Fractality and Size Effect in Fatigue Damage Accumulation: Comparison Between Paris and Wöhler Perspectives

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FRACTALITY AND SIZE EFFECT IN FATIGUE DAMAGE ACCUMULATION: COMPARISON BETWEEN PARIS AND WÖHLER PERSPECTIVES

Francesco Montagnoli[1], Stefano Invernizzi[1], and Alberto Carpinteri[1]

[1]Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129, Torino, Italy
e-mail: francesco.montagnoli@polito.it, stefano.invernizzi@polito.it, alberto.carpinteri@polito.it

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Abstract. *Fatigue life assessment can be carried out according to two well-known approaches. The first is based on the so-called Paris' law that express the sub-critical crack growth rate as a function of the stress-intensity factor range. The second is based on Wöhler's curve that relates the applied stress range with the number of cycles to failure. Both approaches exhibit scale effects, which can be explained in the framework of dimensional analysis and intermediate asymptotics concepts. More recently, the application of fractal geometry concepts provided an alternative way to obtain similar scaling laws. In this new framework, it is possible to conceive that the propagating crack of the Paris formulation is characterised by an invasive fractal roughness, whereas in the Wöhler context the material ligament is represented with a lacunar fractal, in order to account for cross-sectional weakening due to inherent defects. In particular, scaling laws are found for the fatigue threshold* ΔK_{th} *and fatigue limit* $\Delta\sigma_{fl}$*. The former increases with the crack length, whereas the latter decreases with the specimen size. The fractal formulations for the two approaches are presented and put in comparison. Subsequently, the size effects represented by the two fractal formulations are compared with some recent results available in the literature.*

1 INTRODUCTION

The traditional fatigue design of a structural component, known as the cumulative fatigue damage approach, is based on the determination of Wöhler's curve of an un-notched specimen to estimate the fatigue limit of the material. Although this criterion is very easy to apply, the fatigue crack propagation mechanism is not explicitly taken into account.

Since the first pioneering Wöhler's work at the end of the XIX Century [1], much effort has been spent to understand the fatigue failure mechanisms.

In particular, the introduction of the linear elastic fracture mechanics in the 50's led to the development of the well-known Paris law [2] and the so-called damage tolerance approach, which dramatically reduced the number of disasters and fatalities.

In addition, the improvement of the inspection methods furtherly improved the safety levels both in the aeronautical and mechanical fields, making possible to assess when a structural component needed to be substituted.

Subsequently, several researchers reported deviations of the experimental results from the original Paris' law. Among all, the most relevant deviation concerns the anomalous behaviour of short cracks, characterised by higher crack growth rates than expected [3] by decreasing the crack length. In addition, Kitagawa and Takahashi [4] noticed that the fatigue threshold decreases with the crack length, whilst it becomes constant only for sufficiently long cracks. Many conjectures have been put forward including plasticity crack closure and crack roughness, which introduce an effective stress-intensity factor [5].

More recently, Carpinteri [6] exploited fractal geometry concepts in the framework of fracture mechanics. If an invasive fractal fatigue crack surface is considered [7, 8], the so-called short cracks behaviour can be reproduced consistently.

Size effects have been also found for Wöhler's curve, where the fatigue strength depends on the specimen size [9]. This evidence was revealed also thanks to the application of ultrasonic fatigue testing machines, being possible to perform fatigue tests up to 10^9 cycles in less than 14 hours.

Murakami [10] tried to relate the fatigue limit with the dimension of the existing defects in the material, exploiting the weakest-link concept. As a consequence, the larger the structural size, the lower the fatigue limit will be [11]. On the other hand, if the ligament is considered as a lacunar fractal set, similar dependence of the fatigue limit on the structural size can be obtained [12, 13].

In the following sections, the theories of intermediate asymptotics and fractal geometry will be revisited both in the context of Paris' and Wöhler's laws, and compared to some recent experimental results.

2 THE CRACK-SIZE EFFECTS ON PARIS' LAW

2.1 Intermediate asymptotics and fractal geometry

According to experimental evidences, the fatigue crack propagation phenomenon is influenced by different parameters, which can be subdivided in three main categories, i.e. the loading conditions, the material properties, and a geometric parameter. Thus, the fatigue crack propagation rate can be written in terms of six different parameters that govern the phenomenon [14, 15]:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \Pi\left(\Delta K_I, R; K_{IC}, \sigma_u, \Delta K_{th}; a\right). \tag{1}$$

Applying dimensional analysis and intermediate asymptotics theories, in case of incomplete self-similarity, the relationship between the fatigue crack propagation rate and the different parameters of Eq.1 becomes a power-law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{IC}}{\sigma_u}\right)^2 \left(\frac{\Delta K_I}{K_{IC}}\right)^{\alpha_1} (1-R)^{\alpha_2} \left(\frac{\Delta K_{th}}{K_{IC}}\right)^{\alpha_3} \left(a\left(\frac{\sigma_u}{K_{IC}}\right)^2\right)^{\alpha_4} \tag{2}$$

Note that several phenomenological relations, as Forman's or Priddle's equations to account for $(1-R)$ dependency, are empirically provided. Similarly, the short cracks' anomalous behaviour can be herein explained as an incomplete self-similarity of Paris' law with respect to the dimensionless parameter $a\left(\sigma_u/K_{IC}\right)^2$.

Alternatively, if the crack length is provided as an invasive fractal set [6] with dimension $1+d_{\mathcal{G}}$, the crack advancement rate and the stress-intensity factor range become as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{\mathrm{d}a^*}{\mathrm{d}N}\frac{a^{-d_{\mathcal{G}}}}{1+d_{\mathcal{G}}} \tag{3a}$$

$$\Delta K_I = \Delta K_I^* a^{d_{\mathcal{G}}/2} \tag{3b}$$

If these expressions are substituted in the classical Paris' law, the C parameter is no longer constant, but shows the following dependency on the crack length:

$$C\left(a\right) = C^* \frac{a^{-d_{\mathcal{G}}(1+m/2)}}{1+d_{\mathcal{G}}} \tag{4}$$

given that the scale invariant Paris' law is considered [7]:

$$\frac{\mathrm{d}a^*}{\mathrm{d}N} = C^* \Delta K_I^{*m} \tag{5}$$

As a consequence, Eq. 4 implies a negative scaling of the crack length on the fatigue crack growth, i.e. the larger the crack length, the lower the fatigue crack propagation rate for the same value of the stress-intensity factor range. In addition, if the Paris' diagram is traced adopting the introduced fractal quantities, according to Eq. 5 a universal fractal Paris' law is obtained, independently of the initial crack-size.

2.2 Experimental comparisons

Some experimental data obtained investigating Ni-based super-alloy samples with different initial crack size subjected to fourth-point fatigue bending test [16] can be considered, although a proper comparison should require geometrical self-similar specimens. Fig. 1a shows experimental fitting with the classical Paris' Law, thus different power laws are obtained. Subsequently, experimental data are used as fitting parameters for Eq. 4, in order to obtain the dimensional increment $d_{\mathcal{G}}$, the fractal Paris parameter C^*, and the exponent of Paris' law m as reported in Tab. 1. Finally, the experimental data are reported on the fractal Paris' diagram (Fig. 1b), where they all collapse along a single straight line.

3 THE SPECIMEN-SIZE EFFECTS ON WÖHLER'S CURVE

3.1 Intermediate asymptotics and fractal geometry

Since the introduction of Wöhler's curve, fatigue-testing results on smooth specimens have highlighted that the fatigue life of a structural component depends not only on the stress range.

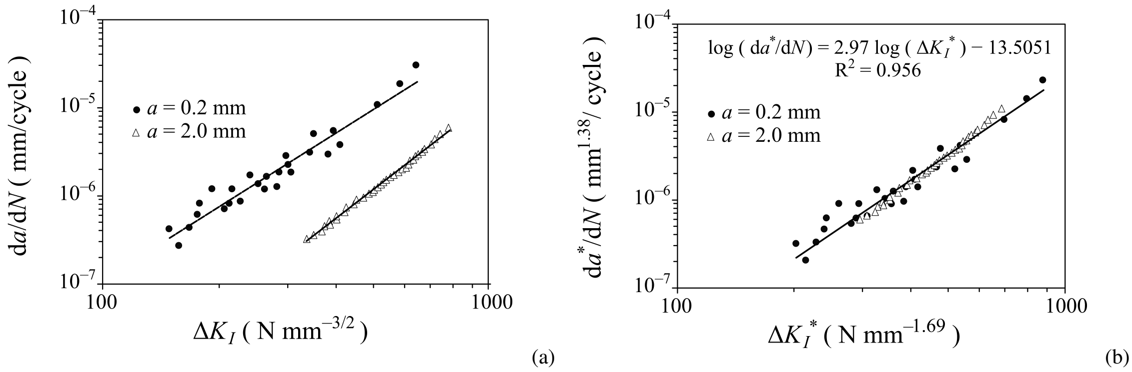

Figure 1: Experimental crack-size effect on Paris' law (a); Experimental assessment of fractal Paris' law (b).

m	C^* ($\mathrm{mm}^{6.3993}\mathrm{N}^{-2.97}$)	$d_{\mathcal{G}}$
2.97	3.125×10^{-14}	0.38

Table 1: Fitting parameters of Eq. 4.

In fact, strong influence of the loading ratio and specimen size on the fatigue strength has been noticed. Therefore, the following functional dependence for the fatigue life of a smooth specimen can be written [17]:

$$N = \Omega \left(\Delta\sigma, R; K_{IC}, \sigma_u, \Delta\sigma_{fl}; b \right) \tag{6}$$

where b represents the characteristic specimen size. The application of dimensional analysis and incomplete self-similarity allows to group the different functional dependencies in the following power-law expression:

$$N = \left(\frac{\Delta\sigma}{\sigma_u} \right)^{\beta_1} (1 - R)^{\beta_2} \left(\frac{\Delta\sigma_{fl}}{\sigma_u} \right)^{\beta_3} \left(b \left(\frac{\sigma_u}{K_{IC}} \right)^2 \right)^{\beta_4} \tag{7}$$

where the specimen-size effect is taken into account by the dimensionless parameter $b\left(\sigma_u/K_{IC}\right)^2$.

Alternatively, fractal geometry concepts can be used to account for the cross-sectional weakening due to inherent defects. If the material ligament is assumed as a lacunar fractal with non-integer dimension $2 - d_\sigma$ [6], the following scaling law for the stress range can be assumed [13]:

$$\Delta\sigma = \Delta\sigma^* b^{-d_\sigma} \tag{8}$$

where $\Delta\sigma^*$ is the fractal stress range. Calling $\Delta\sigma_0$ the intercept of the linear interpolation with the vertical axis (generally higher than the ultimate tensile stress σ_u), the following fractal expression for the power-law regime of the Wöhler's curve can be derived:

$$\Delta\sigma^* = \frac{\Delta\sigma_0^*}{N^{1/n}} \tag{9}$$

In this way, the dependence of the fatigue life on the specimen size is obtained, where the fatigue strength decreases with the specimen size for a given number of cycles. When the fractal stress range is used to represent experimental results, a single one specimen-size invariant Wöhler's curve is obtained in the power-law regime.

3.2 Experimental comparisons

Some experimental results on aluminium alloy flat hourglass samples [18] can be considered to show application of the above concepts. Although geometrical self-similarity of the specimens should be required for proper comparison, three different Wöhler curves are obtained for different specimen sizes, as shown in Fig. 2a.

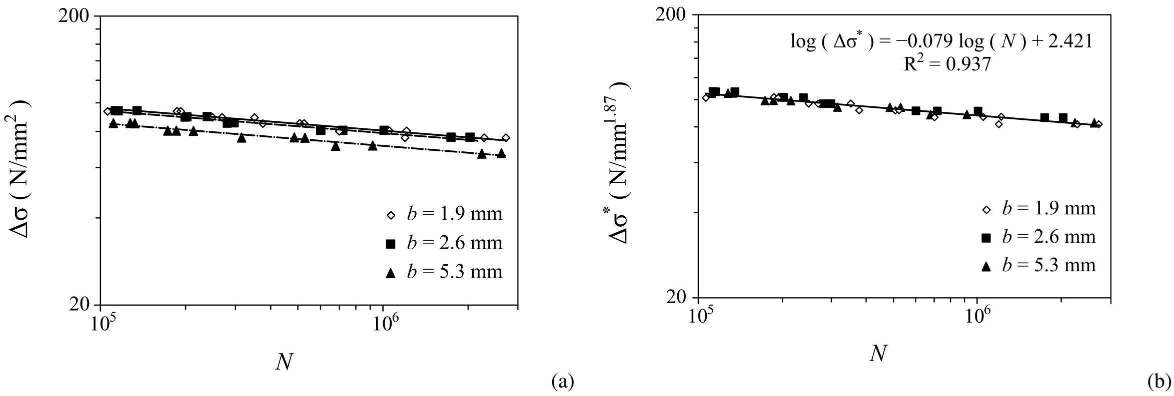

Figure 2: Specimen-size dependent Wöhler's curves in the power-law regime (a); Fractal Wöhler's curve in the power-law regime (b).

Fractal Wöhler parameter $\Delta\sigma_0^*$, n, and d_σ, can be fitted according to Eqns. 8-9, which are reported in Tab. 2. When experimental results are reported in the fractal Wöhler's diagram, they collapse on a single straight line, independently on the specimen size.

n	$\Delta\sigma_0^*$ $(\mathrm{Nmm}^{-1.87})$	d_σ
12.7	263	0.13

Table 2: Fitting parameters of Eqns. 8-9.

4 THE CRACK-SIZE EFFECT ON THE FATIGUE THRESHOLD

4.1 Fractal and Multifractal approaches

The so-called fatigue threshold defines the limit of validity of the Paris' law since the fatigue crack propagation rate shows deviation from the classical power-law regime. In this section, fractal geometry is used to explain the dependence of the fatigue threshold on the crack length. If Eqns. 3a - 3b are written in correspondence of the fatigue threshold, the following scaling laws are obtained:

$$v_{th} = v_{th}^{*} a^{-d_{\mathcal{G}}} \tag{10a}$$

$$\Delta K_{th} = \Delta K_{th}^{*} a^{d_{\mathcal{G}}/2} \tag{10b}$$

On the other hand, Eqns. 3a - 3b can also be written in correspondence of the point of unstable crack propagation:

$$v_{cr} = v_{cr}^{*} a^{-d_{\mathcal{G}}} \tag{11a}$$

$$\Delta K_{cr} = \Delta K_{cr}^{*} a^{d_{\mathcal{G}}/2} \tag{11b}$$

The obtained relationships describe an increment in the fatigue threshold and in the critical stress-intensity factor with crack length [8]. On the contrary, the crack propagation rate at the fatigue threshold, as well as at the critical crack propagation condition, decrease with the crack length [20]. In addition, according to Eq. 10b, the fatigue threshold increases with the crack length at a constant rate equal to $d_{\mathcal{G}}/2$, in the bi-logarithmic diagram. Experiments carried out on a wide range of crack length showed clearly that, when long cracks regime is considered, the fatigue threshold attains a constant value, and can consequently be treated as a material constant. On the other side, when short cracks are considered, the fatigue threshold decreases with the cracks length, until it becomes vanishing for micro-structurally small cracks.

To account for this transition, the Multi-Fractal Scaling Law (MFSL) for the fatigue threshold is introduced in the following way:

$$\Delta K_{th} = \Delta K_{th}^{\infty} \left(1 + \frac{l_{ch}}{a}\right)^{-d_{\mathcal{G}}/2} \tag{12}$$

where ΔK_{th}^{∞} is the fatigue threshold for long cracks, which can be considered the true material property, whereas l_{ch} is a characteristic material length. In this framework, the terms within round brackets can be interpreted as the negative deviation from ΔK_{th}^{∞}, due to fractality.

4.2 Experimental comparisons

The Multi-Fractal Scaling Law for the fatigue threshold can be compared to some recent experimental results available from literature [19]. The parameters to be introduced in Eq. 12 can be easily obtained applying best-fitting algorithms. The MFSL for the fatigue threshold is plotted vs. crack length in Fig. 3 and compared with experimental data.

5 THE SPECIMEN-SIZE EFFECT ON THE FATIGUE LIMIT

5.1 Fractal approach

The power-law range of the Wöhler's curve is bounded in-between two limits, the critical stress range $\Delta\sigma_{cr}$ in correspondence of higher stress ranges and the fatigue limit $\Delta\sigma_{fl}$ in correspondence of smaller stress ranges. When such limits are overcome, Basquin's power-law relation is replaced by constant asymptotic values. If Eq. 8 is written in correspondence of the

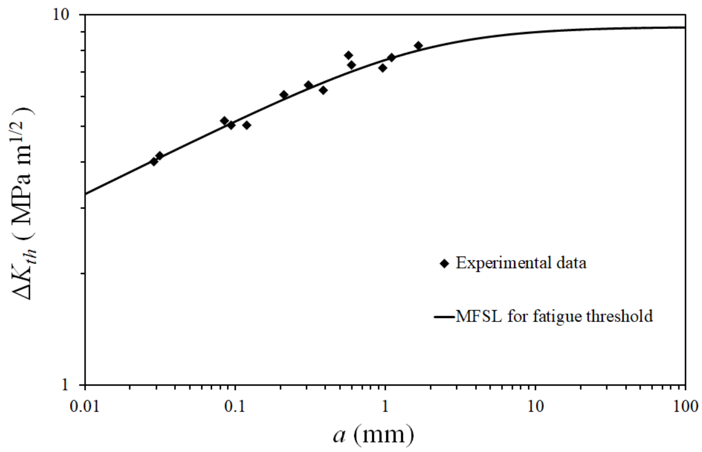

Figure 3: Interpolation of experimental results [19] with the Multi-Fractal Scaling Law Eq. 12.

two limits, the dependence of the fatigue limit and critical stress range on the specimen size is evidenced [12]:

$$\Delta\sigma_{fl} = \Delta\sigma^*_{fl} b^{-d_\sigma} \tag{13a}$$

$$\Delta\sigma_{cr} = \Delta\sigma^*_{cr} b^{-d_\sigma} \tag{13b}$$

For the time being, we will assume that the limit number of cycles, respectively N_{fl} and N_{cr}, would not depend on the specimen size. As a consequence, the Wöhler's curve shifts downwards as the structural size is increased. In this way, negative scaling of both the critical stress range and the fatigue limit is predicted with respect to the dimension of the specimen. In addition, the $\Delta\sigma$ axis can be renormalized in order to obtain a fractal Wöhler's diagram, where Eq. 9 can be plotted independently of the structural size [20].

5.2 Experimental comparisons

In order to fit the parameters of Eq. 13a, some experimental results available from literature [9] are considered. The linear regression in the bi-logarithmic plane $\log\Delta\sigma_{fl}$ vs. $\log b$ is shown in Fig. 4, from which both the dimensional decrement and the fractal fatigue limit can be obtained.

6 CONCLUSIONS

Two classical methods in fatigue, namely Paris' and Wöhler's approaches, are interpreted and compared by means of intermediate asymptotics theory and fractal geometry. In this way, the influence of crack length and structural size is put in evidence, respectively. In addition, fractal geometry allowed formulating non-classical version of the two well-known power-laws, where fractal material properties are invariant with respect to the scale. Finally, some comparison with experimental results are analysed.

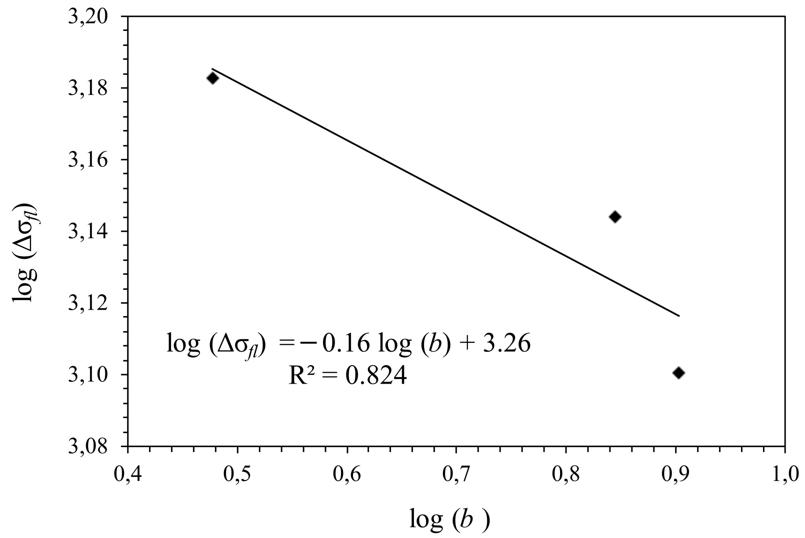

Figure 4: Interpolation of experimental results by Furuya [9] with the fractal scaling law Eq. 13a

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