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# Growth and remodelling from the perspective of Noether's Theorem

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## Abstract

Starting from the observation that the growth of a body breaks the time translation symmetry of the body's dynamics, we determine a scalar field, called *internal time*, that defines an indicator of the intrinsic time scale of the growth-related body's structural evolution. By recasting the theory of growth for monophasic media within a variational framework, we obtain the internal time as the solution of a partial differential equation descending from Noether's Theorem. We do this by considering two approaches, one formulated in terms of internal variables and one adopting the concept of augmented kinematics.

*Keywords:* growth; remodelling; variational methods; Noether's Theorem; endochronic theory

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## 1. Introduction

The mechanics of volumetric growth studies the variation of mass and the concomitant structural evolution of biological tissues [1, 2, 3]. Such processes are often conceived as anelastic, and are described by a generally non-integrable tensor field,  $\mathbf{K}$ , referred to as *growth tensor*.

The role of  $\mathbf{K}$  in the modelling of growth is not unique, and its interpretation depends on the theory within which it is introduced. To the best of our knowledge, there exist at least two ways of interpreting  $\mathbf{K}$ : it can be viewed either as an *internal variable* (see e.g. [3]) or as a *kinematic variable* (see e.g. [4]). The conceptual difference between these two approaches affects all the relations governing the dynamics of a body, especially the one representing the evolution of its internal structure.

The way in which the dissipation is studied in [3] and [4] plays a major role in this work. In the sequel, indeed, we employ the dissipation inequality to show that a growing body possesses an intrinsic time scale, defined by the chosen theory. To this end, we take inspiration from Vakulenko's concept of "*endochronic thermodynamics*" [5, 6], and we demonstrate that the body's intrinsic time scale is related to a generalised force, hereafter denoted by  $\mathcal{F}_0$  and termed *time-like inhomogeneity force* [7]. In our framework,  $\mathcal{F}_0$  plays a role similar to that played by the material inhomogeneity forces in Eshelby's theory of inclusions [8] and, more generally, in the mechanics of materials with inhomogeneities [7], as is the case of growing media [3].

Vakulenko's theory addresses the thermodynamics of anelastic processes [5, 6], and is said to be "*endochronic*" since it associates a given anelastic process with a scalar-valued function, the "*thermodynamic time*", defined from the outset as the time integral of a suitable function of the entropy production [6].

Quite differently, in our work we identify the *internal time* of growth of a body, hereafter denoted by  $\tau$ , with the solution of the partial differential equation [9]

$$\mathcal{N}_0(\tau) := \mathcal{H} \dot{\tau} - (\mathbf{T}^T \mathbf{v}) \text{Grad } \tau - \mathcal{F}_0 \tau = 0, \quad (1)$$

where  $\mathcal{H}$  is the body's total energy density,  $\mathbf{T}$  is the first Piola-Kirchhoff stress tensor and  $\mathbf{v}$  is the Lagrangian velocity field.

Equation (1) was deduced in [9] as a consequence of Noether's Theorem, and  $\tau$  was defined as a deformation of time depending on material points and on time itself. More specifically,  $\tau$  was introduced to highlight how the occurrence of growth in a body is a symmetry breaking, spoiling the invariance of the body's dynamics under time translations and yielding the failure of the conservation of energy [9]. This symmetry breaking results in the arising of  $\mathcal{F}_0$  and manifests itself as the loss of the homogeneity of time.

In this work, we deeply reformulate the mathematical framework of [9] and, after polishing it from some formal imprecisions, we propose the following novelties: (a) we retrieve Equation (1) within the two different pictures of growth given in [3] and [4], respectively; (b) for both pictures, we compute *explicitly* the internal time,  $\tau$ , and we show that the quantity  $\tau_c := 1 - \tau/\tau_0$ , where  $\tau_0$  is a reference value, is analogous to endochronic time in that it increases monotonically in time and may thus represent an intrinsic time-scale associated with growth; (c) within the formulation presented in [4], we describe mechanotransduction through the conceptually systematic approach of Theoretical Mechanics. Our results also apply to remodelling.

## 2. An overview on growth mechanics

We consider the simplest possible formulation of the volumetric growth of a body. In particular, we assume the body to

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be hyperelastic and we employ the Bilby-Kröner-Lee decomposition of the deformation gradient tensor, i.e.,  $\mathbf{F} = \mathbf{\Phi}\mathbf{K}$ , so that the body's material response is described by the strain energy density function

$$\Psi(X, t) = \hat{\Psi}(\mathbf{F}(X, t), \mathbf{K}(X, t)) = J_{\mathbf{K}} \hat{\Psi}_{\nu}(\mathbf{\Phi}(X, t)), \quad (2)$$

where  $\mathbf{\Phi} := \mathbf{F}\mathbf{K}^{-1}$  is the elastic part of the deformation gradient tensor,  $\hat{\Psi}_{\nu}$  is the strain energy density expressed per unit volume of the body in its stress-free state, and  $J_{\mathbf{K}} := \det \mathbf{K} > 0$ .

In local form, and with respect to the body's reference configuration,  $\mathcal{B}$ , the mass balance law is given by  $\varrho_{\mathbf{R}} = \Pi$ , where  $\varrho_{\mathbf{R}}$  is the mass density of the body per unit volume of  $\mathcal{B}$ , the superimposed dot denotes partial differentiation with respect to time, and  $\Pi$  is the source or sink of mass that describes growth. As in [3, 10], we write  $\varrho_{\mathbf{R}} = J_{\mathbf{K}} \varrho_{\nu}$ , where  $\varrho_{\nu}$  is the mass density of the body in its stress-free state, and we require the conditions

$$\frac{J_{\mathbf{K}}}{J_{\mathbf{K}}} = \text{tr}(\mathbf{K}^{-1} \dot{\mathbf{K}}) = \frac{1}{2} \text{tr}(\dot{\mathbf{C}}_{\mathbf{K}} \mathbf{C}_{\mathbf{K}}^{-1}) = \frac{\Pi}{J_{\mathbf{K}} \varrho_{\nu}} =: \Gamma, \quad (3)$$

where  $\mathbf{C}_{\mathbf{K}} := \mathbf{K}^{\mathbf{T}} \mathbf{K}$  is the metric tensor induced by  $\mathbf{K}$ ,  $\Gamma$  measures the relative variation of  $\varrho_{\mathbf{R}}$ , and  $\varrho_{\nu}$  is regarded as a time independent field specified from the outset.

Within the quasi-static limit, and neglecting all inertial and long-range body forces, such as gravity, the local form of the momentum balance law reads

$$\text{Div } \mathbf{T} = \mathbf{0}, \quad (4a)$$

$$\mathbf{T} = \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \circ (\mathbf{F}, \mathbf{K}) = J_{\mathbf{K}} \left[ \frac{\partial \hat{\Psi}_{\nu}}{\partial \mathbf{\Phi}} \circ \mathbf{\Phi} \right] \mathbf{K}^{-\mathbf{T}}, \quad (4b)$$

where  $\text{Div}$  is the material divergence operator and  $\mathbf{T}$  is the first Piola-Kirchhoff stress tensor. The balance law (4a) should be regarded as an equation for the motion of the body,  $\chi$ , whose partial derivatives define the components of  $\mathbf{F}$ . To determine  $\mathbf{K}$ , an additional, independent equation is needed.

### 2.1. Tensor $\mathbf{K}$ viewed as internal variable

The tensor field  $\mathbf{K}$  shares several formal analogies with the inverse of the tensor field referred to as “uniformity mapping” in [3]. Hence, if  $\mathbf{K}$  is regarded as an internal variable, the theory exposed in [3] can be employed to develop a criterion for determining an admissible evolution law for  $\mathbf{K}$ . In particular, by invoking the representation theorem for tensor-valued functions [11], it can be shown that, in the case of isotropy,  $\mathbf{K}$  satisfies

$$\text{sym}[\mathbf{C}_{\mathbf{K}} \mathbf{\Lambda}] = \sum_{n=0}^2 (J_{\mathbf{K}})^{-n} \beta_n \mathbf{H}^n \mathbf{C}_{\mathbf{K}}, \quad (5)$$

where  $\mathbf{\Lambda} := \mathbf{K}^{-1} \dot{\mathbf{K}}$ ,  $\mathbf{H}$  is Eshelby's stress tensor,

$$\mathbf{H} := \Psi \mathbf{I}^{\mathbf{T}} - \mathbf{F}^{\mathbf{T}} \mathbf{T} \equiv \mathbf{K}^{\mathbf{T}} \left( \frac{\partial \hat{\Psi}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) \right), \quad (6)$$

and  $\{\beta_n\}_{n=0}^2$  are to be expressed constitutively through functions of  $J_{\mathbf{K}}$ ,  $\Psi$ , the three principal invariants of  $\mathbf{H}$ , and other quantities, possibly required by phenomenology.

In Equation (5), the convention  $\mathbf{H}^0 = \mathbf{I}^{\mathbf{T}}$  is used, where  $\mathbf{I}^{\mathbf{T}}$  is the transpose of the material identity tensor,  $\mathbf{I}$ . Moreover, because of isotropy,  $\mathbf{H}\mathbf{C}_{\mathbf{K}}$  is symmetric, and so is also  $\mathbf{H}^2 \mathbf{C}_{\mathbf{K}} = \mathbf{H}\mathbf{C}_{\mathbf{K}}\mathbf{H}^{\mathbf{T}}$  [12]. Finally, the functions  $\{\beta_n\}_{n=0}^2$  have to comply with the dissipation inequality

$$\mathcal{D}_{\text{IV}} = \Psi \text{tr}(\mathbf{\Lambda}) - \mathbf{H} : \mathbf{\Lambda} + \mathcal{D}_{\text{nc}} \geq 0. \quad (7)$$

Here,  $\mathcal{D}_{\text{nc}}$  is said to be the “non-compliant” contribution to the dissipation [13] and is attributed to processes accompanying growth but not explicitly accounted for in the model. Moreover, the subscript “IV” in  $\mathcal{D}_{\text{IV}}$  stands for “internal variable” to remark that in Equation (7)  $\mathbf{K}$  is viewed as an internal variable.

In order to model the material inhomogeneities associated with growth, Epstein and Maugin [3] introduce a Lagrangian density function,  $\mathcal{L}$ , whose constitutive representation depends on material points and time through  $\mathbf{K}$ . Hence, within the quasi-static limit, in which the identification  $\mathcal{L} = -\Psi$  applies, and by mimicking the theory of material uniformity [3], we can write

$$\mathcal{L} = \check{\mathcal{L}} \circ (\mathbf{F}, \mathcal{X}, \mathcal{T}) = \hat{\mathcal{L}} \circ (\mathbf{F}, \mathbf{K}) = -\hat{\Psi} \circ (\mathbf{F}, \mathbf{K}), \quad (8)$$

where  $\mathcal{X} : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{B}$  and  $\mathcal{T} : \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R}$  are auxiliary functions defined by  $\mathcal{X}(X, t) = X$  and  $\mathcal{T}(X, t) = t$ , and introduced to account for the explicit dependence of  $\check{\mathcal{L}}$  on material points and time [14], i.e.,

$$\mathcal{L}(X, t) = \check{\mathcal{L}}(\mathbf{F}(X, t), X, t) = -\hat{\Psi}(\mathbf{F}(X, t), \mathbf{K}(X, t)). \quad (9)$$

Equations (8) and (9) permit to determine the *time-like inhomogeneity force*,  $\mathcal{F}_0$  (see also [15], where it is referred to as “energy release rate”), which, recalling the definition  $\mathbf{\Lambda} := \mathbf{K}^{-1} \dot{\mathbf{K}}$ , reads

$$\begin{aligned} \mathcal{F}_0 &:= \frac{\partial \check{\mathcal{L}}}{\partial \mathcal{T}} \circ (\mathbf{F}, \mathcal{X}, \mathcal{T}) = - \left( \frac{\partial \hat{\Psi}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) \right) : \dot{\mathbf{K}} \\ &= - \mathbf{H} : \mathbf{\Lambda} = \mathcal{D}_{\text{IV}} - \mathcal{D}_{\text{nc}} - \Psi \text{tr}(\mathbf{\Lambda}). \end{aligned} \quad (10)$$

Thus,  $\tau$  is determined by Equation (1) with  $\mathcal{H} = \Psi$ .

### 2.2. Tensor $\mathbf{K}$ viewed as kinematic variable

A different approach to the mechanics of growth is provided in [4], where the structural transformation of a body corresponds to the activation of *structural degrees of freedom* describing the body's internal kinematics. From this perspective,  $\mathbf{K}$  and  $\dot{\mathbf{K}}$  acquire the meaning of tensor-valued kinematic descriptors that, together with  $\chi$ ,  $\mathbf{v} = \dot{\chi}$  and  $\text{Grad } \mathbf{v} = \dot{\mathbf{F}}$ , define the overall kinematics of the body.

Restricting our considerations to a material of first grade in  $\chi$  and zeroth grade in  $\mathbf{K}$  [4], it is natural to define the body's configuration manifold as a suitable set of pairs  $(\chi, \mathbf{K})$  describing the overall evolution of the body. Accordingly, the bundle of the body's virtual velocities is given by the set of triples  $(\mathbf{v}, \text{Grad } \mathbf{v}, \mathbf{Z})$  that represent all the admissible realisations of the generalised velocities associated with the “standard” motion, i.e.,  $\mathbf{v}$  and  $\text{Grad } \mathbf{v}$ , and with the structural evolution,  $\mathbf{Z}$ , respectively.

By duality, it is natural to introduce the generalised forces  
 expending virtual power on  $\mathbf{v}$ ,  $\text{Grad } \mathbf{v}$ , and  $\mathbf{Z}$ . Hence, the Principle  
 of Virtual Powers, specialised here to the case of no external  
 forces dual to  $\mathbf{v}$  (i.e., neither inertial nor body forces), reads

$$\int_{\mathcal{B}} \{ \mathbf{T} : \text{Grad } \mathbf{v} + \mathbf{K}^{-\text{T}} \mathbf{Y}_i : \mathbf{Z} \} = \int_{\mathcal{B}} \mathbf{K}^{-\text{T}} \mathbf{Y}_e : \mathbf{Z}, \quad (11)$$

where  $\mathbf{Y}_i$  and  $\mathbf{Y}_e$  are an internal and an external generalised force  
 dual to  $\mathbf{K}^{-1} \mathbf{Z}$ , respectively, and  $\mathbf{Z}$  is the virtual counterpart of  
 $\dot{\mathbf{K}}$ . The strong form of (11) consists of the force balances

$$\text{Div } \mathbf{T} = \mathbf{0}, \quad (12a)$$

$$\mathbf{Y}_i = \mathbf{Y}_e. \quad (12b)$$

To close the model, we prescribe  $\mathbf{Y}_i$  constitutively, in compliance  
 with the dissipation inequality

$$\mathcal{D}_{\text{KV}} = -\mathbf{H} : \mathbf{\Lambda} + \mathbf{Y}_i : \mathbf{\Lambda} = \mathbf{Y}_{\text{id}} : \mathbf{\Lambda} \geq 0, \quad (13)$$

where  $\mathbf{Y}_{\text{id}} := \mathbf{Y}_i - \mathbf{H}$  is said to be the *dissipative part* of  $\mathbf{Y}_i$  [16, 4]  
 and the subscript ‘‘KV’’ reminds that Equation (13) is obtained  
 by regarding  $\mathbf{K}$  as a kinematic variable.

In the sequel, we admit that  $\mathbf{Y}_{\text{id}}$  depends constitutively on  $\mathbf{F}$ ,  
 $\mathbf{K}$  and  $\dot{\mathbf{K}}$ , and, because of isotropy, we express such dependence  
 as a function  $\bar{\mathbf{Y}}_{\text{id}}$  of  $\mathbf{F}$ ,  $\mathbf{C}_{\mathbf{K}}$  and  $\dot{\mathbf{C}}_{\mathbf{K}}$ , i.e.,  $\mathbf{Y}_{\text{id}} = \bar{\mathbf{Y}}_{\text{id}} \circ (\mathbf{F}, \mathbf{C}_{\mathbf{K}}, \dot{\mathbf{C}}_{\mathbf{K}})$ .  
 Thus, we rewrite (12b) as

$$\mathbf{Y}_e - \bar{\mathbf{Y}}_{\text{id}} \circ (\mathbf{F}, \mathbf{C}_{\mathbf{K}}, \dot{\mathbf{C}}_{\mathbf{K}}) = \mathbf{H}, \quad (14)$$

thereby obtaining the equation of ‘‘motion’’ for  $\mathbf{K}$ . To supply  
 an explicit expression for  $\bar{\mathbf{Y}}_{\text{id}}$ , we rewrite it as a function of  $\mathbf{\Lambda}$ ,  
 i.e.,  $\bar{\mathbf{Y}}_{\text{id}} \circ (\mathbf{F}, \mathbf{C}_{\mathbf{K}}, \dot{\mathbf{C}}_{\mathbf{K}}) = \check{\mathbf{Y}}_{\text{id}} \circ (\mathbf{F}, \mathbf{K}, \mathbf{\Lambda})$ , and we notice that,  
 because of isotropy, the tensor  $\mathbf{Y}_e - \mathbf{Y}_{\text{id}}$  in Equation (14) must  
 have the same symmetry property as  $\mathbf{H}$ , i.e.,  $\mathbf{C}_{\mathbf{K}}^{-1} (\mathbf{Y}_e - \mathbf{Y}_{\text{id}}) =$   
 $(\mathbf{Y}_e^{\text{T}} - \mathbf{Y}_{\text{id}}^{\text{T}}) \mathbf{C}_{\mathbf{K}}^{-1}$ . Here, without much loss of generality, we  
 hypothesise that such property holds, independently, both for  
 $\mathbf{Y}_{\text{id}}$  and for  $\mathbf{Y}_e$ , and, by further assuming  $\check{\mathbf{Y}}_{\text{id}}$  to be linear in  $\mathbf{\Lambda}$ ,  
 we prescribe (cf. e.g. [17, 18] and references therein)

$$\mathbf{C}_{\mathbf{K}}^{-1} [\check{\mathbf{Y}}_{\text{id}} \circ (\mathbf{F}, \mathbf{K}, \mathbf{\Lambda})] = \mathbb{D} : \text{sym}(\mathbf{C}_{\mathbf{K}} \mathbf{\Lambda}), \quad (15)$$

where  $\mathbb{D}$  is a fourth-order tensor function given by

$$\mathbb{D} = 3J_{\mathbf{K}} d_{\text{v}} \mathbb{K}^{\sharp} + 2J_{\mathbf{K}} d_{\text{m}} \mathbb{M}^{\sharp}. \quad (16)$$

Here,  $d_{\text{v}}$  and  $d_{\text{m}}$  are scalar constitutive functions to be speci-  
 fied,  $\mathbb{K}^{\sharp}$  and  $\mathbb{M}^{\sharp}$  are defined as (analogous operators have been  
 introduced in [19, 17])

$$\mathbb{K}^{\sharp} = \frac{1}{3} \mathbf{C}_{\mathbf{K}}^{-1} \otimes \mathbf{C}_{\mathbf{K}}^{-1}, \quad (17a)$$

$$\mathbb{M}^{\sharp} = \frac{1}{2} [\mathbf{C}_{\mathbf{K}}^{-1} \underline{\otimes} \mathbf{C}_{\mathbf{K}}^{-1} + \mathbf{C}_{\mathbf{K}}^{-1} \bar{\otimes} \mathbf{C}_{\mathbf{K}}^{-1}] - \mathbb{K}^{\sharp}, \quad (17b)$$

and the tensor products ‘‘ $\underline{\otimes}$ ’’ and ‘‘ $\bar{\otimes}$ ’’ are defined in [20]. By  
 using the identity  $\text{sym}(\mathbf{C}_{\mathbf{K}} \mathbf{\Lambda}) = \frac{1}{2} \dot{\mathbf{C}}_{\mathbf{K}}$ , we find (cf. [21])

$$\mathbb{D} : \frac{1}{2} \dot{\mathbf{C}}_{\mathbf{K}} = \mathbf{C}_{\mathbf{K}}^{-1} [\mathbf{Y}_e - \mathbf{H}], \quad (18)$$

thereby supplying six independent differential equations in the  
 six independent components of  $\mathbf{C}_{\mathbf{K}}$ . Moreover, we split Equa-  
 tion (18) into the two independent equations

$$J_{\mathbf{K}} d_{\text{v}} \text{tr} \left( \frac{1}{2} \dot{\mathbf{C}}_{\mathbf{K}} \mathbf{C}_{\mathbf{K}}^{-1} \right) = \frac{1}{3} \text{tr } \mathbf{Y}_e - \frac{1}{3} \text{tr } \mathbf{H}, \quad (19a)$$

$$2J_{\mathbf{K}} d_{\text{m}} \text{dev} \left( \frac{1}{2} \dot{\mathbf{C}}_{\mathbf{K}} \mathbf{C}_{\mathbf{K}}^{-1} \right) = \text{dev } \mathbf{Y}_e - \text{dev } \mathbf{H}. \quad (19b)$$

Once the external force  $\mathbf{Y}_e$  is identified, and  $\mathbf{C}_{\mathbf{K}}$  is computed by  
 solving (18), the term  $\Gamma$  in the mass balance law (3) is determined  
 by  $\Gamma = \text{tr } \mathbf{\Lambda} = \frac{1}{2} \text{tr}(\dot{\mathbf{C}}_{\mathbf{K}} \mathbf{C}_{\mathbf{K}}^{-1})$ . Finally,  $\mathcal{F}_0$  becomes

$$\mathcal{F}_0 = -\mathbf{H} : \mathbf{\Lambda} = (\mathbf{Y}_{\text{id}} - \mathbf{Y}_e) : \mathbf{\Lambda}, \quad (20)$$

and the equation for  $\tau$  takes on the form

$$\Psi \dot{\tau} - (\mathbf{T}^{\text{T}} \mathbf{v}) \text{Grad } \tau + [(\mathbf{Y}_e - \mathbf{Y}_{\text{id}}) : \mathbf{\Lambda}] \tau = 0. \quad (21)$$

Before proceeding, we remark that Equation (15) is *not* the  
 most general constitutive law relating  $\mathbf{Y}_{\text{id}}$  with  $\mathbf{\Lambda}$ , or  $\dot{\mathbf{C}}_{\mathbf{K}}$ . The  
 main property of (15) is that, being invertible, if  $\mathbf{\Lambda}$  is null,  
 then  $\mathbf{Y}_{\text{id}}$  is null too, thereby implying  $\mathbf{Y}_e = \mathbf{H}$ . Moreover, due  
 to invertibility, it is true that, when  $\mathbf{Y}_{\text{id}}$  is null, also  $\mathbf{\Lambda}$  has to  
 vanish, which means that the balance between  $\mathbf{Y}_e$  and  $\mathbf{H}$  leads  
 to a stop of the growth process. However, in the case of a tumour,  
 this last result need not be true (see e.g. [10]), as it may well  
 happen that, if no nutrients are available for the tumour cells,  
 $\mathbf{\Lambda}$  vanishes also when  $\mathbf{Y}_{\text{id}}$  is not null, a situation that, according  
 to Equation (19a), requires  $d_{\text{v}}$  to diverge for finite values of  
 $\mathbf{Y}_{\text{id}} := \mathbf{Y}_e - \mathbf{H}$ .

### 3. A Noether-like framework

Equation (1) can be obtained by framing growth within a  
 Noether-like approach. To show this, we introduce the *action*

$$\mathcal{A} := \int_{\mathcal{B} \times \mathcal{I}} \mathcal{L},$$

where  $\mathcal{I} \subseteq [0, +\infty[$  is an interval of time, and the notation  
 $\int_{\mathcal{B} \times \mathcal{I}} f \equiv \int_{\mathcal{I}} \left\{ \int_{\mathcal{B}} f \, dV \right\} dt$  applies.

#### 3.1. $\mathbf{K}$ considered as internal variable: internal time

When  $\mathbf{K}$  is regarded as an internal variable, the Lagrangian  
 density function is defined in Equation (9), and the first-order  
 total variation of the action reads

$$D\mathcal{A} = \int_{\mathcal{B} \times \mathcal{I}} [\mathcal{E} \mathbf{h} + \text{Div}(-\mathbf{H}^{\text{T}} \mathbf{W} - \mathbf{T}^{\text{T}} \mathbf{u})], \quad (22)$$

where  $\mathbf{W}$  is a vector field, valued in the tangent bundle of  $\mathcal{B}$ , that  
 at each time  $t$  maps the points  $X$  of  $\mathcal{B}$  into  $\tilde{X} = X + \varepsilon \mathbf{W}(X, t)$ ,  
 with  $\varepsilon$  being a real smallness parameter,  $\mathbf{h}$  is the vector field  
 describing the variation of  $\chi$  when the points  $X$  are held fixed,  
 $\mathbf{u} := \mathbf{h} + \mathbf{F} \mathbf{W}$  is the vector field representing the *total variation* of  
 $\chi$ , and  $\mathcal{E} \mathbf{h} = \mathcal{E}_a h^a$  is the contraction of the co-vector field  $\mathcal{E} :=$   
 $\text{Div } \mathbf{T}$  with  $\mathbf{h}$  (see [14] for a derivation in a notation similar to  
 that adopted here). In addition, we denote by  $\mathcal{J} := -\mathbf{H}^{\text{T}} \mathbf{W} - \mathbf{T}^{\text{T}} \mathbf{u}$

207 *Noether's current density*, which is the sum of a fully material  
 208 current density,  $\mathcal{J}^{(m)} = -\mathbf{H}^T \mathbf{W}$ , and a “spatial” current density,  
 209  $\mathcal{J}^{(s)} = -\mathbf{T}^T \mathbf{u}$  (note that, although  $\mathcal{J}^{(s)}$  is a material field too, we  
 210 call it “spatial” because it is generated by the spatial vector field  
 211  $\mathbf{u}$ ).

212 Upon setting  $\mathbf{W} = \mathbf{0}$  in  $\mathcal{B}$  and  $\mathbf{h}|_{\partial\mathcal{B}} = \mathbf{u}|_{\partial\mathcal{B}} = \mathbf{0}$  for all  
 213 times, Hamilton's Principle of Stationary Action [22] requires  
 214  $D\mathcal{A} = 0$ , which leads to  $\mathcal{E} = \text{Div} \mathbf{T} = \mathbf{0}$  in  $\mathcal{B}$  and  $\mathbf{T} \cdot \mathbf{N} = \mathbf{0}$   
 215 on  $\partial_N \mathcal{B}$ , where  $\mathbf{N}$  is the field of unit vectors normal to the  
 216 Neumann boundary of  $\mathcal{B}$ ,  $\partial_N \mathcal{B}$ .

217 For  $\chi$  and  $\mathbf{K}$  satisfying  $\mathcal{E} = \mathbf{0}$ , we look at Equation (22)  
 218 under the light shed by Noether's Theorem [23]. Hence, we  
 219 search for conservation laws, and we obtain [9]

$$\text{Div} \mathcal{J}^{(s)} = -\mathbf{T} : \text{Grad} \mathbf{u}, \quad (23a)$$

$$\text{Div} \mathcal{J}^{(m)} = \mathcal{F} \mathbf{W} - \mathbf{H} : \text{Grad} \mathbf{W} =: \mathcal{N}(\mathbf{W}), \quad (23b)$$

220 where  $\mathcal{F} := \frac{\partial \hat{\mathcal{L}}}{\partial \chi} \circ (\mathbf{F}, \chi, \mathcal{T}) = -[\frac{\partial \hat{\Psi}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K})] : \text{Grad} \mathbf{K}$  is  
 221 referred to as “*material inhomogeneity force*” [24, 25, 7] and  
 222  $\mathcal{F} \mathbf{W} = \mathcal{F}_A W^A$ . We remark that, more generally, the integrand in  
 223 Equation (22) should feature a summand consisting of the diver-  
 224 gence of a vector field independent of  $\mathbf{F}$ , and descending from  
 225 the so-called “*divergence transformation*” of the Lagrangian  
 226 density function [23, 7, 14]. However, as in [23], this summand  
 227 can be omitted for the type of symmetries addressed here.

228 In Equation (23a),  $\mathbf{T} : \text{Grad} \mathbf{u}$  vanishes identically in three  
 229 cases: when  $\mathbf{u}$  is null, when  $\mathbf{u}$  represents a uniform translation,  
 230 or when  $\mathbf{u}$  takes on the form  $\mathbf{u} = \mathbf{g}^{-1} \boldsymbol{\omega} [\chi - x_0]$ , where  $\boldsymbol{\omega}$  is a  
 231 uniform skew-symmetric tensor,  $x_0$  is a fixed point of space and  
 232  $\mathbf{g}^{-1}$  is the inverse of the spatial metric tensor,  $\mathbf{g}$ . The second  
 233 case is consistent with the fact that  $\hat{\mathcal{L}}$  is independent of  $\chi$ , so  
 234 that the system is invariant under translations in space and, thus,  
 235 linear momentum is conserved. The third case, instead, stems  
 236 from the symmetry of  $\mathbf{g}^{-1} \mathbf{T} \mathbf{F} \mathbf{F}^T$ , which ensures  $\mathbf{T} : \text{Grad} \mathbf{u} =$   
 237  $(\mathbf{g}^{-1} \mathbf{T} \mathbf{F} \mathbf{F}^T) : \boldsymbol{\omega} = 0$  and is equivalent to the conservation of  
 238 angular momentum. In conclusion, for the mentioned choices  
 239 of  $\mathbf{u}$ ,  $\text{Div} \mathcal{J}^{(s)}$  is zero, which implies that  $\mathcal{J}^{(s)}$  is conserved.

240 We turn now to Equation (23b), and we notice that it is ob-  
 241 tained by using the relation  $-\text{Div} \mathbf{H} = \mathcal{F}$ . This result follows  
 242 from the computation of the divergence of  $\mathbf{H}$ , and characterises  
 243 the *fully material* force balance describing the “*inverse dynam-*  
 244 *ics*” of the body [7, 3]. It stipulates that the “spatial” part of the  
 245 body's energy-momentum tensor,  $-\mathbf{H}$ , is not conserved. This  
 246 is a manifestation of the symmetry breaking due to the material  
 247 inhomogeneity of the body, reflected by  $\mathcal{N}(\mathbf{W})$ . This quantity  
 248 plays the role of an *effective* source term for  $\mathcal{J}^{(m)}$  [9] and is such  
 249 that the variation of the action becomes  $D\mathcal{A} = \int_{\mathcal{B} \times \mathcal{I}} \mathcal{N}(\mathbf{W})$ .  
 250 Therefore, in order to search for the class of fields  $\mathbf{W}$  such that  
 251  $\mathcal{J}^{(m)}$  is conserved and the action is invariant, i.e.,  $D\mathcal{A} = 0$ , one  
 252 has to impose [9]

$$\begin{aligned} \mathcal{N}(\mathbf{W}) &= -\mathbf{H} : \text{Grad} \mathbf{W} + \mathcal{F} \mathbf{W} \\ &= -\mathbf{H} : [\text{Grad} \mathbf{W} + (\mathbf{K}^{-1} \text{Grad} \mathbf{K}) \mathbf{W}] = 0. \end{aligned} \quad (24)$$

253 We remark that relations of the type (24) are sometimes referred  
 254 to as “*Noetherian identities*” [26].

255 Apart from the trivial solution  $\mathbf{W} = \mathbf{0}$ , a uniform field  $\mathbf{W}$  does  
 256 not generally satisfy Equation (24) and, thus, the action is not  
 257 invariant under uniform translations of the material points. This  
 258 result is another evidence of the symmetry breaking emerging  
 259 because of  $\mathcal{F}$ . Clearly, if  $\mathbf{K}$  is uniform, so that  $\text{Grad} \mathbf{K} = \mathbf{0}$ ,  
 260 then  $\mathbf{W}$  can be uniform too. When this occurs,  $\mathcal{F}$  vanishes  
 261 identically and, in the jargon of [7], one obtains the conservation  
 262 of “*canonical pseudo-momentum*”. Let us now look at the  
 263 identity

$$\dot{\Psi} - \text{Div}(\mathbf{T}^T \mathbf{v}) = -\mathcal{F}_0, \quad (25)$$

264 which is the *non-conservation* of energy for  $\mathcal{H} = \Psi = -\mathcal{L}$   
 265 (i.e., in the quasi-static limit), and let us multiply Equation (25)  
 266 by a scalar field  $\tau : \mathcal{B} \times \mathcal{I} \rightarrow \mathbb{R}$  describing a point- and  
 267 time-dependent *deformation* of time [9]. Then, recalling the  
 268 definition of  $\mathcal{F}_0$  given in (10), we find (cf. [7])

$$\begin{aligned} \dot{\Psi} \tau + \text{Div}(-\mathbf{T}^T \mathbf{v} \tau) \\ = \Psi \dot{\tau} - (\mathbf{T}^T \mathbf{v}) \text{Grad} \tau + (\mathbf{H} : \boldsymbol{\Lambda}) \tau =: \mathcal{N}_0(\tau). \end{aligned} \quad (26)$$

269 By analogy with Equation (23b), we call  $\mathcal{N}_0(\tau)$  *effective source*  
 270 of Noether's energy current density, defined by the time-like  
 271 component  $\Psi \tau$  and the flux vector  $-\mathbf{T}^T \mathbf{v} \tau$ . As noticed for  $\mathcal{N}(\mathbf{W})$ ,  
 272 the presence of  $\mathcal{F}_0 = -\mathbf{H} : \boldsymbol{\Lambda}$  implies that  $\mathcal{N}_0(\tau)$  does not vanish  
 273 for nonzero constant fields  $\tau$ . Hence, to conserve Noether's  
 274 energy current density, we enforce the condition anticipated by  
 275 Equation (1), i.e.,

$$\mathcal{N}_0(\tau) = \Psi \dot{\tau} - (\mathbf{T}^T \mathbf{v}) \text{Grad} \tau + (\mathbf{H} : \boldsymbol{\Lambda}) \tau = 0, \quad (27)$$

276 in which  $\mathbf{H} : \boldsymbol{\Lambda}$  is now regarded as the generator of  $\tau$ .

### 3.2. $\mathbf{K}$ considered as a kinematic variable: internal time

277 Equations (6), (8) and (14) allow to rephrase the force bal-  
 278 ances (12a) and (12b) as

$$\text{Div} \mathbf{T} \equiv -\text{Div} \left( \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{F}} \circ (\mathbf{F}, \mathbf{K}) \right) = \mathbf{0}, \quad (28a)$$

$$-\mathbf{H} \equiv \mathbf{K}^T \left( \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) \right) = \mathbf{Y}_{\text{id}} - \mathbf{Y}_{\text{e}}. \quad (28b)$$

280 Looking at (28b), we notice that a relevant case occurs when  
 281 there exists a potential  $\hat{\mathcal{U}} = \hat{\mathcal{U}} \circ (\mathbf{F}, \mathbf{K})$  such that

$$\frac{\partial \hat{\mathcal{U}}}{\partial \mathbf{F}} \circ (\mathbf{F}, \mathbf{K}) = \mathbf{0}, \quad \frac{\partial \hat{\mathcal{U}}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) = \mathbf{K}^{-T} \mathbf{Y}_{\text{e}}, \quad (29)$$

282 where the first requirement of Equation (29) prevents  $\hat{\mathcal{U}}$  from  
 283 introducing an unphysical contribution to  $\mathbf{T}$ . Thus, Eqs. (28a)  
 284 and (28b) become

$$-\text{Div} \left( \frac{\partial \hat{\mathcal{L}}_{\text{eff}}}{\partial \mathbf{F}} \circ (\mathbf{F}, \mathbf{K}) \right) = \mathbf{0}, \quad (30a)$$

$$\mathbf{K}^T \left( \frac{\partial \hat{\mathcal{L}}_{\text{eff}}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) \right) = \mathbf{Y}_{\text{id}}, \quad (30b)$$

285 with  $\mathcal{L}_{\text{eff}} := \mathcal{L} + \mathcal{U}$  being referred to as *effective Lagrangian*  
 286 *density function*. Note that, although Equation (29) may be too  
 287 restrictive for biologically meaningful situations, it is possible  
 288 to think of  $\mathbf{Y}_e$  as the sum of an integrable and a non-integrable  
 289 force, with the former one admitting a potential like  $\hat{\mathcal{U}}$ . For  
 290 this reason, in this work we concentrate on the limiting case in  
 291 which  $\mathbf{Y}_e$  is integrable.

292 By defining the effective action,  $\mathcal{A}_{\text{eff}} = \int_{\mathcal{B} \times \mathcal{F}} \mathcal{L}_{\text{eff}}$ , the first-  
 293 order total variation of  $\mathcal{A}_{\text{eff}}$  is given by

$$D\mathcal{A}_{\text{eff}} = \int_{\mathcal{B} \times \mathcal{F}} \left[ \mathbf{K}^{-\text{T}} \mathbf{Y}_{\text{id}} : \mathbf{Y} + \text{Div}(\mathcal{J}^{(s)} + \mathcal{J}_{\text{eff}}^{(m)}) \right], \quad (31)$$

294 with  $\mathcal{J}^{(s)} = -\mathbf{T}^{\text{T}} \mathbf{u}$ ,  $\mathcal{J}_{\text{eff}}^{(m)} = -\mathbf{H}_{\text{eff}}^{\text{T}} \mathbf{W}$ , the effective Eshelby stress  
 295 tensor  $\mathbf{H}_{\text{eff}} = -(\mathcal{L}_{\text{eff}} \mathbf{I}^{\text{T}} + \mathbf{F}^{\text{T}} \mathbf{T})$  and  $\mathbf{Y}$  being the variation of  $\mathbf{K}$   
 296 when the points  $X$  are ‘‘held fixed’’.

297 Upon taking  $\text{Div} \mathcal{J}^{(s)} = 0$ , as done in Section 3.1, a direct  
 298 calculation yields

$$\text{Div} \mathcal{J}_{\text{eff}}^{(m)} = \mathcal{F}_{\text{eff}} \mathbf{W} - \mathbf{H}_{\text{eff}} : \text{Grad} \mathbf{W}, \quad (32)$$

299 where we call  $\mathcal{F}_{\text{eff}} := (\mathbf{K}^{-\text{T}} \mathbf{Y}_{\text{id}} : \text{Grad} \mathbf{K})$  *effective inhomogene-*  
 300 *ity force*, and Equation (31) reduces to

$$D\mathcal{A}_{\text{eff}} = \int_{\mathcal{B} \times \mathcal{F}} [\mathbf{K}^{-\text{T}} \mathbf{Y}_{\text{id}} : \mathbf{Q} - \mathbf{H}_{\text{eff}} : \text{Grad} \mathbf{W}], \quad (33)$$

301 with  $\mathbf{Q} := \mathbf{Y} + (\text{Grad} \mathbf{K}) \mathbf{W}$  being the total variation of  $\mathbf{K}$ . If we  
 302 set  $\mathbf{W} = \mathbf{0}$ , Equation (33) returns Rayleigh-Hamilton Principle  
 303 [22, 27], which states that the first-order variation of the action  
 304 is equal to the integral of the work  $\mathbf{K}^{-\text{T}} \mathbf{Y}_{\text{id}} : \mathbf{Q}$ . Thus, if we  
 305 reinterpret Equation (33) on the basis of this result, we find that  
 306 the class of fields  $\mathbf{W}$  satisfying  $D\mathcal{A}_{\text{eff}} = \int_{\mathcal{B} \times \mathcal{F}} \mathbf{K}^{-\text{T}} \mathbf{Y}_{\text{id}} : \mathbf{Q}$   
 307 is given by all the solutions of the equation

$$-\mathbf{H}_{\text{eff}} : \text{Grad} \mathbf{W} = 0. \quad (34)$$

308 In contrast to (24), Equation (34) is satisfied by nontrivial uni-  
 309 form fields  $\mathbf{W}$ . To see the implications of this result, let us  
 310 consider the situation in which  $\mathbf{Y}_{\text{id}}$  is null. Hence, it follows  
 311 that  $\mathbf{H} = \mathbf{Y}_e$ ,  $\text{Div} \mathcal{J}_{\text{eff}}^{(m)} = -\mathbf{H}_{\text{eff}} : \text{Grad} \mathbf{W}$ , and Equation (33)  
 312 becomes  $D\mathcal{A}_{\text{eff}} = \int_{\mathcal{B} \times \mathcal{F}} [-\mathbf{H}_{\text{eff}} : \text{Grad} \mathbf{W}]$ . In this case, uni-  
 313 form fields  $\mathbf{W}$  leave the action invariant, i.e.,  $D\mathcal{A}_{\text{eff}} = 0$ , and  
 314 represent symmetry transformations. This constitutes a symme-  
 315 try restoration and is due to the fact that, since  $\mathbf{Y}_{\text{id}}$  is null,  $\mathbf{H}$   
 316 is entirely ‘‘balanced’’ by  $\mathbf{Y}_e$ , which plays the role of compensating  
 317 field. In fact, this results follows from Equation (30b), which,  
 318 for  $\mathbf{Y}_{\text{id}} = \mathbf{0}$ , implies  $\mathcal{F}_{\text{eff}} := \left( \frac{\partial \hat{\mathcal{L}}_{\text{eff}}}{\partial \mathbf{K}} \circ (\mathbf{F}, \mathbf{K}) \right) : \text{Grad} \mathbf{K} = \mathbf{0}$  even  
 319 though it holds that  $\mathcal{F} = -\mathbf{H} : \mathbf{K}^{-1} \text{Grad} \mathbf{K} \neq \mathbf{0}$ .

320 As done in Section 3.1, we consider the identity

$$\dot{\Psi}_{\text{eff}} - \text{Div}(\mathbf{T}^{\text{T}} \mathbf{v}) = -\mathbf{Y}_{\text{id}} : \mathbf{\Lambda} = -\mathcal{D}_{\text{KV}}, \quad (35)$$

321 where  $\Psi_{\text{eff}} := -\mathcal{L}_{\text{eff}}$  denotes the effective energy density asso-  
 322 ciated with the body and, by multiplying (35) by  $\tau$ , we obtain

$$\begin{aligned} & \dot{\overline{\Psi}}_{\text{eff}} \tau + \text{Div}(-\mathbf{T}^{\text{T}} \mathbf{v} \tau) \\ &= \Psi_{\text{eff}} \dot{\tau} - (\mathbf{T}^{\text{T}} \mathbf{v}) \text{Grad} \tau - \mathcal{D}_{\text{KV}} \tau =: \mathcal{N}_{0\text{eff}}(\tau). \end{aligned} \quad (36)$$

Equation (35) describes the non-conservation of  $\Psi_{\text{eff}}$ , while  
 Equation (36) defines  $\mathcal{N}_{0\text{eff}}(\tau)$  as the effective source of Noether’s  
 energy current density with time-like component  $\Psi_{\text{eff}} \dot{\tau}$  and flux  
 vector  $-\mathbf{T}^{\text{T}} \mathbf{v} \tau$ . Hence, to conserve Noether’s energy current  
 density, the condition

$$\mathcal{N}_{0\text{eff}}(\tau) = \Psi_{\text{eff}} \dot{\tau} - (\mathbf{T}^{\text{T}} \mathbf{v}) \text{Grad} \tau - \mathcal{D}_{\text{KV}} \tau = 0 \quad (37)$$

has to be imposed. Equation (37) prescribes that  $\mathcal{D}_{\text{KV}}$  is the  
 generator of  $\tau$ . Therefore,  $\mathcal{D}_{\text{KV}}$  can be thought of as an *effective*  
*time-like inhomogeneity force*, i.e.,  $\mathcal{F}_{0\text{eff}} := \mathcal{D}_{\text{KV}}$ , which van-  
 ishes in the non-dissipative limit. If this is the case, a constant  
 field  $\tau$  satisfies  $\mathcal{N}_{0\text{eff}}(\tau) = 0$  and, consequently, Eq. (36) and  
 (37) is satisfied as a conservation law. This is a crucial differ-  
 ence with Equations (21) and (27), in which the generator of  $\tau$   
 is given by  $-\mathcal{F}_0 = \mathbf{Y}_e - \mathbf{Y}_{\text{id}} = \mathbf{H} : \mathbf{\Lambda}$  and need not vanish even  
 when the dissipation is zero.

#### 4. A proof of concept

To supply a proof of concept of the theory discussed so  
 far, we take a benchmark problem from [10]. Specifically, we  
 study a tumour modelled as a monophasic, isotropic, solid body  
 of cylindrical shape, confined by an undeformable lateral wall,  
 and allowed to expand uniformly along its axial direction, with  
 traction-free terminal cross sections. Moreover, we assume the  
 growth tensor,  $\mathbf{K}$ , to be spherical. By using cylindrical coordi-  
 nates, these hypotheses imply that the only nonzero component  
 of the velocity,  $\mathbf{v}$ , is the axial one,  $v^z$ , and that  $\mathbf{F}$ ,  $\mathbf{K}$ ,  $\mathbf{\Lambda} = \mathbf{K}^{-1} \dot{\mathbf{K}}$ ,  
 $\mathbf{T}$  and  $\mathbf{H}$  admit the diagonal matrix representations

$$[\mathbf{F}] = \text{diag}\{1, 1, \mathfrak{f}\}, \quad (38a)$$

$$[\mathbf{K}] = k \text{diag}\{1, 1, 1\}, \quad (38b)$$

$$[\mathbf{\Lambda}] = k^{-1} \dot{k} \text{diag}\{1, 1, 1\}, \quad (38c)$$

$$[\mathbf{T}] = \text{diag}\{T_r^R, T_\varphi^\Phi, T_z^Z\}, \quad (38d)$$

$$[\mathbf{H}] = \text{diag}\{\Psi - T_r^R, \Psi - T_\varphi^\Phi, \Psi - \mathfrak{f} T_z^Z\}. \quad (38e)$$

We remark that, since  $\text{Div} \mathbf{T} = \mathbf{0}$  reduces to  $\partial T_z^Z / \partial Z = 0$ , and  
 the terminal cross sections of the body are free of tractions [10],  
 $T_z^Z$  is zero at all the points of the tumour. This implies that  
 the energy flux  $\mathbf{T}^{\text{T}} \mathbf{v}$  vanishes identically, i.e.,  $\mathbf{T}^{\text{T}} \mathbf{v} = T_z^Z v^z = 0$ .  
 Moreover, as in [10], we adopt the Blatz-Ko strain energy density

$$\Psi = J_{\mathbf{K}} \frac{1}{2} \mu \left[ (I_1 - 3) - \frac{1}{q/2} (I_3^{q/2} - 1) \right], \quad (39)$$

with  $I_1 = \text{tr}(\mathbf{C} \mathbf{C}^{-1})$ ,  $I_3 = J_{\mathbf{K}}^{-2} \det \mathbf{C}$  and material constants  $\mu > 0$   
 and  $q < 0$ . Due to Equation (39), the constitutive expression of  
 $T_z^Z$  is such that [10]

$$T_z^Z = \mu \frac{k^3}{\mathfrak{f}} \left[ \frac{\mathfrak{f}^2}{k^2} - \left( \frac{\mathfrak{f}}{k^3} \right)^q \right] = 0 \Rightarrow \mathfrak{f} = k \frac{2-3q}{2-q}. \quad (40)$$

Therefore, any constitutive function of  $\mathfrak{f}$  and  $k$  can be rephrased  
 as a function of  $k$  alone. For, example, in the case of Eshelby  
 stress, one has  $\mathbf{H} = \hat{\mathbf{H}}(\mathfrak{f}, k) \equiv \mathfrak{H}(k)$  and

$$\mathfrak{H}(k) := \frac{1}{3} \text{tr} \mathfrak{H}(k) = \Psi - \frac{1}{3} (T_r^R + T_\varphi^\Phi) = \frac{1}{3} \text{tr} \mathbf{H}. \quad (41)$$

359 First, we consider the case in which  $\mathbf{K}$  is an internal variable  
 360 [3] and we refer to this model as “IV Model”. We notice  
 361 that, in order to recover the growth law proposed in [10] from  
 362 Equation (5), we have to set  $\beta_n = 0$ , for  $n \neq 0$ , thereby obtaining

$$\dot{k} = \beta_0 k, \quad \beta_0 = \frac{1}{3}\Gamma, \quad (42)$$

363 where, in general,  $\beta_0$  depends on mechanical stress through the  
 364 principal invariants of  $\mathbf{H}$ . However, if  $\beta_0$  is assumed to be a  
 365 positive constant, and if the initial distribution of  $k$ , denoted by  
 366  $k_{\text{in}}$ , is independent of material points,  $k$  is uniform and increases  
 367 exponentially in time [10], i.e.,  $k(t) = k_{\text{in}} \exp(\beta_0 t)$  (see the line  
 368 marked with triangles, and referred to as “IV Model”, in Fig. 1).  
 369 Moreover, according to Equation (40), also  $\mathfrak{f}$  is independent of  
 370 material points. In the case under study, the material inhomogeneity  
 371 force  $\mathcal{F}$  is null, so that uniform fields  $\mathbf{W} = \mathbf{W}_0$  satisfy  
 372 Equation (24) and, since the identity  $\mathbf{H} : \mathbf{\Lambda} = \Psi$  holds true,  
 373 Equation (27) becomes

$$\mathcal{N}_0(\tau) = \Psi \dot{\tau} + \dot{\Psi} \tau = \frac{d}{dt}(\Psi \tau) = 0. \quad (43)$$

374 Coherently with Equation (26), this result implies that the time-  
 375 like component of Noether’s current density,  $\Psi \tau$ , is conserved,  
 376 and the internal time is given by

$$\Psi(t)\tau(t) = \Psi_0 \tau_0 \Rightarrow \tau(t) = \frac{\tau_0 \Psi_0}{\Psi(t)}, \quad (44)$$

377 where  $\Psi_0$  and  $\tau_0$  are reference constant values, and  $\Psi(t)$  is  
 378 rescaled so that  $\Psi(0) = \Psi_0$ . The trend of  $\tau$  is reported in Fig. 2  
 379 and corresponds to the solid line marked with triangles and  
 380 referred to as “ $\tau/\tau_0$  IV Model”. The product  $\Psi_0 \tau_0$  defines the  
 381 negative of a reference value of the action, i.e.,  $\mathcal{A}_0 := -\Psi_0 \tau_0$ ,  
 382 which is invariant.

383 Now, we regard  $\mathbf{K}$  as a kinematic variable [4] and we call  
 384 this model “KV Model”. In this case, the evolution of  $k$  is given  
 385 by Equations (19a) and (19b), which yield

$$\frac{\dot{k}}{k} = \frac{1}{3k^3 d_v} [Y_e - \mathfrak{S}(k)], \quad (45)$$

386 with  $Y_e := \frac{1}{3} \text{tr} \mathbf{Y}_e$  and  $\text{dev} \mathbf{Y}_e = \mathbf{0}$ . Within the present variational  
 387 setting, we choose a constant  $Y_e$ , so that it can be obtained by  
 388 differentiation of the potential  $\dot{\mathcal{U}} \circ (\mathbf{F}, \mathbf{K}) = Y_e \ln(\det \mathbf{K})$ , and  
 389 the numerical solution of Equation (45), obtained for constant  
 390  $d_v$ , is reported in Fig. 1 (see the solid line marked with open  
 391 circles and referred to as “KV Model - Linear Case”).

392 Since it holds true that  $\mathbf{T}^T \mathbf{v} = \mathbf{0}$ , Equation (35) prescribes  
 393  $\mathcal{D}_{\text{KV}} = -\dot{\Psi}_{\text{eff}}$  and, consequently, Equation (37) becomes

$$\mathcal{N}_{0\text{eff}} = \Psi_{\text{eff}} \dot{\tau} + \dot{\Psi}_{\text{eff}} \tau = \frac{d}{dt}(\Psi_{\text{eff}} \tau) = 0. \quad (46)$$

394 Therefore, the internal time,  $\tau$ , is given by

$$\Psi_{\text{eff}}(t)\tau(t) = \Psi_{\text{eff}0} \tau_0 \Rightarrow \tau(t) = \frac{\tau_0 \Psi_{\text{eff}0}}{\Psi_{\text{eff}}(t)}, \quad (47)$$

395 with  $\tau_0$  and  $\Psi_{\text{eff}0}$  being reference constants, and  $\Psi_{\text{eff}}(t)$  rescaled  
 396 so that  $\Psi_{\text{eff}}(0) = \Psi_{\text{eff}0}$ . In spite of the similarity with Equation  
 397 (44), in the present case  $\tau(t)$  depends on  $Y_e$ . Its evolution  
 398 is shown in Fig. 2 and corresponds to the solid line marked with  
 399 open circles.

## 5. Discussion

400 In the IV Model, the coefficient  $\beta_0$  in Equation (42) is assumed  
 401 to be constant. Although this choice may be too restrictive,  
 402 it describes the limit case in which, to activate growth, it  
 403 is sufficient that the nutrient substances in the tumour exceed a  
 404 certain threshold. Clearly, more general models, which include  
 405 the feedback of stress on growth (mechanotransduction), can be  
 406 obtained by considering Equation (5) in full, or by expressing  
 407  $\beta_0$  as a phenomenological function of the stress.

408 In the KV Model, which descends from Equation (15), (19a)  
 409 and (45),  $k$  is coupled with  $Y_{\text{id}} := \frac{1}{3} \text{tr} \mathbf{Y}_{\text{id}} = Y_e - \mathfrak{S}(k)$ , rather than  
 410 with stress alone, and this coupling may appear both directly,  
 411 i.e., in the right-hand-side of Equation (45), and indirectly, i.e.,  
 412 through the coefficient  $d_v$ , which can be taken as a function of the  
 413 principal invariants of  $\mathbf{Y}_{\text{id}}$ . To the best of our understanding, this  
 414 could be a possible interpretation of the “Eshelbian coupling”  
 415 mentioned in [4]. In this respect, we also notice that, even within  
 416 our variational setting, mechanotransduction can be accounted  
 417 for by suitably interpreting  $Y_e$ . This can be achieved by relating  
 418  $\dot{k}/k$  to a term of the type [28, 29]

$$M(H) := 1 - \frac{c_0 H}{c_0 Y_e + H} = 1 - \frac{H}{Y_e} + o\left(\frac{H}{Y_e}\right), \quad (48)$$

420 where  $c_0 \in ]0, 1[$  is a model parameter and  $H := \frac{1}{3} \text{tr} \mathbf{H} = \mathfrak{S}(k)$ .  
 421 By setting  $M_{\text{lin}}(H) := 1 - H/Y_e$ , Equation (45) can be rewritten  
 422 as  $\dot{k}/k = M_{\text{lin}}(H)/3k^3 \bar{\tau}$ , where  $\bar{\tau}$  is a characteristic time scale  
 423 and  $d_v \equiv \bar{\tau} Y_e$ . The solution to this equation, or, equivalently,  
 424 to Equation (45), corresponds to the solid line marked with  
 425 open circles in Fig. 1, where it is compared with the solution  
 426 to the equation  $\dot{k}/k = M(H)/3k^3 \bar{\tau}$ . The latter is represented  
 427 by the solid line marked with triangles in Fig. 1, and refers to  
 428 a phenomenological model in which the mechanotransduction  
 429 term,  $M(H)$ , is not linearised. Looking at the magnified inset  
 430 in Fig. 1, we notice that a constant and integrable  $Y_e$ , although  
 431 being restrictive, leads to reasonable results for the first days in  
 432 which the tumour grows, i.e., as long as the ratio  $H/Y_e$  remains  
 433 sufficiently small. For longer times, however, the solution to  
 434 Equation (45) ceases to be acceptable. Indeed, it tends towards  
 435 a stationary value, corresponding to the force balance  $Y_e = \mathfrak{S}(k)$ ,  
 436 which contradicts the hypothesis  $H/Y_e \rightarrow 0$ . The solution of  
 437 the nonlinear model, instead, keeps increasing in time, and is  
 438 qualitatively closer to the dashed curve marked with open circles  
 439 that describes the trend of  $k$  in the case of a reference model  
 440 available in the literature [29].

441 The main result of this work is the introduction of the internal  
 442 time,  $\tau$ , that, for the considered benchmark problem, is obtained  
 443 by solving Equation (44) for the IV Model and Equation (47) for  
 444 the KV Model. The solutions, expressed in terms of the ratio  
 445  $\tau/\tau_0$ , are reported in Fig. 2 and correspond to the solid lines  
 446 marked with asterisks and open circles, respectively. We notice  
 447 that, since both  $\Psi$  and  $\Psi_{\text{eff}}$  increase with  $k$ , and since  $k$  increases  
 448 with time,  $\tau/\tau_0$  decreases monotonically for both models. In  
 449 particular, since  $k$  is computed by solving Equation (45), which  
 450 admits a stationary solution,  $\tau/\tau_0$  reaches a plateau for long  
 451 times, and the solution predicted by the IV Model tends to con-  
 452 verge to the one supplied by the KV Model. Finally, we notice

453 that the function  $\tau_c = 1 - \tau(t)/\tau_0$  is monotonically increasing,  
 454 and might thus be taken as a natural characteristic time scale of  
 455 growth, just as the endochronic time in Plasticity [6].

## 456 6. Conclusions

457 In this work, we have studied a problem of volumetric growth  
 458 in a continuum body within the quasi-static limit. In doing this,  
 459 we have followed two paths: the one that views the growth tensor,  
 460  $\mathbf{K}$ , as an internal variable, and the one that defines  $\mathbf{K}$  as a  
 461 kinematic variable. We have cast the problem in a variational  
 462 setting and we have employed the framework of Noether's Theorem  
 463 in order to reveal some subtle implications of the two theories of  
 464 growth exploited in the manuscript, especially in terms of material  
 465 inhomogeneities and conservation laws. Hence, we have shown that  
 466 Noether's current is not conserved, in general, for the classes of  
 467 transformations that would represent material symmetries if the body  
 468 were homogeneous. This has been reflected, in fact, by the condition  
 469  $\mathcal{N}(\mathbf{W}) = 0$ , imposed to annihilate the effective source of  
 470 Noether's current [9].

471 We have focussed on the non-conservation of energy. This has led  
 472 us to adopt the conditions  $\mathcal{N}_0(\tau) = 0$  and  $\mathcal{N}_{0\text{eff}}(\tau) = 0$ ,  
 473 respectively, to search for transformations capable of defining a  
 474 characteristic time scale for growth, termed *internal time*.

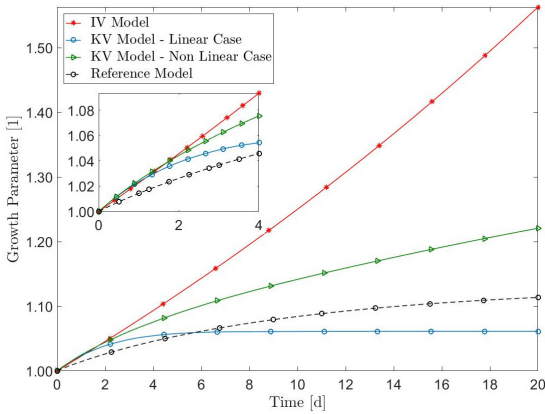


Figure 1: Time evolution of  $k$ . The model parameters are  $\Gamma = 2.68 \cdot 10^{-2} \text{ s}^{-1}$ , for the IV-model, and  $c_0 = 0.7138$ ,  $Y_c = 2.159 \text{ kPa}$  and  $\bar{\tau} = 10^6 \text{ s}$ , for the KV-model. For both models, we set  $\mu = 1.999 \text{ kPa}$ ,  $q = -1$ ,  $k_{\text{in}} = 1$ .

## 475 Conflict of interests

476 The Authors declare no conflict of interests.

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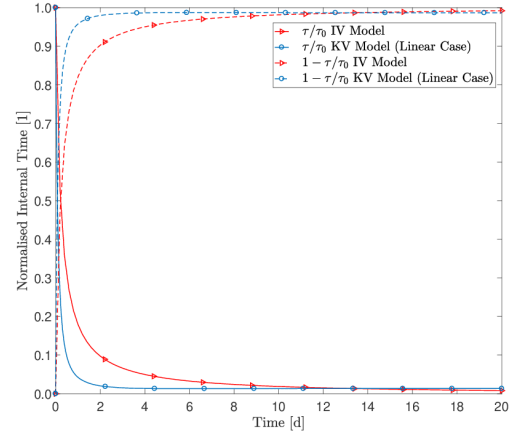


Figure 2: Time evolution of  $\tau$ . The values of the model parameters are declared in the caption of Fig. 1.

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