Schubert Derivations on the Infinite Wedge Power

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Abstract. The *Schubert derivation* is a distinguished Hasse-Schmidt derivation on the exterior algebra of a free abelian group, encoding the formalism of Schubert calculus for all Grassmannians at once. The purpose of this paper is to extend the Schubert derivation to the infinite exterior power of a free $\mathbb{Z}$-module of infinite rank (fermionic Fock space). Classical vertex operators naturally arise from the *integration by parts formula*, that also recovers the generating function occurring in the *bosonic vertex representation* of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{Z})$, due to Date, Jimbo, Kashiwara and Miwa (DJKM). In the present framework, the DJKM result will be interpreted as a limit case of the following general observation: the singular cohomology of the complex Grassmannian $G(r,n)$ is an irreducible representation of the Lie algebra of $n \times n$ square matrices.

Keywords and phrases: Hasse-Schmidt Derivations on Exterior Algebras, Schubert Derivations on infinite wedge powers; Bosonic and Fermionic Fock Spaces; vertex operators, bosonic vertex representation of Date-Jimbo-Kashiwara–Miwa.

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To Israel Vainsencher on the occasion of his seventieth birthday

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Contents

Introduction

0.1 The Goal. Let $r, n \in \mathbb{N} \cup \{\infty\}$ such that $r \leqslant n$. The main characters of this paper are i) the exterior algebra $\bigwedge M_n$ of a free abelian group $M_n := \bigoplus_{0\leqslant j<n} \mathbb{Z}b_j$ and ii) the cohomology ring $B_{r,n}$ of the Grassmann variety $G(r, n)$. By the latter we mean the following. If $0 \leqslant r \leqslant n < \infty$, $B_{r,n}$ stands for the usual singular cohomology ring $H^*(G(r, n), \mathbb{Z})$ of the Grasmannian variety parametrizing r-dimensional subspaces of the complex n-dimensional vector space. If $n = \infty$, the ring $B_r := B_{r,\infty}$ will denote the cohomology of the ind-variety $G(r, \infty)$ (see e.g. [2, p. 302] or [4, 12]), which is a polynomial ring $\mathbb{Z}[e_1, \dots, e_r]$ in r-indeterminates. If $r = n = \infty$, instead, $Gr(\infty) := G(\infty, \infty)$ is the ind-Grassmannian constructed e.g. in [12, Section 3.3] or the Sato's Universal Grassmann Manifold (UGM), as in e.g. [20]. In this case $B := B_{\infty,\infty}$ is the $\mathbb{Z}$-polynomial ring in infinitely many indeterminates. Let $\mathcal{B}_{ij} \in \mathrm{End}_\mathbb{Z}(M_n)$ such that $\mathcal{B}_{ij}(b_k) = b_i\delta_{jk}$, and let

$$gl_n(\mathbb{Z}) := \bigoplus_{0\leqslant i,j<n} \mathbb{Z} \cdot \mathcal{B}_{ij} \subseteq \mathrm{End}_\mathbb{Z}(M_n), \tag{1}$$

which is a Lie algebra with respect to the usual commutator. Clearly $gl_n(\mathbb{Z}) = \mathrm{End}_\mathbb{Z}(M_n)$ if $n < \infty$. This paper is inspired by the following simple observation, for which we have not been able to find an explicit reference in the literature:

The ring $B_{r,n}$ is a module over the Lie algebra $gl_n(\mathbb{Z})$.

If $r = 1$ the claim is obvious, because if e denotes the hyperplane class of $\mathbb{P}^{n-1}$, then $B_{1,n} = \mathbb{Z}[e]/(e^n)$ is a free abelian group of rank n and, therefore, the standard representation of its Lie algebra of endomorphisms. The general case, for $r < \infty$ and arbitrary $n \geqslant r$, follows from noticing that all $A \in \mathfrak{gl}_n(\mathbb{Z})$ induce an even derivation $\delta(A)$ on $\bigwedge M_n$:

$$\begin{cases} \delta(A)\mathbf{u} & := & A\mathbf{u}, & \forall \mathbf{u} \in M_n \\ \delta(A)(\mathbf{v} \wedge \mathbf{w}) & := & \delta(A)\mathbf{v} \wedge \mathbf{w} + \mathbf{v} \wedge \delta(A)\mathbf{w}, & \forall \mathbf{v}, \mathbf{w} \in \bigwedge M_n. \end{cases} \tag{2}$$

Since $\delta([A,B]) = [\delta(A), \delta(B)]$, the map $A \mapsto \delta(A)_{|\bigwedge^r M}$ makes $\bigwedge^r M_n$ into an (irreducible) representation of $\mathfrak{gl}_n(\mathbb{Z})$. Then $B_{r,n}$ gets equipped with a $\mathfrak{gl}_n(\mathbb{Z})$-module structure as well, due to the $\mathbb{Z}$-module isomorphism $B_{r,n} \to \bigwedge^r M_n$. Recall that the latter is the composition of the Poincaré isomorphism, mapping $B_{r,n}$ onto its singular homology $H_*(G(r,n),\mathbb{Z})$, with the natural isomorphism $H_*(G(r,n),\mathbb{Z}) \to \bigwedge^r H_*(\mathbb{P}^{n-1},\mathbb{Z}) \cong \bigwedge^r M_n$, as in [6], or [7, diagramme (5.27)].

For $r = \infty$, the fact that B is a $\mathfrak{gl}_\infty(\mathbb{Z})$-module is well known, and is due to the isomorphism of B with each degree F_m of the fermionic Fock space which, roughly speaking, plays the role of an infinite exterior power. Its structure has been explicitly described by Date, Jimbo, Kashiwara and Miwa (DJKM) in [3], see also [13, Formula (1.17)] and [15, p. 53], by computing the shape of a generating function $\mathcal{B}(z,w)$ encoding the multiplication of any polynomial by elementary matrices $\mathcal{B}_{ij}$ of infinite sizes.

In our contribution [8] we determine the shape of the same generating function in the case $r < \infty$, by using the formalism of Schubert derivations in the sense of [7]. The formula we obtain is new (as far as we know), and has a classical flavor (occurring as a 2-parameter deformation of the Schur determinant occurring in Giambelli's formula). We then felt the need to show that our methods also work in the known case $r = \infty$. The output is the present paper, in which we offer an alternative deduction of the DJKM bosonic vertex representation of $\mathfrak{gl}_\infty(\mathbb{Z})$, based on the extension of the Schubert derivations to an infinite wedge power. Our method to compute the $\mathfrak{gl}_n(\mathbb{Z})$-structure of $B_{r,n}$ then works uniformly for all pairs $r \leqslant n$ ranging over $\mathbb{N} \cup \{\infty\}$.

0.2 Outline. The main tool used in this paper is the notion of *Hasse–Schmidt* (HS) *derivation on an exterior algebra*, quickly recalled in Section 2. Let $M := \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \cdot b_i$ be a free abelian group with basis $\mathbf{b} := (b_i)_{i\in\mathbb{Z}}$. A map $\mathcal{D}(z) : \bigwedge M \to \bigwedge M[[z]]$ is said to be a HS derivation on $\bigwedge M$ if $\mathcal{D}(z)(u\wedge v) = \mathcal{D}(z)u \wedge \mathcal{D}(z)v$. In this paper, we shall be concerned mainly with the *Schubert derivations* (Section 3). They are denoted by $\sigma_+(z), \overline{\sigma}_+(z)$ and by $\sigma_-(z), \overline{\sigma}_-(z)$, where $\sigma_\pm(z) := \sum_{i\geqslant 0} \overline{\sigma}_{\pm i} z^{\pm i} \in \mathrm{End}_{\mathbb{Z}}(\bigwedge M)[[z^{\pm 1}]]$ are the unique HS derivations such that $\sigma_j b_i = b_{i+j}$, for all $i, j \in \mathbb{Z}$, and $\overline{\sigma}_{\pm 1}(z)$ are their inverse in $\mathrm{End}(\bigwedge M)[[z^{\pm 1}]]$.

The reason it is appropriate to call $\sigma_\pm(z)$ and $\overline{\sigma}_\pm(z)$ Schubert derivations is explained in [7, 9]. As a matter of fact, the operator σ_i acting on $\bigwedge^r M$, obeys the same combinatorics enjoyed by the special Schubert cocycles in the cohomology ring of a Grassmannian $G(r,n)$, for r and n big enough.

The Fermionic Fock space (FFS), a graded abelian group $F = \bigoplus_{m\in\mathbb{Z}} F_m$, comes into the game in Section 4, playing the role of something like $\bigwedge^\infty M$ (often denoted in the literature by $\bigwedge^{\infty/2} M$, to signify that is generated by semi-infinite exterior monomials, see e.g. [16, Section 3] or [1, Section 1]). It is a notion for which there are excellent classical references in the literature, such as [5, 14, 15]. However, to keep the exposition as self contained as possible, Section 4 supplies an alternative ad hoc algebraic construction of it, which widely suffices for our purposes and, possibly, may be useful for pedagogical ones.

The extension of the Schubert derivations to the FFS is not entirely trivial, although not difficult. It turns out that the vertex operators occurring in the classical presentation of the Boson–Fermion correspondence can all be recovered by multiplying the four Schubert derivations $\sigma_\pm(z)$ and $\overline{\sigma}_\pm(z)$. For instance $\sigma_+(z)\overline{\sigma}_-(z)$ and $\overline{\sigma}_+(z)\sigma_-(z)$ are basically the bosonic vertex operators acting on the Fock representation of the Heisenberg Lie algebra, [5, 15], and are described in Sections 6–7. Let $\delta(z,w) := \sum_{i,j\in\mathbb{Z}} \delta(\mathcal{B}_{ij})z^i w^{-j}$, where $\mathcal{B}_{ij}$ are as in the first part of this introduction. The main result of this paper is that the action of $\delta(z,w)$ on each degree of the FFS, is proportional to the product

$$\sigma_+(z)\overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w)$$

of the four Schubert derivations, and it coincides with it in degree 0. The product $\overline{\sigma}_-(z)\overline{\sigma}_+(w)$ commutes up to a rational factor, determined in Section 8:

$$\sigma_+(z)\overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w) = \left(1-\frac{w}{z}\right)^{-1}\Gamma(z,w) \tag{3}$$

where we set

$$\Gamma(z,w) := \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)$$

Formula (3) is precisely the DJKM expression of the representation of $\mathfrak{gl}_\infty(\mathbb{Z})$ on F_0, defined over the integers.

To achieve the DJKM expression in its classical form (Section 9), one tensors by $\mathbb{Q}$ and reads the expression as acting on $B\otimes_{\mathbb{Z}}\mathbb{Q}$ via its isomorphism with each degree of the FFS. By essentially the same arguments as in [9, Theorem 7.7], one easily checks that:

$$\Gamma(z,w) = \exp\left(\sum_{i\geqslant 1} x_i(z^i - w^i)\right)\exp\left(-\sum \frac{1}{i}\left(\frac{1}{z^i}-\frac{1}{w^i}\right)\frac{\partial}{\partial x_i}\right), \tag{4}$$

where the sequence $(x_1, x_2, \ldots)$ is defined through the equality $(1 - e_1 z + e_2 z^2 - \cdots)\exp(\sum_i x_i z^i) = 1$. Thus formula (4) is precisely [15, equation (5.33)], i.e. (3) turns into [15, equation (5.32)] for $m = 0$ (Cf. Corollary 9.4).

0.3 We should finally remark that many of the tools employed in this paper within the framework of Schubert derivations have already been reviewed in other contributions (e.g. [6, 7, 9, 10, 11]), which we might well refer to. However, since the vocabulary of HS–derivations is not yet standard, it seems motivated to recall the basic notions and facts in order to keep the paper as self contained as possible.

1 Notation

1.1 A *partition* is a monotone non–increasing sequence $\boldsymbol{\lambda}$ of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ such that all the terms are zero but finitely many. We denote by $\ell(\boldsymbol{\lambda}) := \sharp\{i\,|\,\lambda_i \neq 0\}$ its *length*. We denote by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}_r$ the set of all partitions of weight at most r. The partitions form an additive semigroup: if $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{P}$, then $\boldsymbol{\lambda}+\boldsymbol{\mu}\in\mathcal{P}$. If $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots)$, we denote by $\boldsymbol{\lambda}^{(i)}$ the partition obtained by removing the i-th part:

$$\boldsymbol{\lambda}^{(i)} := (\lambda_1 \geqslant \lambda_{i-1} \geqslant \widehat{\lambda}_i \geqslant \lambda_{i+1} \geqslant \ldots,$$

where $\widehat{}$ means removed. By (1^j) we mean the partition with j parts equal to 1.

1.2 We denote by $\bigwedge M = \bigoplus_{r\geqslant 0} \bigwedge^r M$ the exterior algebra of a free abelian group $M := \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\cdot b_i$ with basis $\mathbf{b} := (b_i)_{i\in\mathbb{Z}}$. A typical element of $\bigwedge^r M$ is a finite linear combination of monomials of the form

$$b_{i_{-1}} \wedge \cdots \wedge b_{i_{-r}}$$

with $\infty > i_{-1} > \cdots > i_{-r} > -\infty$. Given $(m, r, \boldsymbol{\lambda}) \in \mathbb{Z}\times\mathbb{N}\times\mathcal{P}_r$, the following notation will be used:

$$\mathbf{b}^r_{m+\boldsymbol{\lambda}} = b_{m+\lambda_1} \wedge b_{m-1+\lambda_2} \wedge \cdots \wedge b_{m-r+1+\lambda_r} \in \bigwedge^r M_{\geqslant m-r+1} \subseteq \bigwedge M. \tag{5}$$

1.3 We denote by $M_{\geqslant j}$ the sub-module of M spanned by all b_k with $k \geqslant j$, i.e. $M_{\geqslant j} := \bigoplus_{i\geqslant j} \mathbb{Z}\cdot b_i$. In this case

$$\bigwedge^r M_{\geqslant j} := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_r} \mathbb{Z}\mathbf{b}^r_{r-1+j+\boldsymbol{\lambda}} \qquad \text{and} \qquad \bigwedge M_{\geqslant j} = \bigoplus_{r\geqslant 0} \bigwedge^r M_{\geqslant j}.$$

2 Hasse-Schmidt Derivations on Exterior Algebras

Main detailed references for this section are [6, 7, 9].

2.1 Definition. A *Hasse-Schmidt* (HS) *derivation* on $\bigwedge M$ is a $\mathbb{Z}$-linear map $\mathcal{D}(z) : \bigwedge M \to \bigwedge M[[z]]$, such that for all $\mathbf{u}, \mathbf{v} \in \bigwedge M$:

$$\mathcal{D}(z)(\mathbf{u}\wedge\mathbf{v}) = \mathcal{D}(z)\mathbf{u} \wedge \mathcal{D}(z)\mathbf{v}. \tag{6}$$

If $D_i \in \mathrm{End}_{\mathbb{Z}}(\bigwedge M)$ is such that $\mathcal{D}(z) = \sum_{i\geqslant 0} D_i z^i$, then equation (6) is equivalent to

$$D_i(\mathbf{u}\wedge\mathbf{v}) = \sum_{j=0}^{i} \mathcal{D}_i\mathbf{u} \wedge D_{i-j}\mathbf{v}. \tag{7}$$

If D_0 is invertible, up to termwise multiplying $\mathcal{D}(z)$ by D_0^{-1}, we may assume that $D_0 = \mathrm{id}_{\bigwedge M}$. Thus $\mathcal{D}(z)$ is invertible in $\mathrm{End}_{\mathbb{Z}}(\bigwedge M)[[z]]$. An easy check shows that the formal inverse $\overline{\mathcal{D}}(z) := \sum_{j\geqslant 0}(-1)^j \overline{\mathcal{D}}_j z^j$ is a HS-derivation as well.

A main tool of this paper is:

2.2 Proposition. *The integration by parts formulas hold:*

$$\mathcal{D}(z)\mathbf{u}\wedge\mathbf{v} = \mathcal{D}(z)(\mathbf{u}\wedge\overline{\mathcal{D}}(z)\mathbf{v}), \tag{8}$$

$$\mathbf{u}\wedge\overline{\mathcal{D}}(z)\mathbf{v} = \overline{\mathcal{D}}(z)(\mathcal{D}(z)\mathbf{u}\wedge\mathbf{v}). \tag{9}$$

Proof. Straightforward from definition 2.1. ∎

2.3 Duality. Let $\beta_j : M \to \mathbb{Z}$ be the unique linear form such that $\beta_j(b_i) = \delta_{ij}$. The *restricted dual* of M is $M^* := \bigoplus_{j\in\mathbb{Z}} \mathbb{Z} \cdot \beta_j$. Recall the natural identification between $\bigwedge^r M^*$ and $(\bigwedge^r M)^*$:

$$\beta_{i_1} \wedge \cdots \wedge \beta_{i_r}(b_{j_1} \wedge \cdots \wedge b_{j_r}) = \begin{vmatrix} \beta_{i_1}(b_{j_1}) & \cdots & \beta_{i_1}(b_{j_r}) \\ \vdots & \ddots & \vdots \\ \beta_{i_r}(b_{j_1}) & \cdots & \beta_{i_r}(b_{j_r}) \end{vmatrix}.$$

The *contraction* of $u \in \bigwedge^r M$ against $\beta \in M^*$ is the unique vector $\beta \lrcorner \mathbf{u} \in \bigwedge^{r-1} M$ such that the equality

$$\eta(\beta \lrcorner u) = (\beta \wedge \eta)(u),$$

holds for all $\eta \in \bigwedge^{r-1} M^*$.

2.4 Definition. *The map* $\mathcal{D}^T(z) = \sum_{i\geqslant 0} \mathcal{D}_i^T z^i : \bigwedge M^* \to \bigwedge M^*[[z]]$ *such that*

$$(\mathcal{D}^T(z)\eta)(\mathbf{u}) = \eta(\mathcal{D}(z)\mathbf{u}),$$

is called the transpose *of the HS-derivation* $\mathcal{D}(z)$*.*

By [9, Proposition 2.8], it follows that $\mathcal{D}^T(z)(\eta_1 \wedge \eta_2) = \mathcal{D}^T(z)\eta_1 \wedge \mathcal{D}^T(z)(\eta_2)$, for all $\eta_1, \eta_2 \in \bigwedge M^*$, i.e that $\mathcal{D}^T(z)$ is a HS derivation on $\bigwedge M^*$.

3 Schubert Derivations

3.1 Definition. *The* Schubert derivations *are the unique HS-derivations*

$$\sigma_+(z), \sigma_-(z) : \bigwedge M \to \bigwedge M[[z^{\pm 1}]] \tag{10}$$

such that $\sigma_\pm(z)b_j = \sum_{i\geqslant 0} b_{j\pm i} z^{\pm i}$*. Their formal inverses* $\overline{\sigma}_\pm(z) \in \mathrm{End}(\bigwedge M)[[z]]$ *are the unique HS-derivations* $\bigwedge M \to \bigwedge M[[z]]$ *such that*

$$\overline{\sigma}_\pm(z)b_j = b_j - b_{j\pm 1} z^{\pm 1}. \tag{11}$$

3.2 Notational Remark. To save notation, we preferred to write $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ rather than the more precise $\sigma_-(z^{-1})$ and $\overline{\sigma}_-(z^{-1})$, hoping that the subscript "$_-$" to σ may suffice to avoid possible confusions.

Put $\sigma_\pm(z) = \sum_{j\geqslant 0} \sigma_{\pm j} z^{\pm j}$ and $\overline{\sigma}_\pm(z) = \sum_{j\geqslant 0} (-1)^j \overline{\sigma}_{\pm j} z^{\pm j}$. Then:

$$\sigma_i b_j = b_{i+j}, \qquad \forall i, j \in \mathbb{Z}, \tag{12}$$

while $\overline{\sigma}_i u = 0$ if $u \in \bigwedge^{\leqslant |i|-1} M$, for all $i \in \mathbb{Z}$ (Cf. [9, Secs. 3.1–3.2]).

3.3 Remark. The operator σ_i defined on M are precisely the shift operators Λ_i as in [15, p. 32]. The only difference is that i) we extend them to all the exterior algebra of M (and then to the associated fermionic Fock space) embedding them into a Schubert derivation; ii) due to i), we preferred to use the notation σ_i to emphasize the interpretation in terms of Schubert calculus. The shift operators σ_i acts on b_0 as the cap product of the class of a linear space of codimension i with the fundamental class of some $\mathbb{P}^n$ (which is 0 if $i > n$).

3.4 Remark. Let $\sigma_+^*(w) : \bigwedge M^* \to \bigwedge M^*[[w]]$ be the Schubert derivation on $\bigwedge M^*$, i.e. the unique HS-derivation such that

$$\sigma^*(w)\beta_j = \sum_{i\geqslant 0} \sigma_i^*\beta_j \cdot w^i = \sum_{i\geqslant 0} \beta_{j+i}w^i.$$

Its inverse $\overline{\sigma}_+^*(w)$ is the unique HS-derivation on $\bigwedge M^*$ such that

$$\overline{\sigma}_+^*(w)\beta_j = \sum_{i\geqslant 0}(-1)^i\overline{\sigma}_i^*\beta_j w^i = \beta_j - \beta_{j+1}w.$$

An easy check shows that $\sigma_-(z) = \sum \sigma_{-i}z^{-i} = \sigma_+^{*\mathsf{T}}(w)_{|w=z^{-1}}$. Similarly $\overline{\sigma}_-(z) = \overline{\sigma}_+^{*\mathsf{T}}(w)_{|w=z^{-1}} : \bigwedge M \to \bigwedge M[[z^{-1}]]$.

3.5 Proposition. *The following equalities hold:*

$$\sigma_-^{\mathsf{T}}(z)\beta_j = \sum_{i\geqslant 0} \beta_{j+i}z^{-i} = \sigma_+^*(w)\beta_{j_{|w=z^{-1}}}, \tag{13}$$

$$\overline{\sigma}_-^{\mathsf{T}}(z)\beta_j = \beta_j - \beta_{j+1}z^{-1} = \overline{\sigma}^*(w)\beta_{j_{|w=z^{-1}}}. \tag{14}$$

Proof. It follows from the definition. ■

4 Fermionic Fock Space

4.1 There are several excellent references concerning the definition of the fermionic Fock space [5, Ch. 5] or [14, 15]. It amounts to the rigorous formalization of the idea of an infinite exterior power. We propose here an elementary algebraic construction of it, that suffices for our purposes.

Let $[M]$ be the free $\mathbb{Z}$-module generated by the basis $[\mathbf{b}] := ([\mathbf{b}]_m)_{m\in\mathbb{Z}}$. Identify $[M]$ with a submodule of the tensor product $\bigwedge M \otimes_{\mathbb{Z}} [M]$ via the map $[\mathbf{b}]_m \mapsto 1 \otimes [\mathbf{b}]_m$. Let W be the $\bigwedge M$–submodule of $\bigwedge M \otimes_{\mathbb{Z}} [M]$ generated by all the expressions $\{b_m \otimes [\mathbf{b}]_{m-1} - [\mathbf{b}]_m, b_m \otimes [\mathbf{b}]_m\}_{m\in\mathbb{Z}}$. In formulas:

$$W := \bigwedge M \otimes \big(b_m \otimes [\mathbf{b}]_{m-1} - [\mathbf{b}]_m\big) + \bigwedge M \otimes \big(b_m \otimes [\mathbf{b}]_m\big).$$

4.2 Definition. *The fermionic Fock space is the* $\bigwedge M$*-module*

$$F := F(M) := \frac{\bigwedge M \otimes_{\mathbb{Z}} [M]}{W}. \tag{15}$$

Let $\bigwedge M \otimes_{\mathbb{Z}} [M] \to F$ be the canonical projection. The class of $u \otimes [\mathbf{b}]_m$ in F will be denoted $u \wedge [\mathbf{b}]_m$. Thus the equalities $b_m \wedge [\mathbf{b}]_m = 0$ and $b_m \wedge [\mathbf{b}]_{m-1} = [\mathbf{b}]_m$ hold in F. Notation as in (5). For all $m \in \mathbb{Z}$ and $\boldsymbol{\lambda} \in \mathcal{P}$ let, by definition

$$[\mathbf{b}]_{m+\boldsymbol{\lambda}} := \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r},$$

where r is any positive integer such that $\ell(\boldsymbol{\lambda}) \leqslant r$. Then F is a graded $\bigwedge M$-module:

$$F := \bigoplus_{m\in\mathbb{Z}} F_m,$$

where

$$\mathsf{F}_m := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}} \mathbb{Z}[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \bigoplus_{r\geqslant 0}\bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_r} \mathbb{Z}\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r}. \tag{16}$$

4.3 Definition. *The* fermionic Fock space *of charge* m *is the module* F_m *as in* (16) [15, p. 36].

4.4 Proposition. *The equality* $b_j\wedge[\mathbf{b}]_m = 0$ *holds for all* $j\leqslant m$*.*

Proof. Indeed:

$$b_j\wedge[\mathbf{b}]_m = b_j\wedge b_m\wedge\cdots\wedge b_j\wedge[\mathbf{b}]_{j-1} = \pm b_j\wedge b_j\wedge b_m\wedge\cdots\wedge[\mathbf{b}]_{j-1} = 0. \qquad \blacksquare$$

4.5 Proposition. *The image of the map* $\bigwedge^r M\otimes \mathsf{F}_m\to\mathsf{F}$ *given by* $(u,v)\mapsto u\wedge v$ *is contained in* F_{m+r}*.*

Proof. Let $b_{i_1}\wedge\cdots\wedge b_{i_r}\in\bigwedge^r M$ with $i_1>\cdots>i_r$. Write

$$i_1 = m+1+r+\lambda_1,\quad \cdots,\quad i_r = m+1+\lambda_r.$$

If $\lambda_1\geqslant\cdots\geqslant\lambda_r$, then

$$b_{i_1}\wedge\cdots\wedge b_{i_r}\wedge[\mathbf{b}]_m = \mathbf{b}^r_{m+1+r+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_m\in\mathsf{F}_{m+r},$$

otherwise the product is zero. ■

4.6 In particular each $\mathbf{u}\in M$ defines an action $\mathbf{u}\wedge:\mathsf{F}_m\to\mathsf{F}_{m+1}$ given by

$$\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r}\longmapsto(u\wedge\mathbf{b}^r_{m+\boldsymbol{\lambda}})\wedge[\mathbf{b}]_{m-r}. \tag{17}$$

Similarly, one may consider a contraction action of $\bigwedge M^*$ on F, mapping $\mathsf{F}_m\mapsto\mathsf{F}_{m-1}$. Define the contraction of $[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ against β_j as follows: choose r such that $\ell(\boldsymbol{\lambda})\leqslant r$ and $m-r\leqslant j$. Declare that

$$\beta_j\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \left(\beta_j\lrcorner\mathbf{b}^r_{m+\boldsymbol{\lambda}}\right)\wedge[\mathbf{b}]_{m-r}, \tag{18}$$

and extend by linearity. The definition does not depend on the choice of $r>\max(\ell(\boldsymbol{\lambda}),m-j)$. For instance

$$\beta_m\lrcorner[\mathbf{b}]_m = \beta_m\lrcorner(b_m\wedge[\mathbf{b}]_{m-1}) = [\mathbf{b}]_{m-1}.$$

4.7 Remark. Using the *wedging* and *contraction* operators (17) and (18), is easy to show that F is an irreducible representation of a canonical Clifford algebra on $M\oplus M^*$, called Fock fermionic representation in [5, Section 5.], which motivates the terminology we adopted.

4.8 Duality. We denote by F^* the restricted dual of F, contructed out of the restricted dual of M^*, precisely as one did for F. The typical element of F is of the form

$$[\boldsymbol{\beta}]_{m+\boldsymbol{\lambda}} = \boldsymbol{\beta}^r_{m+\boldsymbol{\lambda}}\wedge[\boldsymbol{\beta}]_{m-r}.$$

The duality pairing $F^* \times F \to \mathbb{Z}$ is defined by $[\boldsymbol{\beta}]_{m+\boldsymbol{\lambda}}([\mathbf{b}]_{n+\boldsymbol{\mu}}) = \delta_{m,n}\delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. It extends the natural duality between $\bigwedge^r M$ and $\bigwedge^r M^*$. The contraction of $f \in F_m$ against $\beta \in M^*$ is defined by

$$\eta(\beta \lrcorner f) = (\beta \wedge \eta)(f),$$

for all $\eta \in F^*_{m-1}$. More explicitly, keeping into account that f is a finite sum of elements of the form $[\mathbf{b}]_{m+\boldsymbol{\lambda}}$, one has:

$$\beta \lrcorner [\mathbf{b}]_{m+\boldsymbol{\lambda}} = \beta \lrcorner \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} + (-1)^{r-1}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge (\beta \lrcorner [\mathbf{b}]_{-m-r}).$$

Since β is a finite sum $\sum a_i\beta_i$, the contraction $\beta \lrcorner [\mathbf{b}]_{-m-r}$ is a finite sum.

5 Extending HS-derivations to F

This section is devoted to extend the Schubert derivations (3.1), and their transposed, to suitable maps $F \to F[[z]]$ and $F^* \to F^*[[z^{-1}]]$. The purpose is to (re)-discover the *bosonic* vertex operators as in [15, Theorem 5.1] or [14, p. 92]. This will supply an alternative way to look at the bosonic vertex representation of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{Z})$, due to Date-Jimbo-Kashiwara-Miwa [3, 13]. Although those authors worked over the complex numbers, we work over the integers because it is sufficient for our purposes.

The sought for extension of the Schubert derivations to F will be attained by looking at each degree F_m one at a time.

5.1 Let $\sigma_+(z), \sigma_-(z) : \bigwedge M \to \bigwedge M[[z^{\pm 1}]]$ be the Schubert derivations and $\overline{\sigma}_+(z), \overline{\sigma}_-(z) : \bigwedge M \to \bigwedge M[[z^{\pm 1}]]$ their inverses as in Section 3.1. We first extend them to $\mathbb{Z}$-basis elements of $[M]$ and then we extend to all F by mimicking the typical behavior of an algebra homomorphism.

5.2 Let us begin to extend the definition of $\sigma_\pm(z)$ and $\overline{\sigma}_\pm(z)$ to elements of $[M]$ as follows:

$$\sigma_+(z)[\mathbf{b}]_m := (\sigma_+(z)b_m) \wedge [\mathbf{b}]_{m-1}, \tag{19}$$

and

$$\overline{\sigma}_+(z)[\mathbf{b}]_m := \sum_{j\geqslant 0} (-1)^j [\mathbf{b}]_{m+(1^j)} z^j \tag{20}$$

for all $m \in \mathbb{Z}$. We demand, on the other hand, that $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ act on $[M]$ as the identity:

$$\sigma_-(z)[\mathbf{b}]_m = [\mathbf{b}]_m \qquad \text{and} \qquad \overline{\sigma}_-(z)[\mathbf{b}]_m = [\mathbf{b}]_m, \tag{21}$$

for all $m \in \mathbb{Z}$.

Notice that (20) can be equivalently written in F_m as:

$$\sigma_+(z)[\mathbf{b}]_m = \sum_{j\geqslant 0} (-1)^j [\mathbf{b}]^j_{m+(1^j)} \wedge [\mathbf{b}]_{m-j} z^j.$$

We now extend the Schubert derivations, as in 3.1, to all F_m.

5.3 Definition. *The extension of the Schubert derivations* $\sigma_\pm(z), \overline{\sigma}_\pm(z)$ *to* $\mathbb{Z}$*-linear maps* $F_m \to F_m[[z^{\pm 1}]]$ *is defined by:*

$$\begin{aligned}\sigma_\pm(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= \sigma_\pm(z)(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}) \\ &= \sigma_\pm(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_\pm(z)[\mathbf{b}]_{m-r},\end{aligned} \tag{22}$$

and

$$\begin{aligned}\overline{\sigma}_\pm(z)([\mathbf{b}]_{m+\boldsymbol{\lambda}}) &= \overline{\sigma}_\pm(z)\left(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}\right) \\ &= \overline{\sigma}_\pm(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_\pm(z)[\mathbf{b}]_{m-r}.\end{aligned} \tag{23}$$

5.4 Proposition. *For all* $i \in \mathbb{Z}$*,* $\sigma_i[\mathbf{b}]_m = b_{m+i} \wedge [\mathbf{b}]_{m-1}$ *and is thence zero if* $i < 0$*.*

Proof. If $i \geqslant 0$, $\sigma_i[\mathbf{b}]_m$ is the coefficient of z^i in the expression

$$\sigma_\pm(z)b_m \wedge [\mathbf{b}]_{m-1} = \sum_{i\geqslant 0} \sigma_i b_m \cdot z^i \wedge [\mathbf{b}]_{m-1} = \sum_{i\geqslant 0} b_{m+i} \wedge [\mathbf{b}]_{m-1} z^i.$$

If $-i > 0$, instead, $\sigma_{-i}[\mathbf{b}]_m$ is the coefficient of z^{-i} in the right–hand side of the equation $\sigma_-(z)[\mathbf{b}]_m = [\mathbf{b}]_m$, which is zero as stated. ∎

5.5 Proposition. *For all* $i \geqslant 0$*, the maps* $\sigma_{\pm i} : F_m \to F_m$ *satisfy the* i*-th order Leibniz rule*

$$\sigma_{\pm i}(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}) = \sum_{j=0}^{i} \sigma_{\pm j}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_{\pm i \mp j}[\mathbf{b}]_{m-r}. \tag{24}$$

Proof. In fact, the left–hand side of (24) is the coefficient of $z^{\pm i}$ of the expression $\sigma_\pm(z)(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r})$, which by definition is

$$\sigma_\pm(z)(\mathbf{b}^r_{m+\boldsymbol{\lambda}}) \wedge \sigma_\pm(z)[\mathbf{b}]_{m-r}.$$

The equality

$$\sigma_-(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_-(z)[\mathbf{b}]_{m-r} = \sigma_-(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}$$

implies

$$\sigma_{-i}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} = \sigma_{-i}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r},$$

as all the other terms involving $\overline{\sigma}_{-j}[\mathbf{b}]_{m-r}$ vanish for $j > 0$. Moreover, using Proposition 5.4, one sees that the coefficient of z^i in $\sigma_+(z)(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r})$ is

$$\sigma_i(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}) = \sum_{j=0}^{i} \sigma_j\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_{i-j}[\mathbf{b}]_{m-r} = \sum_{j=0}^{i} \sigma_j\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_{i-j}[\mathbf{b}]_{m-r} \tag{25}$$

as desired. ∎

5.6 Proposition. *The following equalities hold:*

$$\overline{\sigma}_{\pm i}(\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r})=\sum_{j=0}^{i}\overline{\sigma}_{\pm j}\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge\overline{\sigma}_{\pm i\mp j}[\mathbf{b}]_{m-r}$$

Proof. The formula readily follows by comparing the coefficient of $z^{\pm i}$ on either side of formula (23). ■

In particular $\overline{\sigma}_{-j}(\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r})=\overline{\sigma}_{-j}(\mathbf{b}^r_{m+\boldsymbol{\lambda}})\wedge[\mathbf{b}]_{m-r}$

5.7 Lemma. *Let* $\mathbf{z}_r:=(z_1,\ldots,z_r)$ *be indeterminates over* $\mathbb{Z}$*. Then:*

$$\overline{\sigma}_+(z_1)\cdots\overline{\sigma}_+(z_r)b_{m-r}=b_{m-r}+\sum_{j=1}^{r}(-1)^je_j(\mathbf{z}_r)b_{m-r+j},\tag{26}$$

where $e_i(\mathbf{z}_r)$ *is the* i*-th elementary symmetric polynomial in* $(z_1,\ldots,z_r)$*.*

Proof. If $r=1$, $\overline{\sigma}_+(z_1)b_{m-1}=b_{m-1}-b_mz_1=b_{m-1}-e_1(z_1)b_m$, showing that Lemma 5.7 holds in this case. Suppose that (26) holds for all $1\leqslant s\leqslant r-1$. Then

$$\begin{aligned}&\overline{\sigma}_+(z_1)\overline{\sigma}_+(z_2)\cdots\overline{\sigma}_+(z_r)b_{m-r}\\
=\;&\overline{\sigma}_+(z_1)(b_{m-r}-e_1(z_2,\ldots,z_r)b_{m-r+1}+\cdots+(-1)^rz_2\cdots z_rb_{m-1})\\
=\;&\sum_{j=0}(-1)^je_j(z_2,\ldots,z_r)(b_{m-r-j}-z_1b_{m-r-j+1})\\
=\;&b_{m-r}+\sum_{j=1}^{r}(-1)^je_j(\mathbf{z}_r)b_{m-r+j},\end{aligned}$$

as desired. ■

5.8 Proposition. *For all* $m\in\mathbb{Z}$ *and* $r\geqslant 0$*:*

$$\mathbf{b}^r_m\wedge\overline{\sigma}_+(z_1)\cdots\overline{\sigma}_+(z_r)[\mathbf{b}]_{m-r}=[\mathbf{b}]_m.\tag{27}$$

Proof. One has

$$b_m\wedge\cdots\wedge b_{m-r+1}\wedge\sum_{j=0}^{r}(-1)^je_j(\mathbf{z}_r)b_{m-r+j}\wedge\sum_{j=0}^{r}(-1)^je_j(\mathbf{z}_r)b_{m-r-1+j}\wedge\cdots=[\mathbf{b}]_m.$$

5.9 Corollary. *For all* $m\in\mathbb{Z}$*:*

$$b_m\wedge\overline{\sigma}_+(z)[\mathbf{b}]_{m-1}=[\mathbf{b}]_m.$$

Proof. In fact the typical coefficient of z^i, $i>0$, is the sum of monomials of the form $b_m\wedge b_m\wedge\cdots=0$, so that the only surviving summand is $b_m\wedge[\mathbf{b}]_{m-1}=[\mathbf{b}]_m$. ■

5.10 Proposition. *The maps* $\overline{\sigma}_+(z), \sigma_+(z) : F_m \to F_m[[z]]$ *and* $\overline{\sigma}_-(z), \sigma_-(z) : F_m \to F_m[[z^{-1}]]$ *are mutually inverse.*

Proof. The formal power series $\sigma_+(z)$ and $\overline{\sigma}_+(z)$ are both invertible in $F_m[[z]]$, as the constant term is the identity of F_m. It then suffices to show that $\overline{\sigma}_+(z)$ is the left inverse of $\sigma_+(z)$, because in this case it must coincide with its inverse. Now for all $m \in \mathbb{Z}$ and each $\boldsymbol{\lambda} \in \mathcal{P}$,

$$\overline{\sigma}_+(z)\sigma_+(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \overline{\sigma}_+(z)\sigma_+(z)(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}) = \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_+(z)\sigma_+(z)[\mathbf{b}]_{m-r}$$

for all $r \geqslant \ell(\boldsymbol{\lambda})$, where we used the fact that $\overline{\sigma}_+(z), \sigma_+(z)$ are mutually inverse HS–derivations on $\bigwedge M_{\geqslant m-r+1}$. It will then suffice to show that $\overline{\sigma}_+(z)\sigma_+(z)$ acts on $[\mathbf{b}]_m$ as the identity, for all $m \geqslant 0$. Using Corollary 5.4, one easily gets:

$$\overline{\sigma}_+(z)\sigma_+(z)[\mathbf{b}]_m = \overline{\sigma}_+(z)\left(\sigma_+(z)b_m \wedge [\mathbf{b}]_{m-1}\right) = b_m \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-1} = [\mathbf{b}]_m.$$

Similarly, since $\overline{\sigma}_-(z)\sigma_-(z)b_j = b_j$, for all $j \in \mathbb{Z}$, then

$$\overline{\sigma}_-(z)\sigma_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \overline{\sigma}_-(z)[\sigma_-(z)\mathbf{b}]_{m+\boldsymbol{\lambda}} = [\overline{\sigma}_-(z)\sigma_-(z)\mathbf{b}]_{m+\boldsymbol{\lambda}} = [\mathbf{b}]_{m+\boldsymbol{\lambda}}.$$ ∎

5.11 Proposition. *For all partitions* $\boldsymbol{\lambda}$ *of length at most* r *and all* $m \in \mathbb{Z}$, *the integration by parts formulas hold:*

$$\sigma_+(z)\mathbf{b}^r_{m+\boldsymbol{\mu}} \wedge [\mathbf{b}]_{m-r+\boldsymbol{\lambda}} = \sigma_+(z)(\mathbf{b}^r_{m+\boldsymbol{\mu}} \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-r+\boldsymbol{\lambda}}); \tag{28}$$

$$\mathbf{b}^r_{m+\boldsymbol{\mu}} \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-r+\boldsymbol{\lambda}} = \overline{\sigma}_+(z)(\sigma_+(z)\mathbf{b}^r_{m+\boldsymbol{\mu}} \wedge [\mathbf{b}]_{m-r+\boldsymbol{\lambda}}). \tag{29}$$

Proof. Obvious. ∎

5.12 Proposition. Let $i_1, \ldots, i_r$ be an index sequence of length $r \geqslant 1$. Then

$$\sigma_{i_1} \cdots \sigma_{i_r}[\mathbf{b}]_m = (\sigma_{i_1} \cdots \sigma_{i_r}\mathbf{b}^r_m) \wedge [\mathbf{b}]_{m-r}. \tag{30}$$

Proof. Let $(z_1, \ldots, z_r)$ be indeterminates over $\mathbb{Z}$. Then $\sigma_{i_1} \cdots \sigma_{i_r}[\mathbf{b}]_m$ is the coefficient of $z_1^{i_1} \cdots z_r^{i_r}$ in the expansion of $\sigma_+(z_1) \cdots \sigma_+(z_r)[\mathbf{b}]_m$. Now, by virtue of Corollary 5.8:

$$\sigma_+(z_1) \cdots \sigma_+(z_r)[\mathbf{b}]_m = \sigma_+(z_1) \cdots \sigma_+(z_r)\left(\mathbf{b}^r_m \wedge \overline{\sigma}_+(z_1) \cdots \overline{\sigma}_+(z_r)[\mathbf{b}]_{m-r}\right) \tag{31}$$

As $\sigma_+(z_i)$ is a HS-derivation, one obtains

$$\sigma_+(z_1) \cdots \sigma_+(z_r)[\mathbf{b}]_m = \sigma_+(z_1) \cdots \sigma_+(z_r)\mathbf{b}^r_m \wedge [\mathbf{b}]_{m-r}. \tag{32}$$

Comparing the coefficient of $z_1^{i_1} \cdots z_r^{i_r}$ on either side of equality (32), yields formula (30). ∎

5.13 Proposition. *For all* $\boldsymbol{\lambda} \in \mathcal{P}_r$, *let* $\Delta_{\boldsymbol{\lambda}}(\sigma_+) := \det(\sigma_{\lambda_j+j-i})_{1 \leqslant i,j \leqslant r}$. *Then Giambelli's formula holds:*

$$[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(\sigma_+)[\mathbf{b}]_m.$$

Proof. By virtue of Proposition 5.12:

$$\Delta_{\boldsymbol{\lambda}}(\sigma_+)[\mathbf{b}]_m = \Delta_{\boldsymbol{\lambda}}(\sigma_+)[\mathbf{b}]^r_m \wedge [\mathbf{b}]_{m-r}.$$

Now $\Delta_{\boldsymbol{\lambda}}(\sigma_+)[\mathbf{b}]^r_m$ can be read in the exterior power $\bigwedge^r M_{\geqslant m-r+1}$, and then one may invoke [6, Formula (17)] or [7, Corollary 5.8.2]. ∎

5.14 Boson–Fermion Correspondence. Let $\zeta \in \mathrm{End}_{\mathbb{Z}}(\mathsf{F})$ defined by $\zeta[\mathbf{b}]_{m+\boldsymbol{\lambda}} = [\mathbf{b}]_{m+1+\boldsymbol{\lambda}}$. It is an automorphism of F and may be thought of as an extension to F_m of the determinant of the map $\sigma_{1_{|M}}$. Let now $\mathsf{B} := \mathbb{Z}[e_1, e_2, \ldots]$ be the polynomial ring in the infinitely many indeterminates $(e_1, e_2, \ldots)$ with $\mathbb{Z}$-coefficients. Let $\mathsf{E}(z) := 1 + \sum_{j\geqslant 1}(-1)^j e_j z^j$ and $\mathsf{H} := (h_j)_{j\in\mathbb{Z}}$ be the sequence in B defined by the equality

$$\mathsf{H}(z) := \sum_{j\in\mathbb{Z}} h_j z^j := \frac{1}{\mathsf{E}(z)}. \tag{33}$$

From (33) it turns out that $h_j = 0$ for $j < 0$, $h_0 = 1$ and, for $j > 0$, h_j is a $\mathbb{Z}$-polynomial in the e_is, homogeneous of degree j, once each e_i is given degree i. It is well known that [18]

$$\mathsf{B} := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}} \mathbb{Z}\Delta_{\boldsymbol{\lambda}}(\mathsf{H}),$$

where

$$\Delta_{\boldsymbol{\lambda}}(\mathsf{H}) := \det(h_{\lambda_j - j + i})_{1\leqslant i,j\leqslant r},$$

and r is any positive integer bigger or equal than $\ell(\boldsymbol{\lambda})$.

This enables one to equip F_0 with a B-module structure by declaring that $h_i\mathbf{u} = \sigma_i(\mathbf{u})$ for all $\mathbf{u} \in \mathsf{F}_0$. In particular, for all $\boldsymbol{\lambda} \in \mathcal{P}$:

$$\sigma_+(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \frac{1}{\mathsf{E}(z)}[\mathbf{b}]_{m+\boldsymbol{\lambda}}. \tag{34}$$

Since $\mathsf{F}_m = \zeta^m \mathsf{F}_0$, each of them inherits a structure of free B-module generated by $\zeta^m[\mathbf{b}]_0$. Consider the polynomial ring $\mathsf{B}[\zeta, \zeta^{-1}]$.

5.15 Definition. *The Boson–Fermion correspondence is the* $\mathsf{B}[\zeta, \zeta^{-1}]$*-module structure of* F

$$\mathsf{B}[\zeta, \zeta^{-1}] \otimes \mathsf{F} \to \mathsf{F},$$

defined by

$$\Delta_{\boldsymbol{\lambda}}(\mathsf{H})[\mathbf{b}]_m = [\mathbf{b}]_{m+\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(\mathsf{H})\mathbf{b}^r_m \wedge [\mathbf{b}]_{m-r} = \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}, \tag{35}$$

where r *is any integer bigger than* $\ell(\boldsymbol{\lambda})$.

In particular $\Delta_{\boldsymbol{\lambda}}(\mathsf{H})[\mathbf{b}]_0 = [\mathbf{b}]_{0+\boldsymbol{\lambda}}$ and $\zeta^m\Delta_{\boldsymbol{\lambda}}(\mathsf{H})[\mathbf{b}]_0 = \Delta_{\boldsymbol{\lambda}}(\mathsf{H})\zeta^m[\mathbf{b}]_0 = \Delta_{\boldsymbol{\lambda}}(\mathsf{H})[\mathbf{b}]_m$. Notice that, on the dual side, i.e. in $\mathsf{F}(M^*)$, one has $\zeta^m[\boldsymbol{\beta}]_0 = [\boldsymbol{\beta}]_{-m}$. In fact

$$\delta_{m-n,0} = \delta_{m,n} = [\boldsymbol{\beta}]_m([\mathbf{b}]_n) = [\boldsymbol{\beta}]_m(\zeta^n[\mathbf{b}]_0) = \zeta^n[\boldsymbol{\beta}]_m([\mathbf{b}]_0$$

from which $\zeta^n[\boldsymbol{\beta}]_m = [\boldsymbol{\beta}]_{m-n}$.

6 Vertex Operators

Via the Boson–Fermion correspondence 5.15, the Schubert derivations $\overline{\sigma}_-(z), \sigma_-(z)$ induce natural maps $\mathsf{B} \to \mathsf{B}[z^{-1}]$.

6.1 Definition. *Define* $\overline{\sigma}_-(z), \sigma_-(z) : B \to B[z^{-1}]$ *through the equalities*

$$(\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H))\,[\mathbf{b}]_0 := \overline{\sigma}_-(z)\,(\Delta_{\boldsymbol{\lambda}}(H)[\mathbf{b}]_0) = \overline{\sigma}_-(z)[\mathbf{b}]_{0+\boldsymbol{\lambda}}$$

and

$$(\sigma_-(z)\Delta_{\boldsymbol{\lambda}}(H))[\mathbf{b}]_0 := \sigma_-(z)\,(\Delta_{\boldsymbol{\lambda}}(H)[\mathbf{b}]_0) = \sigma_-(z)[\mathbf{b}]_{0+\boldsymbol{\lambda}}.$$

6.2 Proposition.

$$\overline{\sigma}_-(z)h_n = h_n - h_{n-1}z^{-1}, \tag{36}$$

$$\sigma_-(z)h_n = \sum_{j\geqslant 0} h_{n-j}z^{-j}. \tag{37}$$

Proof. Recall that $h_j = 0$ for $j < 0$. Let us prove (36) first.

$$\begin{aligned}(\overline{\sigma}_-(z)h_n)[\mathbf{b}]_0 &= \overline{\sigma}_-(z)(h_n[\mathbf{b}]_0) = \overline{\sigma}_-(z)(b_n \wedge [\mathbf{b}]_{-1})\\ &= (b_n - b_{n-1}z^{-1}) \wedge [\mathbf{b}]_{-1} = (h_n - h_{n-1}z^{-1})[\mathbf{b}]_0,\end{aligned}$$

whence (36). The proof of (37) is analogous:

$$\begin{aligned}(\sigma_-(z)h_n)[\mathbf{b}]_0 &= \sigma_-(z)(h_n[\mathbf{b}]_0) = \sigma_-(z)\sigma_n[\mathbf{b}]_0\\ &= \sigma_-(z)(b_n \wedge [\mathbf{b}]_{-1}) = \sigma_-(z)b_n \wedge [\mathbf{b}]_{-1}\\ &= \sum_{j\geqslant 0} b_{n-j} \wedge [\mathbf{b}]_{-1}z^{-j} = \sum_{j\geqslant 0} h_{n-j}z^{-j}[\mathbf{b}]_0,\end{aligned}$$

whence (37). ■

6.3 Remark. The sum (37) is an infinite sum but its multiplication by $[\mathbf{b}]_m$ is finite, for all $m \in \mathbb{Z}$. For instance $\sum_{j\geqslant 0} h_{n-j}z^{n-j}[\mathbf{b}]_0 = \sum_{0\leqslant j\leqslant n} h_{n-j}z^{n-j}[\mathbf{b}]_0$.

Let $\sigma_-(z)H$ denote the sequence $(\sigma_-(z)h_n)_{n\in\mathbb{Z}}$ (respectively $\overline{\sigma}_-(z)H = (\overline{\sigma}_-(z)h_n)_{n\in\mathbb{Z}}$). The following is one of the main results concerning the combinatorics of the subject.

6.4 Theorem. *Schur determinants commute with taking* $\overline{\sigma}_-(z)$*:*

$$\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H) = \Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H) \qquad \text{and} \qquad \sigma_-(z)\Delta_{\boldsymbol{\lambda}}(H) = \Delta_{\boldsymbol{\lambda}}(\sigma_-(z)H), \tag{38}$$

Proof. By Proposition 5.12, the equality $[\mathbf{b}]_{0+\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]_0$ holds in the exterior power $\bigwedge^r M_{\geqslant -r+1}$. We contend that

$$(\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]_0 = \Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H)[\mathbf{b}]_0,$$

and this is true by [9, Theorem 5.7] that relies on a general determinantal formula in a polynomial ring due to Laksov and Thorup [17, Theorem 0.1]. The same argument holds verbatim for $\sigma_-(z)$.

■

6.5 Corollary. *$\overline{\sigma}_-(z), \sigma_-(z) : B \to B[z^{-1}]$ are rings homomorphisms.*

Proof. In fact $B = \mathbb{Z}[h_1, h_2, \ldots]$. Then

$$\begin{aligned}\sigma_-(z)(h_{i_1}\cdots h_{i_r}) &= \sigma_-(z)(\sum_\lambda a_{\boldsymbol\lambda}\Delta_{\boldsymbol\lambda}(H))\\ &= \sum_{\boldsymbol\lambda} a_\lambda\Delta_{\boldsymbol\lambda}(\sigma_-(z)H) = \sigma_-(z)h_{i_1}\cdots\sigma_-(z)h_{i_r}.\end{aligned}$$

The proof for $\overline{\sigma}_-(z)$ is totally analogous. ∎

In other words, $\sigma_-(z), \overline{\sigma}_-(z) : B \to B[[z^{-1}]]$ are Hasse-Schmidt derivations on B, in the genuine sense of e.g. [19, p. 207].

6.6 Lemma. *For all $m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $\boldsymbol\lambda \in \mathcal{P}_r$:*

$$\mathbf{b}^r_{m+\boldsymbol\lambda+(1^r)} \wedge b_{m-r+1} = [\mathbf{b}]^{r+1}_{m+1+\boldsymbol\lambda}. \tag{39}$$

Proof. The definition of the left–hand side of (39) is

$$b_{m+1+\lambda_1} \wedge \cdots \wedge b_{m+1-r+1+\lambda_r} \wedge b_{m+r-1}$$

$$= b_{m+1+\lambda_1} \wedge \cdots \wedge b_{m+1-(r+1)+2+\lambda_r} \wedge b_{m+1-(r+1)+1}$$

that is precisely the definition of its right–hand side. ∎

6.7 Corollary.

$$\mathbf{b}^r_{m+\boldsymbol\lambda+(1^r)} \wedge [\mathbf{b}]_{m-r+1} = [\mathbf{b}]_{m+1+\boldsymbol\lambda}.$$

Proof. In fact

$$\begin{aligned}\mathbf{b}^r_{m+\boldsymbol\lambda+(1^r)} \wedge [\mathbf{b}]_{m-r+1} &= \mathbf{b}^r_{m+\boldsymbol\lambda+(1^r)} \wedge b_{m-r+1} \wedge [\mathbf{b}]_{m-r}\\ &= [\mathbf{b}]^{r+1}_{m+1+\boldsymbol\lambda} \wedge [\mathbf{b}]_{m-r} = [\mathbf{b}]_{m+1+\boldsymbol\lambda}.\end{aligned}$$ ∎

6.8 Let $R(z) \in \mathrm{End}_{\mathbb{Z}[z,z^{-1}]}(F[[z, z^{-1}]])$ given by $R(z)[\mathbf{b}]_{m+\boldsymbol\lambda} = z^{m+1}[\mathbf{b}]_{m+1+\boldsymbol\lambda}$. It is clearly invertible: $R(z)^{-1}[\mathbf{b}]_{m+\boldsymbol\lambda} = z^{-m}[\mathbf{b}]_{m-1+\boldsymbol\lambda}$, i.e. $R(z)^{-1}$ is an endomorpism of F homogeneous of degree -1. On the bosonic side, define $(R(z)\zeta^m\Delta_{\boldsymbol\lambda}(H))[\mathbf{b}]_0 = z^{m+1}[\mathbf{b}]_{m+1+\boldsymbol\lambda}$, so that $R(z)\zeta^m\Delta_{\boldsymbol\lambda}(H) = z^{m+1}\zeta^{m+1}\Delta_{\boldsymbol\lambda}(H)$. It is a homogeneous operator of degree 1 (and is the fermionic counterpart of the R operator mentioned in [15, Theorem 5.1]). Notice that for all $j \in \mathbb{Z}$, one has $R(z)(\sigma_j[\mathbf{b}]_{m+\boldsymbol\lambda}) = \sigma_j(R(z)[\mathbf{b}]_{m+\boldsymbol\lambda})$, as an immediate check shows, whence the commutativity rules

$$\overline{\sigma}_\pm(z)R(z) = R(z)\overline{\sigma}_{\pm(z)} \qquad \text{and} \qquad \sigma_\pm(z)R(z) = R(z)\sigma_\pm(z). \tag{40}$$

6.9 Proposition. *Let $\mathbf{b}(z) := \sum_{i\in\mathbb{Z}} b_i z^i \in M[[z^{-1}, z]]$. Then for all $\phi \in F$*

$$\mathbf{b}(z) \wedge \phi = R(z)\sigma_+(z)\overline{\sigma}_-(z)\phi$$

Proof. Each $\phi \in F$ is an integral finite linear combination of homogeneous elements, each summand belonging to F_m for some m. Moreover each element of F_m is an integral linear combination of basis elements of the form $[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ of F_m. Hence it suffices to prove the proposition for $\phi = [\mathbf{b}]_{m+\boldsymbol{\lambda}}$.

$$
\begin{aligned}
& \mathbf{b}(z) \wedge [\mathbf{b}]_{m+\boldsymbol{\lambda}} \\
= \;& \sum_{i\in\mathbb{Z}} b_i z^i \wedge \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} && \text{(definition of } \mathbf{b}(z) \text{ and } [\mathbf{b}]_{m+\boldsymbol{\lambda}}) \\
= \;& \sum_{i\geqslant m-r+1} b_i z^i \wedge \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} && \text{(the wedge products vanish for } i < m-r+1) \\
= \;& z^{m-r+1}\sigma_+(z) b_{m-r+1} \wedge \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} && \text{(by definition of } \sigma_+(z)) \\
= \;& z^{m-r+1}\sigma_+(z)(b_{m-r+1} \wedge \overline{\sigma}_+(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-r}) && \text{(integration by parts (8))} \\
= \;& z^{m-r+1}\sigma_+(z)(z^r\overline{\sigma}_-(z)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}+(1^r)} \wedge b_{m-r+1} \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-r}) && \text{(Definition of } \overline{\sigma}_-(z)) \\
= \;& z^{m+1}\sigma_+(z)(\overline{\sigma}_-(z)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}+(1^r)} \wedge [\mathbf{b}]_{m-r-1}) && (z^{m+1} = z^{m-r+1}z^r) \\
= \;& z^{m+1}\sigma_+(z)(\overline{\sigma}_-(z)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}+(1^r)} \wedge \overline{\sigma}_-(z)[\mathbf{b}]_{m-r-1}) && \text{(the } \overline{\sigma}_-(z) \text{ action (21) on } [\mathbf{b}]_{m-r+1}) \\
= \;& z^{m+1}\sigma_+(z)\overline{\sigma}_-(z)([\mathbf{b}]^r_{m+\boldsymbol{\lambda}+(1^r)} \wedge [\mathbf{b}]_{m-r-1}) && (\overline{\sigma}_-(z) \text{ is a HS derivation}) \\
= \;& z^{m+1}\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{b}]_{m+1+\boldsymbol{\lambda}} \\
= \;& R(z)\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} && \text{(by definition of the map } R(z)) \\
= \;& R(z) \cdot \frac{1}{E(z)}\overline{\sigma}_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} && \text{(by the } B\text{-module structure (34) of } F). \quad \blacksquare
\end{aligned}
$$

6.10 Corollary. *Let* $\Gamma(z) : B[\zeta, \zeta^{-1}] \to B[\zeta, \zeta^{-1}][[z]]$ *be given by*

$$(\Gamma(z)\zeta^m\Delta_{\boldsymbol{\lambda}}(H))[\mathbf{b}]_0 := \mathbf{b}(z) \wedge [\mathbf{b}]_{m+\boldsymbol{\lambda}}.$$

Then

$$\Gamma(z) = R(z)\sigma_+(z)\overline{\sigma}_-(z) = R(z)\frac{1}{E(z)}\overline{\sigma}_-(z).$$

Proof. Obvious from the definition. $\blacksquare$

6.11 Lemma. *Let* $R(z)^T : F^*[[z, z^{-1}]] \to F^*[[z, z^{-1}]]$ *be the transpose of the operator* $R(z) : F[[z, z^{-1}]] \to F[[z, z^{-1}]]$ *as in* 6.8. *Then*

$$R(z)^T[\boldsymbol{\beta}]_{m+\boldsymbol{\lambda}} = z^m[\boldsymbol{\beta}]_{m-1+\boldsymbol{\lambda}}, \tag{41}$$

and

$$(R(z)^T)^{-1}[\boldsymbol{\beta}]_{m-1+\boldsymbol{\lambda}} = (R(z)^{-1})^T[\boldsymbol{\beta}]_{m-1+\boldsymbol{\lambda}} = z^{-m}[\boldsymbol{\beta}]_{m+\boldsymbol{\lambda}}. \tag{42}$$

Proof. To prove (41):

$$\begin{aligned}(R(z)^T[\boldsymbol{\beta}]_{m+\boldsymbol{\mu}})([\mathbf{b}]_{m-1+\boldsymbol{\lambda}}) &= [\boldsymbol{\beta}]_{m+\boldsymbol{\mu}}(R(z)[\mathbf{b}]_{m-1+\boldsymbol{\lambda}})\\ &= [\boldsymbol{\beta}]_{m+\boldsymbol{\mu}}(z^m[\mathbf{b}]_{m+\boldsymbol{\lambda}}) = z^m\delta_{\boldsymbol{\mu},\boldsymbol{\lambda}},\end{aligned}$$

and then $R(z)[\boldsymbol{\beta}]_{m+\boldsymbol{\mu}} = z^m[\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}}$ as stated in (41). The proof of (42) is straightforward. ∎

6.12 Corollary. *If* $\boldsymbol{\beta}(z^{-1}) = \sum_{j\in\mathbb{Z}} \beta_j z^{-j-1} \in M^*[[z^{-1}, z]]$, *then*

$$\boldsymbol{\beta}(z^{-1}) \wedge [\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}} = z^{-1}(R(z)^{-1})^T\sigma_-^T(z)\overline{\sigma}_+^T(z)[\boldsymbol{\beta}]_{m+\boldsymbol{\mu}}. \tag{43}$$

Proof. Let $\overline{\sigma}_+^*(w), \sigma_+^*(w)$ be the Schubert derivations on $\bigwedge M^*$ as in Remark 3.4. Let $\beta(w) = w\sum_{j\in\mathbb{Z}} \beta_j w^j$. Then applying Proposition 6.9

$$\begin{aligned}\boldsymbol{\beta}(w) \wedge [\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}} &= w\sum_{j\in\mathbb{Z}} \beta_j w^j \wedge [\boldsymbol{\beta}]_{m-1+\mu}\\ &= wR(w)\sigma_+^*(w)\overline{\sigma}_-^*(w)[\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}} = wR(w)^T\sigma_+^*(w)\overline{\sigma}_-^*(w)[\boldsymbol{\beta}]_{m+\boldsymbol{\mu}}.\end{aligned}$$

Putting $w = z^{-1}$ and observing that $\overline{\sigma}_-^*(w)\beta_{j|w=z^{-1}} = \overline{\sigma}_+(z)^T\beta_j$ and $\sigma_+^*(w)\beta_{j|w=z^{-1}} = \sigma_-(z)^T\beta_j$, for all $j \in \mathbb{Z}$, one finally obtains (43). ∎

6.13 Proposition.

$$\begin{aligned}\boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= z^{-1}R(z)^{-1}\overline{\sigma}_+(z)\sigma_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}}\\ &= z^{-1}R(z)^{-1}E(z)\sigma_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}}.\end{aligned} \tag{44}$$

where the last equality follows from (34).

Proof. For all $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{P} \times \mathcal{P}$ and all $m \in \mathbb{Z}$:

$$[\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}}(\boldsymbol{\beta}(z)\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}) = (\boldsymbol{\beta}(z^{-1}) \wedge [\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}})([\mathbf{b}]_{m+\boldsymbol{\lambda}}).$$

By Corollary 6.12, then

$$\begin{aligned}[\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}}(\boldsymbol{\beta}(z)\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}) &= z^{-1}(R(z)^{-1})^T\sigma_-^T(z)\overline{\sigma}_+^T(z)[\boldsymbol{\beta}]_{m+\boldsymbol{\mu}}\big([\mathbf{b}]_{m+\boldsymbol{\lambda}}\big)\\ &= [\boldsymbol{\beta}]_{m-1+\boldsymbol{\mu}}(z^{-1}\overline{\sigma}_+(z)\sigma_-(z)R(z)^{-1}[\mathbf{b}]_{m+\boldsymbol{\lambda}}),\end{aligned}$$

whence (44), because of (40). ∎

6.14 Corollary. *Let* $\Gamma^*(z) := B[\zeta, \zeta^{-1}] \to B[\zeta, \zeta^{-1}][[z]]$ *be given by*

$$(\Gamma^*(z)\zeta^m\Delta_{\boldsymbol{\lambda}}(H))[\mathbf{b}]_0 := \boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}.$$

Then

$$\Gamma^*(z) = z^{-1}R(z)^{-1}\overline{\sigma}_+(z)\sigma_-(z) = z^{-1}R(z)^{-1}E(z)\sigma_-(z).$$

Proof. A straightforward consequence of the definition. ∎

7 Another expression for $\Gamma^*(z)$

7.1 Notation as in Section 1. Expanding the determinant of the Schur polynomial $\Delta_{\boldsymbol{\lambda}}(H)$ along the first row, according to Laplace's rule, one easily checks that

$$\Delta_{\boldsymbol{\lambda}}(H) = h_{\lambda_1}\Delta_{\boldsymbol{\lambda}^{(1)}(H)} - h_{\lambda_2-1}\Delta_{\boldsymbol{\lambda}^{(2)}+(1)}(H) + \cdots + (-1)^{r-1}h_{\lambda_r-r+1}\Delta_{\boldsymbol{\lambda}^{(r)}+(1^{r-1})}(H). \tag{45}$$

Unable to find a better compact notation for it, we denote by $\Delta_{\boldsymbol{\lambda}}(z^{-\boldsymbol{\lambda}}, H)$ the determinant obtained by $\Delta_{\boldsymbol{\lambda}}(H)$ under the substitution $h_{\lambda_j-i+1} \to z^{-\lambda_j+i-1}$. In other words:

$$\Delta_{\boldsymbol{\lambda}}(z^{-\boldsymbol{\lambda}}, H) := \frac{\Delta_{\boldsymbol{\lambda}^{(1)}}(H)}{z^{-\lambda_1}} - \frac{\Delta_{\boldsymbol{\lambda}^{(2)}}(H)}{z^{\lambda_2-1}} + \cdots + (-1)^{r-1}\frac{\Delta_{\boldsymbol{\lambda}^{(r)}+(1^{r-1})}(H)}{z^{\lambda_r-r+1}} \tag{46}$$

or, more explicitly

$$\Delta_{\boldsymbol{\lambda}}(z^{-\boldsymbol{\lambda}}, H) := \begin{vmatrix} \dfrac{1}{z^{\lambda_1}} & \dfrac{1}{z^{\lambda_2}+1} & \cdots & \dfrac{1}{z^{\lambda_r+r-1}} \\ & & & \\ h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r} \end{vmatrix}. \tag{47}$$

Then, we have:

7.2 Proposition.

$$\boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}} = z^{-m-1}\left(\Delta_{\boldsymbol{\lambda}}(z, H) + (-1)^{r-1}\sum_{j\geqslant 0}(-1)^j\Delta_{\boldsymbol{\lambda}+(1^{r+j})}(H)z^{j+r}\right)[\mathbf{b}]_{m-1}. \tag{48}$$

Proof. First of all we apply directly Leibniz rule enjoyed by the derivation $\boldsymbol{\beta}(z^{-1})\lrcorner$:

$$\begin{aligned} \boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= \boldsymbol{\beta}(z^{-1})\lrcorner(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}) \\ &= \boldsymbol{\beta}(z^{-1})\lrcorner\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} + (-1)^{r-1}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \left(\boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m-r}\right). \end{aligned} \tag{49}$$

We compute separately the two summands occurring in the r.h.s. of (49). Let us begin with the second one:

$$\begin{aligned} \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \left(\boldsymbol{\beta}(z^{-1})\lrcorner[\mathbf{b}]_{m-r}\right) &= \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge z^{-m+r-1}\sigma^T_+(z)\beta_{m-r}\lrcorner[\mathbf{b}]_{m-r} \\ &= z^{-m-1+r}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_+(z)(\beta_{m-r}\lrcorner[\mathbf{b}]_{m-r}) \\ &= z^{m-1+r}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_+(z)(\beta_{m-r}\lrcorner(\sigma_+(z)b_{m-r} \wedge [\mathbf{b}]_{-m-1+r})) \\ &= z^{-m-1+r}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_+(z)[\mathbf{b}]_{m-r-1} \\ &= z^{-m-1+r}\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sum_{j\geqslant 0}(-1)^j[\mathbf{b}]_{m-r-1+(1^j)}z^j \end{aligned}$$

$$
\begin{aligned}
&= z^{-m-1+r}\sum_{j\geqslant 0}(-1)^j \mathbf{b}^r_{m-1+\boldsymbol{\lambda}+(1^r)}\wedge[\mathbf{b}]_{m-1-r+(1^j)}z^j\\
&= z^{-m-1}\sum_{j\geqslant 0}(-1)^j[\mathbf{b}]_{m-1+\boldsymbol{\lambda}+(1^{r+j})}.
\end{aligned} \tag{50}
$$

To compute the first summand, instead, it is sufficient to apply the definition 2.3 of contraction: each b_i occurring in the expression $\mathbf{b}^r_{m+\boldsymbol{\lambda}}$ is replaced, with the appropriate sign, by z^{-i-1}. The straightforward equality,

$$
\mathbf{b}^r_{m+\boldsymbol{\lambda}} = (-1)^j b_{m-j+\lambda_{j+1}}\wedge \mathbf{b}^r_{m+\boldsymbol{\lambda}^{(j)}+(1^{j-1})}, \qquad 1\leqslant j\leqslant r,
$$

holding by the very meaning (5) of the notation $\mathbf{b}^r_{m+\boldsymbol{\lambda}}$, easily implies that

$$
\begin{aligned}
\boldsymbol{\beta}(z^{-1})\lrcorner\mathbf{b}^r_{m+\boldsymbol{\lambda}} &= \frac{1}{z^{m+1}}\frac{\mathbf{b}^r_{m+\boldsymbol{\lambda}^{(1)}}}{z^{\lambda_1}} - \frac{\mathbf{b}^r_{m-1+(\boldsymbol{\lambda}^{(2)}+(1))}}{z^{\lambda_2-1}} + \cdots + (-1)^{r-1}\frac{\mathbf{b}^r_{m-r+1+(\boldsymbol{\lambda}^{(r)}+(1^{r-1}))}}{z^{\lambda_r-r+1}}\\
&= \frac{1}{z^{m+1}}\Delta_{\boldsymbol{\lambda}}(z^{-\boldsymbol{\lambda}},H)[\mathbf{b}]_{m-1},
\end{aligned} \tag{51}
$$

where in the last equality we used the Boson–Fermion correspondence (35). Formula (48) then follows by plugging (50) and (51) into (49), and using again the Boson–Fermion correspondence. ∎

8 Commutation Rules

8.1 Proposition.

i) The following commutation rules hold in $\mathrm{End}_{\mathbb{Z}}(\bigwedge M)$ *for all* $i,j\in\mathbb{Z}$*:*

$$
\sigma_i\sigma_j=\sigma_j\sigma_i,\qquad \overline{\sigma}_i\sigma_j=\sigma_j\overline{\sigma}_i,\qquad \overline{\sigma}_i\overline{\sigma}_j=\overline{\sigma}_j\overline{\sigma}_i. \tag{52}
$$

ii) All the operators $\sigma_\pm(z),\overline{\sigma}_\pm(z):\bigwedge M\to\bigwedge M[[z]]$ *are mutually commuting.*

Proof. i) We check only for the first of (52), the remaining two being similar. One has that for all $k\in\mathbb{Z}$, one has $\sigma_i\sigma_j b_k = b_{k+i+j}=\sigma_j\sigma_i b_k$. Assume commutation holds for $\bigwedge^{r-1}M$. Each $u\in\bigwedge^r M$ is a sum of typical elements of the form $b\wedge v$ for $b\in M$ and $v\in\bigwedge^{r-1}M$. Then

$$
\begin{aligned}
\sigma_i\sigma_j(b\wedge v) &= \sigma_i\sum_{k=0}^{j}\sigma_k b\wedge\sigma_{j-k}v\\
&= \sum_{l=0}^{i}\sum_{k=0}^{j}\sigma_l\sigma_k b\wedge\sigma_{i-l}\sigma_{j-k}v\\
&= \sum_{l=0}^{i}\sum_{k=0}^{j}\sigma_k\sigma_l b\wedge\sigma_{j-k}\sigma_{i-l}v = \sigma_j\sigma_i(b\wedge v),
\end{aligned}
$$

where the third equality follows by induction, which proves i). Item ii) is a straightforward consequence of i). ∎

The commutation rules (52) do not hold when $\sigma_\pm(z)$ and $\overline{\sigma}_\pm(z)$ are extended to the fermionic space F. For example,

$$\sigma_{-1}\sigma_2[\mathbf{b}]_0 = \sigma_{-1}b_2 \wedge [\mathbf{b}]_{-1} = b_1 \wedge [\mathbf{b}]_{-1} \neq 0 = \sigma_2 \cdot 0 = \sigma_2\sigma_{-1}[\mathbf{b}]_{-1}.$$

8.2 Notation. Let $\mathcal{R}(z,w)$ be a rational expression having poles in $z=0$, $w=0$ and $z-w=0$. Following [14, p. 18] we denote by $i_{w,z}R(z,w)$ the expansion of $\mathcal{R}(z,w)$ as a formal power series in (z/w) and by $i_{z,w}$ the expansion of the same expression as a formal power series of w/z.

8.3 Lemma. *For all* $m \in \mathbb{Z}$,

$$\sigma_-(w)\sigma_+(z)[\mathbf{b}]_m = i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)[\mathbf{b}]_m.$$

Proof.

$$\begin{aligned}
\sigma_-(w)\sigma_+(z)[\mathbf{b}]_m &= \sigma_-(w)(\sigma_+(z)b_m \wedge [\mathbf{b}]_{m-1}) && \text{(decomposition of } [\mathbf{b}]_m) \\
&= \sigma_-(w)\sigma_+(z)b_m \wedge [\mathbf{b}]_{m-1} && \text{(distributing } \sigma_-(w) \text{ with respect to } \wedge) \\
&= \sigma_-(w)\textstyle\sum_{j\geqslant 0} b_{m+j}z^j \wedge [\mathbf{b}]_{m-1} && \text{(definition of } \sigma_+(z))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j\geqslant 0}\sum_{i=0}^{j}\frac{b_{m+j}z^j}{w^i} \wedge [\mathbf{b}]_{m-1} \\
&= \left(1+\frac{z}{w}+\frac{z^2}{w^2}+\frac{z^3}{w^3}+\cdots\right)\sigma_+(z)\sigma_-(w)b_m \wedge [\mathbf{b}]_{m-1} \\
&= i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)\cdot[\mathbf{b}]_m.
\end{aligned}$$

■

8.4 Proposition. *Let* $\sigma_\pm(z)$ *and* $\overline{\sigma}_\pm(z)$ *considered as maps from* $\mathsf{F} \to \mathsf{F}[[z^{\pm 1}]]$*. Then the following commutation rules holds*

$$\sigma_-(w)\sigma_+(z) = i_{w,z}\frac{w}{w-z}\cdot\sigma_+(z)\sigma_-(w) \tag{53}$$

$$\overline{\sigma}_-(z)\overline{\sigma}_+(w) = i_{z,w}\frac{z}{z-w}\overline{\sigma}_+(w)\overline{\sigma}_-(z) \tag{54}$$

Proof. Let us prove (53). For all $\boldsymbol{\lambda} \in \mathcal{P}$ we have

$$\sigma_-(w)\sigma_+(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \sigma_-(z)\sigma_+(w)(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r})$$

provided that r is bigger or equal than the length of the partition $\boldsymbol{\lambda}$. Now we use the definition 5.3 of the extension of $\sigma_\pm(z)$ to F. This gives:

$$\sigma_-(w)(\sigma_+(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_+(z)b_{m-r} \wedge [\mathbf{b}]_{m-r-1}) =$$

$$= \sigma_-(w)\sigma_+(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_-(w)\sigma_+(z)b_{m-r} \wedge [\mathbf{b}]_{m-r-1}$$

Proposition 8.1 ensures that $\sigma_-(w)$ and $\sigma_+(z)$ commute on $\bigwedge M$ so obtaining

$$\sigma_+(z)\sigma_-(w)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_-(w)\sigma_+(z)[\mathbf{b}]_{m-r}$$

and now we apply Lemma 8.3 to finally obtain:

$$\begin{aligned}\sigma_+(z)\sigma_-(w)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_-(w)\sigma_+(z)[\mathbf{b}]_{m-r} &= \sigma_+(z)\sigma_-(w)\left(\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \frac{w}{w-z}[\mathbf{b}]_{m-r}\right)\\ &= i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)[\mathbf{b}]_{m+\boldsymbol{\lambda}}\end{aligned}$$

which proves the commutation formula (53). To prove (54) we take the inverse of either side of (53) obtaining

$$\overline{\sigma}_+(z)\overline{\sigma}_-(w) = \left(1-\frac{z}{w}\right)\overline{\sigma}_-(w)\overline{\sigma}_+(z),$$

from which

$$\overline{\sigma}_-(w)\overline{\sigma}_+(z) = i_{w,z}\frac{w}{w-z}\overline{\sigma}_+(z)\overline{\sigma}_-(w).$$

Changing the role of the indeterminates z and w one obtains precisely (54). ∎

9 The DJKM Bosonic Vertex Representation of $gl_\infty(\mathbb{Z})$

9.1 Let $\mathcal{B}_{ij} := b_i \otimes \beta_j \in M \otimes M^* \cong \mathrm{End}(M)$ and $gl_\infty(M) := \bigoplus_{i,j\in\mathbb{Z}} \mathbb{Z}\mathcal{B}_{ij}$. It is a Lie algebra with respect to the obvious Lie bracket $[A, B] = AB - BA$ $(A, B \in gl_\infty(\mathbb{Z}))$. Let

$$\delta : gl_\infty(\mathbb{Z}) \to \mathrm{End}_{\mathbb{Z}}(\bigwedge M)$$

be the representation of $gl_\infty(\mathbb{Z})$ as a sub-algebra of derivations, $A \mapsto \delta(A)$, defined by (2).

9.2 Proposition. *Let* $b \otimes \beta \in M \otimes M^*$ *and* $\mathbf{u} \in \bigwedge M$*. Then*

$$\delta(b \otimes \beta)(\mathbf{u}) = b \wedge (\beta \lrcorner \mathbf{u}).$$

Proof. As $\mathbf{u} \in \bigwedge M$ is a finite sum of homogeneous elements, we may assume without loss of generality that $u \in \bigwedge^r M$. Then we argue by induction on $r \geqslant 1$. If $\mathbf{u} \in M$, $\delta(b \otimes \beta)(\mathbf{u}) = \beta(\mathbf{u})b = b(\beta\lrcorner\mathbf{u})$, and the claim holds for $r = 1$. Assume now the property true for all $\mathbf{u} \in \bigwedge^i M$ and $1 \leqslant i \leqslant r-1$. Each $\mathbf{v} \in \bigwedge^r M$ is a finite sum of monomials of the form $\mathbf{u} \wedge \mathbf{w}$, with $\mathbf{u} \in M$ and $\mathbf{w} \in \bigwedge^{r-1} M$. We may then assume $\mathbf{v} = \mathbf{u} \wedge \mathbf{w}$ and, in this case,

$$\begin{aligned}\delta(b \otimes \beta)(\mathbf{u} \wedge \mathbf{w}) &= \beta(\mathbf{u})b \wedge \mathbf{w} + \mathbf{u} \wedge b \wedge (\beta\lrcorner\mathbf{w})\\ &= b \wedge (\beta(\mathbf{u})\mathbf{w} - \mathbf{u} \wedge \beta\lrcorner\mathbf{w})\\ &= b \wedge (\beta\lrcorner(\mathbf{u} \wedge \mathbf{w})).\end{aligned}$$ ∎

Let us now extend the derivation δ of $\bigwedge M$ to F as follows. Let $[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r} \in F_m \subseteq F$. Each $A \in gl_\infty(\mathbb{Z})$ is a finite linear combination $\sum_{ij} a_{ij} b_i \otimes \beta_j$. Let k be the minimum among all j such that $a_{ij} \neq 0$ and let $r \geqslant 0$ such that $m - r < k$. Thus one defines $\delta : gl_\infty(\mathbb{Z}) \to End_{\mathbb{Z}}(F_m)$ via:

$$\delta(A)_{|F_m}[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \delta(A)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \delta(A)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m-r}.$$

An easy check shows that the definition does not depend on the choice of $r \geqslant 0$ such that $m - r < k$. Let

$$\delta(z, w)_{|F_m} = \sum_{i,j\in\mathbb{Z}} \delta(\mathcal{B}_{ij})_{|F_m} z^i w^{-j} : F_m \to F_m[[z, w^{-1}]]$$

9.3 Theorem [3, Date–Jimbo–Kashiwara–Miwa].

$$\begin{aligned} \delta(z, w)_{|F_m} = \sum_{i,j\in\mathbb{Z}} \delta(\mathcal{B}_{ij})_{|F_m} z^i w^{-j} &= \frac{z^m}{w^m} \sigma_+(z)\overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w) \\ &= \frac{z^m}{w^m} i_{z,w} \frac{z}{z-w} \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w) \\ &= \frac{z^m}{w^m} i_{z,w} \frac{z}{z-w} \frac{E(w)}{E(z)} \overline{\sigma}_-(z)\sigma_-(w). \end{aligned} \tag{55}$$

Proof. We have

$$\begin{aligned} \sum \delta(\mathcal{B}_{ij})_{|F_m}[\mathbf{b}]_{m+\boldsymbol{\lambda}} z^i w^{-j} &= \sum \delta_m(b_i \otimes \beta_j)[\mathbf{b}]_{m+\boldsymbol{\lambda}} z^i w^{-j} \\ &= \mathbf{b}(z) \wedge (w\boldsymbol{\beta}(w^{-1}) \lrcorner [\mathbf{b}]_{m+\boldsymbol{\lambda}})) \\ &= \mathbf{b}(z) \wedge (w^{-m}\overline{\sigma}_+(w)\sigma_-(w)[\mathbf{b}]_{m-1+\boldsymbol{\lambda}}) \\ &= w^{-m}\mathbf{b}(z) \wedge \overline{\sigma}_+(w)\sigma_-(w)[\mathbf{b}]_{m-1+\boldsymbol{\lambda}}. \end{aligned}$$

Now $\overline{\sigma}_+(w)\sigma_-(w)[\mathbf{b}]_{m-1+\boldsymbol{\lambda}}$ is a $\mathbb{Z}[[w, w^{-1}]$-linear combination of elements of F_{m-1} and $b(z)\wedge$ is $\mathbb{Z}[[w, w^{-1}]$ linear. Proposition 6.9 applied to F_{m-1} gives

$$\begin{aligned} & z^m \sigma_+(z)\overline{\sigma}_-(z) w^{-m} \overline{\sigma}_+(w)\sigma_-(w)[\mathbf{b}]_{m+\boldsymbol{\lambda}} \\ = \;& \frac{z^m}{w^m} \sigma_+(z)\overline{\sigma}_-(z)\overline{\sigma}_+(w)\sigma_-(w)[\mathbf{b}]_{m+\boldsymbol{\lambda}} \\ = \;& \frac{z^m}{w^m} i_{z,w} \frac{z}{z-w} \sigma_+(z)\overline{\sigma}_+(w)\overline{\sigma}_-(z)\sigma_-(w)[\mathbf{b}]_{m+\boldsymbol{\lambda}}, \end{aligned}$$

and using the definition of the B-module structure of F_m one gets precisely (55). ∎

Let $(\mathcal{B}^{(m)}(z, w)\Delta_{\boldsymbol{\lambda}}(H))[\mathbf{b}]_m = \delta(z, w)[\mathbf{b}]_{m+\boldsymbol{\lambda}}$.

9.4 Corollary (The DJKM bosonic Vertex Representation of $gl_\infty(\mathbb{Z})$). *Using the Boson–Fermion correspondence 5.14:*

$$\mathcal{B}^{(m)}(z, w) = \frac{z^m}{w^m} \cdot i_{z,w} \frac{z}{z-w} \Gamma(z, w),$$

where the vertex operator $\Gamma(z,w)$ *is given by*

$$\Gamma(z,w)=\frac{E(w)}{E(z)}\overline{\sigma}_-(z)\sigma_-(w). \tag{56}$$

Proof. It follows straightforwardly from the definition and expression of (55) for $\delta(z,w)_{F_m}$. ∎

9.5 Let $\mathcal{A}_\infty(\mathbb{Z})$ be the Lie algebra of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ having only finitely many non-zero diagonals, i.e. $a_{ij}=0$ if $|i-j|>>0$. In this case the representation δ_m must be replaced by a modified representation $\widehat{\delta}_m$ [15, p. 40]:

$$\begin{cases}\widehat{\delta}_m(\mathcal{B}_{ij}) &= \delta_m(\mathcal{B}_{ij}) \quad \text{if} \quad i\neq j \quad \text{or} \quad i=j>0,\\ \widehat{\delta}_m(\mathcal{B}_{ii})-\mathbf{1}_{F_m} &= \delta_m(\mathcal{B}_{ii}) \quad \text{if} \qquad\qquad\quad i=j\leqslant 0.\end{cases}$$

Then, to obtain the generating function of the representation of $\mathcal{B}_{ij}$ via $\widehat{r}_m$ it suffices to subtract from formula (55) the series

$$\sum_{j\leqslant 0} z^j w^{-j}[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\sum_{j\geqslant 0}\left(\frac{w}{z}\right)^j[\mathbf{b}]_{m+\boldsymbol{\lambda}}=i_{z,w}\frac{z}{z-w}[\mathbf{b}]_{m+\boldsymbol{\lambda}},$$

so obtaining

$$\sum_{i,j\in\mathbb{Z}}\widehat{\delta}_m(\mathcal{B}_{ij})z^iw^{-j}=i_{z,w}\,\frac{z}{z-w}\cdot\left(\frac{z^m}{w^m}\Gamma(z,w)-1\right),$$

where $\Gamma(z,w)$ is like in (56).

9.6 Remark. Recall that $\Gamma(z,w)=\dfrac{E(w)}{E(z)}\overline{\sigma}_-(z)\sigma_-(w)$ is a well defined operator $B\to B[[z,w^{-1}]$, which is defined over the integers. In $B_\mathbb{Q}:=B\otimes_\mathbb{Z}\mathbb{Q}$ one can define variables $(x_1,x_2,\ldots)$ through the equalities [15]:

$$\exp\left(\sum_{i\geqslant 1}x_iz^i\right)=\frac{1}{E(z)}\qquad\text{and}\qquad\exp\left(-\sum_{i\geqslant 1}x_iw^i\right)=E(w). \tag{57}$$

Moreover in [9, Theorem 5.7] it is shown that, over the rationals,

$$\overline{\sigma}_-(z)=\exp\left(-\sum_{i\geqslant 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right)\qquad\text{and}\qquad\sigma_-(w)=\exp\left(\sum_{i\geqslant 1}\frac{1}{iw^i}\frac{\partial}{\partial x_i}\right), \tag{58}$$

so that after substituting (57) and (58) into expression (56) returns the traditional form (4).

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