$ on the tree defined for every function $f : \mathcal{V} \to \mathbb{C}$ by

$$\mathcal{L} f(x) = f(x) - \frac{1}{2\sqrt{q}} \sum_{y\in\mathcal{V}: d(x,y)=1} q^{\frac{\ell(y)-\ell(x)}{2}} f(y) \qquad \forall x \in \mathcal{V}\,. \tag{17}$$

The Laplacian $\mathcal{L}$ is bounded on $L^p(\mu)$ for every $p \in [1,\infty]$, it is self-adjoint on $L^2(\mu)$ and its spectrum on $L^2(\mu)$ is $[0,2]$. Suppose that $M : \mathbb{R} \to \mathbb{C}$ is bounded supported in $[0,2)$ and satisfies the following Mikhlin-Hörmander condition of order $s > 3/2$

$$\sup_{t>0} \|(D_t M)\phi\|_{W_2^s} < \infty\,, \tag{18}$$

for some $\phi \in C_c^\infty([\frac{1}{2},4])$, $\phi \neq 0$, where $(D_t M)(\lambda) = M(t\lambda)$ and W_2^s denotes the Sobolev space of order s modelled on $L^2([0,2])$. Then the operator $M(\mathcal{L})$ extends

to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$. Indeed, it was shown in [8, Theorem 2.3] that the integral kernel of the operator $H(\mathcal{L})$ satisfies condition (15). For every function $f : \mathcal{V} \to \mathbb{C}$ we also define the gradient ∇f by the formula:

$$(\nabla f)(x) = \sum_{y \in \mathcal{V}: d(x,y)=1} |f(y) - f(x)| \qquad \forall x \in \mathcal{V}\,. \tag{19}$$

Then the first order Riesz transform $\nabla \mathcal{L}^{-1/2}$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$. Indeed, it was shown in [8, Theorem 2.3] that the integral kernel of this operator satisfies condition (15).

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