(Article begins on next page)

Stability of systems with periodic nonlinearities and external forces: the method of periodic Lyapunov functionals

Vera B. Smirnova[1] and Anton V. Proskurnikov[2]

***Abstract*— Lur'e-type systems with periodic nonlinearities arise in many physical and engineering applications, from the simplest model of a pendulum to large-scale networks of power generators or biological oscillators. Periodic nonlinearities often cause the existence of multiple stable and unstable equilibria, which can lead to presence of "hidden attractors" and other complex phenomena reported in multi-stable systems. Many tools of classical nonlinear control, developed for systems with globally stable equilibria, become inapplicable for systems for pendulum-like systems. To study their asymptotic properties, special Lyapunov techniques have been developed based on special periodic Lyapunov functionals. In this paper, we extend this method to address the problem of robustness against uncertain external disturbances. We are primarily interested in the situation, where the disturbance decays at infinity or, more generally, has a finite limit, which enables the disturbed system to have equilibria. A natural question then arises whether asymptotic properties of the system (e.g. the solutions' convergence and the estimates of slipped cycles' number) are robust against the disturbance. In this paper, we find sufficient frequency-domain conditions ensuring such a robustness.**

***Index Terms*— Nonlinear system, stability, robustness, pendulum-like system**

I. Introduction

The second (or direct) Lyapunov method is recognized as an extremely efficient tool in analysis of nonlinear systems and constructive control design [1]–[3]. In spite of this, relatively "simple" systems often fail to be examined by the classical Lyapunov techniques due to presence of multiple equilibria and other effects of multi-stability [4]–[6]. This is exemplified by Lur'e systems with periodic nonlinearities – from the simple viscously damped pendulum to quite complicated "pendulum-like" systems such as vibrational units, electric motors, power generators and various synchronization circuits such as phase and frequency locked loops (PLL/FLL) [7]–[11]. The latter class of applications has given birth to the terms "synchronization" or "synchronous control" system [12], [13]. Synchronization circuits are naturally described by dynamic models on a smooth manifold (circle, torus or cylinder) [4], [14] or Lur'e systems with periodic nonlinearities and infinite sets of equilibria. In both cases, special methods are needed to examine stability.

One of the central problems concerned with dynamics of pendulum-like systems is the convergence of all solutions to equilibria points, also referred to as the *gradient–like behavior* [12], [15]. To establish this stability property, three non-standard Lyapunov methods have been proposed in the literature. The method of *invariant cones* originates from the works of Noldus [16] and exploits special sign-indefinite quadratic Lyapunov-type functions. The subsequent development of this method [4], [17] has inspired the very recent works on input-to-state stability of periodic systems [18]. The second method is based on the "comparison principle" and reduces a nonlinear system to a simpler "comparison" system, whose trajectories are then explicitly used in the Lyapunov functions design [4], [17], [19]. The third method exploits *periodic* Lyapunov functions (more precisely, functionals) that are designed as a sum of quadratic part and integrals of some periodic functions over the system's trajectory [17], [20], [21]. The existence of the quadratic part is proved analytically via the Kalman–Yakubovich–Popov lemma or numerically by solving LMIs [15], and the positivity of the integral part imposes nonlinear algebraic constraints on the parameters. Stability theorems thus have the form of "frequency–algebraic" or "LMI-algebraic" criteria. Periodic Lyapunov functions also enable one to reveal some relations between the initial condition of the system and the terminal equilibrium, e.g. the number of *slipped cycles* [22]–[24].

In this paper, we extend the method of periodic Lyapunov functions to systems with uncertain *disturbances*. Namely, we address robustness problems, dealing with disturbances that vanish (or, more generally, have finite limits) at infinity and thus enable the disturbed system to have equilibria. First, we obtain conditions ensuring the gradient-like behavior of a disturbed system under *arbitrary* disturbances from this class. Second, we extend the existing results on cycle slipping to such systems. Notice that our results do not follow from the very recent results on ISS robustness analysis of multi-stable systems under bounded excitations [18], [25], [26]. Although the ISS property implies the convergence of solutions under vanishing disturbances, it relies on the constructive design of ISS Lyapunov functions which, up to now, has been proposed only for special situations [25], [27] and remains an open problem for multidimensional systems with periodic nonlinearities. Besides this, the works on ISS analysis primarily deal with continuous-time dynamical systems, whereas we consider discrete-time systems as well.

*The work was supported by Russian Foundation for Basic Research (RFBR) grant 17-08-00715 held by St. Petersburg State University.

[1] St.Petersburg State University of Architecture and Civil Engineering (Department of Mathematics) and St.Petersburg State University (Faculty of Mathematics and Mechanics), St.Petersburg, Russia; smirnova_Vera_b@mail.ru

[2] Department of Electronics and Telecommunications (DET), Politecnico di Torino, Turin, Italy and the Institute for Problems of Mechanical Engineering of the Russian Academy of Sciences (IPME RAS), St.Petersburg, Russia anton.p.1982@ieee.org

II. Gradient-like behavior of continuous-time synchronization systems

We start with a continuous-time MIMO system as follows

$$\begin{aligned} &\dot z(t) = Az(t) + B\xi(t) \in \mathbb{R}^m,\\ &\dot\sigma(t) = C^T z(t) + R\xi(t) \in \mathbb{R}^l, \qquad t \geq 0.\\ &\xi(t) = \psi(\sigma(t)) + f(t) \in \mathbb{R}^l \end{aligned} \tag{1}$$

Here A, B, C, R are real matrices of appropriate dimensions, and $f(t)$ is an uncertain disturbance influencing the system.

The model (1) is typical for phase-locked loops (PLL) [12], [28]. Compared to classical Lur'e systems [4], system (1) has the following important feature. The nonlinear input $\xi(t)$ depends on the system output $\sigma(t)$. This output, however, is not directly influenced by the input $\xi(t)$, which however affects only the derivative $\dot\sigma(t)$. In PLLs, $\sigma(t)$ stands for the phase error, whereas the control input is the *frequency* of a voltage-controlled oscillator. This "indirect" control, along with the periodicity of $\psi(\cdot)$, enables the existence of multiple equilibria in the system and makes many techniques of classical absolute stability theory inapplicable [4].

A. Preliminaries and key assumptions

We start with introducing some assumptions on system (1).

Assumption 1: The linear part of (1) is asymptotically stable, controllable and observable.

Assumption 2: The disturbance $f(t)$ is decomposed as $f(t) = g(t) + L$, where $L = (L_1, \dots, L_l)^\top$ is an (unknown) constant vector and g is absolutely continuous and vanishes at infinity. It is also assumed that $|g|, |\dot g| \in L_2[0, \infty)$.

Assumption 3: The nonlinear feedback is decoupled in the sense that $\psi(\sigma) = (\psi_1(\sigma_1), \dots, \psi_l(\sigma_l))^T$. Each component ψ_j is periodic with the (minimal) period $\Delta_j > 0$. All ψ_j are smooth, in particular, the minima and maxima exist

$$\mu_{1j} \triangleq \min_{\zeta\in[0,\Delta_j]} \frac{d\psi_j(\zeta)}{d\zeta}, \quad \mu_{2j} \triangleq \max_{\zeta\in[0,\Delta_j]} \frac{d\psi_j(\zeta)}{d\zeta}. \tag{2}$$

We also assume that each function $\varphi_j(\zeta) \triangleq \psi_j(\zeta) + L_j$ has simple isolated zeros on $[0, \Delta_j)$ (and hence $\mu_{1j}\mu_{2j} < 0$), so that the disturbed system has equilibria.

For a complex-valued matrix H, H^* denotes its Hermitian (complex-conjugate) transpose. If H is square, let $Re\, H \triangleq (H + H^*)/2$. Identity $m \times m$-matrix is denoted by I_m. The transfer matrix of the linear part of (1) from ξ to $(-\dot\sigma)$ is

$$K(p) \triangleq -R + C^*(A - pI_m)^{-1}B, \qquad p \in \mathbb{C}. \tag{3}$$

Throughout the paper, $\imath = \sqrt{-1}$ denotes the imaginary unit.

All stability criteria presented below involve a set of parameters, that one can arbitrarily choose (from the predefined sets). These parameters are interrelated by two types of constraints: the frequency-domain inequality (which could be replaced by an LMI) and a set of nonlinear algebraic inequalities. These parameters are as follows:

- two diagonal matrices $A_i \triangleq diag\{\alpha_{i1}, \dots, \alpha_{il}\}$, $i = 1, 2$, where $-\infty \leq \alpha_{1j} \leq \mu_{1j}$, $\infty \geq \alpha_{2j} \geq \mu_{2j}$ $\forall j$;
- arbitrary diagonal matrix $\varkappa = diag\{\varkappa_1, \dots, \varkappa_l\}$;
- positive diagonal matrices $\varepsilon = diag\{\varepsilon_1, \dots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \dots, \tau_l\}$, $\delta = diag\{\delta_1, \dots, \delta_l\}$;
- constants $a_1, \dots, a_l \in [0, 1]$.

B. Frequency-domain stability criteria

In this paper, we are interested in the situation where each solution $z(t), \sigma(t)$ converges to some equilibrium point z^0, σ^0. It is easy to show that at any such point we have $z^0 = 0$ and $\varphi(\sigma^0) = \psi(\sigma^0) + L = 0$. We give a definition.

Definition 1: We say that a solution to (1) *converges* (to an equilibrium point) if there exist a vector $\sigma^0 = (\sigma_{10}, \dots, \sigma_{l0})$ such that $\varphi_j(\sigma_{j0}) = 0\, \forall j$ and

$$z(t) \xrightarrow[t\to\infty]{} 0 \tag{4}$$

$$\dot\sigma(t) \xrightarrow[t\to\infty]{} 0 \tag{5}$$

$$\sigma_j(t) \xrightarrow[t\to\infty]{} \sigma_{j0}. \tag{6}$$

The system is *gradient-like* [12] if all its solutions converge.

Our first two results establish "frequency-algebraic" criteria for the gradient-like behavior of (1).

We introduce the functions

$$\Phi_j(\zeta) \triangleq \sqrt{\left(1 - \alpha_{1j}^{-1}\psi_j'(\zeta)\right)\left(1 - \alpha_{2j}^{-1}\psi_j'(\zeta)\right)}, \tag{7}$$

where we put $\alpha_{ij}^{-1} = 0$ if $\alpha_{ij} = \pm\infty$. We also denote

$$\begin{gathered} \nu_j \triangleq \frac{\int_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)|\, d\zeta},\ \nu_{0j} \triangleq \frac{\int_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)|\, \Phi_j(\zeta)d\zeta},\\ \nu_{2j} \triangleq \frac{\int_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)|\sqrt{1 + \frac{\tau_j}{\varepsilon_j}\Phi_j^2(\zeta)}d\zeta}. \end{gathered} \tag{8}$$

Theorem 1: Let parameters $A_1, A_2, \varkappa, \tau, \delta, \varepsilon, a_i$ exist with the aforementioned properties such that

1) the frequency-domain inequality holds for each $\omega \in \mathbb{R}$

$$\begin{aligned} &Re\, \varkappa K(\imath\omega) - K^*(\imath\omega)\varepsilon K(\imath\omega) - \delta -\\ &- \left(K(\imath\omega) + A_1^{-1}\imath\omega\right)^* \tau \left(K(\imath\omega) + A_2^{-1}\imath\omega\right) \geq 0 \end{aligned} \tag{9}$$

2) the following matrices are positive definite

$$\begin{pmatrix} \varepsilon_k & \frac{\varkappa_k a_k \nu_k}{2} & 0\\ \frac{\varkappa_k a_k \nu_k}{2} & \delta_k & \frac{\varkappa_k a_{0k}\nu_{0k}}{2}\\ 0 & \frac{\varkappa_k a_{0k}\nu_{0k}}{2} & \tau_k \end{pmatrix} > 0. \tag{10}$$

Here $a_{0k} \triangleq 1 - a_k$ and $k = 1, \dots, l$.

Then system (1) is gradient-like.

Proof: We start with transforming (1) as follows

$$\begin{aligned} \frac{dy(t)}{dt} &= Qy(t) + L\eta(t),\\ \frac{d\sigma(t)}{dt} &= D^*y(t) \end{aligned} \tag{11}$$

where we put by definition

$$Q \triangleq \begin{bmatrix} A & B\\ 0 & 0 \end{bmatrix}, L \triangleq \begin{bmatrix} 0\\ I_l \end{bmatrix}, D \triangleq \begin{bmatrix} C\\ R^* \end{bmatrix},$$

$$y(t) \triangleq \begin{bmatrix} z(t)\\ \xi(t) \end{bmatrix}, \quad \eta(t) \triangleq \frac{d}{dt}\xi(t).$$

According to the Kalman-Yakubovich-Popov lemma [4], [29], condition 1) entails the existence of a symmetric $(m+l)\times(m+l)$-matrix H such that the following quadratic form

$$G(y,\eta)=2y^*H(Qy+L\eta)+y^*D\varepsilon D^*y+y^*L\varkappa D^*y-\\-(D^*y-A_1^{-1}\eta)\tau(A_2^{-1}\eta-D^*y)+y^*L\delta L^*y$$

is non-negative definite, that is,

$$G(y,\eta)\le 0 \qquad \forall y\in\mathbb{R}^{l+m},\forall\eta\in\mathbb{R}^l. \tag{12}$$

We are going to use periodic functions

$$F_i(\zeta)=\varphi_i(\zeta)-\nu_i\left|\varphi_i(\zeta)\right|, \tag{13}$$

$$\Psi_i(\zeta)=\varphi_i(\zeta)-\nu_{0i}\Phi_i(\zeta)\left|\varphi_i(\zeta)\right|, \tag{14}$$

constructed in a way to have zero average on a period:

$$\int_0^{\Delta_i}F_i(\zeta)d\zeta=\int_0^{\Delta_i}\Psi_i(\zeta)d\zeta=0,\quad \forall i=1,\dots,l. \tag{15}$$

With the help of $F_i(\zeta)$ and $\Psi_i(\zeta)$ we construct a Lyapunov-type function[1]

$$v(t)=y^*(t)Hy(t)+\sum_{k=1}^{l}\varkappa_k\left(a_k\int_{\sigma_k(0)}^{\sigma_k(t)}F_k(\zeta)d\zeta+\right.\\\left.+a_{0k}\int_{\sigma_k(0)}^{\sigma_k(t)}\Psi_k(\zeta)d\zeta\right). \tag{16}$$

The derivative of $v(t)$ in view of (11) is as follows

$$\frac{dv(t)}{dt}=2y^*(t)H(Qy(t)+L\eta(t))+\\+\sum_{k=1}^{l}\varkappa_k(a_kF_k(\sigma_k(t))+a_{0k}\Psi_k(\sigma_k(t))\dot\sigma(t). \tag{17}$$

It follows from (12) that

$$\frac{dv(t)}{dt}\le-\sum_{k=1}^{l}\left(\varepsilon_k\dot\sigma_k^2(t)+\varkappa_k\varphi_k(\sigma_k(t))\dot\sigma_k(t)\right.\\+\delta_k\varphi_k^2(\sigma_k(t))+\tau_k\Phi_k^2(\sigma_k(t))\dot\sigma_k^2(t)-\\-\varkappa_k\left(a_kF_k(\sigma_k(t)+a_{0k}\Psi_k(\sigma_k(t))\right)\dot\sigma_k(t))+U(t) \tag{18}$$

where by definition

$$U(t)\triangleq-\textstyle\sum_{k=1}^{l}\left(\varkappa_kg_k(t)\dot\sigma_k(t)+\delta_kg_k^2(t)+\right.\\+2\delta_kg_k(t)\varphi_k(\sigma_k(t))-\tau_k\alpha_{2k}^{-1}(\dot\sigma_k(t)-\\-\alpha_{1k}^{-1}\dot\varphi_k(\sigma_k(t)))\dot g_k(t)-\tau_k\alpha_{1k}^{-1}(\dot\sigma_k(t)-\\-\alpha_{2k}^{-1}\dot\varphi_k(\sigma_k(t)))\dot g_k(t)+\tau_k\alpha_{1k}^{-1}\alpha_{2k}^{-1}\dot g_k^2(t)). \tag{19}$$

Since $\varphi\in C^1$, one shows that

$$|\dot\sigma_k(t)-\alpha_{ik}^{-1}\dot\varphi_k(\sigma_k(t))|\le D_{0k}|\dot\sigma_k(t)|, \tag{20}$$

where D_{0k} are positive constants. The well-known Cauchy-Bunyakovsky-Schwartz inequality implies that

$$a^\top b\le\bar\varepsilon|a|^2+\bar\varepsilon^{-1}|b|^2 \tag{21}$$

[1]Rigorously speaking, $v(t)=v(y_t(\cdot))$ is a *Lyapunov functional*, depending on the whole trajectory of the system $y_t(\cdot)=y|_{[0,t]}:[0,t]\to\mathbb{R}^{l+m}$. We denote it $v(t)$ with some abuse of notation in order to simplify reading.

for each vectors a,b and a number $\bar\varepsilon>0$. Hence, for an arbitrary $\bar\varepsilon>0$ one has

$$U(t)\le\textstyle\sum_{k=1}^{l}\left\{D_{1k}\bar\varepsilon(\dot\sigma_k(t))^2+D_{2k}\bar\varepsilon(\varphi_k(\sigma_k(t)))^2+\right.\\\left.+D_{3k}\bar\varepsilon^{-1}g_k^2(t)+D_{4k}\bar\varepsilon^{-1}\dot g_k^2(t)\right\} \tag{22}$$

with some positive constants D_{ik} $(i=1,2,3,4)$. Then from (13), (14) and (22) we conclude that

$$\dot v(t)\le-\sum_{k=1}^{l}(\bar Z_k(\dot\sigma_k(t),\varphi(\sigma_k(t)),\Phi_k(\sigma_k(t))\dot\sigma(t))\\+D_{3k}\bar\varepsilon^{-1}g_k^2(t)+D_{4k}\bar\varepsilon^{-1}\dot g_k^2(t)),\\\bar Z_k(x,y,z)\triangleq(\varepsilon_k-D_{1k}\bar\varepsilon)x^2+(\delta_k-D_{2k}\bar\varepsilon)y^2+\\+\tau_kz^2+\varkappa_k\nu_ka_kxy+\varkappa_k\nu_{0k}a_{0k}yz. \tag{23}$$

It follows from condition 2) that for $\bar\varepsilon>0$ being sufficiently small, the quadratic forms $\bar Z_k$ are positive definite, and thus

$$\frac{dv(t)}{dt}\le-\sum_{k=1}^{l}(\delta_{0k}\varphi_k^2(\sigma_k(t))+\varepsilon_{0k}\dot\sigma_k^2(t))+\\+\bar\varepsilon^{-1}\sum_{k=1}^{l}(D_{3k}g_k^2(t)+D_{4k}\dot g_k^2(t)) \tag{24}$$

with $\varepsilon_{0k},\ \delta_{0k}>0$. Integrating from 0 to t, one obtains

$$v(t)-v(0)\le-\sum_{k=1}^{l}\int_0^t(\delta_{0k}\varphi_k^2(\sigma_k(t))+\varepsilon_{0k}\dot\sigma_k^2(t))\,dt+\\+\bar\varepsilon\sum_{k=1}^{l}(D_{3k}\int_0^tg_k^2(t)\,dt+D_{4k}\int_0^t\dot g_k^2(t))\,dt\quad\forall t\ge0.$$

By Assumption 1, matrix A is Hurwitz. Notice now that $v(t)$ is bounded from below since function $\xi(t)$ is bounded (which implies boundedness of the solution) and the integrals in (16) are bounded due to (15) and the periodicity of F_i,Ψ_i. Recalling that $\varepsilon_{0k},\delta_{0k}>0$, one shows that

$$\int_0^\infty\varphi_k^2(\sigma_k(t))dt<+\infty,\qquad\int_0^\infty\dot\sigma_k^2(t)dt<+\infty. \tag{25}$$

Since matrix A is Hurwitz, functions $z(t)$, and $\dot\sigma(t)$ are bounded on $[0,+\infty)$. Functions $\varphi_k(\sigma_k(t))$ and $\dot\sigma_k(t)$ are uniformly continuous on $[0,+\infty)$. Then it follows from (25) according to Barbalat lemma [30] that

$$\varphi_k(\sigma_k(t))\to0\quad as\quad t\to+\infty,\quad(k=1,\dots,l)\\\dot\sigma_k(t)\to0\quad as\quad t\to+\infty,\quad(k=1,\dots,l).$$

Since φ_k has isolated zeros, σ_k converges to one of them, i.e. (6) is valid. By noticing now that $\xi(t)=\varphi(\sigma(t))+g(t)\to0$ and A is Hurwitz, one shows that (4) holds. ■

Theorem 2: Theorem 1 retains its validity, replacing (10) by the inequalities

$$2\sqrt{\varepsilon_j\delta_j}>|\nu_{2j}|\varkappa_j\quad\forall j=1,\dots,l. \tag{26}$$

Proof: Retracing the proof of Theorem 1, consider the Lyapunov-type function

$$v(t)=y^*(t)Hy(t)+\sum_{k=1}^{l}\varkappa_k\int_{\sigma_k(0)}^{\sigma_k(t)}Y_k(\zeta)d\zeta,\\Y_j(\zeta)\triangleq\varphi_j(\zeta)-\nu_{2j}|\varphi_j(\zeta)|\sqrt{1+\frac{\tau_j}{\varepsilon_j}\Phi_j^2(\zeta)}. \tag{27}$$

Using (12) and (26), one can derive the inequality (24) for the Lyapunov function (27). ∎

Notice that the periodicity of nonlinearities is heavily used in the proofs of Theorems 1 and 2, enabling one to establish boundedness of the Lyapunov functional $v(t)$ along each trajectory. Counterparts of Theorems 1 and 2 for undisturbed systems have been used in [20], [31] to compute "stability domains" (the set of parameters for which the system is gradient-like) for undisturbed PLL with proportionally integrating filter and a sine-shaped nonlinearities. Theorems 1,2 show that these domains remain unchanged in presence of disturbances satisfying Assumption 2.

Below we consider a numerical example, dealing with a third-order PLL.

Example 1:
Consider the scalar system ($l = 1$) with the second-order transfer function

$$K(p) = \frac{1}{p^2 + ap + b} \qquad (a, b > 0), \tag{28}$$

assume that the disturbance vanishes at infinity ($L_1 = 0$) and the nonlinearity is

$$\varphi_1(\sigma) = \sin(\sigma) - \frac{1}{2}. \tag{29}$$

To get the stability domain of the system on the plane $\{a, b\}$ Theorem 1 with fixed parameters $\varkappa_1 = |\alpha_{11}| = \alpha_{12} = 1$ and varying parameters $\delta_1, \varepsilon_1, \tau_1, a_1$ has been applied. The stability domain obtained is shown in Fig. 1 (the shaded area). It can be shown that below curve C the system has *no* locally stable equilibria.

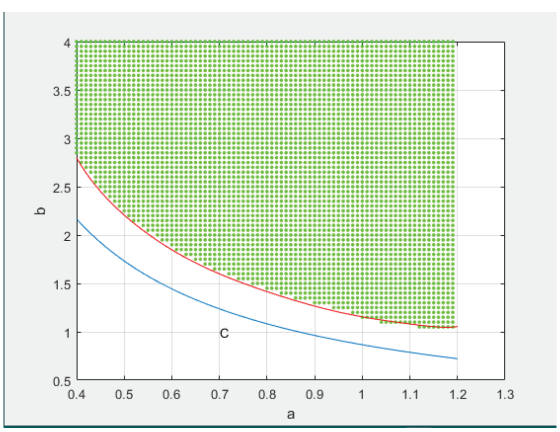

Fig. 1. Stability domain for the third-order system (28),(29), estimated by means of Theorem 1

Remark 1: Instead of using KYP lemma, reducing (12) to the frequency-domain inequality, the existence of matrix $H = H^*$ can be formulated as a condition of some LMI's solvability [32]. This inequality, in fact, is linear not only in H, but also parameters $\varkappa, \varepsilon, \tau, \delta$, but nonlinear in A_1, A_2.

III. Frequency–algebraic stability criterion for discrete-time systems

In this section, we consider discrete-time counterpart of (1)

$$\begin{aligned} &z(n+1) = Az(n) + B\xi(n),\\ &\sigma(n+1) = \sigma(n) + C^* z(n) + R\xi(n),\\ &\xi(n) = \psi(\sigma(n)) + f(n)\\ &(n = 0, 1, 2, \ldots). \end{aligned} \tag{30}$$

The functions and matrices in (30) obey Assumptions 1,3. Assumption 2 is replaced by the discrete-time counterpart.

Assumption 4: The disturbance $f(n)$ is decomposed as $f(n) = g(n) + L$, where L is an (unknown) constant vector and $\sum_{n=1}^{\infty} |g(n)| < \infty$.

Definition 2: We say that a solution to (30) *converges* if there exist a vector $q \in \mathbb{R}^l$ such that $\varphi(q) = 0$ and

$$z(n) \xrightarrow[n\to\infty]{} 0 \tag{31}$$

$$\sigma_j(n) \xrightarrow[n\to\infty]{} q. \tag{32}$$

System (30) is *gradient-like* if all its solutions converge.

Criteria of gradient-like behavior, formulated in this section, also consist of a frequency-domain inequality (replaceable by an LMI in view of the discrete-time KYP lemma). The frequency-domain inequality involves the rational matrix (3), which serves as the transfer function of the linear part of (30) from $\xi(n)$ to $(\sigma(n) - \sigma(n+1))$.

Theorem 3: Suppose there exist parameters $A_1, A_2, \varepsilon, \tau, \varkappa, \delta$, whose properties are specified in Section II-A, such that the following conditions hold:

1) the frequency–domain inequality holds for any $p \in \mathbb{C}$ such that $|p| = 1$

$$\begin{aligned} &0 \le Re\, \varkappa K(p) - \delta - K(p)^* \varepsilon K(p) - \\ &- \left(K(p) - (p-1)A_1^{-1}\right)^* \tau \left(K(p) + (p-1)A_2^{-1}\right); \end{aligned}$$

2) the inequalities hold for all $k = 1, \ldots, l$

$$4(\varepsilon_k - \frac{1}{2}\varkappa_k \alpha_{0k}(1 + |\nu_k|))\delta_k > \varkappa_k^2 \nu_k^2 \tag{33}$$

where ν_k are defined in (8) and

$$\alpha_{0k} \triangleq \begin{cases} \alpha_{2k} & \text{if } \varkappa_k > 0,\\ \alpha_{1k} & \text{if } \varkappa_k < 0. \end{cases} \tag{34}$$

Then system (30) is gradient-like.

Proof: Introduce the matrices

$$P = \begin{bmatrix} A & B\\ 0 & I_l \end{bmatrix}, L = \begin{bmatrix} 0\\ I_l \end{bmatrix} D = \begin{bmatrix} C\\ R^* \end{bmatrix}, y(n) = \begin{bmatrix} z(n)\\ \xi(n) \end{bmatrix}.$$

Denoting $\eta(n) \triangleq \xi(n+1) - \xi(n)$, system (30) shapes into

$$\begin{aligned} &y(n+1) = Py(n) + L\eta(n),\\ &\sigma(n+1) = \sigma(n) + D^* y(n)\\ &\quad (n = 0, 1, 2, \ldots). \end{aligned} \tag{35}$$

Consider a quadratic form of $y \in \mathbb{R}^{m+l}$, $\eta \in \mathbb{R}^l$:

$$\begin{aligned} &G_1(y, \eta) \triangleq (Py + L\eta)^* H(Py + L\eta) - y^* Hy + \\ &+ y^* L\varkappa D^* y + y^* D\varepsilon D^* y + y^* L\delta L^* y - \\ &-(D^* y - A_1^{-1}\eta)\tau(A_2^{-1}\eta - D^* y) \end{aligned}$$

with $H \in \mathbb{R}^{(m+l)\times(m+l)}$, $H = H^*$.

Condition 1) of the theorem guaranties, thanks to the Kalman-Szegö lemma (or discrete-time KYP lemma) [30] that there exist a symmetric matrix H such that

$$G_1(y,\eta) \le 0, \quad \forall y \in \mathbb{R}^{m+l}, \eta \in \mathbb{R}^l. \tag{36}$$

Consider a Lyapunov–type sequence[2]

$$\begin{gathered} V(n) \triangleq W(n) + \sum_{j=1}^{n} \varkappa_j \int_{\sigma_j(0)}^{\sigma_j(n)} F_j(\zeta)\, d\zeta, \\ W(n) \triangleq y^*(n) H y(n). \end{gathered} \tag{37}$$

where F_j are defined by (13). Due to Assumptions 1 and 4, the sequence $y(n)$ (and hence also $W(n)$) is bounded.

From (37) we have

$$\begin{aligned} V(n+1) - V(n) = W(n+1) - W(n) + \\ + \sum_{j=1}^{l} \varkappa_j \int_{\sigma_j(n)}^{\sigma_j(n+1)} F_j(\zeta)\, d\zeta. \end{aligned} \tag{38}$$

We are now going to estimate the difference (38). It follows from (35) and (36) that

$$\begin{aligned} W(n+1) - W(n) = y^*(n+1) H y(n+1) - y^*(n) H y(n) = \\ = (Py(n) + L\eta(n))^* H (Py(n) + L\eta(n))) - y^*(n) H y(n) \le \\ \le -y^*(n) L\varkappa D^* y(n) - y^*(n) D\varepsilon D^* y(n) - y^*(n) L\delta L^* y(n) + \\ + (D^* y(n) - A_1^{-1}\eta(n))\tau(A_2^{-1}\eta(n) - D^* y(n)). \end{aligned}$$

Introduce the first differences

$$\begin{gathered} \bar\sigma(n) \triangleq \sigma(n+1) - \sigma(n), \\ \bar\varphi(n) \triangleq \varphi(\sigma(n+1)) - \varphi(\sigma(n)), \\ \bar g(n) \triangleq g(n+1) - g(n). \end{gathered} \tag{39}$$

Then

$$\begin{aligned} W(n+1) - W(n) \le -\sum_{k=1}^{l} \{\varkappa_k \xi_k(n) \bar\sigma_k(n) + \\ + \varepsilon_k(\bar\sigma_k(n))^2 + \delta_k \xi_k^2(n) + \tau_k(\bar\sigma_k(n) - \\ - \alpha_{1k}^{-1}\eta_k(n))(\bar\sigma_k(n) - \alpha_{2k}^{-1}\eta_k(n))\} = \\ = \sum_{k=1}^{l} \{\varkappa_k \varphi_k(\sigma_k(n)) \bar\sigma_k(n) + \varepsilon_k(\bar\sigma_k(n))^2 + \\ + \tau_k(\bar\sigma_k(n) - \alpha_{1k}^{-1}\bar\varphi_k(n))(\bar\sigma_k(n) - \alpha_{2k}^{-1}\varphi_k(n)) + \\ + \delta_k \varphi_k^2(\sigma_k(n))\} + U_1(n), \end{aligned} \tag{40}$$

where we denote

$$\begin{aligned} U_1(n) \triangleq -\textstyle\sum_{k=1}^{l} \{\varkappa_k g_k(n) \bar\sigma_k(n) + \\ + 2\delta_k g_k(n) \varphi_k(\sigma_k(n)) + \delta_k g_k^2(n) - \tau\alpha_{1k}^{-1} \bar g_k(n)(\bar\sigma_k(n) - \\ - \alpha_{2k}^{-1}\bar\varphi_k(n)) - \tau\alpha_{2k}^{-1}\bar g_k(n)(\bar\sigma_k(n) - \alpha_{1k}^{-1}\bar\varphi_k(n)) + \\ + \tau\alpha_{2k}^{-1}\alpha_{1k}^{-1}\bar g_k^2(n)\} \end{aligned} \tag{41}$$

[2]Similar to the continuous-time case, this sequence can be considered as a Lyapunov functional, depending on the whole trajectory of the system.

It follows from the assumption on the entries of A_1, A_2 (see Section II-A) and (2) that

$$(\bar\sigma_k(n) - \alpha_{1k}^{-1}\bar\varphi_k(n))(\bar\sigma_k(n) - \alpha_{2k}^{-1}\bar\varphi_k(n)) \ge 0 \tag{42}$$

(this estimate relies on the mean-value theorem, stating that

$$\bar\varphi_k(n)) = \varphi_k^{'}(\hat\sigma_{kn})\bar\sigma_k(n),$$

where $\hat\sigma_{kn}$ is a point between $\sigma(n)$ and $\sigma(n+1)$).

It can be shown [20] that

$$\begin{aligned} \varkappa_k \int_{\sigma_k(n)}^{\sigma_k(n+1)} F_k(\zeta)\, d\zeta \le \varkappa_k F_k(\sigma_k(n)) \bar\sigma_k(n) + \\ + \frac{1}{2}\varkappa_k \alpha_{0k}(1 + |\nu_k|)\bar\sigma_k^2(n) \end{aligned} \tag{43}$$

The inequalities (42) and (43) imply that

$$\begin{aligned} V(n+1) - V(n) \le -\sum_{k=1}^{l} \{\varkappa_k \nu_k |\varphi_k(\sigma_k(n))| \bar\sigma_k(n) + \\ + (\varepsilon_k - \frac{1}{2}\varkappa_k \alpha_{0k}(1 + |\nu_k|)\bar\sigma_k^2(n)) + \delta_k \varphi_k^2(\sigma_k(n))\} + \\ + U_1(n). \end{aligned} \tag{44}$$

To finish the estimating of (38) we have to estimate the value of $U_1(n)$. Notice that in virtue of (2)

$$|\bar\sigma_k(n) - \alpha_{ik}^{-1}\bar\varphi_k(n)| \le C_{0k}|\bar\sigma_k(n)|) \quad (i = 1, 2) \tag{45}$$

where C_{0k} does not depend on n. Using the Cauchy-Scwartz inequality (21), for an arbitrary $\bar\varepsilon > 0$ one has

$$\begin{aligned} U_1(n) \le \sum_{k=1}^{l} \{C_{1k}\bar\varepsilon(\bar\sigma_k(n))^2 + C_{2k}\bar\varepsilon(\varphi(\sigma_k(n)))^2 + \\ + C_{3k}\bar\varepsilon^{-1} g^2(n)\}, \end{aligned} \tag{46}$$

where positive constants C_{ik} $(i = 1, 2, 3)$ depend on $\varkappa_k, \tau_k, \varepsilon_k, \delta_k, \alpha_{ik}$ $(i = 1, 2)$ but not on n. Thus

$$V(n+1) - V(n) \le -\sum_{k=1}^{n} Z_k(n) + \sum_{k=1}^{l} C_{3k}\varepsilon^{-1} g_k^2(n) \tag{47}$$

where

$$\begin{aligned} Z_k(n) = (\varepsilon_k - \frac{1}{2}\varkappa_k \alpha_{0k}(1 + |\nu_k|) - C_{1k}\bar\varepsilon)(\bar\sigma_k(n))^2 + \\ + (\delta_k - C_{2k}\bar\varepsilon)\varphi_k^2(\sigma_k(n)) + \varkappa_k \nu_k |\varphi_k(\sigma_k(n))| \bar\sigma_k(n). \end{aligned}$$

Let $\bar\varepsilon$ be chosen so small that the inequalities hold

$$4\left(\varepsilon_k - \frac{1}{2}\varkappa_k \alpha_{0k}(1 + |\nu_k|) - C_{1k}\bar\varepsilon\right)(\delta_k - C_{2k}\bar\varepsilon) > \varkappa_k^2 \nu_k^2.$$

Then $Z_k(n)$ are positive definite quadratic forms of $\bar\sigma_k(n)$ and $|\varphi_k(\sigma_k(n))|$. Therefore,

$$V(n+1) - V(n) \le -\delta_0 |\varphi(\sigma(n))|^2 + C\bar\varepsilon^{-1}|g(n)|^2, \tag{48}$$

where $\delta_0 > 0$ and $C = \max_{k=1,\dots,l} C_{3k}$.

The property (15) and the boundedness of $W(n)$ imply the boundedness of $V(n)$. Then we conclude from (48) and Assumption 4 that

$$\sum_{n=1}^{\infty} |\varphi(\sigma(n))|^2 < \infty, \tag{49}$$

which implies that $\varphi(\sigma(n)) \to 0$ and thus $\xi(n) = \varphi(\sigma(n)) + g(n) \to 0$. In view of the system stability, one obtains (31), also, $\sigma(n+1) - \sigma(n) \to 0$ as $n \to \infty$. Since $\varphi(\cdot)$ has isolated zeros, one arrives at (32), where q is one of such zeros. ■

Example 2: Consider a pulse phase-locked loop (PPLL) [33] with $l = 1, L_1 = 0$,

$$\varphi_1(\sigma) = T_p \Omega_y (\sin(\sigma + \sigma_0) - \sin \sigma_0), \quad \sigma_0 \in (0, \frac{\pi}{2}). \quad (50)$$

and a proportional integrating filter:

$$K(p) = \frac{dp + 1 - b - d}{p - b}. \quad (51)$$

Here $d = 1 - \frac{m}{a}(1 - b)$, $b = e^{-a}$, $a > 0, m \in (0, 1)$. To demonstrate the application of Theorem 3 we fix the parameters $\tau_1 = 0$, $\varkappa_1 = |\alpha_{11}| = \alpha_{12} = 1$, and vary the remaining parameters δ_1, ε_1. Then Theorem 3 requires the frequency-domain inequality

$$Re\, K(p) - \varepsilon_1 |K(p)|^2 - \delta_1 > 0 \quad (52)$$

to hold for $p = \frac{1+i\omega}{1-i\omega}$, $\omega \in \mathbb{R}$, along with the condition

$$4\delta_1(\varepsilon_1 - \frac{1}{2}(1 + \nu_1)) > \nu_1^2. \quad (53)$$

Here ν_1 is given by

$$\nu_1 = \frac{\pi \sin \sigma_0}{2(\cos \sigma_0 + \sigma_0 \sin \sigma_0)}. \quad (54)$$

The condition (52) holds if

$$\varepsilon_1 + \delta_1 = 1. \quad (55)$$

and

$$\varepsilon_1(2d + b - 1)^2 + \delta_1(1 + b)^2 \le (1 + b)(2d + b - 1). \quad (56)$$

We begin with conditions (53) and (55). They can be fulfilled, both of them, if $\nu \in (0, \frac{1}{3})$.

Choosing for every ν

$$\delta_1 = \frac{1 - \nu}{4} \qquad \varepsilon_1 = \frac{\nu + 3}{4}. \quad (57)$$

we can guarantee that inequality (56) is true provided that

$$\nu \ge \frac{2 - b - 3d}{b + d}. \quad (58)$$

So the PPLL is gradient-like if

$$\frac{2 - b - 3d}{b + d} \le \nu < \frac{1}{3}. \quad (59)$$

The stability conditions (59) can be essentially weakened, varying the whole set of parameters.

IV. Conclusion

In this paper we study the asymptotic behavior of forced solutions of multidimensional continuous- and discrete-time Lur'e systems with periodic nonlinearities in presence of uncertain disturbances. It is assumed that this disturbance does not oscillate at infinity, enabling the disturbed system to have equilibria. The results are obtained by means of the "periodic Lyapunov function" method. Notice that in the continuous-time case, some of our results can be also obtained by the method of Popov's integral indices, developed in [34], [35]. The method of Lyapunov functions and sequences, developed in this paper, has however a number of advantages of the latter approach. First, a closer analysis of the proofs reveals the possibility to estimate the convergence rate of the solutions, also, it opens up the perspective of obtaining ISS-like criteria for multidimensional pendulum-like systems [5], [25] in the case of bounded disturbances. Second, it also enables to evaluate some characteristics of transient processes, such as e.g. the number of cycle slipping [22]. These extensions are beyond the scope of this paper and will be included in its extended journal version.

References

[1] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method*. New York: Springer-Verlag, 1977.

[2] D. Siljak, *Large-Scale Dynamic Systems: Stability and Structure*. New York: Dover Publ. Inc., 2007.

[3] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and adaptive control design*. Wiley, 1995.

[4] A. Gelig, G. Leonov, and V. Yakubovich, *Stability of stationary sets in control systems with discontinuous nonlinearities*. World Scientific Publishing Co., 2004.

[5] D. Efimov, "Global Lyapunov analysis of multistable nonlinear systems," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 3132–3154, 2012.

[6] D. Dudkowski, S. Jafari, T. Kapitaniak, N. Kuznetsov, G. Leonov, and A. Prasad, "Hidden attractors in dynamical systems," *Physics Reports*, vol. 637, pp. 1–50, 2016.

[7] G. Baker and J. Blackburn, *The Pendulum: A Physics Case Study*. Oxford University Press, 2005.

[8] J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*. New York: Interscience, 1950.

[9] I. Blekhman, *Vibrational mechanics*. World Scientific Publ. Co., 2000.

[10] W. Margaris, *Theory of the Non-linear Analog Phase Locked Loop*. New York: Springer, 2004.

[11] R. Best, *Phase-locked Loops: Design, Simulation and Applications*. McGraw-Hill, 2003.

[12] G. Leonov, "Phase synchronization. theory and applications," *Autom. Remote Control*, vol. 67, no. 10, pp. 1573–1609, 2006.

[13] F. Hoppensteadt, "An algorithm for approximate solutions to weakly filtered synchronous control systems and nonlinear renewal processes," *SIAM J. Appl. Math.*, vol. 43, no. 4, pp. 834–843, 1983.

[14] J. Kudrewicz and S. Wasowicz, *Equations of Phase-Locked Loops. Dynamics on Circle, Torus and Cylinder*. World Scientific Publ. Co., 2007.

[15] Z. Duan, J. Wang, and L. Huang, "Criteria for dichotomy and gradient-like behavior of a class of nonlinear systems with multiple equilibria," *Automatica*, vol. 43, pp. 1583–1589, 2007.

[16] E. J. Noldus, "New direct Lyapunov-type method for studying synchronization problems," *Automatica*, vol. 13, no. 2, pp. 139 – 151, 1977.

[17] G. A. Leonov, D. Ponomarenko, and V. B. Smirnova, *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications*. Singapore–New Jersey–London–Hong Kong: World Scientific, 1996.

[18] D. Efimov, J. Schiffer, N. Barabanov, and R. Ortega, "A relaxed characterization of ISS for periodic systems with multiple invariant sets," *European Journal of Control*, vol. 37, pp. 1 – 7, 2017.

[19] G. A. Leonov, V. Reitmann, and V. B. Smirnova, *Non-local methods for pendulum-like feedback systems*. Stuttgart-Leipzig: Teubner, 1992.

[20] A. Perkin, V. Smirnova, N. Utina, and A. Shepeljavyi, "Application of the metod of lyapunov periodic functions," *Vestnik SPb Univ. Math.*, vol. 44, no. 3, pp. 197–207, 2011.

[21] A. Perkin, V. Smirnova, and A. Shepeljavyi, "Frequency-algebraic conditions for stability of phase systems with application to phase-locked loops and synchronization systems," *Cybernetics and Physics*, vol. 1, no. 3, pp. 188–197, 2012.

[22] O. B. Ershova and G. A. Leonov, "Frequency estimates of the number of cycle slippings in the phase control systems," *Autom. Remote Control*, vol. 44, no. 5, pp. 600–607, 1983.

[23] Y. Yang and L. Huang, "Cycle slipping in phase synchronization systems," *Phys. Lett. A*, vol. 362, pp. 183–188, 2007.

[24] A. Perkin, A. Proskurnikov, A. Shepeljavyi, and V. Smirnova, "Sharpened estimates for the number of slipped cycles in control systems with periodic differential nonlinearities," *Cybernetics and Physics*, vol. 2, no. 3, pp. 222–231, 2013.

[25] D. Angeli and D. Efimov, "Characterizations of input-to-state stability for systems with multiple invariant sets," *IEEE Transactions on Automatic Control*, vol. 60, no. 12, pp. 3242–3256, 2015.

[26] P. Forni and D. Angeli, "The ISS approach to the stability and robustness properties of nonautonomous systems with decomposable invariant sets: An overview," *European Journal of Control*, vol. 30, pp. 50–60, 2016.

[27] J. Schiffer, D. Efimov, R. Ortega, and N. Barabanov, "An input-to-state stability approach to verify almost global stability of a synchronous-machine-infinite-bus system," *Philos. Trans. A. Math. Phys. Eng. Sci.*, vol. 75, no. 2100, p. 20160304, 2017.

[28] G. Leonov, N. Kuznetsov, M. Yuldashev, and R. Yuldashev, "Hold-in, pull-in, and lock-in ranges of PLL circuits: Rigorous mathematical definitions and limitations of classical theory," *IEEE Trans. Circuits and Systems-I*, vol. 62, no. 10, pp. 2454–2464, 2015.

[29] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Systems and Control Letters*, vol. 28, pp. 7–10, 1996.

[30] V. Popov, *Hyperstability of Control Systems*. Springer Verlag, 1973.

[31] V. Smirnova and A. Proskurnikov, "Phase locking, oscillations and cycle slipping in synchronization systems," in *Proceedings of European Control Conference (ECC)*, Aalborg, Denmark, 2016, pp. 873–878.

[32] S. Boyd and L. Vandenberghe, *Convex optimization*. New York: Cambridge Univ. Press, 2004.

[33] G. A. Leonov and V. B. Smirnova, *Mathematical Problems of Phase Synchroniztion Theory*. Saint-Petersburg: Nauka, (in Russian), 2000.

[34] V. B. Smirnova, A. V. Proskurnikov, and N. V. Utina, "Dichotomy and stability of disturbed systems with periodic nonlinearities," in *Mediterranean Conference on Control and Automation (MED)*, 2018, pp. 1–9.

[35] V. B. Smirnova, A. V. Proskurnikov, N. V. Utina, and R. V. Titov, "Forced solutions of disturbed pendulum-like Lur'e systems," in *2018 IEEE Conference on Control Technology and Applications (CCTA)*, 2018, pp. 748–753.