p-adic denseness of members of partitions of N and their ratio sets

p-ADIC DENSENESS OF MEMBERS OF PARTITIONS OF $\mathbb{N}$ AND THEIR RATIO SETS

PIOTR MISKA AND CARLO SANNA

Abstract. The *ratio set* of a set of positive integers A is defined as $R(A) := \{a/b : a, b \in A\}$. The study of the denseness of $R(A)$ in the set of positive real numbers is a classical topic and, more recently, the denseness in the set of p-adic numbers $\mathbb{Q}_p$ has also been investigated. Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. We prove that for all prime numbers p but at most $\lfloor \log_2 k \rfloor$ exceptions at least one of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_p$. Moreover, we show that for all prime numbers p but at most $k-1$ exceptions at least one of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_p$. Both these results are optimal in the sense that there exist partitions $A_1, \ldots, A_k$ having exactly $\lfloor \log_2 k \rfloor$, respectively $k-1$, exceptional prime numbers; and we give explicit constructions for them. Furthermore, as a corollary, we answer negatively a question raised by Garcia, Hong, *et al.*

1. Introduction

The *ratio set* (or *quotient set*) of a set of positive integers A is defined as

$$R(A) := \{a/b : a, b \in A\}.$$

The study of the denseness of $R(A)$ in the set of positive real numbers $\mathbb{R}_+$ is a classical topic. For example, Strauch and Tóth [10] (see also [11]) showed that $R(A)$ is dense in $\mathbb{R}_+$ whenever A has lower asymptotic density at least equal to $1/2$. Furthermore, Bukor, Šalát, and Tóth [3] proved that if $\mathbb{N} = A \cup B$ for two disjoint sets A and B, then at least one of $R(A)$ or $R(B)$ is dense in $\mathbb{R}_+$. On the other hand, Brown, Dairyko, Garcia, Lutz, and Someck [1] showed that there exist pairwise disjoint sets $A, B, C \subseteq \mathbb{N}$ such that $\mathbb{N} = A \cup B \cup C$ and none of $R(A)$, $R(B)$, $R(C)$ is dense in $\mathbb{R}_+$. See also [2, 4, 7, 8] for other related results.

More recently, the study of when $R(A)$ is dense in the p-adic numbers $\mathbb{Q}_p$, for some prime number p, has been initiated. Garcia and Luca [6] proved that the ratio set of the set of Fibonacci numbers is dense in $\mathbb{Q}_p$, for all prime numbers p. Their result has been generalized by Sanna [9], who proved that the ratio set of the k-generalized Fibonacci numbers is dense in $\mathbb{Q}_p$, for all integers $k \geq 2$ and prime numbers p. Furthermore, Garcia, Hong, Luca, Pinsker, Sanna, Schechter, and Starr [5] gave several results on the denseness of $R(A)$ in $\mathbb{Q}_p$. In particular, they studied $R(A)$ when A is the set of values of a Lucas sequences, the set of positive integers which are sum of k squares, respectively k cubes, or the union of two geometric progressions.

In this paper, we continued the study of the denseness of $R(A)$ in $\mathbb{Q}_p$.

2. Denseness of members of partitions of $\mathbb{N}$

Motivated by the results on partitions of $\mathbb{N}$ mentioned in the introduction, the authors of [5] showed that for each prime number p there exists a partition of $\mathbb{N}$ into two sets A and B such that neither $R(A)$ nor $R(B)$ are dense in $\mathbb{Q}_p$ [5, Example 3.6]. Then, they asked the following question [5, Problem 3.7]:

Question 2.1. Is there a partition of $\mathbb{N}$ into two sets A and B such that $R(A)$ and $R(B)$ are dense in no $\mathbb{Q}_p$?[1]

2010 *Mathematics Subject Classification.* 11A07, 11B05.

Key words and phrases. denseness, p-adic topology, partition, quotient set, ratio set.

[1] Actually, in [5, Problem 3.7] it is erroneously written "such that A and B are dense in no $\mathbb{Q}_p$", so that the answer is obviously: "Yes, pick any partion into two sets!". Question 2.1 is the intended question.

We show that the answer to Question 2.1 is negative. In fact, we will prove even more. Our first result is the following:

Theorem 2.1. *Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $k-1$ exceptions, at least one of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_p$.*

Then, from Theorem 2.1 it follows the next corollary, which gives a strong negative answer to Question 2.1.

Corollary 2.1. *Let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $k-1$ exceptions, at least one of $R(A_1), \ldots, R(A_k)$ is dense in $\mathbb{Q}_p$.*

Proof. It is easy to prove that if A_j is dense in $\mathbb{Z}_p$ then $R(A_j)$ is dense in $\mathbb{Q}_p$. Hence, the claim follows from Theorem 2.1. □

The proof of Theorem 2.1 requires just a couple of easy preliminary lemmas. For positive integers a and b, define $a + b\mathbb{N} := \{a + bk : k \in \mathbb{N}\}$.

Lemma 2.2. *Suppose that $(a + b\mathbb{N}) \subseteq A \cup B$ for some positive integers a, b and some disjoint sets $A, B \subseteq \mathbb{N}$. If p is a prime number such that $p \nmid b$ and A is not dense in $\mathbb{Z}_p$, then there exist positive integers c and j such that $(c + bp^j\mathbb{N}) \subseteq B$.*

Proof. Since A is not dense in $\mathbb{Z}_p$, there exist positive integers d, j such that $(d + p^j\mathbb{N}) \cap A = \varnothing$. Hence, $(a + b\mathbb{N}) \cap (d + p^j\mathbb{N}) \subseteq B$. The claim follows by the Chinese Remainder Theorem, which implies that $(a + b\mathbb{N}) \cap (d + p^j\mathbb{N}) = c + bp^j\mathbb{N}$, for some positive integer c. □

Lemma 2.3. *Let a and b be positive integers. Then, $a + b\mathbb{N}$ is dense in $\mathbb{Z}_p$ for all prime numbers p such that $p \nmid b$.*

Proof. It is follows from the Chinese Remainder Theorem and the fact that $\mathbb{N}$ is dense in $\mathbb{Z}_p$. □

We are now ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. For the sake of contradiction, suppose that $p_1, \ldots, p_k$ are k pairwise distinct prime numbers such that none of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_{p_i}$ for $i = 1, \ldots, k$. Since A_1 is not dense in $\mathbb{Z}_{p_1}$, there exist positive integers c_1 and j_1 such that $(c_1 + p_1^{j_1}\mathbb{N}) \cap A_1 = \varnothing$. Hence, $(c_1 + p_1^{j_1}\mathbb{N}) \subseteq A_2 \cup \cdots \cup A_k$ and, thanks to Lemma 2.2, there exist positive integers c_2 and j_2 such that $(c_2 + p_1^{j_1}p_2^{j_2}\mathbb{N}) \subseteq A_3 \cup \cdots \cup A_k$. Continuing this process, we get that $(c_{k-1} + p_1^{j_1} \cdots p_{k-1}^{j_{k-1}}\mathbb{N}) \subseteq A_k$, for some positive integers $c_{k-1}, j_1, \ldots, j_{k-1}$. By Lemma 2.3, this last inclusion implies that A_k is dense in $\mathbb{Z}_{p_k}$, but this contradicts the hypotheses. □

Remark 2.1. In fact, Theorem 2.1 can be strengthen in the following way: For each partition $A_1, \ldots, A_k$ of $\mathbb{N}$ there exists a member A_j of this partition which is dense in $\mathbb{Z}_p$ for all but at most $k-1$ prime numbers p.

Indeed, for the sake of contradiction, suppose that each member A_j of the partition $A_1, \ldots, A_k$ of $\mathbb{N}$ has at least k prime numbers p such that A_j is not dense in $\mathbb{Z}_p$. Then we can choose prime numbers $p_1, \ldots, p_k$ such that for each $j \in \{1, \ldots, k\}$ the set A_j is not dense in $\mathbb{Z}_{p_j}$. Next, we provide the reasoning from the proof of Theorem 2.1 to reach a contradiction.

The next result shows that the quantity $k-1$ in Theorem 2.1 cannot be improved.

Theorem 2.4. *Let $k \geq 2$ be an integer and let $p_1, \ldots, p_{k-1}$ be pairwise distinct prime numbers. Then, there exists a partition $A_1, \ldots, A_k$ of $\mathbb{N}$ such that none of $A_1, \ldots, A_k$ is dense in $\mathbb{Z}_{p_i}$ for $i = 1, \ldots, k-1$.*

Proof. Let $e_1, \ldots, e_{k-1}$ be positive integers such that $p_i^{e_i} \geq k$ for $i = 1, \ldots, k-1$, and put

$$V := \{0, \ldots, p_1^{e_1} - 1\} \times \cdots \times \{0, \ldots, p_{k-1}^{e_{k-1}} - 1\}.$$

We shall construct a partition $R_0, \dots, R_{k-1}$ of V (note that the indices of R_i start from 0) such that if $(r_1, \dots, r_{k-1}) \in R_j$ then none of the components $r_1, \dots, r_{k-1}$ is equal to j. Then, we define

$$A_j := \{n \in \mathbb{N} : \exists (r_1, \dots, r_{k-1}) \in R_{j-1},\ \forall i = 1, \dots, k-1, \quad n \equiv r_i \pmod{p_i^{e_i}}\},$$

for $j = 1, \dots, k$. At this point, it follows easily that $A_1, \dots, A_k$ is a partition of $\mathbb{N}$, and that none of $A_1, \dots, A_k$ is dense in $\mathbb{Z}_{p_i}$, since A_{j+1} misses the residue class $j \pmod{p_i^{e_i}}$.

The construction of $R_0, \dots, R_{k-1}$ is algorithmic. We start with $R_0, \dots, R_{k-1}$ all empty. Then, we pick a vector $\mathbf{x} \in V$ which is not already in $R_0 \cup \dots \cup R_{k-1}$. It is easy to see that there exists some $j \in \{0, ..., k-1\}$ such that j does not appear as a component of $\mathbf{x}$. We thus throw $\mathbf{x}$ into R_j. We continue this process until all the vectors in V have been picked.

Now, by the construction it is clear that $R_0, \dots, R_{k-1}$ is a partition of V satisfying the desired property. $\square$

3. Denseness of ratio sets of members of partitions of $\mathbb{N}$

The result in Corollary 2.1 is not optimal. Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and write $\log_2$ for the base 2 logarithm. Our next result is the following:

Theorem 3.1. *Let $A_1, \dots, A_k$ be a partition of $\mathbb{N}$ into k sets. Then, for all prime numbers p but at most $\lfloor \log_2 k \rfloor$ exceptions, at least one of $R(A_1), \dots, R(A_k)$ is dense in $\mathbb{Q}_p$.*

Before proving Theorem 3.1, we need to introduce some notation. For a prime number p and a positive integer w, we identify the group $(\mathbb{Z}/p^w\mathbb{Z})^*$ with $\{a \in \{1, \dots, p^w\} : p \nmid a\}$. Moreover, for each $a \in (\mathbb{Z}/p^w\mathbb{Z})^*$ we define

$$(a)_{p^w} := \left\{ x \in \mathbb{Q}_p^* : x/p^{\nu_p(x)} \equiv a \bmod p^w \right\},$$

where, as usual, ν_p denotes the p-adic valuation. Note that the family of sets

$$(a)_{p^w} \cap \nu_p^{-1}(s) = \{(a + rp^w)p^s : r \in \mathbb{Z}_p\}$$

where w is a positive integer, $a \in (\mathbb{Z}/p^w\mathbb{Z})^*$, and $s \in \mathbb{Z}$, is a basis of the topology of $\mathbb{Q}_p^*$. Finally, for all integers $t \leq m$ and for each set $X \subseteq \mathbb{N}$, we define

$$V_{p^w,t,m} := \left\{ (a)_{p^w} \cap \nu_p^{-1}(s) : a \in (\mathbb{Z}/p^w\mathbb{Z})^*,\ s \in \mathbb{Z} \cap [t, m-1] \right\}$$

and

$$V_{p^w,t,m}(X) := \{I \in V_{p^w,t,m} : X \cap I \neq \varnothing\}.$$

Note that the following trivial upper bound holds

$$\# V_{p^w,t,m}(X) \leq \# V_{p^w,t,m} = (m-t)\varphi(p^w),$$

where φ is the Euler's totient function.

Now we are ready to state a lemma that will be crucial in the proof of Theorem 3.1.

Lemma 3.2. *Fix a prime number p, two positive integers w, t, a real number $c > 1/2$, and a set $X \subseteq \mathbb{N}$. Suppose that $\# V_{p^w,0,m}(X) \geq cm\,\varphi(p^w)$ for some positive integer $m > t/(2c-1)$. Then the ratio set $R(X)$ intersects nontrivially with each set in $V_{p^w,0,t}$.*

Proof. Given $(a_0)_{p^w} \cap \nu_p^{-1}(s_0) \in V_{p^w,0,t}$ we have to prove that $R(X) \cap (a_0)_{p^w} \cap \nu_p^{-1}(s_0) \neq \varnothing$. For the sake of convenience, define $A := V_{p^w,t,m}(X)$ and

$$B := \{(a_0 a)_{p^w} \cap \nu_p^{-1}(s_0 + s) : (a)_{p^w} \cap \nu_p^{-1}(s) \in V_{p^w,t-s_0,m-s_0}(X)\}.$$

We have

$$\# A = \# V_{p^w,0,m}(X) - \# V_{p^w,0,t}(X) \geq (cm - t)\varphi(p^w) > \frac{1}{2}(m-t)\varphi(p^w), \tag{1}$$

where we used the inequality $m > t/(2c-1)$. Similarly,

$$\begin{aligned}\#B &= \#V_{p^w,0,m}(X) - \#V_{p^w,0,t-s_0}(X) - \#V_{p^w,m-s_0,m}(X) \\ &\geq (cm - (t-s_0) - s_0)\varphi(p^w) > \frac{1}{2}(m-t)\varphi(p^w). \end{aligned} \tag{2}$$

Now A and B are both subsets of $V_{p^w,t,m}$, while $\#V_{p^w,t,m} = (m-t)\varphi(p^w)$. Therefore, (1) and (2) imply that $A \cap B \neq \varnothing$. That is, there exist $(a_1)_{p^w} \cap \nu_p^{-1}(s_1) \in A$ and $(a_2)_{p^w} \cap \nu_p^{-1}(s_2) \in V_{p^w,t-s_0,m-s_0}(X)$ such that $a_1/a_2 \equiv a_0 \pmod{p^w}$ and $s_1 - s_2 = s_0$, so that $R(X) \cap (a_0)_{p^w} \cap \nu_p^{-1}(s_0) \neq \varnothing$, as claimed. □

Proof of Theorem 3.1. For the sake of contradiction, put $\ell := \lfloor \log_2 k \rfloor + 1$ and suppose that $p_1, \dots, p_\ell$ are ℓ pairwise distinct prime numbers such that none of $R(A_1), \dots, R(A_k)$ is dense in $\mathbb{Q}_{p_i}$ for $i = 1, \dots, \ell$. Hence, there exist positive integers w and t such that for each $i \in \{1, \dots, k\}$ and each $j \in \{1, \dots, \ell\}$ we have $R(A_i) \cap (a_{i,j})_{p_j^w} \cap \nu_{p_j}^{-1}(s_{i,j}) = \varnothing$, for some $a_{i,j} \in (\mathbb{Z}/p_j^w\mathbb{Z})^*$ and some $s_{i,j} \in \{-(t-1), \dots, t-1\}$. Clearly, since ratio sets are closed under taking reciprocals, we can assume $s_{i,j} \geq 0$. Put $c := 1/\sqrt[\ell]{k}$, so that $c > 1/2$, and pick a positive integer $m > t/(2c-1)$. There are

$$N := m^\ell \prod_{j=1}^{\ell} \varphi(p_j^w)$$

sets of the form

$$\bigcap_{j=1}^{\ell} \left((a_j)_{p_j^w} \cap \nu_{p_j}^{-1}(s_j) \right), \tag{3}$$

where $a_j \in (\mathbb{Z}/p_j^w\mathbb{Z})^*$ and $s_j \in \{0, \dots, m-1\}$. Therefore, there exists $i_0 \in \{1, \dots, k\}$ such that A_{i_0} intersects nontrivially with at least N/k of the sets of form (3). Consequently, there exists $j_0 \in \{1, \dots, \ell\}$ such that A_{i_0} intersects nontrivially with at least $cm\varphi(p_{j_0}^w)$ sets of the form $(a)_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s)$, where $a \in (\mathbb{Z}/p_{j_0}^w\mathbb{Z})^*$ and $s \in \{0, \dots, m-1\}$. In other words, $\#V_{p_{j_0}^w,0,m}(A_{i_0}) \geq cm\varphi(p_{j_0}^w)$. Hence, by Lemma 3.2, the set $R(A_{i_0})$ intersects notrivially with all the sets of the form $(a)_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s)$, where $a \in (\mathbb{Z}/p_{j_0}^w\mathbb{Z})^*$ and $s \in \{0, \dots, t-1\}$, but this is in contradiction with the fact that $R(A_{i_0}) \cap (a_{i_0,j_0})_{p_{j_0}^w} \cap \nu_{p_{j_0}}^{-1}(s_{i_0,j_0}) = \varnothing$. □

The bound $\lfloor \log_2 k \rfloor$ in Theorem 3.1 is sharp in the following sense:

Theorem 3.3. *Let $k \geq 2$ be an integer and let $p_1 < \dots < p_\ell$ be $\ell := \lfloor \log_2 k \rfloor$ pairwise distinct prime numbers. Then, there exists a partition of $\mathbb{N}$ into k sets $A_1, \dots, A_k$ such that none of $R(A_1), \dots, R(A_k)$ is dense in $\mathbb{Q}_{p_i}$ for $i = 1, \dots, \ell$.*

Proof. We give two different constructions. Put $h := 2^\ell$ and let $S_1, \dots, S_h$ be all the subsets of $\{1, \dots, \ell\}$. For $j = 1, \dots, h$, define

$$B_j := \{n \in \mathbb{N} : \forall i = 1, \dots, \ell \quad \nu_{p_i}(n) \equiv \chi_{S_j}(i) \pmod{2}\},$$

where χ_{S_j} denotes the characteristic function of S_j. It follows easily that $B_1, \dots, B_h$ is a partition of $\mathbb{N}$, and that none of $R(B_1), \dots, R(B_h)$ is dense in $\mathbb{Q}_{p_i}$, for $i = 1, \dots, \ell$, since each $R(B_j)$ contains only rational numbers with even p_i-adic valuations. Finally, since $h \leq k$, the partition $B_1, \dots, B_h$ can be refined to obtain a partition $A_1, \dots, A_k$ satisfying the desired property.

The second costruction is similar. For $j = 1, \dots, h$, define

$$C_j = \left\{ n \in \mathbb{N} : \left(\frac{n/p_i^{v_{p_i}(n)}}{p_i} \right) = (-1)^{\chi_{S_j}(i)} \text{ for each } i \in \{1, ..., \ell\} \right\},$$

where $\left(\frac{a}{p}\right)$ means the Legendre symbol and in case of $p_1 = 2$ we put $\left(\frac{a}{2}\right) = a \pmod 4$. It follows easily that $C_1, \dots, C_h$ is a partition of $\mathbb{N}$, and that none of $R(C_1), \dots, R(C_h)$ is dense in $\mathbb{Q}_{p_i}$, for $i = 1, \ldots, \ell$, since each $R(C_j)$ contains only products of powers of p_i and quadratic residues modulo p_i (in case of $p_1 = 2$ we have only products of powers of 2 and numbers congruent to 1 modulo 4). Finally, since $h \leq k$, the partition $C_1, \ldots, C_h$ can be refined to obtain a partition $A_1, \ldots, A_k$ satisfying the desired property. □

In the light of Remark 2.1 it is worth to ask a the following question.

Question 3.1. Let us fix a positive integer k. What then is the least number $m = m(k)$ such that for each partition $A_1, \ldots, A_k$ of $\mathbb{N}$ there exists a member A_j of this partition such that $R(A_j)$ is dense in $\mathbb{Q}_p$ for all but at most m prime numbers p?

In virtue of Remark 2.1 we know that $m(k)$ exists and $m(k) \leq k - 1$. On the other hand, by Theorem 3.3 the value $m(k)$ is not less than $\lfloor \log_2 k \rfloor$.

Acknowledgements. C. Sanna is a member of the INdAM group GNSAGA.

References

[1] B. Brown, M. Dairyko, S. R. Garcia, B. Lutz, and M. Someck, *Four quotient set gems*, Amer. Math. Monthly **121** (2014), no. 7, 590–599.

[2] J. Bukor, P. Erdős, T. Šalát, and J. T. Tóth, *Remarks on the* (R)*-density of sets of numbers. II*, Math. Slovaca **47** (1997), no. 5, 517–526.

[3] J. Bukor, T. Šalát, and J. T. Tóth, *Remarks on R-density of sets of numbers*, Tatra Mt. Math. Publ. **11** (1997), 159–165, Number theory (Liptovský Ján, 1995).

[4] J. Bukor and J. T. Tóth, *On accumulation points of ratio sets of positive integers*, Amer. Math. Monthly **103** (1996), no. 6, 502–504.

[5] S. R. Garcia, Y. X. Hong, F. Luca, E. Pinsker, C. Sanna, E. Schechter, and A. Starr, *p-adic quotient sets*, Acta Arith. **179** (2017), no. 2, 163–184.

[6] S. R. Garcia and F. Luca, *Quotients of Fibonacci numbers*, Amer. Math. Monthly **123** (2016), no. 10, 1039–1044.

[7] T. Salat, *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1968/1969), 273–278.

[8] T. Salat, *Corrigendum to the paper "On ratio sets of sets of natural numbers"*, Acta Arith. **16** (1969/1970), 103.

[9] C. Sanna, *The quotient set of k-generalised Fibonacci numbers is dense in* $\mathbb{Q}_p$, Bull. Aust. Math. Soc. **96** (2017), no. 1, 24–29.

[10] O. Strauch and J. T. Tóth, *Asymptotic density of* $A \subset \mathbf{N}$ *and density of the ratio set* $R(A)$, Acta Arith. **87** (1998), no. 1, 67–78.

[11] O. Strauch and J. T. Tóth, *Corrigendum to Theorem 5 of the paper: "Asymptotic density of* $A \subset \mathbb{N}$ *and density of the ratio set* $R(A)$*"*, Acta Arith. **103** (2002), no. 2, 191–200.

Institute of Mathematics, Jagiellonian University in Kraków, Kraków, Poland
E-mail address: piotrmiska91@gmail.com

Department of Mathematics, Università degli Studi di Torino, Torino, Italy
E-mail address: carlo.sanna.dev@gmail.com