Non-local multi-class traffic flow models

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Abstract

We prove the existence for small times of weak solutions for a class of non-local systems in one space dimension, arising in traffic modeling. We approximate the problem by a Godunov type numerical scheme and we provide uniform $\mathbf{L}^1$ and BV estimates for the sequence of approximate solutions. We finally present some numerical simulations illustrating the behavior of different classes of vehicles and we analyze two cost functionals measuring the dependence of congestion on traffic composition.

1 Introduction

We consider the following class of non-local systems of M conservation laws in one space dimension:

$$\partial_t \rho_i(t,x) + \partial_x \left(\rho_i(t,x) v_i((r * \omega_i)(t,x)) \right) = 0, \qquad i = 1, \dots, M, \tag{1.1}$$

where

$$r(t,x) := \sum_{i=1}^{M} \rho_i(t,x), \tag{1.2}$$

$$v_i(\xi) := v_i^{\max} \psi(\xi), \tag{1.3}$$

$$(r * \omega_i)(t,x) := \int_x^{x+\eta_i} r(t,y)\,\omega_i(y-x)\,dy, \tag{1.4}$$

and we assume:

(H1) The convolution kernels $\omega_i \in \mathbf{C}^1([0,\eta_i]; \mathbb{R}^+)$, $\eta_i > 0$, are non-increasing functions such that $\int_0^{\eta_i} \omega_i(y)\,dy = J_i$. We set $W_0 := \max_{i=1,\dots,M} \omega_i(0)$.

(H2) $v_i^{\max}$ are the maximal velocities, with $0 < v_1^{\max} \leq v_2^{\max} \leq \dots \leq v_M^{\max}$.

(H3) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth non-increasing function such that $\psi(0)=1$ and $\psi(r)=0$ for $r \geq 1$ (for simplicity, we can consider the function $\psi(r) = \max\{1-r, 0\}$).

We couple (1.1) with an initial datum

$$\rho_i(0,x) = \rho_i^0(x), \qquad i = 1, \dots, M. \tag{1.5}$$

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Model (1.1) is obtained generalizing the n-populations model for traffic flow described in [3] and it is a multi-class version of the one dimensional scalar conservation law with non-local flux proposed in [4], where ρ_i is the density of vehicles belonging to the i-th class, η_i is proportional to the look-ahead distance and δ_i is the interaction strength. In our setting, the non-local dependence of the speed functions v_i describes the reaction of drivers that adapt their velocity to the downstream traffic, assigning greater importance to closer vehicles, see also [6, 8]. We consider different anisotropic discontinuous kernels for each equation of the system, therefore the results in [1] cannot be applied. The model takes into account the distribution of heterogeneous drivers and vehicles characterized by their maximal speeds and look-ahead visibility in a traffic stream. One of the limitations of the standard LWR traffic flow model [10, 11] is the first in first out rule, conversely in multi-class dynamic faster vehicles can overtake slower ones and slower vehicles slow down the faster ones.

Due to the possible presence of jump discontinuities, solutions to (1.1), (1.5) are intended in the following weak sense.

Definition 1. *A function* $\rho = (\rho_1, \dots, \rho_M) \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)([0,T[\times\mathbb{R};\mathbb{R}^M)$, $T>0$, *is a weak solution of* (1.1), (1.5) *if*

$$\int_0^T\int_{-\infty}^{\infty}\left(\rho_i \partial_t\varphi + \rho_i v_i(r * \omega_i)\partial_x\varphi\right)(t,x)\,dx\,dt + \int_{-\infty}^{\infty}\rho_i^0(x)\varphi(0,x)\,dx = 0$$

for all $\varphi \in \mathbf{C}^1_c(]-\infty, T[\times\mathbb{R};\mathbb{R})$, $i = 1, \dots, M$.

The main result of this paper is the proof of existence of weak solutions to (1.1), (1.5), locally in time.

Theorem 1. *Let* $\rho_i^0(x) \in (\mathbf{BV}\cap\mathbf{L}^\infty)(\mathbb{R};\mathbb{R}^+)$, *for* $i = 1, \dots, M$, *and assumptions* **(H1)** - **(H3)** *hold. Then the Cauchy problem* (1.1), (1.5) *admits a weak solution on* $[0,T[\times\mathbb{R}$, *for some* $T>0$ *sufficiently small.*

The paper is organized as follows. Section 2 is devoted to prove uniform $\mathbf{L}^\infty$ and BV estimates on the approximate solutions obtained through an approximation argument based on a Godunov type numerical scheme, see [7]. We have to point out that these estimates heavily rely on the monotonicity properties of the kernel functions ω_i. In Section 3 we prove the existence in finite time of weak solutions applying Helly's theorem and a Lax-Wendroff type argument, see [9]. In Section 4 we present some numerical simulations for $M = 2$. In particular, we consider the case of a mixed flow of cars and trucks on a stretch of road, and the flow of mixed autonomous and non-autonomous vehicles on a circular road. In this latter case, we analyze two cost functionals measuring the traffic congestion, depending on the penetration ratio of autonomous vehicles. The final Appendix contains an alternative proof of Theorem 1, based on approximate solutions constructed via a Lax-Friedrichs type scheme, which is commonly used in the framework of non-local equations, see [1, 2, 4].

2 Godunov type approximate solutions

First of all, we extend $\omega_i(x) = 0$ for $x > \eta_i$. For $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $x_{j+1/2} = j\Delta x$ be the cells interfaces, $x_j = (j - 1/2)\Delta x$ the cells centers and $t^n = n\Delta t$ the time mesh. We aim at constructing a finite volume approximate solution $\rho^{\Delta x} = \left(\rho_1^{\Delta x}, \dots, \rho_M^{\Delta x}\right)$, with

$\rho_i^{\Delta x}(t,x) = \rho_{i,j}^n$ for $(t,x) \in C_j^n = [t^n, t^{n+1}[\times]x_{j-1/2}, x_{j+1/2}]$ and $i = 1, ..., M$.
To this end, we approximate the initial datum ρ_i^0 for $i = 1, ..., M$ with a piecewise constant function

$$\rho_{i,j}^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_i^0(x)\,\mathrm{d}x, \qquad j \in \mathbb{Z}.$$

Similarly, for the kernel, we set

$$\omega_i^k := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \omega_i^0(x)\,\mathrm{d}x, \qquad k \in \mathbb{N},$$

so that $\Delta x \sum_{k=0}^{+\infty} \omega_i^k = \int_0^{\eta_i} \omega_i(x)\,\mathrm{d}x = J_i$ (the sum is indeed finite since $\omega_i^k = 0$ for $k \geq N_i$ sufficiently large). Moreover, we set $r_{j+k}^n = \sum_{i=1}^{M} \rho_{i,j+k}^n$ for $k \in \mathbb{N}$ and

$$V_{i,j}^n := v_i^{\max}\,\psi\left(\Delta x \sum_{k=0}^{N_i-1} \omega_i^k r_{j+k}^n\right), \qquad i = 1, \dots, M, \quad j \in \mathbb{Z}. \tag{2.1}$$

We consider the following Godunov-type scheme adapted to (1.1), which was introduced in [7] in the scalar case:

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n - \lambda\left(\rho_{i,j}^n V_{i,j+1}^n - \rho_{i,j-1}^n V_{i,j}^n\right) \tag{2.2}$$

where we have set $\lambda = \frac{\Delta t}{\Delta x}$.

2.1 Compactness estimates

We provide here the necessary estimates to prove the convergence of the sequence of approximate solutions constructed via the Godunov scheme (2.2).

Lemma 1.(Positivity) *For any $T > 0$, under the CFL condition*

$$\lambda \leq \frac{1}{v_M^{\max}\|\psi\|_{\infty}}, \tag{2.3}$$

the scheme (2.2) is positivity preserving on $[0, T] \times \mathbb{R}$.

Proof. Let us assume that $\rho_{i,j}^n \geq 0$ for all $j \in \mathbb{Z}$ and $i \in 1, ..., M$. It suffices to prove that $\rho_{i,j}^{n+1}$ in (2.2) is non-negative. We compute

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n\left(1 - \lambda V_{i,j+1}^n\right) + \lambda \rho_{i,j-1}^n V_{i,j}^n \geq 0 \tag{2.4}$$

under assumption (2.3). ∎

Corollary 2. ($\mathbf{L^1}$-bound) *For any $n \in \mathbb{N}$, under the CFL condition (2.3) the approximate solutions constructed via the scheme (2.2) satisfy*

$$\left\|\rho_i^n\right\|_1 = \left\|\rho_i^0\right\|_1, \qquad i = 1, \dots, M, \tag{2.5}$$

where $\left\|\rho_i^n\right\|_1 := \Delta x \sum_j \left|\rho_{i,j}^n\right|$ denotes the $\mathbf{L^1}$ norm of the i-th component of $\rho^{\Delta x}$.

Proof. Thanks to Lemma 1, for all $i \in \{1, \dots, M\}$ we have

$$\left\|\rho_i^{n+1}\right\|_1 = \Delta x \sum_j \rho_{i,j}^{n+1} = \Delta x \sum_j \left(\rho_{i,j}^n - \lambda \rho_{i,j}^n V_{i,j+1}^n + \lambda \rho_{i,j-1}^n V_{i,j}^n\right) = \Delta x \sum_j \rho_{i,j}^n,$$

proving (2.5). □

Lemma 2. ($\mathbf{L^\infty}$-bound) *If $\rho_{i,j}^0 \geq 0$ for all $j \in \mathbb{Z}$ and $i = 1, \dots, M$, and (2.3) holds, then the approximate solution $\rho^{\Delta x}$ constructed by the algorithm (2.2) is uniformly bounded on $[0, T] \times \mathbb{R}$ for any T such that*

$$T < \left(M \left\|\rho^0\right\|_\infty v_M^{\max} \left\|\psi'\right\|_\infty W_0\right)^{-1}.$$

Proof. Let $\bar{\rho} = \max\{\rho_{i,j-1}^n, \rho_{i,j}^n\}$. Then we get

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n \left(1 - \lambda V_{i,j+1}^n\right) + \lambda \rho_{i,j-1}^n V_{i,j}^n \leq \bar{\rho}\left(1 + \lambda\left(V_{i,j}^n - V_{i,j+1}^n\right)\right) \tag{2.6}$$

and

$$\begin{aligned}
V_{i,j}^n - V_{i,j+1}^n &= v_i^{\max}\left(\psi\left(\Delta x \sum_{k=0}^{N-1} \omega_i^k r_{j+k}^n\right) - \psi\left(\Delta x \sum_{k=0}^{N-1} \omega_i^k r_{j+k+1}^n\right)\right) \\
&\leq v_i^{\max} \left\|\psi'\right\|_\infty \Delta x \sum_{k=0}^{N-1} \omega_i^k \left(r_{j+k+1}^n - r_{j+k}^n\right) \\
&= v_i^{\max} \left\|\psi'\right\|_\infty \Delta x \left(-\omega_i^0 r_j^n + \sum_{k=1}^{N-1} \left(\omega_i^{k-1} - \omega_i^k\right) r_{j+k}^n\right) \\
&\leq v_i^{\max} \left\|\psi'\right\|_\infty \Delta x M \left\|\rho^n\right\|_\infty \omega_i(0)
\end{aligned} \tag{2.7}$$

where $\|\rho\|_\infty = \left\|(\rho_1, \dots, \rho_M)\right\|_\infty = \max_{i,j} \rho_{i,j}$. So, until $\left\|\rho^n\right\|_\infty \leq K$, for some $K \geq \left\|\rho^0\right\|_\infty$, we get

$$\left\|\rho^{n+1}\right\|_\infty \leq \left\|\rho^n\right\|_\infty \left(1 + MKv_M^{\max} \left\|\psi'\right\|_\infty W_0 \Delta t\right),$$

which implies

$$\left\|\rho^n\right\|_\infty \leq \left\|\rho^0\right\|_\infty e^{Cn\Delta t},$$

with $C = MKv_M^{\max} \left\|\psi'\right\|_\infty W_0$. Therefore we get that $\left\|\rho(t, \cdot)\right\|_\infty \leq K$ for

$$t \leq \frac{1}{MKv_M^{\max} \left\|\psi'\right\|_\infty W_0} \ln\left(\frac{K}{\left\|\rho^0\right\|_\infty}\right) \leq \frac{1}{Me\left\|\rho^0\right\|_\infty v_M^{\max} \left\|\psi'\right\|_\infty W_0},$$

where the maximum is attained for $K = e\|\rho^0\|_\infty$.

Iterating the procedure, at time t^m, $m \geq 1$ we set $K = e^m\|\rho^0\|_\infty$ and we get that the solution is bounded by K until t^{m+1} such that

$$t^{m+1} \leq t^m + \frac{m}{Me^m\|\rho^0\|_\infty v_M^{\max}\|\psi'\|_\infty W_0}.$$

Therefore, the approximate solution remains bounded, uniformly in x, at least for $t \leq T$ with

$$T \leq \frac{1}{M\|\rho^0\|_\infty v_M^{\max}\|\psi'\|_\infty W_0}\sum_{m=1}^{+\infty}\frac{m}{e^m} \leq \frac{1}{M\|\rho^0\|_\infty v_M^{\max}\|\psi'\|_\infty W_0}.$$

□

Remark 1. Unlike the classical multi-population model [3], the simplex

$$S := \left\{\rho \in \mathbb{R}^M : \sum_{i=1}^{M}\rho_i \leq 1,\ \rho_i \geq 0 \text{ for } i = 1, \dots, M\right\}$$

is not an invariant domain for (1.1), see Figure 1 for a numerical example.

Indeed, let us consider the system

$$\partial_t\rho_i(t,x) + \partial_x\big(\rho_i(t,x)v_i(r(t,x))\big) = 0, \qquad i = 1, ..., M, \tag{2.8}$$

where r and v_i are as in (1.2) and (1.3), respectively. We have the following:

Lemma 3. *Under the CFL condition*

$$\lambda \leq \frac{1}{v_M^{\max}\left(\|\psi\|_\infty + \|\psi'\|_\infty\right)},$$

for any initial datum $\rho_0 \in S$ the approximate solutions to (2.8) computed by the upwind scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda\left[\mathbf{F}(\rho_j^n, \rho_{j+1}^n) - \mathbf{F}(\rho_{j-1}^n, \rho_j^n)\right], \tag{2.9}$$

with $\mathbf{F}(\rho_j^n, \rho_{j+1}^n) = \rho_j^n\,\psi(r_{j+1}^n)$, satisfy the following uniform bounds:

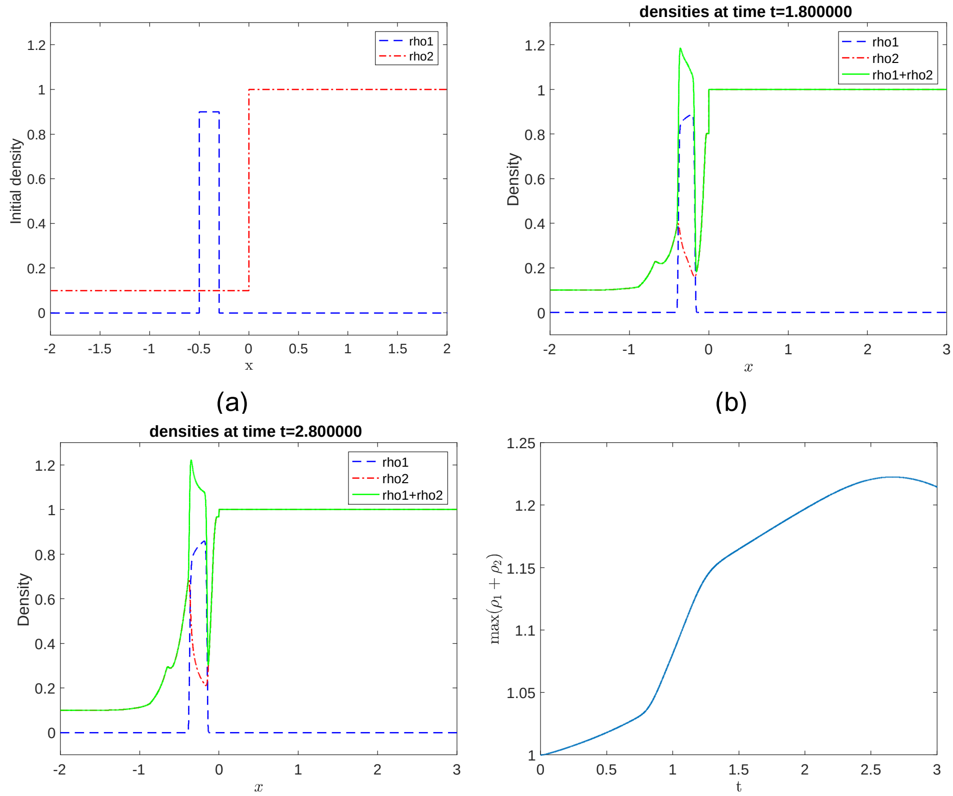
$$\rho_j^n \in S \quad \forall j \in \mathbb{Z},\quad n \in \mathbb{N}.$$

Proof. Assuming that $\rho_j^n \in S$ for all $j \in \mathbb{Z}$, we want to prove that $\rho_j^{n+1} \in S$. Rewriting (2.9), we get

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n - \lambda\left[v_i^{\max}\rho_{i,j}^n\psi(r_{j+1}^n) - v_i^{\max}\rho_{i,j-1}^n\psi(r_j^n)\right].$$

Summing on the index $i = 1, \dots, M$, gives

$$\begin{aligned} r_j^{n+1} = \sum_{i=1}^{M}\rho_{i,j}^{n+1} &= \sum_{i=1}^{M}\rho_{i,j}^n - \lambda\sum_{i=1}^{M}\left[v_i^{\max}\rho_{i,j}^n\psi(r_{j+1}^n) - v_i^{\max}\rho_{i,j-1}^n\psi(r_j^n)\right] \\ &= r_j^n + \lambda\psi(r_j^n)\sum_{i=1}^{M}v_i^{\max}\rho_{i,j-1}^n - \lambda\psi(r_{j+1}^n)\sum_{i=1}^{M}v_i^{\max}\rho_{i,j}^n. \end{aligned}$$



(a) (b) (c) (d)

Figure 1: Numerical simulation illustrating that the simplex S is not an invariant domain for (1.1). We take $M = 2$ and we consider the initial conditions $\rho_1(0, x) = 0.9\,\chi_{[-0.5,-0.3]}$ and $\rho_2(0, x) = 0.1\,\chi_{]-\infty,0]} + \chi_{]0,+\infty[}$ depicted in (a), the constant kernels $\omega_1(x) = \omega_2(x) = 1/\eta$, $\eta = 0.5$, and the speed functions given by $v_1^{max} = 0.2$, $v_2^{max} = 1$, $\psi(\xi) = \max\{1-\xi, 0\}$ for $\xi \geq 0$. The space and time discretization steps are $\Delta x = 0.001$ and $\Delta t = 0.4\,\Delta x$. Plots (b) and (c) show the density profiles of ρ_1, ρ_2 and their sum r at times $t = 1.8,\ 2.8$. The function $\max_{x\in\mathbb{R}} r(t, x)$ is plotted in (d), showing that r can take values greater than 1, even if $r(0, x) = \rho_1(0, x) + \rho_2(0, x) \leq 1$.

Defining the following function of ρ_j^n

$$\Phi(\rho_{1,j}^n, \ldots, \rho_{M,j}^n) = r_j^n + \lambda\,\psi(r_j^n)\sum_{i=1}^{M} v_i^{max}\rho_{i,j}^n - \lambda\,\psi(r_{j+1}^n)\sum_{i=1}^{M} v_i^{max}\rho_{i,j}^n,$$

we observe that

$$\Phi(0,\ldots,0) = \psi(0)\,\lambda \sum_{i} v_i^{max}\rho_{i,j}^n \leq \lambda\,\|\psi\|_1\, v_M^{max} \leq 1$$

if $\lambda \leq 1/v_M^{max}\|\psi\|_1$ and

$$\Phi(\rho_{1,j}^n, \ldots, \rho_{M,j}^n) = 1 - \lambda\,\psi(r_{j+1}^n)\sum_{i=1}^{M} v_i^{max}\rho_{i,j}^n \leq 1$$

for $\rho_j^n \in S$ such that $r_j^n = \sum_{i=1}^{M} \rho_{i,j}^n = 1$. Moreover

$$\frac{\partial \Theta}{\partial \rho_{i,j}^n}(\rho_j^n) = 1 + \lambda \psi'(r_j^n) \sum_{i=1}^{M} v_i^{\max} \rho_{i,j-1}^n - \lambda \psi(r_{j+1}^n) v_i^{\max} \geq 0$$

if $\lambda \leq 1/v_M^{\max}\left(\|\psi\|_\infty + \|\psi'\|_\infty\right)$. This proves that $r_j^{n+1} \leq 1$.

To prove the positivity of (2.9), we observe that

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n\left(1 - \lambda v_i^{\max}\psi(r_{j+1}^n)\right) + \lambda v_i^{\max} \rho_{i,j-1}^n \psi(r_j^n) \geq 0$$

if $\lambda \leq 1/v_M^{\max}\|\psi\|_\infty$. $\square$

Lemma 4.(Spatial BV-bound) *Let $\rho_i^0 \in (\mathbf{BV} \cap \mathbf{L}^1)(\mathbb{R}, \mathbb{R}^+)$ for all $i = 1, ..., M$. If (2.3) holds, then the approximate solution $\rho^{\Delta x}(t, \cdot)$ constructed by the algorithm (2.2) has uniformly bounded total variation for $t \in [0, T]$, for any T such that*

$$T \leq \min_{i=1,\dots,M} \frac{1}{H\,\mathrm{TV}(\rho_i^0) + 1}, \tag{2.10}$$

where $H = \|\rho\|_\infty v_M^{\max} W_0 M \left(6MJ_0\|\rho\|_\infty \|\psi''\|_\infty + \|\psi'\|_\infty\right)$.

Proof. Subtracting the identities

$$\rho_{i,j+1}^{n+1} = \rho_{i,j+1}^n - \lambda\left(\rho_{i,j+1}^n V_{i,j+2}^n - \rho_{i,j}^n V_{i,j+1}^n\right), \tag{2.11}$$

$$\rho_{i,j}^{n+1} = \rho_{i,j}^n - \lambda\left(\rho_{i,j}^n V_{i,j+1}^n - \rho_{i,j-1}^n V_{i,j}^n\right), \tag{2.12}$$

and setting $\Delta_{i,j+1/2}^n = \rho_{i,j+1}^n - \rho_{i,j}^n$, we get

$$\Delta_{i,j+1/2}^{n+1} = \Delta_{i,j+1/2}^n - \lambda\left(\rho_{i,j+1}^n V_{i,j+2}^n - 2\rho_{i,j}^n V_{i,j+1}^n + \rho_{i,j-1}^n V_{i,j}^n\right).$$

Now, we can write

$$\begin{aligned}
\Delta_{i,j+1/2}^{n+1} = {} & \left(1 - \lambda V_{i,j+2}^n\right)\Delta_{i,j+1}^n \\
& + \lambda V_{i,j}^n \Delta_{i,j-1/2}^n \qquad (2.13)\\
& - \lambda \rho_{i,j}^n\left(V_{i,j+2}^n - 2V_{i,j+1}^n + V_{i,j}^n\right). \qquad (2.14)
\end{aligned}$$

Observe that assumption (2.3) guarantees the positivity of (2.13). The term (2.14) can be estimated as

$$\begin{aligned}
& V_{i,j+2}^n - 2V_{i,j+1}^n + V_{i,j}^n = \\
& = v_i^{\max}\left(\psi\left(\sum_{k=0}^{N-1}\omega_i^k r_{j+k+2}^n\right) - 2\psi\left(\sum_{k=0}^{N-1}\omega_i^k r_{j+k+1}^n\right) + \psi\left(\sum_{k=0}^{N-1}\omega_i^k r_{j+k}^n\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= v_i^{\max}\,\psi'(\xi_{j+1})\,\Delta x\left(\sum_{k=0}^{N-1}\omega_i^k r^n_{j+k+2} - \sum_{k=0}^{N-1}\omega_i^k r^n_{j+k+1}\right)\\
&\quad + v_i^{\max}\,\psi'(\xi_j)\,\Delta x\left(\sum_{k=0}^{N-1}\omega_i^k r^n_{j+k} - \sum_{k=0}^{N-1}\omega_i^k r^n_{j+k+1}\right)\\
&= v_i^{\max}\,\psi'(\xi_{j+1})\,\Delta x\left(\sum_{k=1}^{N-1}(\omega_i^{k-1}-\omega_i^k)r^n_{j+k+1} - \omega_i^0 r^n_{j+1}\right)\\
&\quad + v_i^{\max}\,\psi'(\xi_j)\,\Delta x\left(\sum_{k=1}^{N-1}(\omega_i^{k}-\omega_i^{k-1})r^n_{j+k} + \omega_i^0 r^n_{j}\right)\\
&= v_i^{\max}\left(\psi'(\xi_{j+1}) - \psi'(\xi_j)\right)\Delta x\left(\sum_{k=1}^{N-1}(\omega_i^{k-1}-\omega_i^k)r^n_{j+k+1} - \omega_i^0 r^n_{j+1}\right)\\
&\quad + v_i^{\max}\,\psi'(\xi_j)\,\Delta x\left(\sum_{k=1}^{N-1}(\omega_i^{k-1}-\omega_i^k)(r^n_{j+k+1} - r^n_{j+k}) + \omega_i^0(r^n_j - r^n_{j+1})\right)\\
&= v_i^{\max}\,\psi''(\tilde\xi_{j+1/2})(\xi_{j+1}-\xi_j)\,\Delta x\left(\sum_{k=1}^{N-1}\sum_{\ell=1}^{M}\omega_i^k\rho^n_{\ell,j+k+3/2}\right)\\
&\quad + v_i^{\max}\,\psi'(\xi_j)\,\Delta x\left(\sum_{\ell=1}^{M}\sum_{k=1}^{N-1}(\omega_i^{k-1}-\omega_i^k)\rho^n_{\ell,j+k+1/2} - \omega_i^0\rho^n_{\ell,j+1/2}\right),
\end{aligned}
$$

with $\xi_j \in \mathcal{I}\left(\Delta x\sum_{k=0}^{N-1}\omega_i^k r^n_{j+k},\ \Delta x\sum_{k=0}^{N-1}\omega_i^k r^n_{j+k+1}\right)$ and $\tilde\xi_{j+1/2} \in \mathcal{I}\left(\xi_j, \xi_{j+1}\right)$, where we set $\mathcal{I}(a, b) = \left[\min\{a, b\}, \max\{a, b\}\right]$. For some $\theta, \mu \in [0, 1]$, we compute

$$
\begin{aligned}
\xi_{j+1} - \xi_j &= \theta\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+2} + (1-\theta)\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1}\\
&\quad - \mu\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1} - (1-\mu)\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k}\\
&= \theta\,\Delta x\sum_{k=1}^{N-1}\omega_i^{k-1}\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1} + (1-\theta)\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1}\\
&\quad - \mu\,\Delta x\sum_{k=0}^{N-1}\omega_i^k\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1} - (1-\mu)\,\Delta x\sum_{k=-1}^{N-1}\omega_i^{k+1}\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1}\\
&= \Delta x\sum_{k=1}^{N-1}\Big[\theta\omega_i^{k-1} + (1-\theta)\omega_i^k - \mu\omega_i^k - (1-\mu)\omega_i^{k+1}\Big]\sum_{\ell=1}^{M}\rho^n_{\ell,j+k+1}\\
&\quad + (1-\theta)\,\Delta x\,\omega_i^0\sum_{\ell=1}^{M}\rho^n_{\ell,j-1} - \mu\,\Delta x\,\omega_i^0\sum_{\ell=1}^{M}\rho^n_{\ell,j+1}
\end{aligned}
$$

$$
(1-\mu)\,\Delta x\left(\omega_i^0\sum_{\ell=1}^{M}\rho^n_{\ell,j}+\omega_i^1\sum_{\ell=1}^{M}\rho^n_{\ell,j+1}\right).
$$

By monotonicity of ω_i we have

$$
\theta\omega_i^{k-1}+(1-\theta)\omega_i^k-\mu\omega_i^k-(1-\mu)\omega_i^{k+1}\geq 0\,.
$$

Taking the absolute values we get

$$
\begin{aligned}
|\rho_{j+1}-\rho_j| &\leq \Delta x\left\{\sum_{k=2}^{N-1}\left[\theta\omega_i^{k-1}+(1-\theta)\omega_i^k-\mu\omega_i^k-(1-\mu)\omega_i^{k+1}\right]+4\omega_i^0\right\}M\|\rho^n\|_1\\
&\leq \Delta x\left\{\sum_{k=2}^{N-1}\left[\omega_i^{k-1}-\omega_i^{k+1}\right]+4\omega_i^0\right\}M\|\rho^n\|_1\\
&\leq \Delta x\,6W_0M\|\rho^n\|_1\,.
\end{aligned}
$$

Until $\sum_j|\rho^n_{\ell,j}|\leq K_1$ for $\ell=1,\ldots,M$ for some $K_1\geq\sum_j|\rho^0_{\ell,j}|$, taking the absolute values and rearranging the indexes, we have

$$
\sum_j|\Delta^{n+1}_{i,j+1/2}|\leq\sum_j|\Delta^n_{i,j+1/2}|\left(1-\Delta t\left(V^n_{i,j+2}-V^n_{i,j+1}\right)\right)+\Delta t\,HK_1,
$$

where $H=\|\rho\|_1\, v_M^{\max}W_0M\left(6MJ_0\|\rho\|_1\|\omega''\|_1+\|\omega'\|_1\right)$. Therefore, by (2.7) we get

$$
\sum_j|\Delta^{n+1}_{i,j+1/2}|\leq\sum_j|\Delta^n_{i,j+1/2}|(1+\Delta t\,G)+\Delta t\,HK_1,
$$

with $G=v_M^{\max}\|\omega'\|_1W_0M\|\rho\|_1$. We thus obtain

$$
\sum_j|\Delta^n_{i,j+1/2}|\leq e^{Gn\Delta t}\sum_j|\Delta^0_{i,j+1/2}|+e^{HK_1n\Delta t}-1,
$$

that we can rewrite as

$$
\begin{aligned}
\mathrm{TV}(\rho_i^{\Delta x})(n\Delta t,\cdot) &\leq e^{Gn\Delta t}\,\mathrm{TV}(\rho_i^0)+e^{HK_1n\Delta t}-1\\
&\leq e^{HK_1n\Delta t}\left(\mathrm{TV}(\rho_i^0)+1\right)-1\,,
\end{aligned}
$$

since $H\geq G$ and it is not restrictive to assume $K_1\geq 1$. Therefore, we have that $\mathrm{TV}(\rho_i^{\Delta x})\leq K_1$ for

$$
t\leq\frac{1}{HK_1}\ln\left(\frac{K_1+1}{\mathrm{TV}(\rho_i^0)+1}\right),
$$

where the maximum is attained for some $K_1 < e \left(\mathrm{TV}(\rho_i^0) + 1 \right) - 1$ such that

$$\ln \left(\frac{K_1 + 1}{\mathrm{TV}(\rho_i^0) + 1} \right) = \frac{K_1}{K_1 + 1}.$$

Therefore the total variation is uniformly bounded for

$$t \leq \frac{1}{He \left(\mathrm{TV}(\rho_i^0) + 1 \right)}.$$

Iterating the procedure, at time t^m, $m \geq 1$ we set $K_1 = e^m \left(\mathrm{TV}(\rho_i^0) + 1 \right) - 1$ and we get that the solution is bounded by K_1 until t^{m+1} such that

$$t^{m+1} \leq t^m + \frac{m}{He^m \left(\mathrm{TV}(\rho_i^0) + 1 \right)}. \tag{2.15}$$

Therefore, the approximate solution has bounded total variation for $t \leq T$ with

$$T \leq \frac{1}{H \, \mathrm{TV}(\rho_i^0) + 1}.$$

$\square$

Corollary 3. *Let $\rho_i^0 \in (\mathbf{BV} \cap \mathbf{L}^1)(\mathbb{R}; \mathbb{R}^+)$. If (2.3) holds, then the approximate solution $\rho_i^{\Delta x}$ constructed by the algorithm (2.2) has uniformly bounded total variation on $[0, T] \times \mathbb{R}$, for any T satisfying (2.10).*

Proof. If $T \leq \Delta t$, then $\mathrm{TV}(\rho_i^{\Delta x}; [0, T] \times \mathbb{R}) \leq T \, \mathrm{TV}(\rho_i^0)$. Let us assume now that $T > \Delta t$. Let $n_T \in \mathbb{N} \setminus \{0\}$ such that $n_T \Delta t < T \leq (n_T + 1) \Delta t$. Then

$$\begin{aligned}
&\mathrm{TV}(\rho_i^{\Delta x}; [0, T] \times \mathbb{R}) \\
&= \underbrace{\sum_{n=0}^{n_T - 1} \sum_{j \in \mathbb{Z}} \Delta t \left| \rho_{i,j+1}^n - \rho_{i,j}^n \right| + (T - n_T \Delta t) \sum_{j \in \mathbb{Z}} \left| \rho_{i,j+1}^{n_T} - \rho_{i,j}^{n_T} \right|}_{\leq T \sup_{t \in [0,T]} \mathrm{TV}(\rho_i^{\Delta x})(t, \cdot)} + \sum_{n=0}^{n_T - 1} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right|.
\end{aligned}$$

We then need to bound the term

$$\sum_{n=0}^{n_T - 1} \sum_{j \in \mathbb{Z}} \Delta x \left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right|.$$

From the definition of the numerical scheme (2.2), we obtain

$$\begin{aligned}
\rho_{i,j}^{n+1} - \rho_{i,j}^n &= \lambda \left(\rho_{i,j-1}^n V_{i,j}^n - \rho_{i,j}^n V_{i,j+1}^n \right) \\
&= \lambda \left(\rho_{i,j-1}^n \left(V_{i,j}^n - V_{i,j+1}^n \right) + V_{i,j+1}^n \left(\rho_{i,j-1}^n - \rho_{i,j}^n \right) \right).
\end{aligned}$$

Taking the absolute values and using (2.7) we obtain

$$\left| \rho_{i,j}^{n+1} - \rho_{i,j}^n \right| \leq \lambda \left(v_i^{\max} \left\| \psi' \right\|_{\infty} M \left\| \vec{\rho}^n \right\|_1 \left\| \omega_i \right\|_{\infty} \Delta x \, \rho_{i,j-1}^n + v_i^{\max} \left\| \psi \right\|_{\infty} \left| \rho_{i,j-1}^n - \rho_{i,j}^n \right| \right).$$

Summing on j, we get

$$\sum_{j\in\mathbb{Z}} \Delta x \left|\rho^{n+1}_{i,j} - \rho^{n}_{i,j}\right| \le v_i^{\max}\left(\|\omega'\|_1 M\|\rho^n\|_1 + \omega_i(0)\right)\Delta t \sum_{j\in\mathbb{Z}} \Delta x \left|\rho^n_{i,j-1}\right| + v_i^{\max}\|\omega\|_1\,\Delta t\sum_{j\in\mathbb{Z}}\left|\rho^n_{i,j-1} - \rho^n_{i,j}\right|,$$

which yields

$$\sum_{n=0}^{N_T-1}\sum_{j\in\mathbb{Z}} \Delta x \left|\rho^{n+1}_{i,j} - \rho^{n}_{i,j}\right|$$
$$\le v_M^{\max}\|\omega\|_1\, T \sup_{t\in[0,T]} \mathrm{TV}\,(\rho_i^{\Delta x})(t,\cdot) + v_M^{\max}\|\omega'\|_1 M W_0 T \sup_{t\in[0,T]} \left\|\rho_i^{\Delta x}(t,\cdot)\right\|_1 \left\|\rho^{\Delta x}(t,\cdot)\right\|_1$$

that is bounded by Corollary 2, Lemma 2 and Lemma 4. $\square$

3 Proof of Theorem 1

To complete the proof of the existence of solutions to the problem (1.1), (1.5), we follow a Lax-Wendroff type argument as in [4], see also [9], to show that the approximate solutions constructed by scheme (2.2) converge to a weak solution of (1.1). By Lemma 2, Lemma 4 and Corollary 3, we can apply Helly's theorem, stating that for $i = 1,\dots,M$, there exists a subsequence, still denoted by $\rho_i^{\Delta x}$, which converges to some $\rho_i \in (\mathbf{L}^1 \cap \mathbf{BV})([0,T]\times\mathbb{R};\mathbb{R}^+)$ in the $\mathbf{L}^1_{\mathbf{loc}}$-norm. Let us fix $i \in \{1,\dots,M\}$. Let $\varphi \in \mathbf{C}^1_c([0,T[\times\mathbb{R})$ and multiply (2.2) by $\varphi(t^n, x_j)$. Summing over $j \in \mathbb{Z}$ and $n \in \{0,\dots,N_T\}$ we get

$$\sum_{n=0}^{N_T-1}\sum_j \varphi(t^n, x_j)\left(\rho^{n+1}_{i,j} - \rho^n_{i,j}\right)$$
$$= -\lambda\sum_{n=0}^{N_T-1}\sum_j \varphi(t^n, x_j)\left(\rho^n_{i,j}V^n_{i,j+1} - \rho^n_{i,j-1}V^n_{i,j}\right).$$

Summing by parts we obtain

$$\sum_j \varphi((N_T-1)\Delta t, x_j)\rho^{N_T}_{i,j} + \sum_j \varphi(0,x_j)\rho^0_{i,j} + \sum_{n=1}^{N_T-1}\sum_j\left(\varphi(t^n,x_j) - \varphi(t^{n-1},x_j)\right)\rho^n_{i,j}$$
$$+\lambda\sum_{n=0}^{N_T-1}\sum_j\left(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\right)V^n_{i,j+1}\rho^n_{i,j} = 0. \tag{3.1}$$

Multiplying by Δx we get

$$\Delta x\sum_j \varphi((N_T-1)\Delta t, x_j)\rho^{N_T}_{i,j} + \Delta x\sum_j \varphi(0,x_j)\rho^0_{i,j} \tag{3.2}$$
$$+\Delta x\,\Delta t\sum_{n=1}^{N_T-1}\sum_j\frac{\varphi(t^n,x_j) - \varphi(t^{n-1},x_j)}{\Delta t}\rho^n_{i,j} \tag{3.3}$$

$$+\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}V^n_{i,j+1}\,\rho^n_{i,j}=0. \tag{3.4}$$

By $\mathbf{L}^1_{\mathbf{loc}}$ convergence of $\rho_i^{\Delta x}\to\rho_i$, it is straightforward to see that the terms in (3.2), (3.3) converge to

$$\int_{\mathbb{R}}\left(\rho_i^0(x)\varphi(0,x)-\rho_i(T,x)\varphi(T,x)\right)dx+\int_0^T\int_{\mathbb{R}}\rho_i(t,x)\partial_t\varphi(t,x)\,dx\,dt\,, \tag{3.5}$$

as $\Delta x\to 0$. Concerning the last term (3.4), we can rewrite

$$\begin{aligned}
&\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}V^n_{i,j+1}\,\rho^n_{i,j}\\
=\;&\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\left(\rho^n_{i,j}V^n_{i,j+1}-\rho^n_{i,j}V^n_{i,j}\right)\\
+\;&\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\rho^n_{i,j}V^n_{i,j}\,.
\end{aligned} \tag{3.6}$$

By (2.7) we get the estimate

$$\left|\rho^n_{i,j}V^n_{i,j+1}-\rho^n_{i,j}V^n_{i,j}\right|\le v_i^{\max}\,\|\psi'\|_\infty\,\Delta x\,M\|\boldsymbol{\rho}\|^2_\infty\,\omega_i(0).$$

Set $R>0$ such that $\varphi(t,x)=0$ for $|x|>R$ and $j_0, j_1\in\mathbb{Z}$ such that $-R\in\,]x_{j_0-\frac12},x_{j_0+\frac12}]$ and $R\in\,]x_{j_1-\frac12},x_{j_1+\frac12}]$, then

$$\begin{aligned}
&\left|\Delta x\,\Delta t\sum_{n=0}^{N_T}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\left(\rho^n_{i,j}V^n_{i,j+1}-\rho^n_{i,j}V^n_{i,j}\right)\right|\\
\le\;&\Delta x\,\Delta t\,\|\partial_x\varphi\|_\infty\sum_{n=0}^{N_T}\sum_{j=j_0}^{j_1}v_i^{\max}\,\|\psi'\|_\infty\,M\|\boldsymbol{\rho}\|^2_\infty\,\omega_i(0)\,\Delta x\\
\le\;&\|\partial_x\varphi\|_\infty\,v_i^{\max}\,\|\psi'\|_\infty\,M\|\boldsymbol{\rho}\|^2_\infty\,\omega_i(0)\,\Delta x\,2RT\,,
\end{aligned}$$

which goes to zero as $\Delta x\to 0$.

Finally, again by the $\mathbf{L}^1_{\mathbf{loc}}$ convergence of $\rho_i^{\Delta x}\to\rho_i$, we have that

$$\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\rho^n_{i,j}V^n_{i,j}\to\int_0^T\int_{\mathbb{R}}\partial_x\varphi(t,x)\rho_i(t,x)v_i(\boldsymbol{\rho}*\omega_i)\,dx\,dt\,.$$

4 Numerical tests

In this section we perform some numerical simulations to illustrate the behaviour of solutions to (1.1) for $M=2$ modeling two different scenarios. In the following, the space mesh is set to $\Delta x=0.001$.

4.1 Cars and trucks mixed traffic

In this example, we consider a stretch of road populated by cars and trucks. The space domain is given by the interval $[-2, 3]$ and we impose absorbing conditions at the boundaries, adding $N_1 = \eta_1/\Delta x$ ghost cells for the first population and $N_2 = \eta_2/\Delta x$ for the second one at the right boundary, and just one ghost cell for both populations at the left boundary, where we extend the solution constantly equal to the last value inside the domain. The dynamics is described by the following 2×2 system

$$
\begin{cases}
\partial_t \rho_1(t,x) + \partial_x \left(\rho_1(t,x) v_1^{\max} \psi((r * \omega_1)(t,x)) \right) = 0, \\
\partial_t \rho_2(t,x) + \partial_x \left(\rho_2(t,x) v_2^{\max} \psi((r * \omega_2)(t,x)) \right) = 0,
\end{cases} \tag{4.1}
$$

with

$$
\begin{aligned}
&\omega_1(x) = \frac{2}{\eta_1}\left(1 - \frac{x}{\eta_1}\right), && \eta_1 = 0.3, \\
&\omega_2(x) = \frac{2}{\eta_2}\left(1 - \frac{x}{\eta_2}\right), && \eta_2 = 0.1, \\
&\psi(\xi) = \max\{1 - \xi, 0\}, && \xi \geq 0, \\
&v_1^{max} = 0.8, \quad v_2^{max} = 1.3.
\end{aligned}
$$

In this setting, ρ_1 represents the density of trucks and ρ_2 is the density of cars on the road. Trucks moves at lower maximal speed than cars and have grater view horizon, but of the same order of magnitude. Figure 2 describes the evolution in time of the two population densities, correspondent to the initial configuration

$$
\begin{cases}
\rho_1(0,x) = 0.5\,\chi_{[-1.1,\,-1.6]}, \\
\rho_2(0,x) = 0.5\,\chi_{[-1.6,\,-1.9]},
\end{cases}
$$

in which a platoon of trucks precedes a group of cars. Due to their higher speed, cars overtake trucks, in accordance with what observed in the local case [3].

4.2 Impact of connected autonomous vehicles

The aim of this test is to study the possible impact of the presence of Connected Autonomous Vehicles (CAVs) on road traffic performances. Let us consider a circular road modeled by the space interval $[-1, 1]$ with periodic boundary conditions at $x = \pm 1$. In this case, we assume that autonomous and non-autonomous vehicles have the same maximal speed, but the interaction radius of CAVs is two orders of magnitude grater than the one of human-driven cars. Moreover, we assume CAVs have constant convolution kernel, modeling the fact that they have the same degree of accuracy on information about surrounding traffic, independent from the distance. In this case, model (1.1) reads

$$
\begin{cases}
\partial_t \rho_1(t,x) + \partial_x \left(\rho_1(t,x) v_1^{\max} \psi((r * \omega_1)(t,x)) \right) = 0, & i = 1, \dots, M, \\
\partial_t \rho_2(t,x) + \partial_x \left(\rho_2(t,x) v_2^{\max} \psi((r * \omega_2)(t,x)) \right) = 0, \\
\rho_1(0,x) = \alpha (0.5 + 0.3 \sin(5\pi x)), \\
\rho_2(0,x) = (1 - \alpha)(0.5 + 0.3 \sin(5\pi x)),
\end{cases} \tag{4.2}
$$

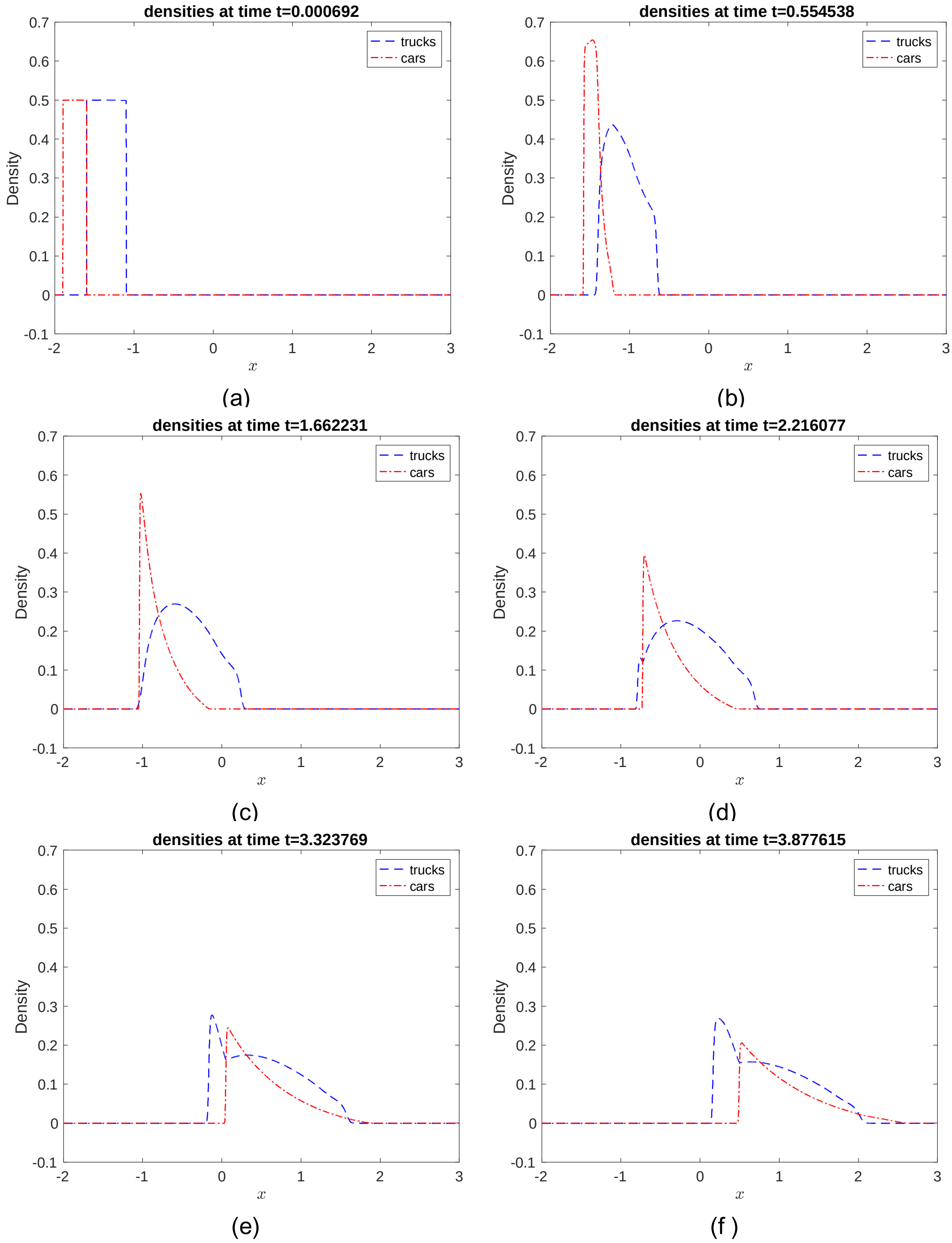


Figure 2: Density profiles of cars and trucks at increasing times corresponding to the non-local model (4.1).

with

$$!_1(x) = \frac{1}{\mathscr{H}_1} \qquad \mathscr{H}_1 = 1,$$

$$
\begin{aligned}
&\omega_2(x) = \frac{2}{\eta_2}\left(1 - \frac{x}{\eta_2}\right), && \eta_2 = 0.01,\\
&\psi(\rho) = \max\{1 - \rho, 0\}, && \rho \geq 0,\\
&v_1^{max} = v_2^{max} = 1.
\end{aligned}
$$

Above ρ_1 represents the density of autonomous vehicles, ρ_2 the density of non-autonomous vehicles and $\alpha \in [0, 1]$ is the penetration rate of autonomous vehicle. Figure 3 displays the traffic dynamics in the case $\alpha = 0.9$.

As a metric of traffic congestion, given a time horizon $T > 0$, we consider the two following functionals:

$$
J(\alpha) = \int_0^T \mathrm{d}|\partial_x r|\, \mathrm{d}t, \tag{4.3}
$$

$$
\Phi(\alpha) = \int_0^T \left(\rho_1(t, \bar{x}) v_1^{\max}\, \psi((r * \omega_1)(t, \bar{x})) + \rho_2(t, \bar{x}) v_2^{\max}\, \psi((r * \omega_2)(t, \bar{x}))\right) \mathrm{d}t, \tag{4.4}
$$

where $\bar{x} = x_0 - 0$. The functional J measures the integral with respect to time of the spatial total variation of the total traffic density, see [5]. Instead, the functional Φ measures the integral with respect to time of the traffic flow at a given point $\bar{x}$, corresponding to the number of cars that have passe through $\bar{x}$ in the studied time interval. Figure 4 displays the values of the functionals J and Φ for different values of $\alpha = 0, 0.1, 0.2, \ldots, 1$. We can notice that the functionals are not monotone and present minimum and maximum values. The traffic evolution patterns corresponding these stationary velues are reported in Figure 5, showing the (t, x)-plots of the total traffic density $r(t, x)$ corresponding to these values of α.

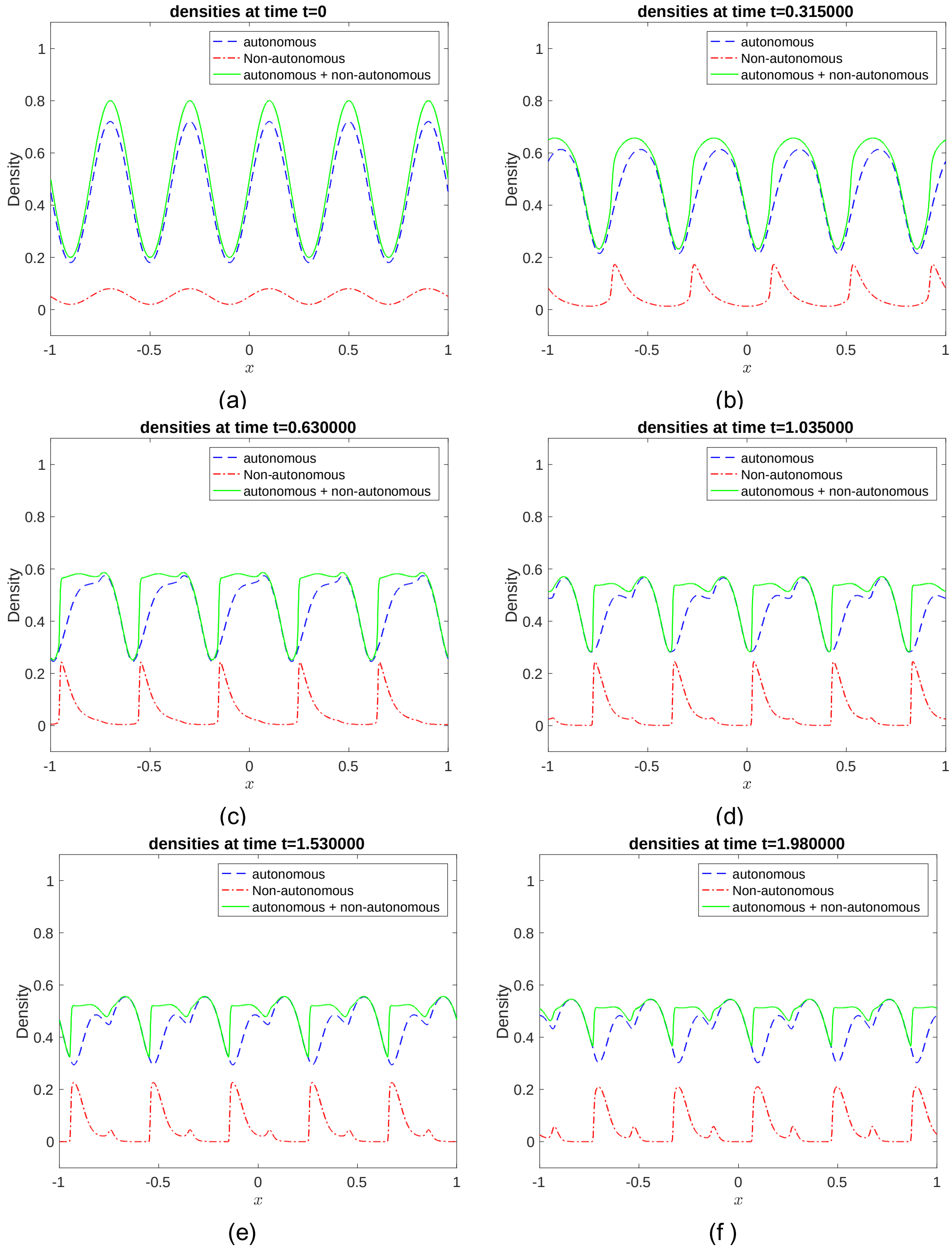


Figure 3: Density profiles corresponding to the non-local problem (4.2) with = 0.9 at different times.

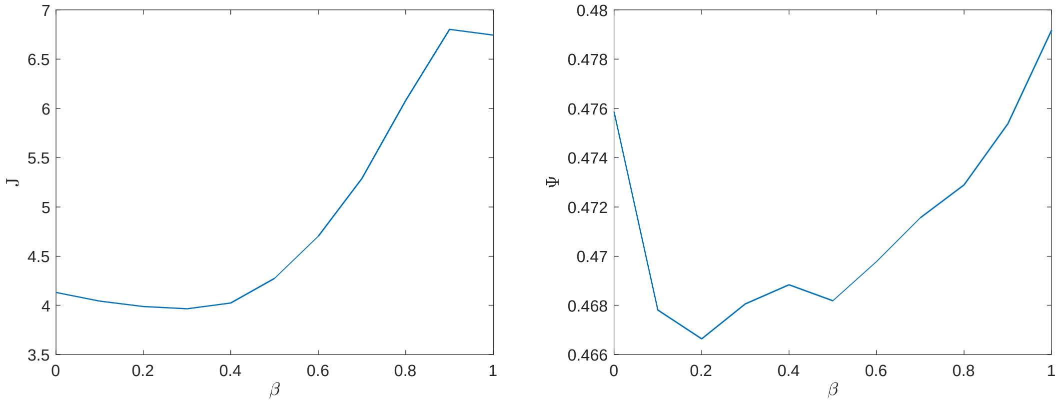
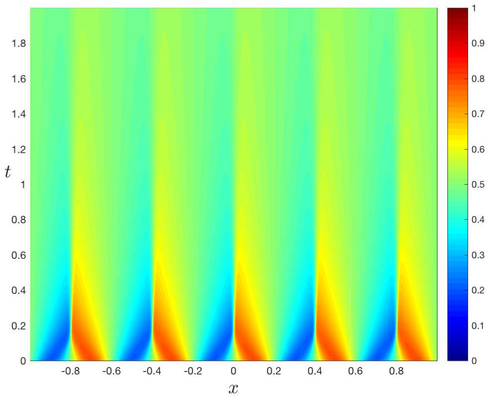
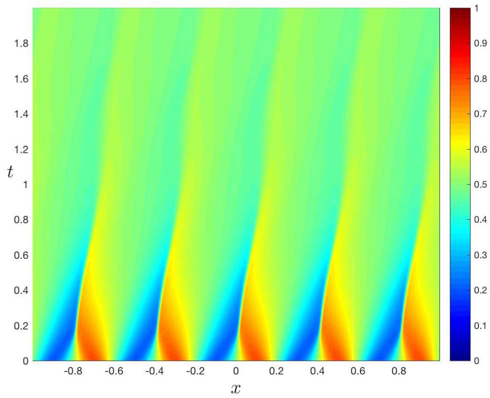


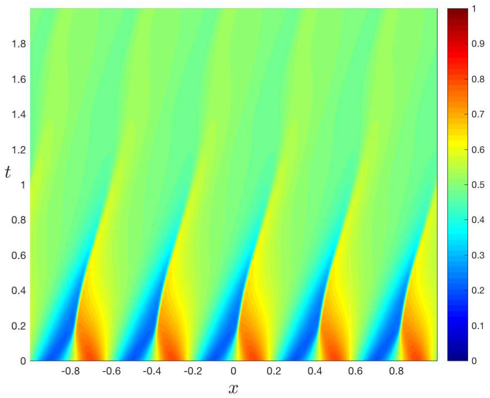
Figure 4: Functional J (left) and (right)



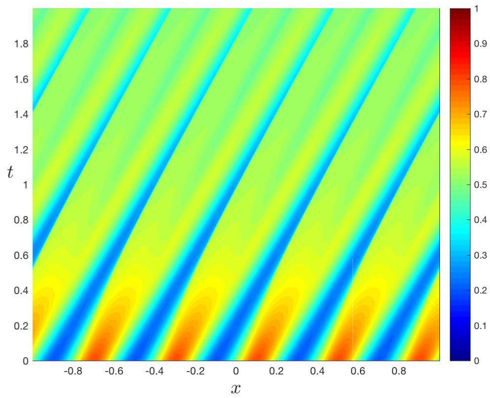
(a) = 0



(b) = 0.2



(c) = 0.3



(d) = 0.9

Figure 5: (t, x)-plots of the total traffic density $r(t, x) = \rho_1(t, x) + \rho_2(t, x)$ in (4.2) corresponding to different values of : (a) no autonomous vehicles are present; (b) point of minimum for ; (c) point of minimum for J; (d) point of maximum for J.

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Appendix A Lax-Friedrichs numerical scheme

We provide here an alternative existence proof for (1.1), based on approximate solutions constructed via the following adapted Lax-Friedrichs scheme:

$$\rho^{n+1}_{i,j} = \rho^n_{i,j} - \lambda \left(F^n_{i,j+1/2} - F^n_{i,j-1/2} \right), \tag{A.1}$$

with

$$F^n_{i,j+1/2} := \frac{1}{2}\rho^n_{i,j} V^n_{i,j} + \frac{1}{2}\rho^n_{i,j+1} V^n_{i,j+1} + \frac{\alpha}{2\lambda}\left(\rho^n_{i,j} - \rho^n_{i,j+1} \right), \tag{A.2}$$

where $\alpha \geq 1$ is the viscosity coefficient and $\lambda = \frac{\Delta t}{\Delta x}$.

Lemma 5. *For any $T > 0$, under the CFL conditions*

$$\lambda\alpha < 1, \tag{A.3}$$

$$\alpha \geq \lambda v_M^{\max} \| \omega \|_1 , \tag{A.4}$$

the scheme (A.2)-(A.1) is positivity preserving on $[0, T] \times \mathbb{R}$.

Proof. Let us assume that $\rho^n_{i,j} \geq 0$ for all $j \in \mathbb{Z}$ and $i \in \{1, ..., M\}$. It suffices to prove that $\rho^{n+1}_{i,j}$ in (A.1) is non-negative. Compute

$$\rho^{n+1}_{i,j} = \rho^n_{i,j} + \frac{1}{2}\lambda(\rho^n_{i,j+1} - 2\rho^n_{i,j} + \rho^n_{i,j-1}) + \frac{1}{2}\left[\rho^n_{i,j-1}V^n_{i,j-1} - \rho^n_{i,j+1}V^n_{i,j+1}\right] \tag{A.5}$$

$$= \rho^n_{i,j-1}\frac{1}{2}\left[\lambda + V^n_{i,j-1}\right] + \rho^n_{i,j}(1-\lambda) + \rho^n_{i,j+1}\frac{1}{2}\left[\lambda - V^n_{i,j+1}\right], \tag{A.6}$$

under assumptions (A.3) and (A.4), we obtain that $\rho^{n+1}_{i,j}$ is positive. ∎

Corollary 4. ($\mathbf{L^1}$ bound) *For any $T > 0$, under the CFL conditions (A.3)-(A.4) the scheme (A.2)-(A.1) preserves the $\mathbf{L^1}$ norm of the i-th component of $\rho^{\Delta x}$.*

Proof. See proof of Corollary 2. ∎

Lemma 6. ($\mathbf{L^1}$-bound) *If $\rho^0_{i,j} \geq 0$ for all $j \in \mathbb{Z}$ and $i = 1, ..., M$, and the CFL conditions (A.3)-(A.4) hold, the approximate solution $\rho^{\Delta x}$ constructed by the algorithm (A.2)-(A.1) is uniformly bounded on $[0, T] \times \mathbb{R}$ for any T such that*

$$T < \left(M\|\rho^0\|_1 v^{max}_M \|\omega'\|_1 W_0\right)^{-1}. \tag{A.7}$$

Proof. From (A.6) we can define

$$\rho^{n+1}_{i,j} = \frac{1}{2}\rho^n_{i,j-1}\left[\lambda + V^n_{i,j-1}\right] + (1-\lambda)\rho^n_{i,j} + \frac{1}{2}\rho^n_{i,j+1}\left[\lambda - V^n_{i,j+1}\right]$$

Let $\bar{\rho} = \max\left\{\rho^n_{i,j-1}, \rho^n_{i,j}, \rho^n_{i,j+1}\right\}$. Then we get

$$\rho^{n+1}_{i,j} \leq \bar{\rho}\left(1 + \frac{1}{2}\left[V^n_{i,j-1} - V^n_{i,j+1}\right]\right)$$

and by (2.7)

$$V^n_{i,j-1} - V^n_{i,j+1} \leq 2v^{max}_M \|\omega'\|_1 \Delta x\, \omega_i(0) M \|\rho\|_1. \tag{A.8}$$

Therefore, until $\|\rho^n\|_1 \leq K$, for some $K \geq \|\rho^0\|_1$, we get

$$\|\rho^{n+1}\|_1 \leq \|\rho^n\|_1\left[1 + MKv^{max}_M \|\omega'\|_1 W_0 \Delta t\right],$$

and we can reason as in the proof of Lemma 2. ∎

Lemma 7. (BV estimates) *Let* $\rho_i^0 \in (\mathbf{BV} \cap \mathbf{L}^\infty)(\mathbb{R}, \mathbb{R}^+)$ *for all* $i = 1, \ldots, M$. *If* (A.4) *holds and*

$$\Delta t \leq \frac{2}{2\gamma + \Delta x \|\omega'\|_1 W_0 v_M^{\max} \|\rho\|_1} \Delta x, \tag{A.9}$$

then the solution constructed by the algorithm (A.2)-(A.1) *has uniformly bounded total variation for any* T *such that*

$$T \leq \min_{i=1,\ldots,M} \frac{1}{D \, \mathrm{TV}(\rho_i^0) + 1}, \tag{A.10}$$

where $D = \|\rho\|_1 v_M^{\max} W_0 M \left(3MJ_0 \|\rho\|_1 \|\omega''\|_1 + 2\|\omega'\|_1\right)$.

Proof. Subtracting the following expressions

$$\rho^{n+1}_{i,j+1} = \rho^n_{i,j+1} + \frac{\gamma}{2}\left(\rho^n_{i,j} - 2\rho^n_{i,j+1} + \rho^n_{i,j+2}\right) + \frac{\lambda}{2}\left(\rho^n_{i,j} V^n_{i,j} - \rho^n_{i,j+2} V^n_{i,j+2}\right),$$

$$\rho^{n+1}_{i,j} = \rho^n_{i,j} + \frac{\gamma}{2}\left(\rho^n_{i,j-1} - 2\rho^n_{i,j} + \rho^n_{i,j+1}\right) + \frac{\lambda}{2}\left(\rho^n_{i,j-1} V^n_{i,j-1} - \rho^n_{i,j+1} V^n_{i,j+1}\right),$$

we get

$$\begin{aligned}\Delta^{n+1}_{i,j+1/2} = {} & \frac{\gamma}{2}\Delta^{n+1}_{i,j-1/2} + (1-\gamma)\Delta^n_{i,j+1/2} + \frac{\gamma}{2}\Delta^{n+1}_{i,j+3/2} \\ & + \frac{\lambda}{2}\left(V^n_{i,j}\Delta^n_{i,j-1/2} + \rho^n_{i,j-1}\left(V^n_{i,j} - V^n_{i,j-1}\right) - V^n_{i,j+2}\Delta_{i,j+3/2} + \rho_{i,j+1}\left(V^n_{i,j+1} - V^n_{i,j+2}\right)\right).\end{aligned}$$

Now, we can write

$$\begin{aligned}V^n_{i,j} - V^n_{i,j-1} &= v_i^{\max}\, \psi'(\xi_{j-1/2})\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \left(\rho^n_{\ell,j+k} - \rho^n_{\ell,j+k-1}\right) \\ &= v_i^{\max}\, \psi'(\xi_{j-1/2})\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \Delta^n_{\ell,j+k-1/2} \\ &= v_i^{\max}\, \psi'(\xi_{j-1/2})\, \Delta x \left(\sum_{\ell=1}^{M} \sum_{k=0}^{N-1} (\omega_i^k - \omega_i^{k+1})\rho_{\ell,j+k} - \omega_i^0 \rho_{\ell,j-1}\right),\end{aligned}$$

and

$$\begin{aligned}V^n_{i,j+2} - V^n_{i,j+1} &= v_i^{\max}\, \psi'(\xi_{j+3/2})\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \left(\rho^n_{\ell,j+k+2} - \rho^n_{\ell,j+k+1}\right) \\ &= v_i^{\max}\, \psi'(\xi_{j+3/2})\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \Delta^n_{\ell,j+k+3/2} \\ &= v_i^{\max}\, \psi'(\xi_{j+3/2})\, \Delta x \left(\sum_{\ell=1}^{M} \sum_{k=1}^{N-1} (\omega_i^{k-1} - \omega_i^{k})\rho_{\ell,j+k+1} - \omega_i^0 \rho_{\ell,j+1}\right).\end{aligned}$$

We get

$$ {}^{n+1}_{i,j+1/2} = $$

$$ = \frac{\lambda}{2}\left(\alpha + V^n_{i,j}\right){}^n_{i,j-1/2} + (1-\alpha)\,{}^n_{i,j+1/2} + \frac{\lambda}{2}\left(\alpha - V^n_{i,j+2}\right){}^n_{i,j+3/2} $$

$$ + \frac{\lambda}{2}\rho^n_{i,j-1}\left(v_i^{\max}\,{}'(\xi_{j-1/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=0}^{\ell-1}(\omega_i^k - \omega_i^{k+1})\rho^n_{i,j+k} - \omega_i^0\rho^n_{i,j-1}\right)\right) $$

$$ - \frac{\lambda}{2}\rho^n_{i,j+1}\left(v_i^{\max}\,{}'(\xi_{j+3/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=1}^{\ell-1}(\omega_i^{k-1} - \omega_i^{k})\rho^n_{i,j+k+1} - \omega_i^0\rho^n_{i,j+1}\right)\right) $$

$$ = \frac{\lambda}{2}\left(\alpha + V^n_{i,j}\right){}^n_{i,j-1/2} + (1-\alpha)\,{}^n_{i,j+1/2} + \frac{\lambda}{2}\left(\alpha - V^n_{i,j+2}\right){}^n_{i,j+3/2} $$

$$ + \frac{\lambda}{2}(\rho^n_{i,j-1} - \rho^n_{i,j+1})\left(V^n_{i,j} - V^n_{i,j-1}\right) $$

$$ + \frac{\lambda}{2}\rho^n_{i,j+1}\Bigg[v_i^{\max}\,{}'(\xi_{j-1/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=0}^{\ell-1}(\omega_i^k - \omega_i^{k+1})\rho^n_{i,j+k} - \omega_i^0\rho^n_{i,j-1}\right) $$

$$ \pm v_i^{\max}\,{}'(\xi_{j+3/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=0}^{\ell-1}(\omega_i^k - \omega_i^{k+1})\rho^n_{i,j+k} - \omega_i^0\rho^n_{i,j-1}\right) $$

$$ - v_i^{\max}\,{}'(\xi_{j+3/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=1}^{\ell-1}(\omega_i^{k-1} - \omega_i^{k})\rho^n_{i,j+k+1} - \omega_i^0\rho^n_{i,j+1}\right)\Bigg] $$

$$ = \frac{\lambda}{2}\left(\alpha + V^n_{i,j}\right){}^n_{i,j-1/2} + (1-\alpha)\,{}^n_{i,j+1/2} + \frac{\lambda}{2}\left(\alpha - V^n_{i,j+2}\right){}^n_{i,j+3/2} $$

$$ - \frac{\lambda}{2}\left(V^n_{i,j} - V^n_{i,j-1}\right)\left({}^n_{i,j-1/2} + {}^n_{i,j+1/2}\right) $$

$$ + \frac{\lambda}{2}\rho^n_{i,j+1}\Bigg[v_i^{\max}\,{}''(\xi_{j+1})(\xi_{j-1/2} - \xi_{j+3/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=0}^{\ell-1}(\omega_i^k - \omega_i^{k+1})\rho^n_{i,j+k} - \omega_i^0\rho^n_{i,j-1}\right) $$

$$ + v_i^{\max}\,{}'(\xi_{j+3/2}) \times \left(\sum_{\ell=1}^{M}\sum_{k=0}^{\ell-1}(\omega_i^k - \omega_i^{k+1})(\rho^n_{i,j+k} - \rho^n_{i,j+k+2}) - \omega_i^0(\rho^n_{i,j-1} - \rho^n_{i,j+1})\right)\Bigg] $$

$$ = \frac{\lambda}{2}\left(\alpha + V^n_{i,j-1}\right){}^n_{i,j-1/2} \tag{A.11} $$

$$ + \left(1 - \alpha - \frac{\lambda}{2}\left(V^n_{i,j} - V^n_{i,j-1}\right)\right){}^n_{i,j+1/2} \tag{A.12} $$

$$ + \frac{\lambda}{2}\left(\alpha - V^n_{i,j+2}\right){}^n_{i,j+3/2} \tag{A.13} $$

$$ + \frac{\lambda}{2}\rho^n_{i,j+1}\Bigg[v_i^{\max}\,{}''(\tilde{\xi}_{j+1})(\xi_{j-1/2} - \xi_{j+3/2}) \times \left(\sum_{k=0}^{M-1}\omega_i^k\sum_{\ell=k+1}^{M}{}^n_{i,j+k-1/2}\right) $$

$$+\, v_i^{\max}\, V'(\tilde\rho_{j+3/2})\, \Delta x \left(\sum_{k=0}^{N-1} (\omega_i^{k+1} - \omega_i^{k}) \sum_{\ell=1}^{M} \left(\rho^n_{\ell,j+k+1/2} + \rho^n_{\ell,j+k+3/2} \right) + \omega_i^{0} \sum_{\ell=1}^{M} \left(\rho^n_{\ell,j-1/2} + \rho^n_{\ell,j+1/2} \right) \right) \Bigg] .$$

where $\tilde\rho_{j+1} \in I(\rho_{j+1/2}, \rho_{j+3/2})$. For some $\alpha, \beta \in [0, 1]$, we compute

$$\begin{aligned}
\rho_{j-1/2} - \rho_{j+3/2} &= \alpha\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} + (1-\alpha)\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \rho^n_{\ell,j+k-1} \\
&\quad - \beta\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \rho^n_{\ell,j+k+2} - (1-\beta)\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \rho^n_{\ell,j+k+1} \\
&= \alpha\, \Delta x \sum_{k=0}^{N-1} \omega_i^k \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} + (1-\alpha)\, \Delta x \sum_{k=-1}^{N-1} \omega_i^{k+1} \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} \\
&\quad - \beta\, \Delta x \sum_{k=2}^{N-1} \omega_i^{k-2} \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} - (1-\beta)\, \Delta x \sum_{k=1}^{N-1} \omega_i^{k-1} \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} \\
&= \Delta x \sum_{k=2}^{N-1} \left[\alpha\omega_i^k + (1-\alpha)\omega_i^{k+1} - \beta\omega_i^{k-2} - (1-\beta)\omega_i^{k-1} \right] \sum_{\ell=1}^{M} \rho^n_{\ell,j+k} \\
&\quad + \alpha\, \Delta x \left(\omega_i^0 \sum_{\ell=1}^{M} \rho^n_{\ell,j} + \omega_i^1 \sum_{\ell=1}^{M} \rho^n_{\ell,j+1} \right) \\
&\quad + (1-\alpha)\, \Delta x \left(\omega_i^0 \sum_{\ell=1}^{M} \rho^n_{\ell,j-1} + \omega_i^1 \sum_{\ell=1}^{M} \rho^n_{\ell,j} + \omega_i^2 \sum_{\ell=1}^{M} \rho^n_{\ell,j+1} \right) \\
&\quad - (1-\beta)\, \Delta x \left(\omega_i^0 \sum_{\ell=1}^{M} \rho^n_{\ell,j+1} \right) .
\end{aligned}$$

By monotonicity of ω_η we have

$$\alpha\omega_i^k + (1-\alpha)\omega_i^{k+1} - \beta\omega_i^{k-2} - (1-\beta)\omega_i^{k-1} \le 0.$$

Taking the absolute values we get

$$\begin{aligned}
|\rho_{j-1/2} - \rho_{j+3/2}| &\le \Delta x \left\{ \sum_{k=2}^{N-1} \left[\beta\omega_i^{k-2} + (1-\beta)\omega_i^{k-1} - \alpha\omega_i^k - (1-\alpha)\omega_i^{k+1} \right] + 3\omega_i(0) \right\} M \|\rho^n\|_1 \\
&\le \Delta x \left\{ \sum_{k=2}^{N-2} \left[\omega_i^{k-2} - \omega_i^{k+1} \right] + 3\omega_i(0) \right\} M \|\rho^n\|_1 \\
&\le \Delta x\, 6\, \omega_i(0) M \|\rho^n\|_1 .
\end{aligned}$$

Observe that assumption (A.4) guarantees the positivity of (A.11) and (A.13). Similarly, (A.9) ensures the positivity of (A.12).

Until $\sum_j {}^{n}_{,j} \leq K_1$ for $= 1, \dots, M$ for some $K_1 \quad \sum_j {}^{0}_{,j}$, taking the absolute values and rearranging the indexes, we have

$$\sum_j {}^{n+1}_{i,j+1/2} \leq \sum_j {}^{n}_{i,j+1/2} \left(1 + \frac{}{2} \quad V_{i,j} \quad {}_1 \quad V_{i,j+1} \right) + \quad t\, DK_1$$

where $D = k k_1 \quad v^{\max}_M W_0 M \left(3MJ \,{}_0 k k_1 \quad {}^{00}_{1} + 2 \quad {}^{0}_{1} \right)$. Therefore, by (A.8) we get

$$\sum_j {}^{n+1}_{i,j+1/2} \leq \sum_j {}^{n}_{i,j+1/2} \quad (1 + \quad t\, C) + \quad t\, DK_1$$

with $C = v^{\max}_M \quad {}^{0}_{1} \; W_0 M k k_1$. In this way we obtain

$$\sum_j {}^{n+1}_{i,j+1/2} \leq e^{Cn \quad t} \sum_j {}^{0}_{i,j+1/2} + e^{DK_1 n \quad t} \quad 1,$$

that we can rewrite as

$$\begin{aligned} \mathrm{TV}\,(_{i}, \quad _x)(n \quad t, \cdot) &\leq e^{Cn \quad t}\, \mathrm{TV}\,(^0_i) + e^{DK_1 n \quad t} \quad 1 \\ &\leq e^{DK_1 n \quad t} \left(\mathrm{TV}\,(^0_i) + 1 \right) \quad 1, \end{aligned} \tag{A.14}$$

since $D \quad 2C$ and it is not restrictive to assume $K_1 \quad \frac{1}{2}$. Therefore we have that $\mathrm{TV}\,(_{,} \quad _x) \leq K_1$ for

$$t \leq \frac{1}{DK_1} \ln \left(\frac{K_1 + 1}{\mathrm{TV}\,(^0_i) + 1} \right),$$

where the maximum is attained for some $K_1 \leq e \left(\mathrm{TV}\,(^0_i) + 1 \right) \quad 1$ such that

$$\ln \left(\frac{K_1 + 1}{TV(^0_i) + 1} \right) = \frac{K_1}{K_1 + 1}.$$

Therefore the total variation is uniformly bounded for

$$t \leq \frac{1}{De \left(\mathrm{TV}\,(^0_i) + 1 \right)}.$$

Iterating the procedure, at time t^m, $m \quad 1$ we set $K_1 = e^m \left(\mathrm{TV}\,(^0_i) + 1 \right) \quad 1$ and we get that the solution is bounded by K_1 until t^{m+1} such that

$$t^{m+1} \leq t^m + \frac{m}{De^m \left(\mathrm{TV}\,(^0_i) + 1 \right)}. \tag{A.15}$$

Therefore, the approximate solution satisfies the bound (A.14) for $t \leq T$ with

$$T \leq \frac{1}{D\, \mathrm{TV}\,(^0_i) + 1}.$$

Corollary 5. *Let $\bar{\rho} \in$* **BV**$(\mathbb{R};[0,1])$. *If (A.3)-(A.4) holds, then the approximate solution $\rho^{\Delta x}$ constructed by the algorithm (A.2)-(A.1) has uniformly bounded total variation on $[0,T] \times \mathbb{R}$, for any T satisfying (A.10).*

Proof. Let us fix $T \in \mathbb{R}^+$ such that (A.10) and (A.7) hold. If $T \leq \Delta t$, then $\mathrm{TV}(\rho_i^{\Delta x}; \mathbb{R} \times [0,T]) \leq T\, \mathrm{TV}(\rho_{i,0})$. Let us assume now that $T > \Delta t$. Let $M \in \mathbb{N}\setminus\{0\}$ such that $n_T \Delta t < T \leq (n_T + 1)\Delta t$. Then

$$
\begin{aligned}
&\mathrm{TV}(\rho_i^{\Delta x}; \mathbb{R} \times [0,T]) && \text{(A.16)}\\
&= \underbrace{\sum_{n=0}^{n_T-1}\sum_{j\in\mathbb{Z}} \Delta t \left|\rho^n_{i,j+1} - \rho^n_{i,j}\right| + (T - n_T \Delta t)\sum_{j\in\mathbb{Z}} \left|\rho^{n_T}_{i,j+1} - \rho^{n_T}_{i,j}\right|}_{\leq T \sup_{t\in[0,T]} \mathrm{TV}(\rho_i^{\Delta x})(t,\cdot)} + \sum_{n=0}^{n_T-1}\sum_{j\in\mathbb{Z}} \Delta x \left|\rho^{n+1}_{i,j} - \rho^n_{i,j}\right| && \text{(A.17)}
\end{aligned}
$$

We then need to bound the term

$$
\sum_{n=0}^{n_T-1}\sum_{j\in\mathbb{Z}} \Delta x \left|\rho^{n+1}_{i,j} - \rho^n_{i,j}\right|. \tag{A.18}
$$

Let us make use of the definition of the numerical scheme (A.2)-(A.1), we obtain

$$
\begin{aligned}
\rho^{n+1}_{i,j} - \rho^n_{i,j} &= \frac{\lambda}{2}\left(\alpha + V_{i,j+1}\right)\left(\rho_{i,j-1} - \rho_{i,j}\right) - \frac{\lambda}{2}\left(\alpha - V_{i,j+1}\right)\left(\rho_{i,j} - \rho_{i,j+1}\right)\\
&\quad + \frac{\lambda}{2}\rho_{i,j-1}\left(V_{i,j-1} - V_{i,j+1}\right).
\end{aligned}
$$

If (A.4) holds, we can take the absolute value

$$
\begin{aligned}
\left|\rho^{n+1}_{i,j} - \rho^n_{i,j}\right| &\leq \frac{\lambda}{2}\left(\alpha + V_{i,j+1}\right)\left|\rho_{i,j-1} - \rho_{i,j}\right| + \frac{\lambda}{2}\left(\alpha - V_{i,j+1}\right)\left|\rho_{i,j} - \rho_{i,j+1}\right|\\
&\quad + \frac{\lambda}{2}\rho_{i,j-1}\left|V_{i,j-1} - V_{i,j+1}\right|.
\end{aligned}
$$

Summing on j and rearranging the indexes we get

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \Delta x \left|\rho^{n+1}_{i,j} - \rho^n_{i,j}\right| &\leq \frac{\Delta t}{2}\sum_{j\in\mathbb{Z}}\left|\rho_{i,j+1} - \rho_{i,j}\right|\left(2\alpha + V_{i,j+2} - V_{i,j+1}\right)\\
&\quad + \frac{\Delta t}{2}\sum_{j\in\mathbb{Z}}\rho_{i,j-1}\left|V_{i,j-1} - V_{i,j+1}\right|\\
&\leq \frac{\Delta t}{2}\sum_{j\in\mathbb{Z}}\left|\rho_{i,j+1} - \rho_{i,j}\right|\left(2\alpha + v^{\max}_M\left(\left\|\omega'\right\|_1 + \Delta x\, \omega(0)\right)M\left\|\rho\right\|_1\right)
\end{aligned}
$$

$$+ \Delta t \sum_{j\in\mathbb{Z}} \|\rho_{i,j}\|_1 \, \Delta x \, \|v_M^{\max}\|_{\infty} \|W_0\| M \|\vec{k}\|_1$$

which yields

$$\sum_{n=0}^{n_T-1}\sum_{j\in\mathbb{Z}} \Delta x \left|\rho_{i,j}^{n+1}-\rho_{i,j}^{n}\right| \tag{A.19}$$

$$\le T e^{C_1 n_T \Delta t}\left(\mathrm{TV}(\rho_i^0)+1\right)\left(\lambda + \frac{1}{2}\|v_M^{\max}\|_{\infty} \Delta x \, W_0 M \|\vec{k}\|_1\right) \tag{A.20}$$

$$+ T \|\vec{\rho}\|_{\mathbf{L^1}} \|v_M^{\max}\|_{\infty} W_0 M \|\vec{k}\|_1 . \tag{A.21}$$

$\square$

Proof of Theorem 1. Let us define

$$g(\rho_{i,j}^n, \ldots, \rho_{i,j+N}^n) := \frac{1}{2}\rho_{i,j}^n V_{i,j}^n + \frac{1}{2}\rho_{i,j+1}^n V_{i,j+1}^n + \frac{\alpha}{2}\left(\rho_{i,j}^n - \rho_{i,j+1}^n\right).$$

Fix $i \in \{1, \ldots, M\}$. Let $\varphi \in \mathbf{C}_c^1([0, T] \times \mathbb{R})$ and multiply (A.1) by $\varphi(t^n, x_j)$. Summing over $j \in \mathbb{Z}$ and $n \in \{0, 1, \ldots, n_T\}$ we get

$$\sum_{n=0}^{n_T-1}\sum_j \varphi(t^n, x_j)\left(\rho_{i,j}^{n+1} - \rho_{i,j}^n\right)$$

$$= -\lambda \sum_{n=0}^{n_T-1}\sum_j \varphi(t^n, x_j)\left(g(\rho_{i,j}^n, \ldots, \rho_{i,j+N}^n) - g(\rho_{i,j-1}^n, \ldots, \rho_{i,j+N-1}^n)\right).$$

Summing by parts we obtain

$$\sum_j \varphi((n_T - 1)\Delta t, x_j)\rho_{i,j}^{n_T} + \sum_j \varphi(0, x_j)\rho_{i,j}^0 + \sum_{n=1}^{n_T-1}\sum_j \left(\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j)\right)\rho_{i,j}^n$$

$$+ \lambda \sum_{n=0}^{n_T-1}\sum_j \left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\right) g(\rho_{i,j}^n, \ldots, \rho_{i,j+N}^n) = 0. \tag{A.22}$$

Multiplying by Δx

$$\Delta x \sum_j \varphi((n_T - 1)\Delta t, x_j)\rho_{i,j}^{n_T} + \Delta x \sum_j \varphi(0, x_j)\rho_{i,j}^0 + \Delta x \, \Delta t \sum_{n=1}^{n_T-1}\sum_j \frac{\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j)}{\Delta t}\rho_{i,j}^n \tag{A.23}$$

$$+ \Delta x \, \Delta t \sum_{n=0}^{n_T-1}\sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} g(\rho_{i,j}^n, \ldots, \rho_{i,j+N}^n) = 0. \tag{A.24}$$

By $\mathbf{L^1_{loc}}$ convergence of $\rho_{i,\Delta x} \to \rho_i$, it is straightforward to see that the first two terms in (A.23) converge to

$$\int_{\mathbb{R}} \left(\rho_i^0(x)\varphi(0, x) - \rho_i(T, x)\varphi(T, x)\right) dx + \int_0^T\int_{\mathbb{R}} \rho_i(t, x)\partial_t\varphi(t, x)\, dx\, dt \tag{A.25}$$

as $\Delta x \to 0$. Concerning the last term, we can observe that

$$
\begin{aligned}
& g(\rho^n_{i,j},\dots,\rho^n_{i,j+N}) - \rho^n_{i,j}V^n_{i,j} \\
\le\; & \frac{\alpha}{2}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right| + \frac12\left(\rho^n_{i,j+1}-\rho^n_{i,j}\right)V^n_{i,j+1} + \rho^n_{i,j}\left(V^n_{i,j+1}-V^n_{i,j}\right) \\
\le\; & \frac{\alpha + v^{\max}_M\|k\|_1}{2}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right| + \frac12 W_0\,\Delta x\,\mathrm{TV}\left(\rho_{i,\Delta x}(t^n,\cdot)\right)v^{\max}_M\|\omega'\|_1 \\
\le\; & \frac{\alpha + v^{\max}_M\|k\|_1}{2}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right| + J\,\Delta x.
\end{aligned}
$$

where $J = \frac12 v^{\max}_M\|\omega'\|_1 W_0\,\mathrm{TV}(\rho_{i,\Delta x}(T,\cdot))$. Therefore, the last term in (A.22) can be rewritten as

$$
\begin{aligned}
& \Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\, g(\rho^n_{i,j},\dots,\rho^n_{i,j+N}) \\
=\; & \Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\,\rho^n_{i,j}V^n_{i,j} \\
+\; & \Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\left(g(\rho^n_{i,j},\dots,\rho^n_{i,j+N}) - \rho^n_j V^n_{i,j}\right).
\end{aligned}
$$

By $\mathbf{L}^1_{\mathbf{loc}}$ convergence of $\rho_{i,\Delta x}\to\rho_i$ and boundedness of ω, the first term in the above decomposition converges to

$$
\int_0^T\int_{\mathbb R}\rho_i(t,x)v(r\ast\omega)\partial_x\varphi(t,x)\,dx\,dt\,.
$$

Set $R>0$ such that $\varphi(t,x)=0$ for $|x|>R$ and $j_0, j_1\in\mathbb Z$ such that $-R\in\,]x_{j_0-1/2},x_{j_0+1/2}[$ and $R\in\,]x_{j_1-1/2},x_{j_1+1/2}[$, then

$$
\begin{aligned}
& \Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\left(g(\rho^n_{i,j},\dots,\rho^n_{i,j+N}) - \rho^n_{i,j}V^n_{i,j}\right) \\
\le\; & \Delta x\,\Delta t\|\partial_x\varphi\|_1\sum_{n=0}^{N_T-1}\sum_{j=j_0}^{j_1}\left(\frac{\alpha+v^{\max}_M\|k\|_1}{2}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right| + J\,\Delta x\right) \\
=\; & \frac{\alpha+v^{\max}_M\|k\|_1}{2}\|\partial_x\varphi\|_1\,\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_{j=j_0}^{j_1}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right| + \|\partial_x\varphi\|_1\,J\,\Delta x\,2R \\
\le\; & \frac{\alpha+v^{\max}_M\|k\|_1}{2}\|\partial_x\varphi\|_1\,\mathrm{TV}(\rho_{i,\Delta x}(T,\cdot))\,\Delta x + \|\partial_x\varphi\|_1\,J\,\Delta x\,2R
\end{aligned}
$$

which goes to zero when $\Delta x\to 0$. Finally, again by the $\mathbf{L}^1_{\mathbf{loc}}$ convergence of $\rho_i^{\Delta x}\to\rho_i$, we have that

$$
\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\,\rho^n_{i,j}V^n_{i,j} \to \int_0^T\int_{\mathbb R}\partial_x\varphi(t,x)\rho_i(t,x)v_i(r\ast\omega_i)\,dx\,dt\,.
$$