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Growth and remodelling from the perspective of Noether's Theorem

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Abstract

Starting from the observation that the growth of a body breaks the time translation symmetry of the body's dynamics, we determine a scalar field, called *internal time*, that defines an indicator of the intrinsic time scale of the growth-related body's structural evolution. By recasting the theory of growth for monophasic media within a variational framework, we obtain the internal time as the solution of a partial differential equation descending from Noether's Theorem. We do this by considering two approaches, one formulated in terms of internal variables and one adopting the concept of augmented kinematics.

Keywords: growth; remodelling; variational methods; Noether's Theorem; endochronic theory

1. Introduction

The mechanics of volumetric growth studies the variation of mass and the concomitant structural evolution of biological tissues [1, 2, 3]. Such processes are often conceived as anelastic, and are described by a generally non-integrable tensor field, $\boldsymbol{K}$, referred to as *growth tensor*.

The role of $\boldsymbol{K}$ in the modelling of growth is not unique, and its interpretation depends on the theory within which it is introduced. To the best of our knowledge, there exist at least two ways of interpreting $\boldsymbol{K}$: it can be viewed either as an *internal* variable (see e.g. [3]) or as a *kinematic* variable (see e.g. [4]). The conceptual difference between these two approaches affects all the relations governing the dynamics of a body, especially the one representing the evolution of its internal structure.

The way in which the dissipation is studied in [3] and [4] plays a major role in this work. In the sequel, indeed, we employ the dissipation inequality to show that a growing body possesses an intrinsic time scale, defined by the chosen theory. To this end, we take inspiration from Vakulenko's concept of *"endochronic thermodynamics"* [5, 6], and we demonstrate that the body's intrinsic time scale is related to a generalised force, hereafter denoted by $\mathcal{F}_0$ and termed *time-like inhomogeneity force* [7]. In our framework, $\mathcal{F}_0$ plays a role similar to that played by the material inhomogeneity forces in Eshelby's theory of inclusions [8] and, more generally, in the mechanics of materials with inhomogeneities [7], as is the case of growing media [3].

Vakulenko's theory addresses the thermodynamics of anelastic processes [5, 6], and is said to be *"endochronic"* since it associates a given anelastic process with a scalar-valued function, the *"thermodynamic time"*, defined from the outset as the time integral of a suitable function of the entropy production [6].

Quite differently, in our work we identify the *internal time* of growth of a body, hereafter denoted by τ, with the solution of the partial differential equation [9]

$$\mathcal{N}_0(\tau) := \mathcal{H}\,\dot{\tau} - (\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v})\,\mathrm{Grad}\,\tau - \mathcal{F}_0\,\tau = 0, \tag{1}$$

where $\mathcal{H}$ is the body's total energy density, $\boldsymbol{T}$ is the first Piola-Kirchhoff stress tensor and $\boldsymbol{v}$ is the Lagrangian velocity field.

Equation (1) was deduced in [9] as a consequence of Noether's Theorem, and τ was defined as a deformation of time depending on material points and on time itself. More specifically, τ was introduced to highlight how the occurrence of growth in a body is a symmetry breaking, spoiling the invariance of the body's dynamics under time translations and yielding the failure of the conservation of energy [9]. This symmetry breaking results in the arising of $\mathcal{F}_0$ and manifests itself as the loss of the homogeneity of time.

In this work, we deeply reformulate the mathematical framework of [9] and, after polishing it from some formal imprecisions, we propose the following novelties: (a) we retrieve Equation (1) within the two different pictures of growth given in [3] and [4], respectively; (b) for both pictures, we compute *explicitly* the internal time, τ, and we show that the quantity $\tau_{\mathrm{c}} := 1 - \tau/\tau_0$, where τ_0 is a reference value, is analogous to endochronic time in that it increases monotonically in time and may thus represent an intrinsic time-scale associated with growth; (c) within the formulation presented in [4], we describe mechanotransduction through the conceptually systematic approach of Theoretical Mechanics. Our results also apply to remodelling.

2. An overview on growth mechanics

We consider the simplest possible formulation of the volumetric growth of a body. In particular, we assume the body to

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be hyperelastic and we employ the Bilby-Kröner-Lee decomposition of the deformation gradient tensor, i.e., $\boldsymbol{F} = \boldsymbol{\Phi}\boldsymbol{K}$, so that the body's material response is described by the strain energy density function

$$\Psi(X,t) = \hat{\Psi}(\boldsymbol{F}(X,t), \boldsymbol{K}(X,t)) = J_{\boldsymbol{K}} \hat{\Psi}_\nu(\boldsymbol{\Phi}(X,t)), \tag{2}$$

where $\boldsymbol{\Phi} := \boldsymbol{F}\boldsymbol{K}^{-1}$ is the elastic part of the deformation gradient tensor, $\hat{\Psi}_\nu$ is the strain energy density expressed per unit volume of the body in its stress-free state, and $J_{\boldsymbol{K}} := \det \boldsymbol{K} > 0$.

In local form, and with respect to the body's reference configuration, $\mathscr{B}$, the mass balance law is given by $\dot{\varrho}_{\mathrm{R}} = \Pi$, where ϱ_{R} is the mass density of the body per unit volume of $\mathscr{B}$, the superimposed dot denotes partial differentiation with respect to time, and Π is the source or sink of mass that describes growth. As in [3, 10], we write $\varrho_{\mathrm{R}} = J_{\boldsymbol{K}}\varrho_\nu$, where ϱ_ν is the mass density of the body in its stress-free state, and we require the conditions

$$\frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = \tfrac{1}{2}\mathrm{tr}(\dot{\boldsymbol{C}}_{\boldsymbol{K}}\boldsymbol{C}_{\boldsymbol{K}}^{-1}) = \frac{\Pi}{J_{\boldsymbol{K}}\varrho_\nu} =: \Gamma, \tag{3}$$

where $\boldsymbol{C}_{\boldsymbol{K}} := \boldsymbol{K}^{\mathrm{T}}.\boldsymbol{K}$ is the metric tensor induced by $\boldsymbol{K}$, Γ measures the relative variation of ϱ_{R}, and ϱ_ν is regarded as a time independent field specified from the outset.

Within the quasi-static limit, and neglecting all inertial and long-range body forces, such as gravity, the local form of the momentum balance law reads

$$\mathrm{Div}\,\boldsymbol{T} = \boldsymbol{0}, \tag{4a}$$

$$\boldsymbol{T} = \frac{\partial\hat{\Psi}}{\partial\boldsymbol{F}} \circ (\boldsymbol{F},\boldsymbol{K}) = J_{\boldsymbol{K}}\left[\frac{\partial\hat{\Psi}_\nu}{\partial\boldsymbol{\Phi}} \circ \boldsymbol{\Phi}\right]\boldsymbol{K}^{-\mathrm{T}}, \tag{4b}$$

where Div is the material divergence operator and $\boldsymbol{T}$ is the first Piola-Kirchhoff stress tensor. The balance law (4a) should be regarded as an equation for the motion of the body, χ, whose partial derivatives define the components of $\boldsymbol{F}$. To determine $\boldsymbol{K}$, an additional, independent equation is needed.

2.1. Tensor $\boldsymbol{K}$ viewed as internal variable

The tensor field $\boldsymbol{K}$ shares several formal analogies with the inverse of the tensor field referred to as *"uniformity mapping"* in [3]. Hence, if $\boldsymbol{K}$ is regarded as an internal variable, the theory exposed in [3] can be employed to develop a criterion for determining an admissible evolution law for $\boldsymbol{K}$. In particular, by invoking the representation theorem for tensor-valued functions [11], it can be shown that, in the case of isotropy, $\boldsymbol{K}$ satisfies

$$\mathrm{sym}[\boldsymbol{C}_{\boldsymbol{K}}\boldsymbol{\Lambda}] = \sum_{n=0}^{2}(J_{\boldsymbol{K}})^{-n}\beta_n \boldsymbol{H}^n\boldsymbol{C}_{\boldsymbol{K}}, \tag{5}$$

where $\boldsymbol{\Lambda} := \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, $\boldsymbol{H}$ is Eshelby's stress tensor,

$$\boldsymbol{H} := \Psi\boldsymbol{I}^{\mathrm{T}} - \boldsymbol{F}^{\mathrm{T}}\boldsymbol{T} \equiv \boldsymbol{K}^{\mathrm{T}}\left(\frac{\partial\hat{\Psi}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F},\boldsymbol{K})\right), \tag{6}$$

and $\{\beta_n\}_{n=0}^{2}$ are to be expressed constitutively through functions of $J_{\boldsymbol{K}}$, Ψ, the three principal invariants of $\boldsymbol{H}$, and other quantities, possibly required by phenomenology.

In Equation (5), the convention $\boldsymbol{H}^0 = \boldsymbol{I}^{\mathrm{T}}$ is used, where $\boldsymbol{I}^{\mathrm{T}}$ is the transpose of the material identity tensor, $\boldsymbol{I}$. Moreover, because of isotropy, $\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{K}}$ is symmetric, and so is also $\boldsymbol{H}^2\boldsymbol{C}_{\boldsymbol{K}} = \boldsymbol{H}\boldsymbol{C}_{\boldsymbol{K}}\boldsymbol{H}^{\mathrm{T}}$ [12]. Finally, the functions $\{\beta_n\}_{n=0}^{2}$ have to comply with the dissipation inequality

$$\mathcal{D}_{\mathrm{IV}} = \Psi\,\mathrm{tr}(\boldsymbol{\Lambda}) - \boldsymbol{H} : \boldsymbol{\Lambda} + \mathcal{D}_{\mathrm{nc}} \geq 0. \tag{7}$$

Here, $\mathcal{D}_{\mathrm{nc}}$ is said to be the *"non-compliant"* contribution to the dissipation [13] and is attributed to processes accompanying growth but not explicitly accounted for in the model. Moreover, the subscript "IV" in $\mathcal{D}_{\mathrm{IV}}$ stands for "internal variable" to remark that in Equation (7) $\boldsymbol{K}$ is viewed as an internal variable.

In order to model the material inhomogeneities associated with growth, Epstein and Maugin [3] introduce a Lagrangian density function, $\mathcal{L}$, whose constitutive representation depends on material points and time through $\boldsymbol{K}$. Hence, within the quasi-static limit, in which the identification $\mathcal{L} = -\Psi$ applies, and by mimicking the theory of material uniformity [3], we can write

$$\mathcal{L} = \check{\mathcal{L}} \circ (\boldsymbol{F}, \mathcal{X}, \mathcal{T}) = \hat{\mathcal{L}} \circ (\boldsymbol{F},\boldsymbol{K}) = -\hat{\Psi} \circ (\boldsymbol{F},\boldsymbol{K}), \tag{8}$$

where $\mathcal{X} : \mathscr{B} \times \mathbb{R} \to \mathscr{B}$ and $\mathcal{T} : \mathscr{B} \times \mathbb{R} \to \mathbb{R}$ are auxiliary functions defined by $\mathcal{X}(X,t) = X$ and $\mathcal{T}(X,t) = t$, and introduced to account for the explicit dependence of $\check{\mathcal{L}}$ on material points and time [14], i.e.,

$$\mathcal{L}(X,t) = \check{\mathcal{L}}(\boldsymbol{F}(X,t), X, t) = -\hat{\Psi}(\boldsymbol{F}(X,t), \boldsymbol{K}(X,t)). \tag{9}$$

Equations (8) and (9) permit to determine the *time-like inhomogeneity force*, $\mathcal{F}_0$ (see also [15], where it is referred to as *"energy release rate"*), which, recalling the definition $\boldsymbol{\Lambda} := \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, reads

$$\begin{aligned} \mathcal{F}_0 :=& \frac{\partial\check{\mathcal{L}}}{\partial\mathcal{T}} \circ (\boldsymbol{F},\mathcal{X},\mathcal{T}) = -\left(\frac{\partial\hat{\Psi}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F},\boldsymbol{K})\right) : \dot{\boldsymbol{K}} \\ =& -\boldsymbol{H} : \boldsymbol{\Lambda} = \mathcal{D}_{\mathrm{IV}} - \mathcal{D}_{\mathrm{nc}} - \Psi\,\mathrm{tr}(\boldsymbol{\Lambda}). \end{aligned} \tag{10}$$

Thus, τ is determined by Equation (1) with $\mathcal{H} = \Psi$.

2.2. Tensor $\boldsymbol{K}$ viewed as kinematic variable

A different approach to the mechanics of growth is provided in [4], where the structural transformation of a body corresponds to the activation of *structural degrees of freedom* describing the body's internal kinematics. From this perspective, $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$ acquire the meaning of tensor-valued kinematic descriptors that, together with χ, $\boldsymbol{v} = \dot{\chi}$ and $\mathrm{Grad}\,\boldsymbol{v} = \dot{\boldsymbol{F}}$, define the overall kinematics of the body.

Restricting our considerations to a material of first grade in χ and zeroth grade in $\boldsymbol{K}$ [4], it is natural to define the body's configuration manifold as a suitable set of pairs $(\chi, \boldsymbol{K})$ describing the overall evolution of the body. Accordingly, the bundle of the body's virtual velocities is given by the set of triples $(\mathfrak{v}, \mathrm{Grad}\,\mathfrak{v}, \boldsymbol{Z})$ that represent all the admissible realisations of the generalised velocities associated with the "standard" motion, i.e., $\mathfrak{v}$ and $\mathrm{Grad}\,\mathfrak{v}$, and with the structural evolution, $\boldsymbol{Z}$, respectively.

By duality, it is natural to introduce the generalised forces expending virtual power on $\mathfrak{v}$, $\operatorname{Grad}\mathfrak{v}$, and $\boldsymbol{Z}$. Hence, the Principle of Virtual Powers, specialised here to the case of no external forces dual to $\mathfrak{v}$ (i.e., neither inertial nor body forces), reads

$$\int_{\mathscr{B}} \{\boldsymbol{T} : \operatorname{Grad}\mathfrak{v} + \boldsymbol{K}^{-\mathrm{T}}\boldsymbol{Y}_{\mathrm{i}} : \boldsymbol{Z}\} = \int_{\mathscr{B}} \boldsymbol{K}^{-\mathrm{T}}\boldsymbol{Y}_{\mathrm{e}} : \boldsymbol{Z}, \tag{11}$$

where $\boldsymbol{Y}_{\mathrm{i}}$ and $\boldsymbol{Y}_{\mathrm{e}}$ are an internal and an external generalised force dual to $\boldsymbol{K}^{-1}\boldsymbol{Z}$, respectively, and $\boldsymbol{Z}$ is the virtual counterpart of $\dot{\boldsymbol{K}}$. The strong form of (11) consists of the force balances

$$\operatorname{Div}\boldsymbol{T} = \boldsymbol{0}, \tag{12a}$$

$$\boldsymbol{Y}_{\mathrm{i}} = \boldsymbol{Y}_{\mathrm{e}}. \tag{12b}$$

To close the model, we prescribe $\boldsymbol{Y}_{\mathrm{i}}$ constitutively, in compliance with the dissipation inequality

$$\mathcal{D}_{\mathrm{KV}} = -\boldsymbol{H} : \boldsymbol{\Lambda} + \boldsymbol{Y}_{\mathrm{i}} : \boldsymbol{\Lambda} = \boldsymbol{Y}_{\mathrm{id}} : \boldsymbol{\Lambda} \geq 0, \tag{13}$$

where $\boldsymbol{Y}_{\mathrm{id}} := \boldsymbol{Y}_{\mathrm{i}} - \boldsymbol{H}$ is said to be the *dissipative part* of $\boldsymbol{Y}_{\mathrm{i}}$ [16, 4] and the subscript "KV" reminds that Equation (13) is obtained by regarding $\boldsymbol{K}$ as a kinematic variable.

In the sequel, we admit that $\boldsymbol{Y}_{\mathrm{id}}$ depends constitutively on $\boldsymbol{F}$, $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$, and, because of isotropy, we express such dependence as a function $\bar{\boldsymbol{Y}}_{\mathrm{id}}$ of $\boldsymbol{F}$, $\boldsymbol{C}_{\boldsymbol{K}}$ and $\dot{\boldsymbol{C}}_{\boldsymbol{K}}$, i.e., $\boldsymbol{Y}_{\mathrm{id}} = \bar{\boldsymbol{Y}}_{\mathrm{id}} \circ (\boldsymbol{F}, \boldsymbol{C}_{\boldsymbol{K}}, \dot{\boldsymbol{C}}_{\boldsymbol{K}})$. Thus, we rewrite (12b) as

$$\boldsymbol{Y}_{\mathrm{e}} - \bar{\boldsymbol{Y}}_{\mathrm{id}} \circ (\boldsymbol{F}, \boldsymbol{C}_{\boldsymbol{K}}, \dot{\boldsymbol{C}}_{\boldsymbol{K}}) = \boldsymbol{H}, \tag{14}$$

thereby obtaining the equation of "motion" for $\boldsymbol{K}$. To supply an explicit expression for $\bar{\boldsymbol{Y}}_{\mathrm{id}}$, we rewrite it as a function of $\boldsymbol{\Lambda}$, i.e., $\bar{\boldsymbol{Y}}_{\mathrm{id}} \circ (\boldsymbol{F}, \boldsymbol{C}_{\boldsymbol{K}}, \dot{\boldsymbol{C}}_{\boldsymbol{K}}) = \check{\boldsymbol{Y}}_{\mathrm{id}} \circ (\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{\Lambda})$, and we notice that, because of isotropy, the tensor $\boldsymbol{Y}_{\mathrm{e}} - \boldsymbol{Y}_{\mathrm{id}}$ in Equation (14) must have the same symmetry property as $\boldsymbol{H}$, i.e., $\boldsymbol{C}_{\boldsymbol{K}}^{-1}(\boldsymbol{Y}_{\mathrm{e}} - \boldsymbol{Y}_{\mathrm{id}}) = (\boldsymbol{Y}_{\mathrm{e}}^{\mathrm{T}} - \boldsymbol{Y}_{\mathrm{id}}^{\mathrm{T}})\boldsymbol{C}_{\boldsymbol{K}}^{-1}$. Here, without much loss of generality, we hypothesise that such property holds, independently, both for $\boldsymbol{Y}_{\mathrm{id}}$ and for $\boldsymbol{Y}_{\mathrm{e}}$, and, by further assuming $\check{\boldsymbol{Y}}_{\mathrm{id}}$ to be linear in $\boldsymbol{\Lambda}$, we prescribe (cf. e.g. [17, 18] and references therein)

$$\boldsymbol{C}_{\boldsymbol{K}}^{-1}[\check{\boldsymbol{Y}}_{\mathrm{id}} \circ (\boldsymbol{F}, \boldsymbol{K}, \boldsymbol{\Lambda})] = \mathbb{D} : \operatorname{sym}(\boldsymbol{C}_{\boldsymbol{K}}\boldsymbol{\Lambda}), \tag{15}$$

where $\mathbb{D}$ is a fourth-order tensor function given by

$$\mathbb{D} = 3J_{\boldsymbol{K}}\, d_{\mathrm{v}}\mathbb{K}^{\sharp} + 2J_{\boldsymbol{K}}\, d_{\mathrm{m}}\mathbb{M}^{\sharp}. \tag{16}$$

Here, d_{v} and d_{m} are scalar constitutive functions to be specified, $\mathbb{K}^{\sharp}$ and $\mathbb{M}^{\sharp}$ are defined as (analogous operators have been introduced in [19, 17])

$$\mathbb{K}^{\sharp} = \tfrac{1}{3}\boldsymbol{C}_{\boldsymbol{K}}^{-1} \otimes \boldsymbol{C}_{\boldsymbol{K}}^{-1}, \tag{17a}$$

$$\mathbb{M}^{\sharp} = \tfrac{1}{2}[\boldsymbol{C}_{\boldsymbol{K}}^{-1} \underline{\otimes} \boldsymbol{C}_{\boldsymbol{K}}^{-1} + \boldsymbol{C}_{\boldsymbol{K}}^{-1} \overline{\otimes} \boldsymbol{C}_{\boldsymbol{K}}^{-1}] - \mathbb{K}^{\sharp}, \tag{17b}$$

and the tensor products "$\underline{\otimes}$" and "$\overline{\otimes}$" are defined in [20]. By using the identity $\operatorname{sym}(\boldsymbol{C}_{\boldsymbol{K}}\boldsymbol{\Lambda}) = \frac{1}{2}\dot{\boldsymbol{C}}_{\boldsymbol{K}}$, we find (cf. [21])

$$\mathbb{D} : \tfrac{1}{2}\dot{\boldsymbol{C}}_{\boldsymbol{K}} = \boldsymbol{C}_{\boldsymbol{K}}^{-1}[\boldsymbol{Y}_{\mathrm{e}} - \boldsymbol{H}], \tag{18}$$

thereby supplying six independent differential equations in the six independent components of $\boldsymbol{C}_{\boldsymbol{K}}$. Moreover, we split Equation (18) into the two independent equations

$$J_{\boldsymbol{K}}\, d_{\mathrm{v}} \operatorname{tr}\left(\tfrac{1}{2}\dot{\boldsymbol{C}}_{\boldsymbol{K}}\boldsymbol{C}_{\boldsymbol{K}}^{-1}\right) = \tfrac{1}{3}\operatorname{tr}\boldsymbol{Y}_{\mathrm{e}} - \tfrac{1}{3}\operatorname{tr}\boldsymbol{H}, \tag{19a}$$

$$2J_{\boldsymbol{K}}\, d_{\mathrm{m}} \operatorname{dev}\left(\tfrac{1}{2}\dot{\boldsymbol{C}}_{\boldsymbol{K}}\boldsymbol{C}_{\boldsymbol{K}}^{-1}\right) = \operatorname{dev}\boldsymbol{Y}_{\mathrm{e}} - \operatorname{dev}\boldsymbol{H}. \tag{19b}$$

Once the external force $\boldsymbol{Y}_{\mathrm{e}}$ is identified, and $\boldsymbol{C}_{\boldsymbol{K}}$ is computed by solving (18), the term Γ in the mass balance law (3) is determined by $\Gamma = \operatorname{tr}\boldsymbol{\Lambda} = \frac{1}{2}\operatorname{tr}(\dot{\boldsymbol{C}}_{\boldsymbol{K}}\boldsymbol{C}_{\boldsymbol{K}}^{-1})$. Finally, $\mathcal{F}_0$ becomes

$$\mathcal{F}_0 = -\boldsymbol{H} : \boldsymbol{\Lambda} = (\boldsymbol{Y}_{\mathrm{id}} - \boldsymbol{Y}_{\mathrm{e}}) : \boldsymbol{\Lambda}, \tag{20}$$

and the equation for τ takes on the form

$$\Psi\,\dot{\tau} - (\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v}) \operatorname{Grad}\tau + [(\boldsymbol{Y}_{\mathrm{e}} - \boldsymbol{Y}_{\mathrm{id}}) : \boldsymbol{\Lambda}]\tau = 0. \tag{21}$$

Before proceeding, we remark that Equation (15) is *not* the most general constitutive law relating $\boldsymbol{Y}_{\mathrm{id}}$ with $\boldsymbol{\Lambda}$, or $\dot{\boldsymbol{C}}_{\boldsymbol{K}}$. The main property of (15) is that, being invertible, if $\boldsymbol{\Lambda}$ is null, then $\boldsymbol{Y}_{\mathrm{id}}$ is null too, thereby implying $\boldsymbol{Y}_{\mathrm{e}} = \boldsymbol{H}$. Moreover, due to invertibility, it is true that, when $\boldsymbol{Y}_{\mathrm{id}}$ is null, also $\boldsymbol{\Lambda}$ has to vanish, which means that the balance between $\boldsymbol{Y}_{\mathrm{e}}$ and $\boldsymbol{H}$ leads to a stop of the growth process. However, in the case of a tumour, this last result need not be true (see e.g. [10]), as it may well happen that, if no nutrients are available for the tumour cells, $\boldsymbol{\Lambda}$ vanishes also when $\boldsymbol{Y}_{\mathrm{id}}$ is not null, a situation that, according to Equation (19a), requires d_{v} to diverge for finite values of $\boldsymbol{Y}_{\mathrm{id}} := \boldsymbol{Y}_{\mathrm{e}} - \boldsymbol{H}$.

3. A Noether-like framework

Equation (1) can be obtained by framing growth within a Noether-like approach. To show this, we introduce the *action*

$$\mathcal{A} := \int_{\mathscr{B}\times\mathscr{I}} \mathcal{L},$$

where $\mathscr{I} \subseteq [0, +\infty[$ is an interval of time, and the notation $\int_{\mathscr{B}\times\mathscr{I}} f \equiv \int_{\mathscr{I}} \left\{\int_{\mathscr{B}} f \,\mathrm{d}V\right\} \mathrm{d}t$ applies.

3.1. $\boldsymbol{K}$ considered as internal variable: internal time

When $\boldsymbol{K}$ is regarded as an internal variable, the Lagrangian density function is defined in Equation (9), and the first-order total variation of the action reads

$$D\mathcal{A} = \int_{\mathscr{B}\times\mathscr{I}} \left[\boldsymbol{\mathcal{E}}\boldsymbol{h} + \operatorname{Div}(-\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W} - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{u})\right], \tag{22}$$

where $\boldsymbol{W}$ is a vector field, valued in the tangent bundle of $\mathscr{B}$, that at each time t maps the points X of $\mathscr{B}$ into $\tilde{X} = X + \varepsilon\boldsymbol{W}(X, t)$, with ε being a real smallness parameter, $\boldsymbol{h}$ is the vector field describing the variation of χ when the points X are held fixed, $\boldsymbol{u} := \boldsymbol{h} + \boldsymbol{F}\boldsymbol{W}$ is the vector field representing the *total variation* of χ, and $\boldsymbol{\mathcal{E}}\boldsymbol{h} = \mathcal{E}_a h^a$ is the contraction of the co-vector field $\boldsymbol{\mathcal{E}} := \operatorname{Div}\boldsymbol{T}$ with $\boldsymbol{h}$ (see [14] for a derivation in a notation similar to that adopted here). In addition, we denote by $\boldsymbol{\mathfrak{J}} := -\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W} - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{u}$

Noether's current density, which is the sum of a fully material current density, $\mathfrak{J}^{(\mathrm{m})} = -\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W}$, and a "spatial" current density, $\mathfrak{J}^{(\mathrm{s})} = -\boldsymbol{T}^{\mathrm{T}}\boldsymbol{u}$ (note that, although $\mathfrak{J}^{(\mathrm{s})}$ is a material field too, we call it "spatial" because it is generated by the spatial vector field $\boldsymbol{u}$).

Upon setting $\boldsymbol{W} = \boldsymbol{0}$ in $\mathscr{B}$ and $\boldsymbol{h}_{|\partial\mathscr{B}} = \boldsymbol{u}_{|\partial\mathscr{B}} = \boldsymbol{0}$ for all times, Hamilton's Principle of Stationary Action [22] requires $D\mathcal{A} = 0$, which leads to $\boldsymbol{\mathcal{E}} = \mathrm{Div}\,\boldsymbol{T} = \boldsymbol{0}$ in $\mathscr{B}$ and $\boldsymbol{T}.\boldsymbol{N} = \boldsymbol{0}$ on $\partial_N\mathscr{B}$, where $\boldsymbol{N}$ is the field of unit vectors normal to the Neumann boundary of $\mathscr{B}$, $\partial_N\mathscr{B}$.

For χ and $\boldsymbol{K}$ satisfying $\boldsymbol{\mathcal{E}} = \boldsymbol{0}$, we look at Equation (22) under the light shed by Noether's Theorem [23]. Hence, we search for conservation laws, and we obtain [9]

$$\mathrm{Div}\,\mathfrak{J}^{(\mathrm{s})} = -\boldsymbol{T} : \mathrm{Grad}\,\boldsymbol{u}, \tag{23a}$$

$$\mathrm{Div}\,\mathfrak{J}^{(\mathrm{m})} = \mathfrak{F}\boldsymbol{W} - \boldsymbol{H} : \mathrm{Grad}\,\boldsymbol{W} =: \mathcal{N}(\boldsymbol{W}), \tag{23b}$$

where $\mathfrak{F} := \frac{\partial\check{\mathcal{L}}}{\partial\mathcal{X}} \circ (\boldsymbol{F}, \mathcal{X}, \mathcal{T}) = -[\frac{\partial\hat{\Psi}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K})] : \mathrm{Grad}\,\boldsymbol{K}$ is referred to as *"material inhomogeneity force"* [24, 25, 7] and $\mathfrak{F}\boldsymbol{W} = \mathcal{F}_A W^A$. We remark that, more generally, the integrand in Equation (22) should feature a summand consisting of the divergence of a vector field independent of $\boldsymbol{F}$, and descending from the so-called *"divergence transformation"* of the Lagrangian density function [23, 7, 14]. However, as in [23], this summand can be omitted for the type of symmetries addressed here.

In Equation (23a), $\boldsymbol{T} : \mathrm{Grad}\,\boldsymbol{u}$ vanishes identically in three cases: when $\boldsymbol{u}$ is null, when $\boldsymbol{u}$ represents a uniform translation, or when $\boldsymbol{u}$ takes on the form $\boldsymbol{u} = \boldsymbol{g}^{-1}\boldsymbol{\omega}[\chi - x_0]$, where $\boldsymbol{\omega}$ is a uniform skew-symmetric tensor, x_0 is a fixed point of space and $\boldsymbol{g}^{-1}$ is the inverse of the spatial metric tensor, $\boldsymbol{g}$. The second case is consistent with the fact that $\check{\mathcal{L}}$ is independent of χ, so that the system is invariant under translations in space and, thus, linear momentum is conserved. The third case, instead, stems from the symmetry of $\boldsymbol{g}^{-1}\boldsymbol{T}\boldsymbol{F}^{\mathrm{T}}$, which ensures $\boldsymbol{T} : \mathrm{Grad}\,\boldsymbol{u} = (\boldsymbol{g}^{-1}\boldsymbol{T}\boldsymbol{F}^{\mathrm{T}}) : \boldsymbol{\omega} = 0$ and is equivalent to the conservation of angular momentum. In conclusion, for the mentioned choices of $\boldsymbol{u}$, $\mathrm{Div}\,\mathfrak{J}^{(\mathrm{s})}$ is zero, which implies that $\mathfrak{J}^{(\mathrm{s})}$ is conserved.

We turn now to Equation (23b), and we notice that it is obtained by using the relation $-\mathrm{Div}\,\boldsymbol{H} = \mathfrak{F}$. This result follows from the computation of the divergence of $\boldsymbol{H}$, and characterises the *fully material* force balance describing the *"inverse dynamics"* of the body [7, 3]. It stipulates that the "spatial" part of the body's energy-momentum tensor, $-\boldsymbol{H}$, is not conserved. This is a manifestation of the symmetry breaking due to the material inhomogeneity of the body, reflected by $\mathcal{N}(\boldsymbol{W})$. This quantity plays the role of an *effective* source term for $\mathfrak{J}^{(\mathrm{m})}$ [9] and is such that the variation of the action becomes $D\mathcal{A} = \int_{\mathscr{B}\times\mathscr{I}} \mathcal{N}(\boldsymbol{W})$. Therefore, in order to search for the class of fields $\boldsymbol{W}$ such that $\mathfrak{J}^{(\mathrm{m})}$ is conserved and the action is invariant, i.e., $D\mathcal{A} = 0$, one has to impose [9]

$$\begin{aligned}\mathcal{N}(\boldsymbol{W}) &= -\boldsymbol{H} : \mathrm{Grad}\,\boldsymbol{W} + \mathfrak{F}\boldsymbol{W} \\ &= -\boldsymbol{H} : \left[\mathrm{Grad}\,\boldsymbol{W} + (\boldsymbol{K}^{-1}\mathrm{Grad}\,\boldsymbol{K})\boldsymbol{W}\right] = 0.\end{aligned} \tag{24}$$

We remark that relations of the type (24) are sometimes referred to as *"Noetherian identities"* [26].

Apart from the trivial solution $\boldsymbol{W} = \boldsymbol{0}$, a uniform field $\boldsymbol{W}$ does not generally satisfy Equation (24) and, thus, the action is not invariant under uniform translations of the material points. This result is another evidence of the symmetry breaking emerging because of $\mathfrak{F}$. Clearly, if $\boldsymbol{K}$ is uniform, so that $\mathrm{Grad}\,\boldsymbol{K} = \boldsymbol{0}$, then $\boldsymbol{W}$ can be uniform too. When this occurs, $\mathfrak{F}$ vanishes identically and, in the jargon of [7], one obtains the conservation of *"canonical pseudo-momentum"*. Let us now look at the identity

$$\dot{\Psi} - \mathrm{Div}(\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v}) = -\mathcal{F}_0, \tag{25}$$

which is the *non*-conservation of energy for $\mathcal{H} = \Psi = -\mathcal{L}$ (i.e., in the quasi-static limit), and let us multiply Equation (25) by a scalar field $\tau : \mathscr{B} \times \mathscr{I} \to \mathbb{R}$ describing a point- and time-dependent *deformation* of time [9]. Then, recalling the definition of $\mathcal{F}_0$ given in (10), we find (cf. [7])

$$\begin{aligned}&\dot{\overline{\Psi\tau}} + \mathrm{Div}(-\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v}\tau) \\ &\quad = \Psi\dot{\tau} - (\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v})\,\mathrm{Grad}\,\tau + (\boldsymbol{H} : \boldsymbol{\Lambda})\tau =: \mathcal{N}_0(\tau).\end{aligned} \tag{26}$$

By analogy with Equation (23b), we call $\mathcal{N}_0(\tau)$ *effective source* of Noether's energy current density, defined by the time-like component $\Psi\tau$ and the flux vector $-\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v}\tau$. As noticed for $\mathcal{N}(\boldsymbol{W})$, the presence of $\mathcal{F}_0 = -\boldsymbol{H} : \boldsymbol{\Lambda}$ implies that $\mathcal{N}_0(\tau)$ does not vanish for nonzero constant fields τ. Hence, to conserve Noether's energy current density, we enforce the condition anticipated by Equation (1), i.e.,

$$\mathcal{N}_0(\tau) = \Psi\dot{\tau} - (\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v})\mathrm{Grad}\,\tau + (\boldsymbol{H} : \boldsymbol{\Lambda})\tau = 0, \tag{27}$$

in which $\boldsymbol{H} : \boldsymbol{\Lambda}$ is now regarded as the generator of τ.

*3.2. **K** considered as a kinematic variable: internal time*

Equations (6), (8) and (14) allow to rephrase the force balances (12a) and (12b) as

$$\mathrm{Div}\,\boldsymbol{T} \equiv -\mathrm{Div}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K})\right) = \boldsymbol{0}, \tag{28a}$$

$$-\boldsymbol{H} \equiv \boldsymbol{K}^{\mathrm{T}}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K})\right) = \boldsymbol{Y}_{\mathrm{id}} - \boldsymbol{Y}_{\mathrm{e}}. \tag{28b}$$

Looking at (28b), we notice that a relevant case occurs when there exists a potential $\mathcal{U} = \hat{\mathcal{U}} \circ (\boldsymbol{F}, \boldsymbol{K})$ such that

$$\frac{\partial\hat{\mathcal{U}}}{\partial\boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K}) = \boldsymbol{0}, \quad \frac{\partial\hat{\mathcal{U}}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}) = \boldsymbol{K}^{-\mathrm{T}}\boldsymbol{Y}_{\mathrm{e}}, \tag{29}$$

where the first requirement of Equation (29) prevents $\hat{\mathcal{U}}$ from introducing an unphysical contribution to $\boldsymbol{T}$. Thus, Eqs. (28a) and (28b) become

$$-\mathrm{Div}\left(\frac{\partial\hat{\mathcal{L}}_{\mathrm{eff}}}{\partial\boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K})\right) = \boldsymbol{0}, \tag{30a}$$

$$\boldsymbol{K}^{\mathrm{T}}\left(\frac{\partial\hat{\mathcal{L}}_{\mathrm{eff}}}{\partial\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K})\right) = \boldsymbol{Y}_{\mathrm{id}}, \tag{30b}$$

with $\mathcal{L}_{\rm eff} := \mathcal{L} + \mathcal{U}$ being referred to as *effective Lagrangian density function*. Note that, although Equation (29) may be too restrictive for biologically meaningful situations, it is possible to think of $\boldsymbol{Y}_{\rm e}$ as the sum of an integrable and a non-integrable force, with the former one admitting a potential like $\hat{\mathcal{U}}$. For this reason, in this work we concentrate on the limiting case in which $\boldsymbol{Y}_{\rm e}$ is integrable.

By defining the effective action, $\mathcal{A}_{\rm eff} = \int_{\mathscr{B}\times\mathscr{I}} \mathcal{L}_{\rm eff}$, the first-order total variation of $\mathcal{A}_{\rm eff}$ is given by

$$D\mathcal{A}_{\rm eff} = \int_{\mathscr{B}\times\mathscr{I}} \left[\boldsymbol{K}^{-\rm T}\boldsymbol{Y}_{\rm id} : \boldsymbol{\Upsilon} + {\rm Div}(\boldsymbol{\mathfrak{J}}^{(\rm s)} + \boldsymbol{\mathfrak{J}}^{(\rm m)}_{\rm eff})\right], \tag{31}$$

with $\boldsymbol{\mathfrak{J}}^{(\rm s)} = -\boldsymbol{T}^{\rm T}\boldsymbol{u}$, $\boldsymbol{\mathfrak{J}}^{(\rm m)}_{\rm eff} = -\boldsymbol{H}^{\rm T}_{\rm eff}\boldsymbol{W}$, the effective Eshelby stress tensor $\boldsymbol{H}_{\rm eff} = -(\mathcal{L}_{\rm eff}\boldsymbol{I}^{\rm T} + \boldsymbol{F}^{\rm T}\boldsymbol{T})$ and $\boldsymbol{\Upsilon}$ being the variation of $\boldsymbol{K}$ when the points X are "held fixed".

Upon taking ${\rm Div}\,\boldsymbol{\mathfrak{J}}^{(\rm s)} = 0$, as done in Section 3.1, a direct calculation yields

$${\rm Div}\,\boldsymbol{\mathfrak{J}}^{(\rm m)}_{\rm eff} = \boldsymbol{\mathfrak{F}}_{\rm eff}\boldsymbol{W} - \boldsymbol{H}_{\rm eff} : {\rm Grad}\,\boldsymbol{W}, \tag{32}$$

where we call $\boldsymbol{\mathfrak{F}}_{\rm eff} := \left(\boldsymbol{K}^{-\rm T}\boldsymbol{Y}_{\rm id} : {\rm Grad}\,\boldsymbol{K}\right)$ *effective inhomogeneity force*, and Equation (31) reduces to

$$D\mathcal{A}_{\rm eff} = \int_{\mathscr{B}\times\mathscr{I}} [\boldsymbol{K}^{-\rm T}\boldsymbol{Y}_{\rm id} : \boldsymbol{Q} - \boldsymbol{H}_{\rm eff} : {\rm Grad}\,\boldsymbol{W}], \tag{33}$$

with $\boldsymbol{Q} := \boldsymbol{\Upsilon} + ({\rm Grad}\boldsymbol{K})\boldsymbol{W}$ being the total variation of $\boldsymbol{K}$. If we set $\boldsymbol{W} = \boldsymbol{0}$, Equation (33) returns Rayleigh-Hamilton Principle [22, 27], which states that the first-order variation of the action is equal to the integral of the work $\boldsymbol{K}^{-\rm T}\boldsymbol{Y}_{\rm id} : \boldsymbol{Q}$. Thus, if we reinterpret Equation (33) on the basis of this result, we find that the class of fields $\boldsymbol{W}$ satisfying $D\mathcal{A}_{\rm eff} = \int_{\mathscr{B}\times\mathscr{I}} \boldsymbol{K}^{-\rm T}\boldsymbol{Y}_{\rm id} : \boldsymbol{Q}$ is given by all the solutions of the equation

$$-\boldsymbol{H}_{\rm eff} : {\rm Grad}\,\boldsymbol{W} = 0. \tag{34}$$

In contrast to (24), Equation (34) is satisfied by nontrivial uniform fields $\boldsymbol{W}$. To see the implications of this result, let us consider the situation in which $\boldsymbol{Y}_{\rm id}$ is null. Hence, it follows that $\boldsymbol{H} = \boldsymbol{Y}_{\rm e}$, ${\rm Div}\,\boldsymbol{\mathfrak{J}}^{(\rm m)}_{\rm eff} = -\boldsymbol{H}_{\rm eff} : {\rm Grad}\,\boldsymbol{W}$, and Equation (33) becomes $D\mathcal{A}_{\rm eff} = \int_{\mathscr{B}\times\mathscr{I}} [-\boldsymbol{H}_{\rm eff} : {\rm Grad}\,\boldsymbol{W}]$. In this case, uniform fields $\boldsymbol{W}$ leave the action invariant, i.e., $D\mathcal{A}_{\rm eff} = 0$, and represent symmetry transformations. This constitutes a symmetry restoration and is due to the fact that, since $\boldsymbol{Y}_{\rm id}$ is null, $\boldsymbol{H}$ is entirely "balanced" by $\boldsymbol{Y}_{\rm e}$, which plays the role of compensating field. In fact, this results follows from Equation (30b), which, for $\boldsymbol{Y}_{\rm id} = \boldsymbol{0}$, implies $\boldsymbol{\mathfrak{F}}_{\rm eff} := \left(\frac{\partial \hat{\mathcal{L}}_{\rm eff}}{\partial \boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K})\right) : {\rm Grad}\,\boldsymbol{K} = \boldsymbol{0}$ even though it holds that $\boldsymbol{\mathfrak{F}} = -\boldsymbol{H} : \boldsymbol{K}^{-1}{\rm Grad}\,\boldsymbol{K} \neq \boldsymbol{0}$.

As done in Section 3.1, we consider the identity

$$\dot{\Psi}_{\rm eff} - {\rm Div}(\boldsymbol{T}^{\rm T}\boldsymbol{v}) = -\boldsymbol{Y}_{\rm id} : \boldsymbol{\Lambda} = -\mathcal{D}_{\rm KV}, \tag{35}$$

where $\Psi_{\rm eff} := -\mathcal{L}_{\rm eff}$ denotes the effective energy density associated with the body and, by multiplying (35) by τ, we obtain

$$\begin{aligned} &\overline{\Psi_{\rm eff}\tau}^{\boldsymbol{\cdot}} + {\rm Div}(-\boldsymbol{T}^{\rm T}\boldsymbol{v}\tau) \\ &\quad = \Psi_{\rm eff}\dot{\tau} - (\boldsymbol{T}^{\rm T}\boldsymbol{v})\,{\rm Grad}\,\tau - \mathcal{D}_{\rm KV}\,\tau =: \mathcal{N}_{0\rm eff}(\tau). \end{aligned} \tag{36}$$

Equation (35) describes the non-conservation of $\Psi_{\rm eff}$, while Equation (36) defines $\mathcal{N}_{0\rm eff}(\tau)$ as the effective source of Noether's energy current density with time-like component $\Psi_{\rm eff}\tau$ and flux vector $-\boldsymbol{T}^{\rm T}\boldsymbol{v}\tau$. Hence, to conserve Noether's energy current density, the condition

$$\mathcal{N}_{0\rm eff}(\tau) = \Psi_{\rm eff}\dot{\tau} - (\boldsymbol{T}^{\rm T}\boldsymbol{v})\,{\rm Grad}\,\tau - \mathcal{D}_{\rm KV}\,\tau = 0 \tag{37}$$

has to be imposed. Equation (37) prescribes that $\mathcal{D}_{\rm KV}$ is the generator of τ. Therefore, $\mathcal{D}_{\rm KV}$ can be thought of as an *effective time-like inhomogeneity force*, i.e., $\mathcal{F}_{0\rm eff} := \mathcal{D}_{\rm KV}$, which vanishes in the non-dissipative limit. If this is the case, a constant field τ satisfies $\mathcal{N}_{0\rm eff}(\tau) = 0$ and, consequently, Eq. (36) and (37) is satisfied as a conservation law. This is a crucial difference with Equations (21) and (27), in which the generator of τ is given by $-\mathcal{F}_0 = \boldsymbol{Y}_{\rm e} - \boldsymbol{Y}_{\rm id} = \boldsymbol{H} : \boldsymbol{\Lambda}$ and need not vanish even when the dissipation is zero.

4. A proof of concept

To supply a proof of concept of the theory discussed so far, we take a benchmark problem from [10]. Specifically, we study a tumour modelled as a monophasic, isotropic, solid body of cylindric shape, confined by an undeformable lateral wall, and allowed to expand uniformly along its axial direction, with traction-free terminal cross sections. Moreover, we assume the growth tensor, $\boldsymbol{K}$, to be spherical. By using cylindrical coordinates, these hypotheses imply that the only nonzero component of the velocity, $\boldsymbol{v}$, is the axial one, v^z, and that $\boldsymbol{F}$, $\boldsymbol{K}$, $\boldsymbol{\Lambda} = \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, $\boldsymbol{T}$ and $\boldsymbol{H}$ admit the diagonal matrix representations

$$[F] = {\rm diag}\{1, 1, \mathfrak{f}\}, \tag{38a}$$

$$[K] = k\,{\rm diag}\{1, 1, 1\}, \tag{38b}$$

$$[\Lambda] = k^{-1}\dot{k}\,{\rm diag}\{1, 1, 1\}, \tag{38c}$$

$$[T] = {\rm diag}\{T_r{}^R, T_\varphi{}^\Phi, T_z{}^Z\}, \tag{38d}$$

$$[H] = {\rm diag}\{\Psi - T_r{}^R, \Psi - T_\varphi{}^\Phi, \Psi - \mathfrak{f}\,T_z{}^Z\}. \tag{38e}$$

We remark that, since ${\rm Div}\,\boldsymbol{T} = \boldsymbol{0}$ reduces to $\partial T_z{}^Z/\partial Z = 0$, and the terminal cross sections of the body are free of tractions [10], $T_z{}^Z$ is zero at all the points of the tumour. This implies that the energy flux $\boldsymbol{T}^{\rm T}\boldsymbol{v}$ vanishes identically, i.e., $\boldsymbol{T}^{\rm T}\boldsymbol{v} = T_z{}^Z v^z = 0$. Moreover, as in [10], we adopt the Blatz-Ko strain energy density

$$\Psi = J_{\boldsymbol{K}}\,\tfrac{1}{2}\mu\left[(I_1 - 3) - \tfrac{1}{q/2}(I_3{}^{q/2} - 1)\right], \tag{39}$$

with $I_1 = {\rm tr}(\boldsymbol{C}\boldsymbol{C}_{\boldsymbol{K}}^{-1})$, $I_3 = J_{\boldsymbol{K}}^{-2}{\rm det}\boldsymbol{C}$ and material constants $\mu > 0$ and $q < 0$. Due to Equation (39), the constitutive expression of $T_z{}^Z$ is such that [10]

$$T_z{}^Z = \mu\frac{k^3}{\mathfrak{f}}\left[\frac{\mathfrak{f}^2}{k^2} - \left(\frac{\mathfrak{f}}{k^3}\right)^q\right] = 0 \;\Rightarrow\; \mathfrak{f} = k^{\frac{2-3q}{2-q}}. \tag{40}$$

Therefore, any constitutive function of $\mathfrak{f}$ and k can be rephrased as a function of k alone. For, example, in the case of Eshelby stress, one has $\boldsymbol{H} = \hat{\boldsymbol{H}}(\mathfrak{f}, k) \equiv \boldsymbol{\mathfrak{H}}(k)$ and

$$\mathfrak{H}(k) := \tfrac{1}{3}{\rm tr}\,\boldsymbol{\mathfrak{H}}(k) = \Psi - \tfrac{1}{3}(T_r{}^R + T_\varphi{}^\Phi) = \tfrac{1}{3}{\rm tr}\,\boldsymbol{H}. \tag{41}$$

First, we consider the case in which $\boldsymbol{K}$ is an internal variable [3] and we refer to this model as "IV Model". We notice that, in order to recover the growth law proposed in [10] from Equation (5), we have to set $\beta_n = 0$, for $n \neq 0$, thereby obtaining

$$\dot{k} = \beta_0 k, \quad \beta_0 = \tfrac{1}{3}\Gamma, \tag{42}$$

where, in general, β_0 depends on mechanical stress through the principal invariants of $\boldsymbol{H}$. However, if β_0 is assumed to be a positive constant, and if the initial distribution of k, denoted by k_{in}, is independent of material points, k is uniform and increases exponentially in time [10], i.e., $k(t) = k_{\text{in}} \exp(\beta_0 t)$ (see the line marked with triangles, and referred to as "IV Model", in Fig. 1). Moreover, according to Equation (40), also $\mathfrak{f}$ is independent of material points. In the case under study, the material inhomogeneity force $\mathfrak{F}$ is null, so that uniform fields $\boldsymbol{W} = \boldsymbol{W}_0$ satisfy Equation (24) and, since the identity $\boldsymbol{H} : \boldsymbol{\Lambda} = \dot{\Psi}$ holds true, Equation (27) becomes

$$\mathcal{N}_0(\tau) = \Psi\dot{\tau} + \dot{\Psi}\tau = \dot{\overline{\Psi\tau}} = 0. \tag{43}$$

Coherently with Equation (26), this result implies that the time-like component of Noether's current density, $\Psi\tau$, is conserved, and the internal time is given by

$$\Psi(t)\tau(t) = \Psi_0\tau_0 \;\Rightarrow\; \tau(t) = \frac{\tau_0\Psi_0}{\Psi(t)}, \tag{44}$$

where Ψ_0 and τ_0 are reference constant values, and $\Psi(t)$ is rescaled so that $\Psi(0) = \Psi_0$. The trend of τ is reported in Fig. 2 and corresponds to the solid line marked with triangles and referred to as "τ/τ_0 IV Model". The product $\Psi_0\tau_0$ defines the negative of a reference value of the action, i.e., $\mathcal{A}_0 := -\Psi_0\tau_0$, which is invariant.

Now, we regard $\boldsymbol{K}$ as a kinematic variable [4] and we call this model "KV Model". In this case, the evolution of k is given by Equations (19a) and (19b), which yield

$$\frac{\dot{k}}{k} = \frac{1}{3k^3 d_{\text{v}}}\left[Y_{\text{e}} - \mathfrak{H}(k)\right], \tag{45}$$

with $Y_{\text{e}} := \frac{1}{3}\text{tr}\,\boldsymbol{Y}_{\text{e}}$ and $\text{dev}\,\boldsymbol{Y}_{\text{e}} = \boldsymbol{0}$. Within the present variational setting, we choose a constant Y_{e}, so that it can be obtained by differentiation of the potential $\hat{\mathfrak{U}} \circ (\boldsymbol{F}, \boldsymbol{K}) = Y_{\text{e}} \ln(\det \boldsymbol{K})$, and the numerical solution of Equation (45), obtained for constant d_{v}, is reported in Fig. 1 (see the solid line marked with open circles and referred to as "KV Model - Linear Case").

Since it holds true that $\boldsymbol{T}^{\text{T}}\boldsymbol{v} = \boldsymbol{0}$, Equation (35) prescribes $\mathcal{D}_{\text{KV}} = -\dot{\Psi}_{\text{eff}}$ and, consequently, Equation (37) becomes

$$\mathcal{N}_{0\text{eff}} = \Psi_{\text{eff}}\dot{\tau} + \dot{\Psi}_{\text{eff}}\tau = \dot{\overline{\Psi_{\text{eff}}\tau}} = 0. \tag{46}$$

Therefore, the internal time, τ, is given by

$$\Psi_{\text{eff}}(t)\tau(t) = \Psi_{\text{eff}0}\tau_0 \;\Rightarrow\; \tau(t) = \frac{\tau_0\Psi_{\text{eff}0}}{\Psi_{\text{eff}}(t)}, \tag{47}$$

with τ_0 and $\Psi_{\text{eff}0}$ being reference constants, and $\Psi_{\text{eff}}(t)$ rescaled so that $\Psi_{\text{eff}}(0) = \Psi_{\text{eff}0}$. In spite of the similarity with Equation (44), in the present case $\tau(t)$ depends on Y_{e}. Its evolution is shown in Fig. 2 and corresponds to the solid line marked with open circles.

5. Discussion

In the IV Model, the coefficient β_0 in Equation (42) is assumed to be constant. Although this choice may be too restrictive, it describes the limit case in which, to activate growth, it is sufficient that the nutrient substances in the tumour exceed a certain threshold. Clearly, more general models, which include the feedback of stress on growth (mechanotransduction), can be obtained by considering Equation (5) in full, or by expressing β_0 as a phenomenological function of the stress.

In the KV Model, which descends from Equation (15), (19a) and (45), k is coupled with $Y_{\text{id}} := \frac{1}{3}\text{tr}\,\boldsymbol{Y}_{\text{id}} = Y_{\text{e}} - \mathfrak{H}(k)$, rather than with stress alone, and this coupling may appear both directly, i.e., in the right-hand-side of Equation (45), and indirectly, i.e., through the coefficient d_{v}, which can be taken as a function of the principal invariants of $\boldsymbol{Y}_{\text{id}}$. To the best of our understanding, this could be a possible interpretation of the *"Eshelbian coupling"* mentioned in [4]. In this respect, we also notice that, even within our variational setting, mechanotransduction can be accounted for by suitably interpreting Y_{e}. This can be achieved by relating $\dot{k}/k$ to a term of the type [28, 29]

$$M(H) := 1 - \frac{c_0 H}{c_0 Y_{\text{e}} + H} = 1 - \frac{H}{Y_{\text{e}}} + o\left(\frac{H}{Y_{\text{e}}}\right), \tag{48}$$

where $c_0 \in]0, 1[$ is a model parameter and $H := \frac{1}{3}\text{tr}\,\boldsymbol{H} = \mathfrak{H}(k)$. By setting $M_{\text{lin}}(H) := 1 - H/Y_{\text{e}}$, Equation (45) can be rewritten as $\dot{k}/k = M_{\text{lin}}(H)/3k^3\bar{\tau}$, where $\bar{\tau}$ is a characteristic time scale and $d_{\text{v}} \equiv \bar{\tau}Y_{\text{e}}$. The solution to this equation, or, equivalently, to Equation (45), corresponds to the solid line marked with open circles in Fig. 1, where it is compared with the solution to the equation $\dot{k}/k = M(H)/3k^3\bar{\tau}$. The latter is represented by the solid line marked with triangles in Fig. 1, and refers to a phenomenological model in which the mechanotransduction term, $M(H)$, is not linearised. Looking at the magnified inset in Fig. 1, we notice that a constant and integrable Y_{e}, although being restrictive, leads to reasonable results for the first days in which the tumour grows, i.e., as long as the ratio H/Y_{e} remains sufficiently small. For longer times, however, the solution to Equation (45) ceases to be acceptable. Indeed, it tends towards a stationary value, corresponding to the force balance $Y_{\text{e}} = \mathfrak{H}(k)$, which contradicts the hypothesis $H/Y_{\text{e}} \to 0$. The solution of the nonlinear model, instead, keeps increasing in time, and is qualitatively closer to the dashed curve marked with open circles that describes the trend of k in the case of a reference model available in the literature [29].

The main result of this work is the introduction of the internal time, τ, that, for the considered benchmark problem, is obtained by solving Equation (44) for the IV Model and Equation (47) for the KV Model. The solutions, expressed in terms of the ratio τ/τ_0, are reported in Fig. 2 and correspond to the solid lines marked with asterisks and open circles, respectively. We notice that, since both Ψ and Ψ_{eff} increase with k, and since k increases with time, τ/τ_0 decreases monotonically for both models. In particular, since k is computed by solving Equation (45), which admits a stationary solution, τ/τ_0 reaches a plateau for long times, and the solution predicted by the IV Model tends to converge to the one supplied by the KV Model. Finally, we notice that the function $\tau_c = 1 - \tau(t)/\tau_0$ is monotonically increasing, and might thus be taken as a natural characteristic time scale of growth, just as the endochronic time in Plasticity [6].

6. Conclusions

In this work, we have studied a problem of volumetric growth in a continuum body within the quasi-static limit. In doing this, we have followed two paths: the one that views the growth tensor, $\boldsymbol{K}$, as an internal variable, and the one that defines $\boldsymbol{K}$ as a kinematic variable. We have cast the problem in a variational setting and we have employed the framework of Noether's Theorem in order to reveal some subtle implications of the two theories of growth exploited in the manuscript, especially in terms of material inhomogeneities and conservation laws. Hence, we have shown that Noether's current is not conserved, in general, for the classes of transformations that would represent material symmetries if the body were homogeneous. This has been reflected, in fact, by the condition $\mathcal{N}(\boldsymbol{W}) = 0$, imposed to annihilate the effective source of Noether's current [9].

We have focussed on the non-conservation of energy. This has led us to adopt the conditions $\mathcal{N}_0(\tau) = 0$ and $\mathcal{N}_{0\text{eff}}(\tau) = 0$, respectively, to search for transformations capable of defining a characteristic time scale for growth, termed *internal time*.

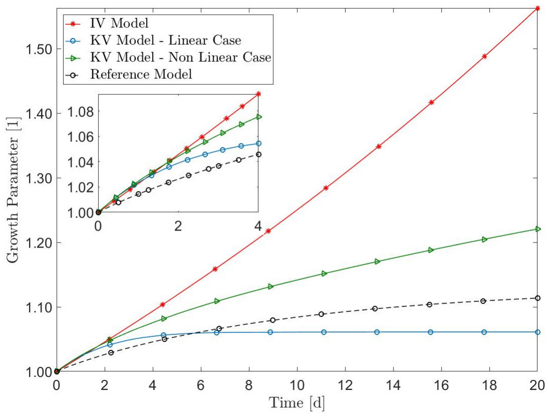

Figure 1: Time evolution of k. The model parameters are $\Gamma = 2.68 \cdot 10^{-2}\,\mathrm{s}^{-1}$, for the IV-model, and $c_0 = 0.7138$, $Y_e = 2.159\,\mathrm{kPa}$ and $\bar{\tau} = 10^6\,\mathrm{s}$, for the KV-model. For both models, we set $\mu = 1.999\,\mathrm{kPa}$, $q = -1$, $k_{in} = 1$.

Conflict of interests

The Authors declare no conflict of interests.

Acknowledgments

This work was supported, partially, by the *Dipartimento di Eccellenza 2018-2022*, Politecnico di Torino (Italy), Project no. E11G18000350001 [AG,SDiS] and, partially, by the Natural Sciences and Engineering Research Council of Canada, through the NSERC Discovery Programme [SF]. We thank Vito Napoli, Luigi Preziosi and Gabriel Wittum for useful discussions.

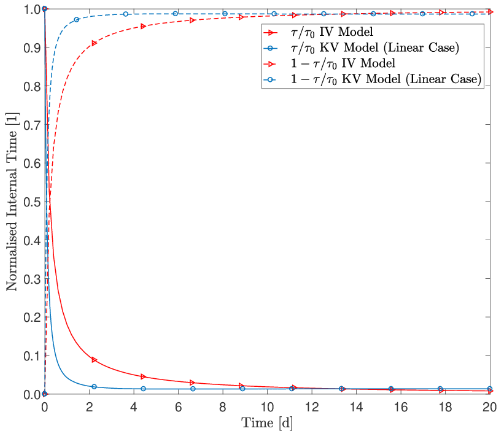

Figure 2: Time evolution of τ. The values of the model parameters are declared in the caption of Fig. 1.

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