Long-term Behavior of Mean-field Noisy Bounded Confidence Models with Distributed Radicals

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M. A. S. Kolarijani, P. Mohajerin Esfahani and A. V. Proskurnikov

Abstract— **In this paper, we consider the mean-field model of noisy bounded confidence opinion dynamics under exogenous influence of static radical opinions. The long-term behavior of the model is analyzed by providing a sufficient condition for exponential convergence of the dynamics to stationary state. The stationary state is also characterized by a global estimate for sufficiently large noises. Furthermore, we consider the order-disorder transition in the model in order to identify the effect of the relative mass of the radicals on the critical noise level at which this transition occurs. A numerical scheme for approximating the critical noise level is provided and validated through numerical simulations of the mean-field model and the corresponding agent-based model for a particular distribution of radical opinions.**

I. INTRODUCTION

Long before the recent "boom" in the study of complex systems, the necessity of mathematical models that can capture the diversity of clustering behaviors in real-world networks was realized in mathematical sociology [1]. The problem of disclosing these mechanisms is nowadays referred to as the community cleavage problem or Abelson's diversity puzzle [2], [3] and primarily concerned with temporal mechanisms of *opinion formation* under social influence. The interdisciplinary area of social dynamics [3], [4], [5], [6], [7], [8] has attracted enormous attention of the research community.

One feature observed in social and biological systems is the *homophily* [9], or tendency of individuals to bond with similar ones. In other words, like-minded individuals influence a social actor "stronger" than different-minded ones. Mathematically, social influence weights should be non-constant, but rather *opinion-dependent*. That is, the underlying interaction graph is dynamic and varies as the opinions of the individuals change. A class of nonlinear models resembling this feature are the so-called *bounded confidence* models proposed as extensions of the deterministic [10] and randomized gossip-based [11] consensus algorithms for multi-agent networks. Bounded confidence models stipulate that a social actor is insensitive to opinions beyond its bounded *confidence set* (usually, this set is an open or closed ball, centered at the actor's own opinion). A detailed survey of bounded confidence models and relevant mathematical results can be found in [8]. Bounded confidence models exhibit convergence of the opinions to some steady values, which can reach consensus or split into several disjoint clusters. Opinions in real social groups, however, usually do not terminate at steady values, which is usually explained by two factors.

The first reason explaining opinion fluctuation is exogenous influence, which can be interpreted as some "truth" available to some individuals [12] or a position shared by a group of close-minded opinion leaders ("radicals") [13], [14]. Typically, the exogenous signal are supposed to change slowly compared to the opinion evolution and is thus replaced by a constant; the main concern is the dependence between the constant input and the resulting opinion profile. The second culprit of persistent opinion fluctuation is uncertainty in the opinion dynamics, usually modeled as a random drift of each opinion. Whereas these models are still waiting for clear sociopsychological interpretation, they are broadly adopted in statistical physics [15], [16], [17], [18] to study phase transitions in systems of interacting particles.

In spite of some progress in analysis of "noisy" bounded confidence models [19], [20], in particular, the interplay of confidence ranges and noise levels, all consequences of a noise and exogenous influence in nonlinearly coupled networks are far from being understood. Even for the classical models from [10], [11], disclosing the relation between the initial and the terminal opinion profiles remains a challenging problem (including e.g. the $2R$-conjecture [21], [22]). This motivates examination of the corresponding *mean-field* models as the number of actors $N \to \infty$. The arising macroscopic approximations of microscopic models describe the evolution of the *distribution* (a probability measure or a density) of opinion over some domain. In continuous-time models (considered in this paper), the density obeys a nonlinear Fokker-Planck (FP) equation. This models are known as density-based [23], continuum-agent [24], [25], Eulerian [26], [27], kinetic [28] or mean-field [22], [29].

In this paper, we advance the theory of macroscopic modeling of noisy bounded confidence dynamics by considering "radical" opinions (equivalently, exogenous influences) that are not necessarily concentrated at a single point (as in [13], [26]) but have their own distribution. Following [22], [30], [31], we consider the model on a circle by considering *periodic* boundary conditions. However, we employ an *even 2-periodic* extension of the system in order the preserve the extreme opinions explicitly. Our main theoretical result shows that for sufficiently large noises the dynamic will converge exponentially fast to a stationary state that can be

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made arbitrarily close to uniform distribution by increasing the noise level.

Exploiting the periodic nature of system, we derive a system of quadratic ordinary differential equations (ODEs) describing the evolution of the Fourier coefficients of the solution to the FP equation. For a uniform initial data, we use the linearized ODEs to study the order-disorder transition in the system. In particular, we provide a approximation scheme for computing the critical noise level at which the transition occurs. Our analysis shows that there is a direct relation between the mass of radicals and the critical noise level. This result is then validated by numerical simulations of the mean-field partial differential equation (PDE) and the corresponding agent-based stochastic differential equations (SDEs) for a particular distribution of radicals.

Notations. We use $\star$ to denote the convolution of two functions. First and second order differentiation with respect to (w.r.t.) x are denoted by $(\cdot)_x$ and $(\cdot)_{xx}$, respectively. Similar notations are used for differentiation w.r.t. t. $\mathscr{P}(X)$ is the the space probability densities on $X=[0,L]$, that is, for all $\rho \in \mathscr{P}(X)$ we have $\int_X \rho(x)\ \mathrm{d}x = 1$ and $\rho(x) \geq 0$. $\mathscr{P}_e(\tilde{X})$ is the space of *even* density functions on $\tilde{X}=[-L,L]$ such that for all $\rho \in \mathscr{P}_e(\tilde{X})$ we have $\int_{\tilde{X}} \rho(x)\ \mathrm{d}x = 2$, $\rho(x) \geq 0$ and $\rho(x)=\rho(-x)$. The subscript ep is used to denote the subspace of *even* $2L$*-periodic* functions in the corresponding function space; e.g., $L^p_{ep}(\tilde{X}) \subset L^p(\tilde{X})$ is the corresponding subspace of even $2L$-periodic functions on $\tilde{X}=[-L,L]$ for which the p-th power of the absolute value is Lebesgue integrable.

II. The Model

The model includes N interacting *normal* agents with opinions $x_i, i=1,2,\ldots,N$, where each agent i is influenced by agents j resided in its confidence bound, that is, $j \in \mathscr{N}_i = \{j : |x_i - x_j| \leq R\}$. We also consider exogenous influence in form of N_r *radical* agents with static opinions $x_{r_i}, i=1,\ldots,N_r$, introduced to the original population of normal agents. The same bounded confidence mechanism is assumed for interaction between a normal agent and the radicals. The model also includes noise representing the uncertainties [30] or the effect of free will [32] in agents' opinions. The following system of interacting SDEs describes the evolution of normal agents' opinions

$$\begin{cases} \mathrm{d}x_i = \frac{1}{N}\left[\sum\limits_{j\in\mathscr{N}_i}(x_j - x_i) + \sum\limits_{j\in\mathscr{N}_i}(x_{r_j}-x_i)\right]\mathrm{d}t + \sigma \mathrm{d}W^i_t, \\ x_i(0) = x_{i_0}, \end{cases} \tag{1}$$

where $x_{i_0}, i=1,2,\ldots,N$, are the initial opinions of the normal agents and $\sigma > 0$ is the noise level with W^i_t representing independent Wiener processes.

Following [30], we pass to the mean-field limit ($N \to \infty$) of the agent-based model described by (1). Then, the density of the normal opinions $\rho(x,t)$ satisfies the following nonlinear FP equation

$$\begin{cases} \rho_t = [\rho\ \ (w \star (\rho + M\rho_r))]_x + \frac{\sigma^2}{2}\rho_{xx}, \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{2}$$

where $w(x) = x\ \mathbf{1}_{|x|\leq R}$ is the interaction kernel. The function $\rho(x,t)$ is the limit measure of the empirical measure

$$\rho^N(dx,t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}(dx),$$

as $N \to \infty$ (here, $\delta_x(dx)$ denotes a Dirac measure centered at x). The exogenous influence of radicals is included in the mean-field model by introducing the radical opinions density function $M\rho_r(x)$ to (2). Here, $M = \frac{N_r}{N}$ is a constant denoting the relative mass (the zeroth moment) of the radical opinions density and $\rho_r(x)$ is the corresponding time-invariant probability density function.

We assume that the opinions reside in the bounded interval $X=[0,L]$. Notice, however, the diffusion term in (1) and (2) allows the opinions to leave $[0,L]$. To avoid this, following [17], [22], [30], [31], we consider a periodic boundary condition. However, unlike the usual periodic extensions, we consider an *even* $2L$*-periodic* extension of system. In effect, we treat the same mean-field model as in [31] with an extra constraint on the initial condition (and the newly introduced radical density) to be even. This particular extension has the advantage that explicitly preserves the two extreme opinion values at $x=0$ and $x=L$.

The corresponding PDE with the even $2L$-periodic extension is formally expressed as

$$\begin{cases} \rho_t = (\rho\ \ G_\rho)_x + \frac{\sigma^2}{2}\rho_{xx} & \textit{in} \quad \tilde{X}\times(0,T], \\ \rho(-L,\cdot) = \rho(L,\cdot) & \textit{on} \quad \partial\tilde{X}\times[0,T], \\ \rho(x,\cdot) = \rho_0(x) & \textit{on} \quad \tilde{X}\times\{t=0\}, \end{cases} \tag{3}$$

where $\tilde{X}=[-L,L]$ and

$$G_\rho(x,t) := w(x) \star (\rho(x,t) + M\rho_r(x)). \tag{4}$$

Notice that ρ_0 and ρ_r are the even $2L$-periodic extensions of the initial density of normal opinions and the radical opinions density, respectively. The evenness of ρ_0 and ρ_r implies that the solution to (3) is also even. Moreover, due to periodicity, the mass is preserved in (3), that is, $\int_X \rho(x,t)\ \mathrm{d}x = \int_X \rho_0(x)\ \mathrm{d}x = 1$ for all $t \geq 0$. PDE (3) fully describes the macroscopic model considered in this paper.

Remark 1 (Well-posedness of (3)*):* The mean-field PDE (3) has a unique weak solution $\rho \in C^1(0,\infty; C^2_{ep}(\tilde{X}))$ with $\rho(t) \in \mathscr{P}_e(\tilde{X})$ for all $t \geq 0$. See [33, Theorem 3.1] for details on the required regularity on data ρ_0 and ρ_r.

III. Analysis of the Long-term Behavior

In this section we study the long-term behavior of the system by considering the equilibria of system, that is, the solution to the stationary equation corresponding to (3) given by

$$\frac{\sigma^2}{2}\rho_{xx} + (\rho\ \ G_\rho)_x = 0. \tag{5}$$

In this regard, notice that, unlike the autonomous systems (without radicals) considered in previous studies [31], [34], the uniform distribution $\rho = \frac{1}{L}$ is not an equilibrium of the system. There, the authors provide sufficient condition for exponential convergence towards uniform distribution.

Here, we extend this result to our model with the exogenous influence, i.e., the radicals.

Remark 2 (Well-posedness of (5)*):* A fixed-point characterization of the solution to the stationary equation (5) can be used to derive existence result for a classical solution $\rho \in C^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$ for equation (5). For details of this result and the corresponding required regularity on data ρ_r see [33, Theorem 4.1].

A. Main Result

Let us start by presenting two main theoretical results regarding the long-term behavior of the system. The following Proposition provides an estimate that characterizes the equilibria of the system for noises larger than a particular value.

Proposition 3 (Characterization of Equilibria): Assume $\rho_s \in C^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$ is a classical solution to the stationary equation (5) with $\rho_r \in L^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$. Then, it holds that $\|\rho_s - \frac{1}{L}\|_{L^2} < \frac{1}{\eta}\|\rho_r\|_{L^2}$ if $\sigma^2 > \sigma_b^2 + \eta c_b$, where

$$\begin{cases} \sigma_b^2 := \frac{4RL}{\pi}\left(M + \frac{R}{\sqrt{3}L} + 2\right), \\ c_b := \frac{4R^2 M}{\pi\sqrt{3}}. \end{cases} \tag{6}$$

Proposition 3 implies that, even in presence of radical opinions, the stationary solution can be made arbitrarily close to the uniform distribution by increasing the noise level. Particularly, one notices that the minimum noise level σ_b is directly related to the confidence bound R and the relative mass M. Also, as the "energy" $M\|\rho_r\|_{L^2}$ of the radicals increases, in order to counteract their effect and keep the system in a somewhat uniform state, one must increase the noise level further beyond σ_b.

Proposition 3, by itself, cannot be used for describing the long-term behavior of the dynamics as it only characterizes the equiliblria of the system. To put this result in use, we need to also consider the stability of these equilibria. Our next theoretical result provides a sufficient condition for exponential convergence of the dynamics to stationary state for arbitrary (and sufficiently smooth) initial and radical densities.

Proposition 4 (Stability of Equilibria): Assume that PDE (3) has a classical solution $\rho \in C^1(0,\infty; C^2_{ep}(\tilde{X}))$, with $\rho(t) \in \mathscr{P}_e(\tilde{X})$ for all $t \geq 0$. Then, $\rho(t)$ converges to a stationary state $\rho_s \in C^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$ exponentially in L^2 as $t \to \infty$ if $\sigma > \sigma_s$, where $\sigma_s > 0$ uniquely solves

$$\sigma_s^2 = \frac{4RL(3+M)}{\pi} + \frac{4R^2}{\pi\sqrt{3}} \exp\left(\frac{8RL(1+M)}{\sigma_s^2}\right), \tag{7}$$

for any $\rho_0 \in L^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$ and $\rho_r \in L^2_{ep}(\tilde{X}) \cap \mathscr{P}_e(\tilde{X})$.

Combining the results of Propositions 3 and 4, we obtain our main theoretical result.

Theorem 5 (Main Result): Let $\rho \in C^1(0,\infty; C^2_{ep}(\tilde{X}))$, with $\rho(t) \in \mathscr{P}_e(\tilde{X})$ for all $t \geq 0$, be the classical solution to PDE (3). If $\sigma^2 > \max\{\sigma_b^2 + \eta c_b,\ \sigma_s^2\}$, where σ_b and c_b are defined in (6) and σ_s uniquely solves (7), then it holds that

$$\|\rho(t) - 1/L\|_{L^2} \leq \beta e^{-\lambda t} + \frac{1}{\eta}\|\rho_r\|_{L^2}, \tag{8}$$

where $\beta, \lambda > 0$ depend on system data.

Proof: Notice that

$$\|\rho(t) - 1/L\|_{L^2} \leq \|\rho(t) - \rho_s\|_{L^2} + \|\rho_s - 1/L\|_{L^2}.$$

Then, the inequality (8) immediately follows from Propositions 3 and 4. ■

This result implies that, for sufficiently large noises, the dynamics will converge to a stationary state that can be made arbitrarily close to uniform distribution by increasing the noise level, equation (3)

B. Technical Proofs

This section is devoted to the proofs of Propositions 3 and 4. In the following, all the norms are w.r.t. the domain $\tilde{X} = [-L, L]$, unless indicated otherwise.

1) Proof of Proposition 3: Let $\psi = \rho_s - \frac{1}{L}$. From the stationary equation (5), we obtain

$$-\frac{\sigma^2}{2}\psi_{xx} = \left(\psi\ G_\psi\right)_x + \frac{1}{L}\ (G_\psi)_x,$$

where we used the fact that $w \star \frac{1}{L} = 0$. Multiplying this equation by ψ and integrating by part over $\tilde{X}$, we obtain (the extra terms are zero due to periodicity)

$$\frac{\sigma^2}{2}\|\psi_x\|^2_{L^2} = -\int_{\tilde{X}} \psi_x\ \psi\ G_\psi\ \mathrm{d}x - \frac{1}{L}\int_{\tilde{X}} \psi_x\ G_\psi\ \mathrm{d}x.$$

Hence,

$$\begin{aligned} \frac{\sigma^2}{2}\|\psi_x\|^2_{L^2} &\leq \left|\int_{\tilde{X}} \psi_x\ \psi\ G_\psi\ \mathrm{d}x\right| + \frac{1}{L}\left|\int_{\tilde{X}} \psi_x\ G_\psi\ \mathrm{d}x\right| \\ &\leq \|G_\psi\|_{L^\infty}\ \|\psi_x\|_{L^2}\ \|\psi\|_{L^2} + \frac{1}{L}\ \|\psi_x\|_{L^2}\ \|G_\psi\|_{L^2}, \end{aligned}$$

where for the second inequlity we used the Cauchy-Schwarz inequality. Hence,

$$\frac{\sigma^2}{2}\|\psi_x\|^2_{L^2} \leq \|G_\psi\|_{L^\infty}\ \|\psi\|_{L^2} + \frac{1}{L}\ \|G_\psi\|_{L^2}, \tag{9}$$

where we used the fact that $\|\psi_x\|_{L^2} \neq 0$ (recall that uniform distribution is not an equilibrium of the system). Now, notice

$$\begin{aligned} |G_\psi(x,t)| &= |w \star (\psi + M\rho_r)| \\ &\leq \int |x-y|\ \mathbf{1}_{|x-y|\leq R}\ |\psi(y,t) + M\rho_r(y)|\ \mathrm{d}y \\ &\leq R\int_{\tilde{X}} |\psi(y,t) + M\rho_r(y)|\ \mathrm{d}y \\ &\leq 2R\left(\int_X |\psi(y,t)|\ \mathrm{d}y + M\right). \end{aligned} \tag{10}$$

Thus,

$$\begin{aligned} \|G_\psi\|_{L^\infty} &\leq 2R\left(\|\psi\|_{L^1(X)} + M\right) \\ &= 2R\left(\|\rho - 1/L\|_{L^1(X)} + M\right) \\ &\leq 2R\left(\|\rho\|_{L^1(X)} + 1 + M\right) \leq 2R(M+2). \end{aligned} \tag{11}$$

Also, we have

$$
\begin{aligned}
|G_\psi(x)|^2 &= \left(\int_{x-R}^{x+R} (x-y)\ (\psi(y)+M\rho_r(y))\ \mathrm{d}y \right)^2 \\
&\le \int_{x-R}^{x+R} (x-y)^2\ \mathrm{d}y\ \int_{x-R}^{x+R} (\psi(y)+M\rho_r(y))^2\ \mathrm{d}y \\
&\le \frac{2}{3} R^3 \int_{x-R}^{x+R} (\psi(y)+M\rho_r(y))^2\ \mathrm{d}y. \qquad (12)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|G_\psi\|_{L^2}^2 &\le \frac{2}{3} R^3 \int_{\tilde{X}} \int_{x-R}^{x+R} (\psi(y)+M\rho_r(y))^2\ \mathrm{d}y\mathrm{d}x \\
&= \frac{2}{3} R^3 \int_{\tilde{X}} \int_{-R}^{R} (\psi(x+y)+M\rho_r(x+y))^2\ \mathrm{d}y\mathrm{d}x \\
&= \frac{2}{3} R^3 \int_{-R}^{R} \int_{\tilde{X}} (\psi(x+y)+M\rho_r(x+y))^2\ \mathrm{d}x\mathrm{d}y \\
&= \frac{4}{3} R^4 \|\psi + M\rho_r\|_{L^2}^2. \qquad (13)
\end{aligned}
$$

Using estimates (11) and (13) for (9), we obtain

$$
\begin{aligned}
\frac{\sigma^2}{2} \|\psi_x\|_{L^2} &\le 2R(M+2)\ \|\psi\|_{L^2} + \frac{2R^2}{\sqrt{3L}}\ \|\psi + M\rho_r\|_{L^2} \\
&\le 2R(M+2)\ \|\psi\|_{L^2} + \frac{2R^2}{\sqrt{3L}} (\|\psi\|_{L^2} + M\|\rho_r\|_{L^2}) \\
&= 2R\left(M + \frac{R}{\sqrt{3L}} + 2 \right)\ \|\psi\|_{L^2} + \frac{2R^2 M \|\rho_r\|_{L^2}}{\sqrt{3L}}. \qquad (14)
\end{aligned}
$$

Now, notice that since $\int_X \psi(x)\ \mathrm{d}x = 0$, we can employ Poincaré inequality [35] (Section 5.8.1, Theorem 1) to obtain

$$
\|\psi\|_{L^2} \le C\ \|\psi_x\|_{L^2}. \qquad (15)
$$

The optimal value for the Poincaré constant for $\tilde{X} = [-L, L]$ is $C = \frac{L}{\pi}$. Combining inequalities (14) and (15), we have

$$
\left(\sigma^2 - \frac{4RL}{\pi} \left(M + \frac{R}{\sqrt{3L}} + 2 \right) \right) \|\psi\|_{L^2} \le \frac{4R^2 M \|\rho_r\|_{L^2}}{\pi\sqrt{3}}. \qquad (16)
$$

Defining σ_b and c_b as in (6) gives the desired inequality

$$
\|\psi\|_{L^2} \le \frac{c_b}{\sigma^2 - \sigma_b^2} \|\rho_r\|_{L^2}.
$$

2) Proof of Proposition 4: Let $\psi = \rho - \rho_s$. From the dynamic equation (3), we obtain

$$
\psi_t = \left[\psi\ (w \star \psi + G_{\rho_s}) \right]_x + [\rho_s\ (w \star \psi)]_x + \frac{\sigma^2}{2}\ \psi_{xx}, \qquad (17)
$$

where we used the fact that ρ_s is a solution to the stationary equation (5), that is,

$$
\left[\rho_s\ G_{\rho_s} \right]_x + \frac{\sigma^2}{2}\ \rho_{s_{xx}} = 0.
$$

Multiplying (17) by ψ and integrating by part over $\tilde{X}$ we obtain (the extra terms are zero due to periodicity)

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\psi\|_{L^2}^2 + \frac{\sigma^2}{2} \|\psi_x\|_{L^2}^2 \\
&\quad \le \left| \int_{\tilde{X}} \psi_x\ \psi\ (w \star \psi + G_{\rho_s})\ \mathrm{d}x \right| + \left| \int_{\tilde{X}} \psi_x\ \rho_s\ (w \star \psi)\ \mathrm{d}x \right| \\
&\quad \le \left(\|w \star \psi\|_{L^\infty} + \|G_{\rho_s}\|_{L^\infty} \right)\ \|\psi_x\|_{L^2}\ \|\psi\|_{L^2} \\
&\qquad + \|\rho_s\|_{L^\infty}\ \|\psi_x\|_{L^2}\ \|w \star \psi\|_{L^2}, \qquad (18)
\end{aligned}
$$

where the second inequality follows from Cauchy-Schwarz inequality. Following a similar procedure as in (10) and (11) and also as in (12) and (13) (with $M = 0$ for the second and third inequalities below), we obtain

$$
\begin{cases}
\|G_{\rho_s}\|_{L^\infty} \le 2R(1+M), \\
\|w \star \psi\|_{L^\infty} \le 4R, \\
\|w \star \psi\|_{L^2} \le \frac{2R^2}{\sqrt{3}}\ \|\psi\|_{L^2}.
\end{cases} \qquad (19)
$$

Now, we need an estimate for $\|\rho_s\|_{L^\infty}$ to proceed with the proof. To this end, let us provide an alternative representation of the stationary solution ρ_s as a fixed point of a nonlinear operator. Observe that integrating (5) once, we obtain

$$
\frac{\sigma^2}{2} \rho_x + \rho\ G_\rho = C. \qquad (20)
$$

Here we use the particular *even 2L-periodic* extension considered for the model. Notice that we can set $C = 0$ since we are interested in even solutions to (20). Indeed, from (20) we have

$$
\frac{\sigma^2}{2} \rho_x(-x) + \rho(-x)[w(-x) \star (\rho(-x) + M\rho_r(-x))] = C.
$$

Hence, for an even solution, we obtain

$$
-\frac{\sigma^2}{2} \rho_x(x) - \rho(x)[w(x) \star (\rho(x) + M\rho_r(x))] = C,
$$

where we used the fact that w is an odd function. This implies $C = 0$. Rearranging and integrating (20) once again, we have

$$
\rho(x) = \frac{1}{K} \exp\left(-\frac{2}{\sigma^2} \int_0^x G_\rho(z)\ \mathrm{d}z \right) \qquad (21)
$$

where the normalizing condition gives the constant K as

$$
K = \int_0^L \exp\left(-\frac{2}{\sigma^2} \int_0^x G_\rho(z)\ \mathrm{d}z \right) \mathrm{d}x.
$$

The implicit equation (21) represents the stationary solution as the fixed point of an operator. Using this representation, we have

$$
\begin{aligned}
|\rho_s| &= \frac{\exp\left\{ -\frac{2}{\sigma^2} \int_0^x G_{\rho_s}(z)\ \mathrm{d}z \right\}}{\int_0^L \exp\left\{ -\frac{2}{\sigma^2} \int_0^x G_{\rho_s}(z)\ \mathrm{d}z \right\} \mathrm{d}x} \\
&\le \frac{\exp\left\{ \frac{4RL(1+M)}{\sigma^2} \right\}}{L \exp\left\{ -\frac{4RL(1+M)}{\sigma^2} \right\}} = \frac{1}{L} \exp\left\{ \frac{8RL(1+M)}{\sigma^2} \right\} =: \alpha, \qquad (22)
\end{aligned}
$$

where for the second inequality we used the first estimate in (19). Thus, we have $\|\rho_s\|_{L^\infty} \leq \alpha$. Using this result and the inequalities in (19), we can rewrite (18) as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 + \frac{\sigma^2}{2}\|\psi_x\|_{L^2}^2 &\\ \leq \left(2R(3+M)+\frac{2R^2}{\sqrt{3}}\alpha\right)\ \|\psi_x\|_{L^2}\ \|\psi\|_{L^2} &\\ \leq \frac{1}{\sigma^2}\ \left(2R(3+M)+\frac{2R^2}{\sqrt{3}}\ \alpha\right)^2\|\psi\|_{L^2}^2 + \frac{\sigma^2}{4}\ \|\psi_x\|_{L^2}^2,\end{aligned}$$

where for the second inequality, we used Young's inequality. Hence,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 \leq &\frac{1}{\sigma^2}\ \left(2R(3+M)+\frac{2R^2}{\sqrt{3}}\ \alpha\right)^2\|\psi\|_{L^2}^2 \\ &- \frac{\sigma^2}{4}\ \|\psi_x\|_{L^2}^2.\end{aligned}$$

Once again, since $\int_X \psi\ \mathrm{d}x = 0$, we can employ the Poincaré inequality [35] (Section 5.8.1, Theorem 1) $\|\psi\|_{L^2} \leq C\ \|\psi_x\|_{L^2}$ with optimal Poincaré constant $C = \frac{L}{\pi}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 \leq \left\{\frac{2}{\sigma^2}\ \left[2R(3+M)+\frac{2R^2}{\sqrt{3}}\ \alpha\right]^2 - \frac{\pi^2\sigma^2}{2L^2}\right\}\|\psi\|_{L^2}^2.$$

Then, by Grönwall's inequality, we have

$$\begin{aligned}&\|\psi(t)\|_{L^2}^2 \leq \\ &\|\psi(0)\|_{L^2}^2 \exp\left[\left\{\frac{2}{\sigma^2}\ \left[2R(3+M)+\frac{2R^2}{\sqrt{3}}\ \alpha\right]^2 - \frac{\pi^2\sigma^2}{2L^2}\right\}t\right].\end{aligned}$$

Now, notice that by assumption

$$\|\psi(0)\|_{L^2} = \|\rho_0-\rho_s\|_{L^2} \leq \|\rho_0\|_{L^2} + \|\rho_s\|_{L^2} < \infty.$$

Thus, if the multiplicative term in the exponential is negative, then $\|\psi(t)\|_{L^2}^2 \to 0$ exponentially fast as $t \to \infty$. Negativity of this multiplicative term corresponds to the condition $\sigma > \sigma_s$ where $\sigma_s > 0$ solves (7). Notice that the constant α is defined in (22).

IV. Order-disorder Transition

A common behavior in noisy interactive particle systems is the so-called order-disorder transition. For large values of σ, the effect of diffusion process can overcome the attracting forces among agents preventing the system from forming any cluster. This behavior has been analyzed and observed in several noisy bounded confidence models for opinion dynamics. Pineda et. al. used linear stability analysis in [32], [36] to compute the critical noise level above which the clustering behavior diappears for a modified version of Defuant model. The same technique of linear stability analysis was used in [22] and [30] to compute the critical noise level for a noisy Hegselmann-Krause system similar to our model, except without radicals. In particular, [22] considered the interplay between the confidence bound R and the critical noise level σ_c at which the system experiences the order-disorder transition.

In this section, we study the effect of the relative mass M of radicals on the critical noise level σ_c. We emphasize that we consider a uniform distribution for initial normal opinions. In the sequel, we make use of the *order parameter* [22]

$$\begin{cases} Q_d(x) = \frac{1}{N^2}\sum_{i,j=1}^N \mathbf{1}_{|x_i(t)-x_j(t)|\leq R}, \\ Q_c(\rho) = \frac{1}{L^2}\iint_{X^2} \rho(x,t)\rho(y,t)\mathbf{1}_{|x-y|\leq R}\ \mathrm{d}x\mathrm{d}y, \end{cases} \tag{23}$$

to quantify the clustering behavior of the agent-based and mean-field models, respectively. The order parameter provides a measure for orderedness in opinions: for a uniform distribution of opinions - absolute disorder, we have $Q = 0.2$; and, for a single cluster distribution with all agents residing in an interval of width R or less - complete order, we have $Q = 1$. Roughly speaking, in case of emergence of a clustered profile, the inverse of the order parameter is equal to the number of clusters.

A. Approximating the Critical Noise Level σ_c

In this section, we provide a method for approximating the critical noise level σ_c. To do so, exploiting the periodic nature of the system, we derive the ODEs describing the evolution of the Fourier coefficients of the solution ρ to (3). Then, we use the linearized ODEs to present a numerical scheme for approximating σ_c, without actually solving the dynamics.

1) Fourier ODEs: The even $2L$-periodic extension of the probability densities in the model allows us to consider the Fourier expansions of ρ and ρ_r expressed as

$$\begin{cases} \rho(x,t) = \sum_{n=0}^\infty\ p_n(t)\ \cos\left(\frac{\pi n x}{L}\right), \\ \rho_r(x) = \sum_{n=0}^\infty\ q_n\ \cos\left(\frac{\pi n x}{L}\right). \end{cases}$$

Inserting these expansions into (3) and setting the inner product of the residual with elements of the basis to zero, we can obtain a system of quadratic ODEs describing the evolution of Fourier coefficients $p_n(t)$. For $n = 1,\ldots,N_f$, these ODEs are expressed as

$$\dot{p}_n = c_n + b_n^T p + p^T Q_n p, \tag{24}$$

where $p = (p_1, p_2, \ldots, p_{N_f})^T$. Note that for $n = 0$ (the constant term in the Fourier expansion), we obtain $\dot{p}_0 = 0$ which corresponds to the periodic nature of the system that preserves the zeroth moment. The coefficients of (24) are given by

$$c_n := \frac{2MR}{L}\ f_n\ q_n \tag{25}$$

$$(b_n)_k := \begin{cases} \frac{2R}{L}\ f_n + \frac{MR}{2}\ f_{2n}\ q_{2n} - \frac{\pi^2\sigma^2 n^2}{2L^2}, & k = n, \\ nMR\left\{\frac{q_{n+k}\ f_{n+k}}{n+k} + \frac{q_{|n-k|}\ f_{n-k}}{n-k}\right\}, & k \neq n, \end{cases} \tag{26}$$

$$(Q_n)_{k,l} := \begin{cases} nR\frac{f_k}{k}, & l = n-k > 1, \\ nR\left\{\frac{f_k}{k} + \frac{f_{n-k}}{n-k}\right\}, & l = k-n > 1, \\ 0, & Otherwise, \end{cases} \tag{27}$$

where $q_n, n \in \mathbb{N}$ are the Fourier coefficients of ρ_r and

$$f_n := -\cos\left(\frac{n\pi R}{L}\right) + \mathrm{sinc}\left(\frac{n\pi R}{L}\right).$$

2) Approximation Scheme: Linearizing the system at $t=0$ (which is equivalent to disregarding the quadratic terms in (24) since the initial condition is uniform, i.e., $p_n(0)=0$ for $n \neq 0$) we obtain the system of linear ODEs

$$\dot{p} = c + Bp, \tag{28}$$

where the vector c and matrix B are defined accordingly using the objects c_n and b_n in (25) and (26) for $n=1,\ldots,N_f$.

Looking at coefficients of the quadratic ODEs (24), we notice that the noise level σ only appears in the diagonal entries of B such that by increasing σ, these diagonal entries decrease. That is, for a large enough σ, all eigenvalues of B are negative and the linearized system (28) is stable. This will be our first criterion for determining the critical noise level σ_c: the noise level above which all eigenvalues of B are negative.

In order to consider the effect of the constant linear growth rates c in (28), we further require the stationary values $\bar{p}_n, n = 1,\ldots,N_f$ of the linearized system (28) (i.e., the solutions of the equations $c+Bp=0$) to be relatively small. In order to quantify this description, we use Parseval's identity to set our second criteria as

$$\sum_{n=1}^{N_f} \bar{p}_n^2 < \gamma, \tag{29}$$

where the constant $\gamma > 0$ determines the level of similarity between ρ and uniform distribution (disordered state). To sum up, we solve numerically for the level of noise above which the eigenvalues of B are negative and the inequality (29) holds.

B. Numerical Simulations

In this section, we numerically study the effect of the relative mass of radicals M on the order-disorder transition. Furthermore, we use our simulation results to examine the approximation scheme presented in Section IV-A.2. To this end, we consider a particular distribution of radical opinions, namely, a triangular one with average A and width $2S$

$$\rho_r(x) = \begin{cases} \frac{1}{S^2}(S - |x-A|), & |x-A| \leq S, \\ 0, & o.w. \end{cases} \tag{30}$$

The corresponding Fourier coefficients of ρ_r are given by

$$q_n = \frac{2}{L} \cos\left(\frac{n\pi A}{L}\right) \operatorname{sinc}^2\left(\frac{n\pi S}{2L}\right). \tag{31}$$

In all the simulations of this section, the average and width of radical distribution are fixed at $A = 0.7L$ and $S = 0.1L$, respectively, with $L = 1$.

In order to solve the mean-field model described by PDE (3) numerically, we use the ODEs (24) to compute the coefficients of Fourier expansion of normal opinion density ρ cut at $N_f = 128$. In particular, we use the Fourier coefficients of the triangular ρ_r given in (31) for solving ODEs (24). To be precise, we need the Fourier coefficients q_n of ρ_r for $1 \leq n \leq 2N_f$, that is, twice the length of Fourier expansion of ρ (see the coefficients in (26)). For the initial condition, we consider uniform distribution $\rho_0 = \frac{1}{L}$ which corresponds to $p_0 = \frac{1}{L}$ and $p_n(0) = 0$ for the Fourier coefficients.

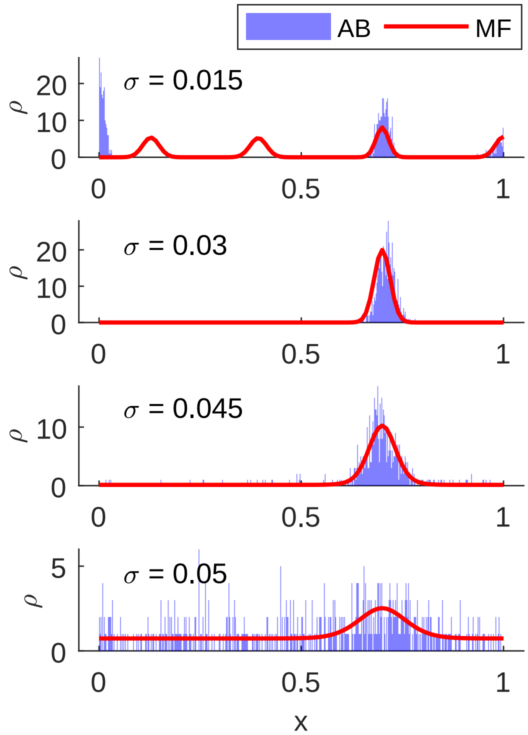
For the agent-based model, the SDEs (1) are solved numerically using the Euler-Maruyama method for $N = 500$ normal agents with time step $\Delta t = 0.01$. For the radical agents, we produce a random sample of size $N_r = MN$ from the triangular distribution (30). The initial opinions of normal agents are randomly sampled from a uniform distribution on the interval $[0, L]$. For complete correspondence between the agent-based and mean-field models, we also consider the effect of even $2L$-periodic extension in the simulations of the agent-based model.

1) An Illustrative Example: Fig. 1 shows the result of numerical simulation of the mean-field and agent-based models for different noise levels with the relative mass of radicals fixed at $M = 0.15$. Indeed, for σ larger than a critical level the clustering behavior almost disappears (see the lower panel corresponding to $\sigma = 0.05$ in Fig. 1a).

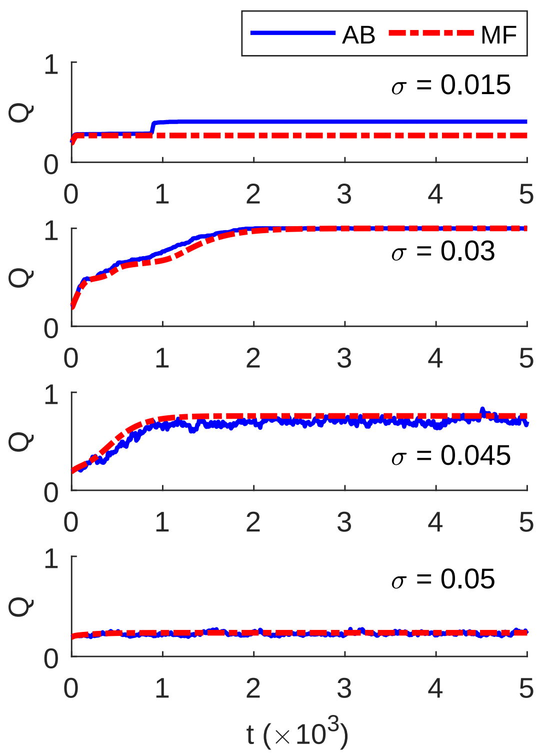
In this regard, we observe that as the level of noise increases, the number of clusters in the possible clustering behavior of the system decreases. To be more precise, a higher level of noise decreases the life-time of clustering behaviors with larger number of clusters. This effect can be particularly seen in the evolution of order parameter in Fig. 1b. In this regard, notice that, for noises smaller than the critical noise level (here, $\sigma = 0.015, 0.030, 0.045$) the flat areas in the order parameter in Fig. 1b correspond to a clustered behavior where the number of clusters is equal to the inverse of the order parameter. To illustrate, observe that for $\sigma = 0.03$ and $\sigma = 0.04$, the system reaches a single-cluster profile around the average radical opinion $A = 0.7$. Notice, however, for $\sigma = 0.03$ the system goes through 3-cluster and 4-cluster profiles as depicted in Fig. 1b (the almost flat areas in the order parameter). For $\sigma = 0.015$, we observe 4-cluster and 3-cluster profiles in mean-field and agent-based models, respectively, at $t = 10^4$. This particular difference between mean-field and agent-based models has been also mentioned and explained by [22]. Finally, we also note that for $M = 0.15$, the approximation scheme results in $\sigma_c = 0.046$ for $\gamma = 1$ and $\sigma_c = 0.057$ for $\gamma = 0.1$ (see (29) for influence of γ).

2) Effect of M on σ_c: Fig. 2 shows the order parameter derived numerically by simulating the mean-field and agent-based models. Notice how for each M, the system experiences a transition form order (clustered phase with $Q = 1$ in the yellow strip) to the disorder (with $Q = 0.2$ in the blue area in the upper part of the plots) as noise increases. Also, we note that the blue strip in the lower part of plots in Fig. 2 represents clustering behaviors with larger number of clusters (similar to the behavior seen for $\sigma = 0.015$ in Fig. 1).

This result shows that as the relative mass of radicals M increases, the corresponding critical noise level σ_c, above which the system is in disordered state, also increases. The dependence of σ_c on M is in the form of a concave function. Furthermore, for small values of M, the transition seems to be discrete, signaling a first-order transition. However, for large values of M the transition becomes blurry. This

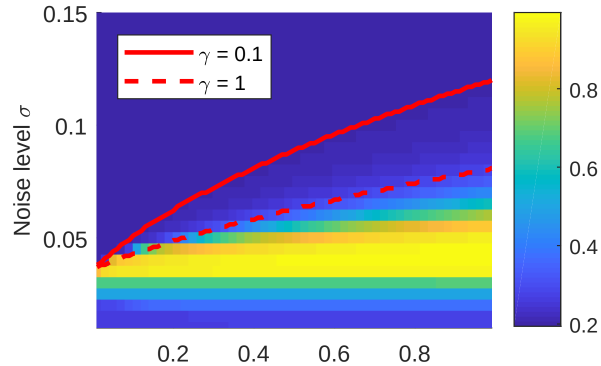


(a) Distribution of opinions/agents at $t = 10^4$

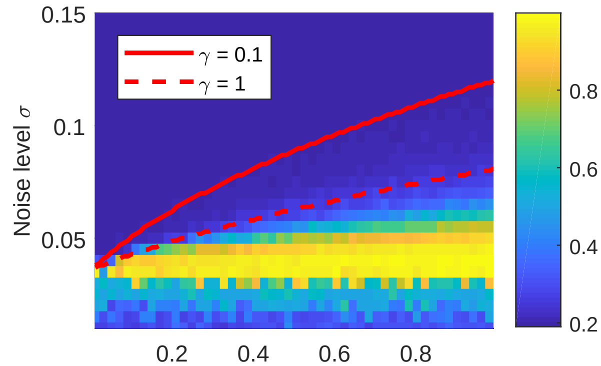


(b) Evolution of order parameter (23)

Fig. 1: Numerical simulation of the mean-field (*MF*) and agent-based (*AB*) models for increasing values of noise σ with $M = 0.15$ and uniform initial state. See Section IV-B.1 for details.



(a) Mean-field model (PDE)



(b) Agent-based model (SDE)

Fig. 2: The order parameter (23) at $t = 10^3$ from numerical simulation of the mean-field and agent-based models starting form uniform initial distribution. For the Agent-based model, the average of order parameter over the time window $[900, 1000]$ is reported. The plot covers the region $\sigma \times M \in [0.01, 0.15] \times [0.01, 1]$ with step sizes $\Delta\sigma = 0.005$ and $\Delta M = 0.02$. The red lines show the result of the numerical scheme described in Section IV-A.2 for approximating the critical noise level for different values of γ w.r.t. the second criterion (29). See Section IV-B.2 for details.

phenomenon was also observed and reported in [22] for the dependence of the critical noise level on the confidence bound R. Notice that as M increases, the required noise level for disordered behavior also increases. This increase in the noise level leads to wider clusters (see Fig. 1) which, in turn, makes differentiating the clustered (ordered) and not clustered (disordered) behaviors difficult.

Also shown in Fig. 2 (red lines) is the result of scheme provided in Section IV-A.2 for approximating the critical noise level. As can be seen, the scheme indeed provides a good approximation of the critical noise level, in particular, for $\gamma = 1$.

V. Conclusions

In this paper, we considered a mean-field model for bounded confidence opinion dynamics with environmental noise. The model also included exogenous influence by adding a mass of radical (continuum) agents to the original population of the normal agents. Two main theoretical results were provided that characterized the long-term behavior of the system. In particular, we showed that, for sufficiently large noises, the dynamics will exponentially convergence to a stationary state that can be made arbitrarily close to uniform distribution by increasing the noise level.

We then used Fourier analysis to provide a numerical scheme for approximating the critical noise level corresponding to order-disorder transition in the model. This scheme was then validated through numerical simulations of the mean-field model and the corresponding agent-based model. The results showed that for a larger mass of radicals, the order-disorder transition occurs at higher levels of noise. The numerical simulations also revealed a first-order transition for small values of the radical's relative mass, while for large values the transition becomes blurry.

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