Considerations about the choice of layerwise and through-thickness global functions of 3-D physically-based zig-zag theories

Abstract

A generalization of physically-based fixed degrees of freedom 3-D zig-zag theories is developed, which allows for any arbitrary choice of layerwise and representation functions. Thereby users can choose arbitrarily the representation case-by-case. This paper aims to prove that the choice of global and layerwise functions is immaterial whenever coefficients are recalculated exactly (via symbolic calculus) by the enforcement of interfacial stress continuity, boundary conditions and equilibrium in point form, as prescribed by the elasticity theory. Vice versa, accuracy of theories partially fulfilling constraints will prove to largely depend on the assumptions made and to be inadequate when strong layerwise effects rise, under distributed/localized step loading and boundary conditions other than simply supported edges (tests carried out in closed-form). The present and previous authors' theories are tested with the aim to understand in which cases an adequate level of accuracy is still achieved by lower-order theories that are derived as particularizations.

Keywords

Physically-based zig-zag theories generalized formulation, Arbitrary through-thickness representation of global and layerwise functions with fixed d.o.f., Transverse shear and normal stresses, Laminated and soft-core sandwich plates and beams, Layers stiffness ratio, Analytical solutions

1. Introduction

As well known, laminated fibre-reinforced and sandwich composites offer excellent specific strength and stiffness, fatigue and energy absorption properties, better resistance to corrosion than metals and greater design flexibility. To ward off any possible catastrophic failure or intolerable loss of performance of these materials, an accurate prediction of through-thickness displacement, strain and stress fields is crucial [1]. Displacements have to be C° continuous at interfaces and with the suited slope (zig-zag effect) so that out-of-plane stresses are continuous and satisfy local equilibrium equations.

So far many laminated plate and shell theories with varying degrees of accuracy, computational costs and memory storage occupation have been developed, as explained in detail in the book by Reddy [2] and in the papers by Reddy and co-workers [1], [3], Vasilive and Lur'e [4], Lur'e, and Shumova [5], Noor et al. [6], Altenbach [7], Carrera [8-12], Qatu [13], Qatu et al. [14], Wanji and Zhen [15], Khandan et al. [16] and Kapuria and Nath [17].

Equivalent single-layer [18]-[21], discrete-layer [22], [23], zig-zag [11], [24], hierarchical [25], [26] and axiomatic/asymptotic [27], [28] theories can be categorized, which further subdivide into displacement-based and mixed formulations (displacements, strains and stress fields chosen separately).

Equivalents single layer theories are of limited validity, not even being able to predict overall behaviour quantities, but even this limited goal could be disregarded, as shown among many others by Icardi and Sola [29], Icardi and Urraci [30,31], Kapuria et al. [32], Zhen and Wanji [33], Burlayenko et al. [34] and Jun et al. [35] for cases with a strong transverse anisotropy. A forcibly schematic description like that of this article leads to summarize discrete-layer theories as potentially accurate irrespective of lay-up, property variation across the thickness, loading and boundary conditions, but also problematic because they could overwhelm the computational capacity when structures of industrial interest are analyzed, owing to their too much number of variables. Recent accurate, versatile and efficient theories like Carrera's unified formulation [8] and refined zig-zag theories [29-31,36] are instead suitable for analysis of industrial structures. Theory [8] allows displacements to take arbitrary forms that can be chosen by the user as an input of the analysis and it is able to get existing theories as particularizations. This paper intends to pursue the same purpose with a new generalized zig-zag theory.

Zig-zag theories subdivide into: (i) Di Sciuva's like physically-based theories (see [37-39] as examples), or (ii) Murakami's like kinematic-based mixed theories, (see, e.g. [11],[40], [41]), given the different type of zig.zag functions used. Both strike the right balance between accuracy and cost saving, allowing designers' demand of theories in a simple already accurate form to be met.

Theories (i) generally have a fixed number of unknowns irrespective of the number of constituent layers. Their layerwise contributions are the product of linear [37] (or nonlinear [42]) zig-zag functions and unknown zig-zag amplitudes, inferred by enforcing the fulfilment of stress continuity conditions at layer interfaces.

Theories (ii) assume zig-zag functions featuring a periodic change of sign of the slope of displacements at interfaces, as occurring for periodic laminations, therefore without taking into account orientation angle, material properties and thickness of constituent layers. Their number of variables (kinematic and stress quantities) depends upon the number of constituent/computational layers. They can be inaccurate [30, 31, 43, 44], but they enable a more easily obtainment of C° formulations of plate theories for the development of efficient finite elements. Mixed multilayered theories accurately predict stresses, with the merit of keeping simpler kinematics as is evident from the papers by Tessler et al. [36], Kim and Cho [45], Barut et al. [46], Iurlaro et al. [47] and Zhen and Wanji [48]. When strong layerwise effects rise, (ii) require many more degrees of freedom and/or a higher expansion order than (i), as it can be seen, for example, comparing the results of [30] and [49].

Efficient and accurate theories where zig-zag functions are not explicitly incorporated based on a global-local superposition of displacement fields were proposed by Li and Liu [50] and refined by Zhen and Wanji, e.g. [51], which in the version of Shariyat [52] consider a non-uniform transverse displacement across the thickness.

Theories with a hierarchical set of locally defined polynomials where neither zig-zag contributions are incorporated in the kinematics, nor post-processing steps are required (see, [25], [26], Catapano et al. [53]

and de Miguel et al. [54]) have been developed, for favouring numerical efficiency. Anyway, they require a quite large number of variables to mitigate the effects of the omitted enforcement of interfacial stress contact conditions, so to avoid small interfacial stress jumps.
Physically-based 3D zig-zag theories like [29-31] also allow a variable representation arbitrarily chosen by the user, without increasing their only 5 d.o.f. , while Strain Energy Updating Technique [55] can be used to obtain a C° formulation. Because their coefficients are redefined across the thickness, they can be counted among those with variable-kinematics (see, e.g. Vescovini and Dozio [56]).
So far, power series expansion, hierarchic polynomials, Taylor's series, trigonometric and exponential functions, a combination of both and radial basis functions (see [57] to [66]) have been used to represent variables across the thickness and a high sensitivity to the different assumptions made was shown.
Going into more detail regarding physically-based 3D zig-zag theories, in Icardi and Sola [29] a displacement-based laminated plate theory with piecewise cubic in-plane displacements, a fourth-order transverse displacement and only 5 d.o.f. (mid-plane displacements and rotations), referred as ZZA, was developed, whose coefficients are redefined across the thickness by imposing the fulfilment of elasticity theory constraints (interfacial stress compatibility conditions, stress boundary conditions and local equilibrium equations at arbitrary selected points across the thickness).
Variants of ZZA were developed in [30], assuming different layerwise functions, in order to evaluate how the different choices affect accuracy. For the same purpose, different forms of representation of global functions and various layerwise functions (as well as without them) were assumed in [67]-[71]. In an attempt of lowering the computational cost, a mixed variant of the already very efficient ZZA theory was developed in [31] within the framework of Hu-Washizu variational theorem, so to retain separately only the essential contributions of displacement, strain and stress fields. Theories [29,30,67-71], all developed by using symbolic calculus to exactly fulfil the elasticity theory constraints, were proven to be efficient and suitable to analyse challenging elastostatic and dynamic cases, including pumping modes which are usually solved through FEM. In effects, their accuracy was proven to be similar to that of discrete layer and layerwise models with a very high through-thickness expansion order of variables and so a larger computational burden.
A not negligible advantage of these theories is that there is no need to approximate loading with a series expansion, because its mathematical expression is used to exactly compute the work of external forces via symbolic calculus. Another advantage is that also in-plane displacement and stress continuity can be enforced, with the aim to analyse also structures with step variable properties along in-plane directions and not only across the thickness. Moreover, stresses of theories [29,30,67-71] are obtained from constitutive relations and there is no need to post-processing.
Summing up the features of theories [29,30,67-71], similarly to [50] - [52], it is not necessary to include specific zigzag functions to respect the interfacial out-of-plane stress compatibility, because a number of coefficients can be redefined for each layer for this purpose. However, unlike [50] - [52] and all conventional plate theories, in [29,30,67-71] all coefficients can be redefined across the thickness so that stress-boundary conditions are met at the outer layers and equilibrium can be satisfied in a strong point form within inner layers. Similarly to hierarchical and asymptotic theories, the representation can be

enriched in order to achieve a better accuracy, but the expansion order and the number of variables do not grow once the piecewise cubic-quartic representation adopted is fragmented into computational layers and/or changing the type of representation from layer to layer and differently for each displacement.
Unlike the theories that today are very popular for their accuracy and versatility (e.g. [8] and subsequent developments), [29,30,67-71] are formulated by respecting physical constraints exactly (from which the appellation of physically-based theories follows) so as to limit the computational burden, rather respecting them in a limit sense increasing the expansion order and / or the number of variables.
A generalization of previous 3-D zig-zag theories is developed in this paper, whose formulation allows user to arbitrarily choose layerwise and representation functions, also differently for each displacement and for each layer. Nevertheless any number of d.o.f. could be assumed (not necessarily just mid-plane displacements and rotations), it is here limited to five to carry out comparisons with ZZA and other previous theories under the same conditions. From the present generalized theory an approximate non-plate theory (number of d.o.f. not fixed that depends from the expansion order imposed) AT-3D is derived, which can be used as reference solution when exact one is unavailable. Given its generality, also hierarchical theories could be obtained by the present theory, which however won't be considered in this research.
Six theories different from those of [29]-[31] and [67]-[71] are particularized in (3.1) assuming sinusoidal, exponential, power series or a combination of them (which can be different for each displacement and from layer to layer) to describe the variation of displacements across the thickness.
Fourteen additional lower order theories are developed in (3.2)-(3.8) to test along with the former six if a plate theory with fixed d.o.f. can enable a kinematic-variable representation across the thickness requiring a lower computational burden than existing theories with same features, and to understand in which cases it is still allowed to reach an adequate level of accuracy. The results will show that a piecewise cubic and a fourth-order polynomial for in-plane and transverse displacements respectively achieve this goal. In one theory, equilibrium equations are enforced in weak form, instead than in a point form as for all others theories mentioned, in order to verify the degree of accuracy achievable. Results will confirm on a broader basis what preliminarily demonstrated in [29]-[31] and [67]-[71], that the choice of zig-zag and representation functions is immaterial if coefficients are redefined across the thickness and calculated by imposing full set of physical constraints.
Table 1 provides a quick reference pattern of all cases considered in the numerical applications and lists details on length-to-thickness ratio, lay-up, loading and boundary conditions. Mechanical properties of materials are in Table 2, Table 3 reports trial functions, expansion order and equilibrium points position across the thickness for each case, Table 4 contains normalizations, Table 5 records a brief description of theories, while the processing time is listed in Table 6.

2. Theoretical Framework

Notations and basic assumptions, that are common to all the theories, are discussed in the following section. Features of the new theories introduced in this paper are examined in details. Readers are referred to the literature quoted for previously developed theories.

2.1 Basic assumptions and main notations

This study is restricted to multilayered elastic plates subject to small deformations, as representative of laminated and sandwiches. Loading is assumed conservative, strains are assumed to be infinitesimal, while constituent layers are assumed to be linearly elastic, to have orthotropic properties, a uniform arbitrary thickness h^k and to be perfectly bonded to each other (bonding resin interlayer is disregarded). A rectangular, right-handed Cartesian coordinate reference system Ω on the middle reference plane is assumed as the reference frame. As easily allowed by symbolic calculus used to develop theories, Ω could be assumed differently to any other position, so to prevent an eventual zeroing of coefficients for certain lay-ups and theories. The thickness coordinate is ς ($\varsigma \in \left[-\frac{h}{2}, \frac{h}{2}\right]$, h being the overall thickness) and a comma is used to indicate spatial derivatives, e.g. $(.)_{,\alpha} = \partial(.)/\partial_\alpha$. Lα and Lβ symbolize the plate side-length in the α- and β-directions. Strains and stresses are indicated respectively as ε_{ij}, σ_{ij} $(i, j = 1,2,3;\ 1 \equiv \alpha, 2 \equiv \beta, 3 \equiv \zeta)$. The upper and lower positions of layer interfaces are indicated with ${}^{(k)}\varsigma^{+}$ and ${}^{(k)}\varsigma^{-}$, respectively, the superscript (or in other cases the subscript) *k* being used to indicate that a quantity belongs to the layer *k*. To indicate that a quantity is evaluated on upper and lower faces of the laminate, the markers *u* and *l* are used, respectively.

Except for a couple of theories for which the appropriate specifications will be given below, all remaining theories have the following five d.o.f. u_α^0, u_β^0, w^0, Γ_α^0, Γ_β^0, consisting of midle plane displacement components and shear rotations of the normal

2.2 Solutions search

Solution is searched in a closed form within the framework of Rayleigh-Ritz method, in conjunction with Lagrange multipliers method. Accordingly, d.o.f. are expressed as a truncated series expansion of unknown amplitudes and trial functions that individually satisfy the prescribed boundary conditions. The trial functions are explicitly defined in Table 3 for each specific case along with the expansion order and normalizations used. The same trial functions and expansion order are shared by all theories, in order to compare them under the same conditions. The methodology to satisfy the mechanical boundary conditions (transverse shear stress resultant force equals the constraint force and resultant couple of in-plane stresses, where they are not identically satisfied) based on Lagrange multipliers method is the same of [31], where readers can find all details here omitted. The derivation of governing equations will be omitted to contain the paper length because they are obtainable in a straightforward way with standard techniques.

2.3 ZZA displacement-based theory

Hereafter the main features of zig-zag adaptive theory (ZZA [29]) are summarized, it being the basis from which all theories developed and assessed in this paper evolved. The following through-thickness displacement field is postulated:

$$u_{\alpha}(\alpha,\beta,\varsigma)=[\ u_{\alpha}{}^{0}(\alpha,\beta)+\varsigma(\Gamma_{\alpha}^{0}(\alpha,\beta)-w^{0}(\alpha,\beta)_{,\alpha})\ ]_{0}+[\ F_{\alpha}^{u}(\alpha,\beta,\varsigma)\ ]_{i}+$$
$$[\ \sum_{k=1}^{n_{i}}\Phi_{\alpha}^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(z)+\sum_{j=1}^{n_{\Im}}{}_{\alpha}C_{u}^{j}(\alpha,\beta)H_{j}(\varsigma)\ ]_{c}$$

$$u_{\varsigma}(\alpha,\beta,\varsigma)=[\ w^{0}(\alpha,\beta)\ ]_{0}+[\ F^{\varsigma}(\alpha,\beta,\varsigma)\ ]_{i}+[\ \sum_{k=1}^{n_{i}}\Psi^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(\varsigma)+$$
$$\sum_{k=1}^{n_{i}}\Omega^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})^{2}H_{k}(\varsigma)+\sum_{j=1}^{n_{\Im}}C_{\varsigma}^{j}(\alpha,\beta)H_{j}(\varsigma)\ ]_{c} \tag{1}$$

Symbols n_i and $n_{\Im}$ are used to distinguish the number of physical interfaces from that of mathematical layer interfaces, respectively.

It could be noticed that linear-$[...]_0$, higher-$[...]_i$ and layerwise-$[...]_c$ contributions are incorporated, whose purpose is defined as follows. The first term $[...]_0$ introduces the functional degrees of freedom, while the second one $[...]_i$ contains higher-order contributions, while the third contribution is characteristic of physically-based zig-zag theories.

Although $[\ F_{\alpha}^{u}(\alpha,\beta,\zeta)\ ]_i$, $[\ F^{\zeta}(\alpha,\beta,\zeta)\ ]_i$ could be chosen entirely arbitrarily, the following power series expansion is chosen:

$$[\ F_{\alpha}^{u}(\alpha,\beta,\zeta)\ ]_{i}=[\ C_{\alpha}^{i}(\alpha,\beta)\zeta^{2}+D_{\alpha}^{i}(\alpha,\beta)\zeta^{3}+(O\zeta^{4}...)\ ]_{i}=[\ {}_{3}(\tilde{.})_{\alpha}\ ]_{i}+[\ (O\zeta^{4}...)\ ]_{i}=$$
$$=[A_{\alpha2}(\alpha,\beta)\zeta^{2}+A_{\alpha3}(\alpha,\beta)\zeta^{3}]+A_{\alpha4}(\alpha,\beta)\zeta^{4}+...+A_{\alpha n}(\alpha,\beta)\zeta^{n} \tag{2}$$

$$[\ F^{\zeta}(\alpha,\beta,\zeta)\ ]_{i}=[\ b^{i}(\alpha,\beta)\zeta+c^{i}(\alpha,\beta)\zeta^{2}+d^{i}(\alpha,\beta)\zeta^{3}+e^{i}(\alpha,\beta)\zeta^{4}+(O\zeta^{5}...)\ ]_{i}=$$
$$=[\ {}_{4}(\tilde{.})_{\zeta}\ ]_{i}+[\ (O\zeta^{5}...)\ ]_{i}=[A_{\varsigma1}(\alpha,\beta)\zeta+A_{\varsigma2}(\alpha,\beta)\zeta^{2}+A_{\varsigma3}(\alpha,\beta)\zeta^{3}+$$
$$+A_{\varsigma4}(\alpha,\beta)\zeta^{4}]+A_{\varsigma5}(\alpha,\beta)\zeta^{5}+...+A_{\varsigma n}(\alpha,\beta)\zeta^{n}$$

so to include theory [42] as a particularization of ZZA. Contributions $[\ (Oz^{4}...)\ ]_i$, $[\ (Oz^{5}...)\ ]_i$ are characteristic of ZZA, while $[\ {}_{3}(\tilde{.})_{\alpha}\ ]_i$, $[\ {}_{4}(\tilde{.})_{\zeta}\ ]_i$ are the same as in [42]. Note that functions $[\ F_{\alpha}^{u}\ ]_i$, $[\ F^{\zeta}\ ]_i$ aren't just depending on ς because apexes and subscripts ${}_{\alpha}^{u}$, ${}^{\varsigma}$ represent the functional dependence on the d.o.f. that are function of in-plane coordinates. Expressions of coefficients C_{α}^{i}, D_{α}^{i}, b^i to e^i are obtained by enforcing the fulfilment of stress boundary conditions:

$$\sigma_{\alpha\varsigma} = \sigma_{\varsigma\varsigma,\varsigma} = 0; \quad \sigma_{\varsigma\varsigma} = p^0(\pm) \tag{3}$$

$p^0(\pm)$ being the distributed loading acting on at the upper (+) and lower (-) bounding faces, and of local equilibrium equations:

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha\varsigma,\varsigma} = b_\alpha; \quad \sigma_{\alpha\varsigma,\alpha} + \sigma_{\varsigma\varsigma,\varsigma} = b_\varsigma \tag{4}$$

at points across the thickness, positioned where residual are the largest. The through-thickness redefinition of coefficients by (3) and (4) determines the adaptive appellation attributed to ZZA theory. Applied distributed loadings are managed via symbolic calculus like general (continuous or discontinuous) functions $\chi(\alpha,\beta)$ acting at upper and/or lower faces, so it can be avoided to express them as a series expression that could limit accuracy.

Layerwise contributions $[...]_c$ allow a priori fulfilment of stress compatibility conditions at physical and mathematical layer interfaces, so Φ_α^k, Ψ^k, Ω^k are determined by imposing

$$\sigma_{\alpha\varsigma}({}^{(k)}z^+) = \sigma_{\alpha\varsigma}({}^{(k)}z^-); \quad \sigma_{\varsigma\varsigma}({}^{(k)}z^+) = \sigma_{\varsigma\varsigma}({}^{(k)}z^-); \quad \sigma_{\varsigma\varsigma,\varsigma}({}^{(k)}z^+) = \sigma_{\varsigma\varsigma,\varsigma}({}^{(k)}z^-) \tag{5}$$

Here - and + indicate the position just before and just after the interface, respectively). Contributions ${}_\alpha C_u^j H_j$ and $C_\varsigma^j H_j$ in (1) restore the continuity of displacements:

$$u_\alpha({}^{(k)}z^+) = u_\alpha({}^{(k)}z^-); \quad u_\varsigma({}^{(k)}z^+) = u_\varsigma({}^{(k)}z^-) \tag{6}$$

since (2) which is contained within $[...]_i$ can be assumed differently from point to point across the thickness. At the end of this brief discussion it is reminded that symbolic calculus enables that expressions of coefficients and zig-zag amplitudes to be obtained in exact form and once and for all. It is also reminded that just a third/fourth order representation is required to obtain accurate results [30].

In order to assess whether and when the choice of zig-zag functions can be immaterial, in the numerical applications a new variant of ZZA called ZZA_MHR is considered, wherein Murakami's zig-zag function multiplied for amplitudes that are recalculated across the thickness are assumed as the layerwise functions, instead of $[...]_c$ by ZZA.

2.4 Previously developed theories that are considered for sake of comparisons

They comprise theories HWZZ [31], HWZZM [30], MHR [31], ZZA* [30] and ZZ [42], whose features are briefly summarized as follows. HWZZ is a mixed HW theory [3,4], neglecting quadratic zig-zag and higher-order adaptive contribution of (1) from displacement field, out-of-plane strains are the same of (1), while out-of-plane stresses by integrating local equilibrium equation. HWZZM has the same features of HWZZ, except zig-zag functions which are modified versions of Murakami's layrwise functions with amplitudes that are redefined by enforcing (5).

MHR assumes a piecewise cubic in-plane displacement including original Murakami's layerwise functions, namely amplitudes are not redefined across the thickness, and a fourth-order polynomial transverse displacement. ZZA* has the same features of ZZA, except zig-zag functions are replaced by appropriate additional contributions whose amplitudes are redefined across the thickness. ZZ is based on a piecewise cubic in-plane representation a piecewise fourth-order for transverse displacement, the same as (1), except that $[...]_i$ are not redefined across the thickness.

3. New theories of this paper

New theories, which constitutes the main theoretical contribution of this paper, are proposed as a generalization of ZZA and all previously developed theories by the authors [29-31, 42, 55, 68-71], in order to prove the objective set in the introductory section. Initially, the displacement field is thought in the following form:

$$u_\alpha{}^j = \sum_{i=0}^{n_\alpha=3} \left[{}^jC_\alpha^i(\alpha,\beta) \;\; F_\alpha^i(\varsigma) \right]$$
$$u_\varsigma{}^j = \sum_{i=0}^{n_\varsigma=4} \left[{}^jC_\varsigma^i(\alpha,\beta) \;\; G^i(\varsigma) \right] \tag{7}$$

that does not contain zig-zag functions because coefficients are redefined across the thickness, being calculated by imposing the full set of physical constraints (3)-(6) in order to get accurate results using a low order of expansion. Subsequently, particularizations are developed from (7) in the next section (3.1). Symbol j represents the mathematical layers, so, ${}^jC_\varsigma^i$, ${}^jC_\alpha^i$, $F_\alpha^i(\varsigma)$ and $G^i(\varsigma)$ can be represented differently across the thickness. Coefficients ${}^1C_\alpha^0$, ${}^1C_\alpha^1$ and ${}^1C_\varsigma^0$ are assumed as d.o.f. To obtain (1) as a particularization of (7) it is necessary that ${}^1C_\alpha^0 = u_\alpha^0$, ${}^1C_\alpha^1 = \Gamma_\alpha^0 - w_{,\alpha}^0$ and ${}^1C_\varsigma^0 = w^0$ ($F^0(\varsigma) = G^0(\varsigma) = 1$, $F^1(\varsigma) = \varsigma$), while the remaining terms are calculated by enforcing physical constraints (3)-(6). Of course, the expansion order n_α and n_ς, which reflects in the number of unknowns from one interface (physical or computational) to another, must be chosen in accordance with the physical constraints to be imposed. Once the interface has passed, the coefficients are redefined and then it is possible that the representation changes. Functions $F^i(\varsigma)$ and $G^i(\varsigma)$ can be freely chosen by the user (symbolic calculations being performed automatically regardless of the choices made).

In numerical applications, theory AT-3D with $F^i(\varsigma) = \varsigma^i$ and $G^i(\varsigma) = \varsigma^i$ is considered, wherein n_α and n_ς can be arbitrarily assumed, but anyway so that ${}^jC_\alpha^i$ and ${}^jC_\varsigma^i$ are in number greater than physical constraints (3)-(6). In the form described in this section, (7) constitutes a 3-D theory which owing to its generality can be used as a surrogate of the exact solution when it is not available [71].

3.1 Particularizations with u_α^0, $\Gamma_\alpha^0 - w_{,\alpha}^0$, w^0 as d.o.f.

Particularizations of (7) having same d.o.f. of ZZA are developed in this section assuming $F^i(\varsigma)$ and $G^i(\varsigma)$ as a mixture of trigonometric and exponential functions randomly selected for each theory, to demonstrate that their choice is immaterial whenever (3)-(6) are enforced and exact relations are found from such enforcement via symbolic calculus.

Theory name	Function	thickness range	
NOZZG	$F_\alpha^i(\varsigma)=G^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ e^{(i\,\varsigma_j/h_j)} & \text{for } i>1\end{cases}$	$(-h/2\le\varsigma\le h/2)$	(8a)
ZZA_PP34	$F_\alpha^i(\varsigma)=G^i(\varsigma)=(\varsigma)^i$	$(-h/2\le\varsigma\le h/2)$	(8b)
ZZA_PT34	$F_\alpha^i(\varsigma)=G^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ \cos(i\pi\varsigma/2h) & \text{for } i=2,4\\ \sin((i+1)\pi\varsigma/2h) & \text{for } i=3\end{cases}$	$(-h/2\le\varsigma\le h/2)$	(8c)
ZZA_PM34	$F_\alpha^i(\varsigma)=G^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ \exp(\varsigma/h) & \text{for } i=2\\ \sin(\pi\varsigma/2h) & \text{for } i=3\\ \cos(\pi\varsigma/2h) & \text{for } i=4\end{cases}$	$(-h/2\le\varsigma\le h/2)$	(8d)
ZZA_PMTP34	$F_\alpha^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ \exp(\varsigma/h) & \text{for } i=2\\ \sin(\pi\varsigma/2h) & \text{for } i=3\\ \cos(\pi\varsigma/2h) & \text{for } i=4\end{cases}$	$(-h/2\le\varsigma\le h/2)$	(8e)
	$F_\beta^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ \cos(i\pi\varsigma/2h) & \text{for } i=2,4\\ \sin((i+1)\pi\varsigma/2h) & \text{for } i=3\end{cases}$	$(-h/2\le\varsigma\le h/2)$	
	$G^i(\varsigma)=(\varsigma)^i$	$(-h/2\le\varsigma\le h/2)$	
ZZA_PPM34	$F_\alpha^i(\varsigma)=G^i(\varsigma)=(\varsigma)^i$	$(-h/2\le\varsigma\le -0.45h)$	(8f)
	$F_\alpha^i(\varsigma)=G^i(\varsigma)=\begin{cases}1 & \text{for } i=0\\ \varsigma & \text{for } i=1\\ \exp(\varsigma/h) & \text{for } i=2\\ \sin(\pi\varsigma/2h) & \text{for } i=3\\ \cos(\pi\varsigma/2h) & \text{for } i=4\end{cases}$	$(-0.45h<\varsigma\le 0.4h)$	
	$F_\alpha^i(\varsigma)=G^i(\varsigma)=(\varsigma)^i$	$(0.4h<\varsigma\le h/2)$	

Theories that only partially satisfy physical constraints are set hereafter, which are considered for the purpose of comparisons.

3.2 ZZAM_P3P4 theory

It has the following representation form:

$$u_{\alpha}(\alpha,\beta,\varsigma)=[\ u_{\alpha}{}^{0}(\alpha,\beta)+\varsigma(\Gamma_{\alpha}^{0}(\alpha,\beta)-w^{0}(\alpha,\beta)_{,\alpha})\ ]_{0}+[\ \sum_{k=1}^{3}C_{k\_\alpha}^{i}(\alpha,\beta)F^{\alpha}{}_{k}(\varsigma)+C_{\alpha}^{i}\ ]_{i+c}$$
$$u_{\varsigma}(\alpha,\beta,\varsigma)=[\ w^{0}(\alpha,\beta)\ ]_{0}+[\ \sum_{k=1}^{4}D_{k}^{i}(\alpha,\beta)G_{k}(\varsigma)+C_{\varsigma}^{i}\ ]_{i+c}$$
$$C_{k\_\alpha}^{i=1}=0;\quad C_{\alpha}^{i=1}=0;\quad C_{\varsigma}^{i=1}=0;\quad F^{\alpha}{}_{k}(\varsigma)=G_{k}(\varsigma)=(\varsigma)^{k} \qquad (9)$$

It does not contain zig-zag contributions, but still uses a power function representation and coefficients are still redefined across the thickness through the enforcement of (3)-(6). The substantial difference with respect to all previous theories is that equilibrium conditions are imposed in integral form:

$$\int_{\varsigma_i^-}^{\varsigma_i^+}(\sigma_{\alpha\beta,\beta}+\sigma_{\alpha\varsigma,\varsigma})\Psi_{\alpha}d\varsigma \quad \int_{\varsigma_i^-}^{\varsigma_i^+}(\sigma_{\alpha\varsigma,\alpha}+\sigma_{\varsigma\varsigma,\varsigma})\Psi_{\varsigma}d\varsigma \qquad (10)$$

so, is no longer punctual, that is computationally more advantageous.

3.3 Theory PP23

This theory assume a lower-order of expansion, because less equilibrium points are considered (in-plane displacements are parabolic, while the transverse one is cubic).For the rest, coefficients are redefined across the thickness by enforcing (3)-(6).

$$u_{\alpha}(\alpha,\beta,\varsigma)=[\ u_{\alpha}{}^{0}(\alpha,\beta)+\varsigma(\Gamma_{\alpha}^{0}(\alpha,\beta)-w^{0}(\alpha,\beta)_{,\alpha})\ ]_{0}+[\ \sum_{k=1}^{2}C_{k\_\alpha}^{i}(\alpha,\beta)F^{\alpha}{}_{k}(\varsigma)+C_{\alpha}^{i}\ ]_{i+c}$$
$$u_{\varsigma}(\alpha,\beta,\varsigma)=[\ w^{0}(\alpha,\beta)\ ]_{0}+[\ \sum_{k=1}^{3}D_{k}^{i}(\alpha,\beta)G_{k}(\varsigma)+C_{\varsigma}^{i}\ ]_{i+c} \qquad (11)$$
$$C_{k\_\alpha}^{i=1}=0;\quad C_{\alpha}^{i=1}=0;\quad C_{\varsigma}^{i=1}=0;\quad F^{\alpha}{}_{k}(\varsigma)=G_{k}(\varsigma)=(\varsigma)^{k}$$

3.4 ZS1, ZS1_1, ZS1_2 , ZS1_3 and ZS1_4

For these theories with a partial fulfilment of (3)-(6), various contributions are cut off, in order to highlight whether they are still accurate.

- ZS1.

$$u_{\alpha}(\alpha,\beta,\varsigma)=[\ u_{\alpha}{}^{0}(\alpha,\beta)+\varsigma(\Gamma_{\alpha}^{0}(\alpha,\beta)-w^{0}(\alpha,\beta)_{,\alpha})\ ]_{0}+\sum_{k=1}^{n_i}\Phi_{\alpha}^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(\varsigma)$$
$$u_{\varsigma}(\alpha,\beta,\varsigma)=[\ w^{0}(\alpha,\beta)\ ]_{0}+\sum_{k=1}^{n_i}\Psi^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(\varsigma)+\sum_{k=1}^{n_i}\Omega^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})^{2}H_{k}(\varsigma) \qquad (12)$$

Zig-zag amplitudes Φ_{α}^{k}, Ψ^{k} and Ω^{k} are calculated by imposing the fulfilment of stresses compatibility (5) at interfaces Therefore the only redefined coefficients are zig-zag amplitudes.

- ZS1_1. Layerwise functions of (12) are substituted with $C_{\alpha}^{i}(\alpha,\beta)\varsigma$, $\Psi^{i}(\alpha,\beta)\varsigma$ and $\Omega^{i}(\alpha,\beta)\varsigma^{2}$,

- ZS1_2. It is developed as a mixed variant of ZS1 wherein the same displacements are assumed but out-of-plane stresses are assumed apart by integrating local equilibrium equations, within the framework of HR variational theorem.

- ZS1_3. Is a mixed HR theory derived from ZS1 assuming the same displacement field, while out-of-plane stresses are assumed apart being those coming from ZZAM_P3P4.

- ZS1_4. Considers the same displacement field of ZS1_1 but different Γ_α^0 d.o.f. are used, that are referred to the top face instead of middle reference plane:

$$
\begin{aligned}
u_\alpha(\alpha,\beta,\varsigma) &= [\ u_\alpha{}^0(\alpha,\beta)+(\varsigma-h/2)\Gamma_\alpha^0(\alpha,\beta)-\varsigma\ w^0(\alpha,\beta)_{,\alpha}\ ]_0 + C_\alpha^i(\alpha,\beta)\varsigma + D_\alpha^i \\
u_\varsigma(\alpha,\beta,\varsigma) &= [\ w^0(\alpha,\beta)\ ]_0 + \Psi^i(\alpha,\beta)\varsigma + \Omega^i(\alpha,\beta)\varsigma^2 + C_\varsigma^i \qquad (C_\alpha^{i=1}=0)
\end{aligned}
\tag{13}
$$

3.5 Theory ZS2

Displacement field is:

$$
\begin{aligned}
u_\alpha(\alpha,\beta,\varsigma) &= [\ u_\alpha{}^i(\alpha,\beta)\ L_1(\varsigma)+u_\alpha{}^{i+1}(\alpha,\beta)\ L_2(\varsigma)\ ]_0 \\
u_\varsigma(\alpha,\beta,\varsigma) &= [\ w^i(\alpha,\beta)\ L_1(\varsigma)+w^{i+1}(\alpha,\beta)\ L_2(\varsigma)\ ]_0 + \Omega^k(\alpha,\beta)\varsigma^2
\end{aligned}
\tag{14}
$$

L1 and L2 being linear Lagrange polynomials

$$
L_1(\varsigma)=1-\frac{\varsigma-\varsigma^i{}_b}{\varsigma^i{}_t-\varsigma^i{}_b}; \qquad L_2(\varsigma)=\frac{\varsigma-\varsigma^i{}_b}{\varsigma^i{}_t-\varsigma^i{}_b}
\tag{15}
$$

The suffix b and t are used to indicate upper and lower coordinates of each generic lamina i and displacements at upper and lower bounding faces ($u_\alpha{}^{top}=u_\alpha{}^{n_l}$, $u_\alpha{}^{bottom}=u_\alpha{}^1$, $w^{top}=w^{n_l}$, $w^{bottom}=w^1$, n_l number of computational layers) are assumed as the functional d.o.f. So, other coefficients $u_\alpha{}^i$ and w^i are obtained as functions of $u_\alpha{}^{top}$, $u_\alpha{}^{bottom}$, w^{top} , w^{bottom} by imposing (5).

3.6 Theory ZS3

The displacement field is structured as follows:

$$
\begin{aligned}
u_\alpha(\alpha,\beta,\varsigma) &= [\ u_\alpha{}^0(\alpha,\beta)+\varsigma(\Gamma_\alpha^0(\alpha,\beta)-w^0(\alpha,\beta)_{,\alpha})\ ]_0 + C_\alpha^i(\alpha,\beta)\varsigma + D_\alpha^i(\alpha,\beta)\varsigma^2 + C_{\alpha\_u}^i \\
u_\varsigma(\alpha,\beta,\varsigma) &= [\ w^0(\alpha,\beta)\ ]_0
\end{aligned}
\tag{16}
$$

Terms C_α^i D_α^i are obtained by imposing (3)-(6), as well as the boundary condition on transverse shear $T=\int_{-h/2}^{h/2}\sigma_{\alpha\varsigma}d\varsigma$, as considered in certain theories in literature while, $C_{\alpha\_u}^i$ by imposing (6).

3.7 Theories ZS3_1 and ZS3_2

Theory ZS3_1 assumes a piecewise parabolic representation for all displacements, free of zig-zag contributions, whose coefficients are calculated even imposing (5)-(6):

$$
\begin{aligned}
u_\alpha(\alpha,\beta,\varsigma) &= [\ u_\alpha{}^0(\alpha,\beta)+\varsigma(\Gamma_\alpha^0(\alpha,\beta)-w^0(\alpha,\beta)_{,\alpha})\ ]_0 + C_\alpha^i(\alpha,\beta)\varsigma + D_\alpha^i(\alpha,\beta)\varsigma^2 + C_{\alpha\_u}^i \\
u_\varsigma(\alpha,\beta,\varsigma) &= [\ w^0(\alpha,\beta)\ ]_0 + \Psi^i(\alpha,\beta)\varsigma + \Omega^i(\alpha,\beta)\varsigma^2 + C_\varsigma^i
\end{aligned}
\tag{17}
$$

Theory ZS3_2 is developed assuming the same displacements field of ZS3_1 but out-of-plane stresses are assumed apart by integrating local equilibrium equations.

3.8 Theories ZZAS1 to ZZAS4

The in-plane displacement is assumed linear while the transverse displacement is uniform for layers ($i = i_j$). For $i \neq i_j$ they are assumed as the same of (1).

$$u_\alpha(\alpha,\beta,\varsigma) = [\ u_\alpha{}^0(\alpha,\beta) + \varsigma(\Gamma^0_\alpha(\alpha,\beta) - w^0(\alpha,\beta)_{,\alpha})\ ]_0 + \\ + \begin{cases} [\ F^u_\alpha(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Phi^k_\alpha(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \sum_{k=1}^{n_i}{}_\alpha C^k_u(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ +C^i_\alpha(\alpha,\beta)\varsigma + B^i_\alpha(\alpha,\beta) & \text{if } i = i_j \end{cases}$$

$$u_\varsigma(\alpha,\beta,\varsigma) = [\ w^0(\alpha,\beta)\ ]_0 + \\ + \begin{cases} [\ F^\varsigma(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Psi^k(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \\ \qquad + \sum_{k=1}^{n_i} \Omega^k(\alpha,\beta)(\varsigma-\varsigma_k)^2 H_k(\varsigma) + \sum_{k=1}^{n_i} C^k_\varsigma(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ B^i_\varsigma(\alpha,\beta) & \text{if } i = i_j \end{cases} \tag{18}$$

ZZAS2 assumes a parabolic transverse displacement for layers $i = i_j$, while the same in-plane displacement of ZZAS1 is maintained:

$$u_\alpha(\alpha,\beta,\varsigma) = [\ u_\alpha{}^0(\alpha,\beta) + \varsigma(\Gamma^0_\alpha(\alpha,\beta) - w^0(\alpha,\beta)_{,\alpha})\ ]_0 + \\ + \begin{cases} [\ F^u_\alpha(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Phi^k_\alpha(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \sum_{k=1}^{n_i}{}_\alpha C^k_u(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ +C^i_\alpha(\alpha,\beta)\varsigma + B^i_\alpha(\alpha,\beta) & \text{if } i = i_j \end{cases}$$

$$u_\varsigma(\alpha,\beta,\varsigma) = [\ w^0(\alpha,\beta)\ ]_0 + \\ + \begin{cases} [\ F^\varsigma(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Psi^k(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \\ \qquad + \sum_{k=1}^{n_i} \Omega^k(\alpha,\beta)(\varsigma-\varsigma_k)^2 H_k(\varsigma) + \sum_{k=1}^{n_i} C^k_\varsigma(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ C^i_\varsigma(\alpha,\beta)\varsigma + D^i_\varsigma(\alpha,\beta)\varsigma^2 + B^i_\varsigma(\alpha,\beta) & \text{if } i = i_j \end{cases} \tag{19}$$

ZZAS3 assumes a parabolic-cubic displacements field for i_j layers and the same in-plane field of ZZAS1 and ZZAS2:

$$u_\alpha(\alpha,\beta,\varsigma) = [\ u_\alpha{}^0(\alpha,\beta) + \varsigma(\Gamma^0_\alpha(\alpha,\beta) - w^0(\alpha,\beta)_{,\alpha})\ ]_0 + \\ + \begin{cases} [\ F^u_\alpha(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Phi^k_\alpha(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \sum_{k=1}^{n_i}{}_\alpha C^k_u(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ C^i_\alpha(\alpha,\beta)\varsigma + D^i_\alpha(\alpha,\beta)\varsigma^2 + B^i_\alpha(\alpha,\beta) & \text{if } i = i_j \end{cases}$$

$$u_\varsigma(\alpha,\beta,\varsigma) = [\ w^0(\alpha,\beta)\ ]_0 + \\ + \begin{cases} [\ F^\varsigma(\alpha,\beta,\varsigma)\ ]_i + [\ \sum_{k=1}^{n_i} \Psi^k(\alpha,\beta)(\varsigma-\varsigma_k)H_k(\varsigma) + \\ \qquad + \sum_{k=1}^{n_i} \Omega^k(\alpha,\beta)(\varsigma-\varsigma_k)^2 H_k(\varsigma) + \sum_{k=1}^{n_i} C^k_\varsigma(\alpha,\beta)H_k(\varsigma)\ ]_c & \text{if } i \neq i_j \\ C^i_\varsigma(\alpha,\beta)\varsigma + D^i_\varsigma(\alpha,\beta)\varsigma^2 + E^i_\varsigma(\alpha,\beta)\varsigma^3 + B^i_\varsigma(\alpha,\beta) & \text{if } i = i_j \end{cases} \tag{20}$$

Because of limitative assumptions, theories from 3.3 to the previous one may turn out to be inaccurate. ZZAS4 is developed considering the same representation of displacements as in section 2.3 for layers $i \neq i_j$, while for $i = i_j$ zig-zag functions are omitted, their sole being played by C_{α}^{i} to B_{ς}^{i}:

$$u_{\alpha}(\alpha,\beta,\varsigma)=[\ u_{\alpha}{}^{0}(\alpha,\beta)+\varsigma(\Gamma_{\alpha}^{0}(\alpha,\beta)-w^{0}(\alpha,\beta)_{,\alpha})\ ]_{0}+$$

$$+\begin{cases}[\ F_{\alpha}^{u}(\alpha,\beta,\varsigma)\ ]_{i}+[\ \sum_{k=1}^{n_i}\Phi_{\alpha}^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(\varsigma)+\sum_{k=1}^{n_i}{}_{\alpha}C_{u}^{k}(\alpha,\beta)H_{k}(\varsigma)\ ]_{c} & \text{if } i\neq i_j\\ C_{\alpha}^{i}(\alpha,\beta)\varsigma+D_{\alpha}^{i}(\alpha,\beta)\varsigma^{2}+E_{\alpha}^{i}(\alpha,\beta)\varsigma^{3}+B_{\alpha}^{i}(\alpha,\beta) & \text{if } i=i_j\end{cases}$$

(21)

$$u_{\varsigma}(\alpha,\beta,\varsigma)=[\ w^{0}(\alpha,\beta)\ ]_{0}+$$

$$+\begin{cases}[\ F^{\varsigma}(\alpha,\beta,\varsigma)\ ]_{i}+[\ \sum_{k=1}^{n_i}\Psi^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})H_{k}(\varsigma)+\\ \qquad +\sum_{k=1}^{n_i}\Omega^{k}(\alpha,\beta)(\varsigma-\varsigma_{k})^{2}H_{k}(\varsigma)+\sum_{k=1}^{n_i}C_{\varsigma}^{k}(\alpha,\beta)H_{k}(\varsigma)\ ]_{c} & \text{if } i\neq i_j\\ C_{\varsigma}^{i}(\alpha,\beta)\varsigma+D_{\varsigma}^{i}(\alpha,\beta)\varsigma^{2}+E_{\varsigma}^{i}(\alpha,\beta)\varsigma^{3}+F_{\varsigma}^{i}(\alpha,\beta)\varsigma^{4}+B_{\varsigma}^{i}(\alpha,\beta) & \text{if } i=i_j\end{cases}$$

Unlike the previous theories, the full set of physical constraints (3)-(6)is imposed in (21), so, there is no loss of accuracy.

In conclusion, a generalization of physically-based theories has been proposed, that enables users to arbitrarily choose representation and layerwise functions, without affecting the accuracy of the results. On the contrary, numerical results will show that this choice strongly influence the accuracy of theories that partially satisfy physical constraints, according to [57] to [66].

4. Numerical assessments and discussion

Accuracy of previous theories is assessed considering different benchmarks, loading and boundary conditions, some of which exhibit strong layerwise effects. The aim is to evaluate whether accuracy of results is independent on the choice of global and layerwise functions. Conversely, it will be shown that accuracy of theories only partially satisfy physical constraints, is highly dependent on the choices made. Low length-to-thickness ratios with strong layerwise effects in most cases, but also slender ones are analysed, so to assess if findings hold in general. FEA-3D results [72] are used as reference solutions in addition to exact ones where available.

4.1 Cases a and b

They concern [90/0] [53] and [0/90/0] [73] simply-supported beams under sinusoidal loading, respectively, so, to assess the correct implementation of theories and to preliminarily prove their accuracy under mild layerwise effects. The results of Tables 7 and 8 demonstrate that the choice of the zig-zag functions is immaterial and can even be omitted, since adaptive theories ZZA, ZZAS4, NOZZG, ZZA_PP34, ZZA_PT34, ZZA_PM34, ZZA_PMTP34, ZZA_PPM34, ZZA *, HWZZ, HWZZM, ZZA_MHR, AT-3D whose coefficients are redefined by full satisfying (3)-(6) are indistinguishable from one another. Considering for example u_{α} at ς=-h/2, the results of the theories are in sequence

4.5486,4.5486, 4.4951, 4.5486, 4.5220, 4.5479, 4.5522, 4.5522, 4.5486, 4.5261, 4.5261, 4.5486, 4.5159. Consequently, they are no longer reported individually in subsequent cases.
It could be noticed that an increase in the number of degrees of freedom as in AT-3D, does not constitute any advantage in terms of accuracy in the case of adaptive theories, hence the maximum accuracy degree has already been obtained through the redefinition of the coefficients. Even theories with a uniform transverse displacement appear rather accurate for these cases. Because of results of $\sigma_{\varsigma\varsigma}$ are very accurately predicted by all theories, they are not reported in Tables 7 and 8. In these and following tables, the through thickness representation of a quantity, for which the greatest discrepancies among the theories occur, is reported as an example. Results reported as percentage errors are calculated as $\left|\left(q_{ref}-q_{th}\right)/q_{ref}\right|$; $q_{ref}=q_{exact/FEA}$. Accuracy of theories ZZAS1, ZZAS2 and ZZAS3 is strongly dependent by the choice of the representation form and of zig-zag functions as well as of their position across the thickness. Indeed, ZZAS1 (ij=2) is not reported for case b, because it is too inaccurate.

4.2 Cases c to h

Case c and d, retaken respectively from [74] and [44] concern simply-supported laminated beams under sinusoidal loading. Case c is a double core sandwich beam with two thick weak cores and laminated faces, which is simulated as a [(0/90/0) / 0 / (0/90/0) / 0 / (0/90/0)] laminate. This case is selected because in [74] it is shown that it cannot be simulated by equivalent single layer theories and so, it is suited for checking if only theories of sections 2.3 to 3.1 or even lower order ones of sections 3.2 to 3.9 could be adequate. Case d is a [0/90/0_5/90] laminate whose displacements do not satisfy Murakami's rule, so kinematic-based models are not adequate for this case [31].
Cases e and f, retaken from [49] and [31] concern simply supported rectangular sandwich plates under bisinusoidal loading. To enhance layerwise effects, case f has a damaged lower face (components of tensor of elastic moduli E_{1111}, E_{1122}, E_{2222}, E_{1212} reduced by a factor $2\cdot10^{-1}$) and a rather stiffer core. Case g is a propped cantilever sandwich beam under a uniform load retaken from [31], whose peculiarity is to require an accurate piecewise description of the transverse displacement at the supported edge, otherwise stresses are misestimate. Even though a length-to-thickness ratio of 20 is considered, still strong layerwise effects as shown in [31], so, equivalent single layers and other lower-order theories cannot be adequate. Case h [31] is a simply-supported square sandwich plate, that because of a uniform local loading on the upper face, suffers from strong layerwise effects and strongly asymmetric transverse shear stresses across the thickness. Because of lay-up, loading and boundary conditions are symmetric along in-plane directions, the following equalities $u_\alpha=u_\beta$, $\sigma_{\alpha\alpha}=\sigma_{\beta\beta}$, $\sigma_{\alpha\varsigma}=\sigma_{\beta\varsigma}$ apply.
Regarding case c, adaptive theories are still accurate, while lower-order theories make mistakes from 7 to 55%, except PP23, ZZAS2 (ij=6), ZZAS3 (ij=6), ZZAS1 to ZZAS3 (ij>6), whose errors range from 0.5 to 5.6%. For brevity, these results are not reported.
The results of Tables 9 to 13, that refer to cases d to h, confirm the previous findings, about adaptive and lower order theories. Because of too high percentage errors, ZZAS1 with ij=2 is not reported in Table 10, ZS3_2 in Tables 10 and 11, ZS1 and PP23 in Table 12. Regarding case e, it should be noticed that most

of lower-order theories, quite accurately predict displacements and stresses in the top face, while quantities in the bottom face are imprecise, because of their simple representation. It could be noticed that ZS1 and ZS1_1 obtain good predictions of $\sigma_{\alpha\varsigma}$ (even if less accurate than adaptive theories) for cases d and e, but not for cases f to h and beyond up to m. Although ZS1 contains zig-zag functions and ZS1_1 is devoid of them, their results are indistinguishable, so, it is demonstrated that the choice of zig-zag functions is immaterial. ZS1_3 obtains better results than ZS1, ZS2 accurately calculate stresses for cases d, e and g, while ZS3, ZS3_1 and ZS3_2 are accurate only for case g. Theory PP23 is not adequate for cases f to h, while ZZAS1, ZZAS2 and ZZAS3 are unreliable for cases e, g and h.

4.3 Case i

A simply-supported eleven-layers sandwich beam under sinusoidal loading, is considered, whose laminated faces are made of layers with different thickness and material properties [29], [42]. It represents a very severe test for theories because of the low elastic modulus of the lower face (E_{3333} reduced by a factor of 10^{-2}). The results of Figure 1, show that only higher-order adaptive theories can obtain accurate results. Whether the exact solution is not available, 3-D FEA is used as reference solution, here as well as in subsequent cases.

Assumption of uniform transverse displacement is now totally inadequate because anyway stress fields are wrong, even for mixed theories. Redefinition of coefficients is reconfirmed to improve precision, as well as zig-zag layerwise contributions can be explicitly omitted and accuracy of theories PP23, ZS1, ZS1_1, ZS1_2, ZS1_3, ZS1_4, ZS2, ZS3, ZS3_1, ZS3_2, ZZAS1, ZZAS2, ZZAS3 with a partial fulfilment of physical constraints is lost. For the sake of that, they will no longer be reported next, where stronger layerwise effects rise, being too inaccurate. The comparison between MHR and ZZA_MHR, shows the advantages of redefining amplitudes even when using Murakami's zig-zag function. For case i and subsequent j to m, theories MHR [31] and ZZ [42] are considered, to further support that the redefinition of coefficients dramatically improves accuracy.

4.4 Cases j to m

A three-layer simply-supported sandwich plate under bi-sinusoidal loading is considered as case j, where the lower face (E_{1111}, E_{1122}, E_{2222}, E_{1212} reduced by a factor $1\cdot10^{-2}$) and the core are damaged (the core is partially damaged up to 0.15h from below, E_{1122}, E_{2222}, E_{1212}, E_{1313}, E_{2323} reduced by a factor $2\cdot10^{-1}$); as a consequence, strong 3-D effects rise. Case k differs from case i as concerns loading, a step compressive loading on the upper and lower faces of the two halves of the undamaged beam [31] being applied. Nevertheless lay-up is symmetric, displacements and stresses are strongly asymmetric across the thickness, because of loading. Case l is a modified version of beam [75],with simply-supported edges under a two-step loading and a length-to-thickness ratio of 25 is assumed. Additionally, it considers a damaged core (E_{1122}, E_{2222}, E_{1212}, E_{1313}, E_{2323} reduced by a factor $1\cdot10^{-1}$) and upper face (E_{1111}, E_{1122}, E_{2222}, E_{1212} reduced by a factor $4\cdot10^{-2}$). Case m regards a propped-cantilever sandwich beam with a uniform transverse loading on the upper face and a length-to-thickness ratio of 5.714.. An accurate description of transverse deformability is mandatory for this case, otherwise inaccurate results are obtained [31].

The findings of Figures 2 to 5 confirm that the choice functions of representation and the zig-zag layerwise contributions is immaterial for adaptive theories.
In particular, the results of Figure 2 which refer to the case j, show that only in-plane stresses are quite accurately predicted by all theories and the same apply for case k (Figure 3). Results of Figure 4 for case l show that relevant discrepancies among theories still exist also for this rather thin sandwich and that only adaptive theories can be really accurate [31]. Results of case m by Figure 5 highlight the strong layerwise effects of this case, to which follow that displacement and stress fields are strongly asymmetric across the thickness and consequently only higher-order adaptive theories can obtain accurate results.
Note that ZZAM_P3P4, that impose the same number of equilibrium points of ZZA, but in integral form (computationally more effective), obtains indistinguishable results for cases a to j, while for cases k to m is inaccurate due to numerical problems and for this reason it is not reported.
Computational burden of all theories of this paper (Table 6) is still comparable with that of FSDT, which being inaccurate, as well as the lower-order theories of this paper, do not result in any convenience. This table shows the slightly lower computational costs of lower-order theories, but considering that they are not always accurate, this advantage cannot be exploited.

4.5 Convergence assesstments

A convergence study of theories with respect to the expansion order of trial function and of 3-D FEA with respect to meshing is performed in this section. For the former case, results are reported in Table 14 and Figures 6 to 7. Because a quite large number of theories is considered, to limit the amount of data reported just the number of components that ensures convergence for theory ZZA_PMTP34 are reported since it has the slowest convergence rate among adaptive theories ZZAS4, NOZZG, ZZA_PP34, ZZA_PT34, ZZA_PM34, ZZA_PPM34, AT-3D. It must be considered that however there is no wide variation of the order of various theories because convergence orders differ only a few units, so the results for ZZA_PMTP34 are representative of all adaptive theories. They are representative also for all lower-order theories, which however converge to different and less accurate results.
A quite large number of components is required for cases h and m, which refer to a simply-supported plate under localized loading and to a propped cantilever beam, respectively (see Figures 6 to 7), while for cases i to l one component is sufficient (see Table 14). The largest error, which is obtained considering just the first component, is reported in an inset inside Figures 6 to 7, so the convergence curves can be normalized to it, i.e. all curves start from one for reasons of uniformity. To provide more details about the convergence behaviour, stresses and displacements across the thickness are reported for M=4 and 20 (convergence value) in Figure 6, for M=5 and 10 (convergence value) in Figure 7.
In Figure 8 u_{α} of case b is considered as an example of through the thickness convergence behavior. For this purpose, the converging solution u_{α} is firstly expanded in a Fourier series across the thickness, then the number of components necessary to represent the converging solution is determined. Subsequently, the number of components is progressively reduced, then the percentage error committed is calculated. All adaptive theories behave the same way, because of they require the same expansion order for the

convergent solution, and provide the same results for all lower expansion orders considered. Results from N = 1 to N = 100 are reported in Figure 8, but an acceptable error is obtained for N = 14. Results allow us to guess that if a Fourier series expansion with 14 or more components is used as the representation within theories, whose amplitudes are assumed as d.o.f., similarly to asymptotic and hierarchical theories, the same results of the present adaptive theories with just five d.o.f., would be obtained (it seems to agree with the expansion order declared in [49]).

The error made by the 3-D FEA when the mesh density increases is shown in Figures 9 (as an example case m is considered). The three numbers indicated in the figure represent the discretization respectively along α, β, ς. Percentage error of stresses and displacements, reported in the figures are evaluated at specific points where less accurate results are obtained, whose position is specified in the figures.

5. Concluding Remarks

This study illustrated numerically that the degree of accuracy of higher-order physically-based zig-zag theories (in displacement-based and mixed form) is independent on the choice of global and layerwise functions, once all stress continuity, boundary and equilibrium conditions are enforced at the same time. On the contrary, if coefficients are not redefined or physical constraints are partially satisfied, results are strongly dependent by choices made, as demonstrated by assessments of the lower-order theories, which are sometimes adequate, but not always.

Six new higher-order physically-based adaptive theories have been developed as particularizations of a novel generalized theory which keeps the same advantages of theories previously developed by the authors and enables users to choose arbitrarily the representation and layerwise contributions. In these theories, zig-zag contributions can be omitted (their role can be played by the redefinition of coefficients at layer interfaces) and displacements can be assumed differently each other and for each region across the thickness. Accordingly, these theories become similar to hierarchical and axiomatic/asymptotic theories, but are more efficient, because they require a lower expansion order and only five d.o.f. to get accurate results. Indeed, a piecewise cubic/fourth-order representation for in-plane and transverse displacements, respectively, is sufficient to get the maximal accuracy.

Also an approximate 3-D solution is obtained as particularization of general formulation, whose results are always precise, demonstrating that it can be used as reference solution when exact one is unavailable. Closed form solutions have been presented using Rayleigh-Ritz method and the same trial functions for all theories. Elastostatic benchmarks with distributed/ localized step loading, different boundary conditions and material properties of layers that generate strong layerwise effects were considered. A convergence study has been carried out, in order to determine the minimum expansion order along in-plane directions, beyond which errors don’t appreciably decrease.

Data Availability

All data generated or analyse during the study are included in the manuscript.

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